Lecture notes

Geometry, Topology and Physics I

136.007 winter term 09/10

Maximilian Kreuzer Institut für Theoretische Physik, TU–Wien

+43 – 1 – 58801-13621 Wiedner Hauptstraße 8–10, A-1040 Wien

kreuzer@hep.itp.tuwien.ac.at http://hep.itp.tuwien.ac.at/~kreuzer/inc/gtp1.pdf

Contents

Bibliography

[AH60] L.V.Ahlfors, L.Sario, *Riemann surfaces* (Princeton Univ. Press, Princteon 1960)

[BO82] R.Bott, L.W.Tu, *Differential Forms in Algebraic Topology* (Springer, New York 1982)

[CH77] Y.Choquet-Bruhat, C.DeWitt-Morette, M.Dillard-Bleick, *Analysis, Manifolds and Physics,* 2nd ed. 1982 (North-Holland, Amsterdam 1977)

[CO98] D.A.Cox, J.Little, D.O'Shea, *Using Algebraic Geometry* (GTM, Springer 1998)

[CR78] F.H.Croom, *Basic Concepts of Algebraic Topology* (Springer, New York 1978)

[FA80] H.M.Farkas, I.Kra, *Riemann Surfaces,* GTM 71 (Springer, New York 1980)

[GO62] S.I.Goldberg, *Curvature and Homology,* Pure and Applied Mathematics Vol. 11 (Academic Press, New York 1962)

[KO63] S.Kobayashi, K.Nomizu, *Foundations of Differential Geometry*, vol I (Interterscience, New York 1963), vol II (Interterscience, New York 1969)

[MA91] W.S.Massey, *A basic course in algebraic topology* (Springer, New York 1991)

[NA83] C.Nash, S.Sen, *Topology and Geometry for Physicists* (Academic Press, London 1983)

[NA90] M.Nakahara, *Geometry, topology, and physics* (IOP Publishing, Bristol 1990)

[eg80] T.Eguchi, P.B.Gilkey, A.J.Hanson, *Gravitation, gauge theories and differential geometry,* Phys. Rep. **66** (1980) 213

Chapter 1

Topology

If we are interested in global aspects of geometry then concepts like distances or even smoothness are not important. What we investige in topology is just a very basic set of structures that allow us to identify global data of a space, like the number of holes of a surface, but also local properties, like the dimension of a manifold. Similarly to the definition of a smooth manifold, whose differentiable structure will be fixed by a consistent declaration of which functions we call differentiable, we might define the topological structure of a set of points by a consistend choice of which (real) functions on that set are declared to be continuous.

A somewhat more suggestive approach is to start with what we call a neighborhood of a point. As we will see the standard definition of a topological space, which uses the concept of open sets, is closely related to this, but is slightly more efficient as it only requires us to declare what subsets of a space X are open instead of declaring for each point $x \in X$ what subsets of X we call neighorhoods of x.

After introducing basic concepts like compactness and connectedness we will find some data that are topological invariants, so that they help us to decide whether two topological spaces are isomorphic (i.e. 'essentially the same'). Such data often can be equipped with an algebraic structure, like a group or a ring; then we enter realm of algebraic topology. Next we will turn to our main interest, which is manifolds, and show how algebraic operations allow us to classify the topology of surfaces. In the last part we consider homology groups and their dimensions, the Betti numbers of a manifold.

The mathematical symbols that we use include '$\wedge$' for the logical *and*, '$A \setminus B$' for set-theoretic *complement* of B in A, and '*iff*' for 'if and only if'.

1.1 Definitions

A **metric** on a set X of points is a non-negative function $d : X \times X \to \mathbb{R}$ satisfying

$$d(x,y) = d(y,x), \qquad d(x,y) > 0 \Leftrightarrow x \neq y, \qquad d(x,y) + d(y,z) \geq d(x,z). \tag{1.1}$$

$d(x,y)$ is called the distance between x and y. An (open) ε-neighborhood of a point $x \in X$ is the open ball $U_\varepsilon(x) = \{y \in X : d(x,y) < \varepsilon\}$ where $\varepsilon > 0$ is a positive real number. A function $f : X \to Y$ between two metric spaces (X,d) and (Y,d') is called **continuous** at the point $x \in X$ if there exists a positive number $\delta > 0$ for each $\varepsilon > 0$ such that $d'(f(x), f(p)) < \varepsilon$ for all $p \in X$ with $d(x,p) < \delta$. If f is continuous at all points it is called continous.
A subset $\mathcal{O} \subset X$ of a metric space is called **open** if each of its points has an ε-neighborhood that is contained in $\mathcal{O}$, i.e. if for each $x \in \mathcal{O}$ there exists a positive number ε with $U_\varepsilon(x) \subseteq \mathcal{O}$.

Examples: The discrete metric on a arbitrary set is $d(x,y) = 1$ for $x \neq y$ and $d(x,x) = 0$. For the n-sphere $S^n = \{\vec{x} \in \mathbb{R}^{n+1} \ : \ |\vec{x}| = 1\}$ we can consider the geodesic distance on S^n or the distance induced by the embedding in the Euclidean space $\mathbb{R}^{n+1}$ with its natural metric. $\mathbb{R}$ can be given a bounded metric by "pulling back" the metric from the circle $z = x + iy = e^{i\varphi}$ along the stereographic projection $y = \cot\frac{\varphi}{2}$.

Exercise 1: Show that a function $f : X \to Y$ between metric spaces is continuous *iff* the inverse image of every open set in Y is an open set in X, where the inverse image of a set $U \subset Y$ is the set of points x whose image $f(x)$ is in U.

The assertion of this exercise shows that the notion of continuity only depends on the system of open sets of a metric space and not of the actual distances between points. This fact can be taken as a motivation for the more abstract definition of a topological space, which waives the concept of a distance and only keeps the notion of open sets, whose collection we call *topology*:

A family $\mathcal{T}$ of subsets of a set X is called a **topology** on X if it contains X and the empty set, as well as finite intersections and arbitrary unions of elements of $\mathcal{T}$:

$$\textbf{Top1: } \emptyset \in \mathcal{T},\ X \in \mathcal{T}, \qquad \textbf{Top2: } \mathcal{O}_1 \cap \mathcal{O}_2 \in \mathcal{T}, \qquad \textbf{Top3: } \bigcup_{i \in I} \mathcal{O}_i \in \mathcal{T} \quad \forall\, \mathcal{O}_i \in \mathcal{T}. \tag{1.2}$$

The sets $\mathcal{O}_i$ are called **open** and their complements $\mathcal{A}_i = X \setminus \mathcal{O}_i$ are the **closed sets** of the topological space $(X, \mathcal{T})$.

Remarks: A set can be open *and* closed, like $\emptyset$ and X. Typically, however, most sets are *neither open nor closed* (like semi-open intervals in $\mathbb{R}$).
Note that finite unions and arbitrary intersections of closed sets are closed. We thus could have defined a topology using the "dual" axioms for closed sets and then defining the open sets as their complements. Sometimes proofs can be made much simpler by first dualizing the statement to an assertion for the complements of the relevant sets.

Examples: The power set $\mathcal{P}(X)$ and $\{\emptyset, X\}$ are called **discrete** and **indiscrete** topology, respectively. With every metric space there comes the **natural topology**, whose open sets are the unions of open balls. The natural topology on $\mathbb{R}^n$ is given by the unions of the open balls $U_\varepsilon(x) = \{y \in X : d(x,y) < \varepsilon\}$ for some $\varepsilon \in \mathbb{R}_+$. (Unless stated differently we will assume that $\mathbb{R}^n$ is equipped with the usual distance and the respective natural topology.)

Two topological spaces $(X_1, \mathcal{T}_1)$ and $(X_2, \mathcal{T}_2)$ are called homeomorphic iff there exists a bijection $f : X_1 \to X_2$ that induces a bijective map $f : \mathcal{T}_1 \to \mathcal{T}_2$ of the topologies. The map f is called a **homeomorphism**. A map $g : X \to X'$ is called **continuous** iff the sets $g^{-1}(\mathcal{O}') \in \mathcal{T}$ are open in $(X, \mathcal{T})$ for all open sets $\mathcal{O}' \in \mathcal{T}'$. Equivalently, g is continuous iff all inverse images of closed sets are closed. A bijective map h is a homeomorphism iff it is bi-continuous, i.e. iff h and h^{-1} are both continuous.

For each subset $M \subseteq X$ we can define the **induced topology**, which is given by the set of intersections $\mathcal{O}_i \cap M$ with $\mathcal{O}_i \in \mathcal{T}$. For two topological spaces $(A, \mathcal{T}_A)$ and $(B, \mathcal{T}_B)$ we can define the product topology on $A \times B$ by declaring the products of open sets to be open. These products thus form a basis of the product topology, i.e. every open set is a union of finite intersections of the basic open sets. (In fact, every subset of $\mathcal{P}(X)$ is a basis of the topology that it generates in the described way.) Similar definitions can be used to define the image or preimage topology with respect to some function if the domain or the image of the function is equipped with a topology, respectively.

The smallest closed subset of X that contains $M \subset X$ is called the **closure** $\overline{M}$ of M in X. The **interior** M^0 of M is defined to be the largest open set contained in M. ($\overline{M}$ and M^0 are given by the intersection/union of all closed/open sets containing/contained in M.) A subset $M \subset X$ is called **dense** in X iff $X = \overline{M}$. The difference $\partial M := \overline{M} \setminus M^0$ is called the **boundary** of M. A subset $\mathcal{U} \subset X$ is called a **neighborhood** of $x \in X$ iff x is contained in an open subset of $\mathcal{U}$. Note that a subset $\mathcal{S} \subseteq X$ is open iff it is a neighborhood of all of it points. (Using this correspondence, topological spaces may equivalently be defined by a set of axioms for neighborhoods, with open sets being a derived concept.)

A family of subsets $M_i \subseteq X$ is called a **covering** of X if $X = \bigcup M_i$. A topological space $(X, \mathcal{T})$ is called **Hausdorff space** iff any two points can be separated by neighborhoods (i.e. iff $\forall x \neq y \in X$ there exist $\mathcal{U}, \mathcal{V} \in \mathcal{T}$ with $x \in \mathcal{U} \wedge y \in \mathcal{V} \wedge \mathcal{U} \cap \mathcal{V} = \emptyset$). A subset $\mathcal{A} \subseteq X$ is called quasi-compact if each covering of $\mathcal{A}$ by open sets $\mathcal{O}_i$ contains a finite subcovering, it is called **compact** if it is quasi-compact and Hausdorff, and it is called **locally compact** if each point $x \in X$ has a compact neighborhood. More generally, any property of a topological space with the adjective 'locally' will mean that each point has a basis of neighorhoods with that property.

Example: Let $A^n = \mathbb{C}[x_1, \ldots, x_n]$ be the polynomial ring in n complex varibles and regard x_i

as coordinates of $\mathbb{C}^n$. For any subset $T \subseteq A^n$ of polynomials the vanishing set is defined as

$$Z(T) = \{x \in \mathbb{C}^n | P(x) = 0 \ \ \forall P \in T\} \tag{1.3}$$

A subset of $\mathbb{C}^n$ is called algebraic if it is equal to the set of solutions $Z(T)$ of some set T of polynomials. Unions $Z(T_1) \cup Z(T_2)$ of algebaic sets are algebraic, with $T = \{P_i(x) \cdot P_j(x) : P_i \in T_1, \ P_j \in T_2\}$, and arbitrary intersections are also algebraic. They can thus be taken to be the closed sets of a topology, called the **Zariski topology** $\mathcal{T}_Z$, which is frequently used in algebraic geometry. $(\mathbb{C}^n, Z(T))$ is not Hausdorff, but quasicompact because any algebraic set is the zero set of a *finite set* of polynomials, and all but a finite set of its defining polynomial equations are redundant (this is essentially the content of Hilbert's basis theorem [CO98]).

Theorems: • Compact subsets $\mathcal{A} \subseteq X$ of a Hausdorff space X are closed.
• Closed subspaces and continuous images of compact spaces are compact.
• Metric spaces M are compact iff every sequence in M contains a convergent subsequence.
• A subset of $\mathbb{R}^n$ is compact iff it is bounded and closed [Heine-Borel].

A compact topological space $(\tilde{X}, \tilde{\mathcal{T}})$ is called a **compactification** of $(X, \mathcal{T})$ if X is (homeomorphic to) a dense subset of $\tilde{X}$ and if $\mathcal{T}$ is the topology that is induced on X by $\tilde{\mathcal{T}}$. All locally compact spaces can be compactified in the following way by adding only a single point [Alexandroff]: Define $\tilde{X} = X \cup \{\omega\}$ with $\mathcal{O} \notin X$ and let $\tilde{\mathcal{T}}$ contain all open sets $\mathcal{O} \in \mathcal{T}$ and, in addition, all subsets $\tilde{\mathcal{O}} \subseteq \tilde{X}$ that contain ω and whose complement $\tilde{X} - \tilde{\mathcal{O}}$ is a compact subset of X. It can be checked that $\tilde{\mathcal{T}}$ is a topology on $\tilde{X}$ and that the resulting topological space is compact. (X is dense in $\tilde{X}$ if X is not compact. It can be shown that the one point compactification is unique up to homeomorphisms.)

Examples: The one point compactification of $\mathbb{R}^n$ is homeomorphic to the n-dimensional sphere S^n (consider, for example, the stereographic projection).
An alternative compactification of $\mathbb{R}^n$ is the (real) **projective space** $\mathbb{RP}^n$. It is the set of equivalence classes $[v] = \{\lambda v \ : \ v \in \mathbb{R}^{n+1} - \{0\} \ \wedge \ \lambda \in \mathbb{R}^*\}$ of non-vanishing vectors $v \in \mathbb{R}^{n+1}$ modulo scaling by non-vanishing real numbers λ. Taking a vector of length $|v| = 1$ as a representative of the class it is easy to see that $\mathbb{RP}^n$ is homeomorphic to the sphere S^n modulo the $\mathbb{Z}_2$ identification $v \to -v$. Alternatively, we can describe $\mathbb{RP}^n$ as the space of 1-dimensional linear subspaces[1] of $\mathbb{R}^{n+1}$. The points of projective space can be described by homogeneous coordinates $(x_0 : x_1 : \ldots : x_n)$, which are identified with $(\lambda x_0 : \lambda x_1 : \ldots : \lambda x_n)$ for $\lambda \neq 0$. The set $\mathcal{U}_i$ of points $(x_0 : x_1 : \ldots : 1 : x_{i+1} : x_n)$ that can be respresented by homogeneous coordinates with $x_i = 1$ for some $i \geq 0$ form a subspace that can be identified with $\mathbb{R}^n$ in a natural way. Since at least one of the homogeneous coordinates is nonzero we can alway scale one of the coordinates to 1. This shows that $\mathbb{RP}^n$ can be covered by $n + 1$ coordinate patches

[1] An generalization of this are the spaces of s-dimesional linear subspaces of $\mathbb{R}^{n+s}$, which are called **Grassmannian manifolds**. A further generalization are **flag manifolds**, whose points are chains of linear subspaces.

$\mathcal{U}_i$ that are isomorphic to $\mathbb{R}^n$. For each patch $\mathbb{RP}^n - \mathcal{U}_i$ is isomorphic to $\mathbb{RP}^{n-1}$. Projective space is therefore a disjoint union $\mathbb{R}^n \cup \mathbb{R}^{n-1} \cup \ldots \cup \mathbb{R}^1 \cup \mathbb{R}^0$ of affine spaces. Its topology can be defined as the set of unions of open sets in the patches $\mathcal{U}_i$.

Exercise 2: Show that S^n and $\mathbb{RP}^n$ are compactifications of $\mathbb{R}^n$. (For projective space use the fact that a finite union of compact subspaces is compact and that homogeneous coordinates can always be choses to have $|x_i| \leq 1$).

A topological space is called disconnected iff X can be decomposed into two disjoint open sets $\mathcal{O}_i \in \mathcal{T}$, $X = \mathcal{O}_1 \cup \mathcal{O}_2$, $\mathcal{O}_1 \cap \mathcal{O}_2 = \emptyset$ with $\mathcal{O}_i \neq \emptyset$ and $\mathcal{O}_i \neq X$. If such sets don't exist then $(X, \mathcal{T})$ is called **connected**. Since the closed sets are the complements of open sets we could replace all open sets by closed sets in this definition. There is a related notion, which is equivalent to connectedness except for 'pathological' situations. For this we first need the concept of a path in a topological space: A continuous map $f : I \to X$ from the closed interval $I = [0, 1]$ to X is called a **path** from $a \in X$ to $b \in X$ if $f(0) = a$ and $f(1) = b$. A topological space is called **arcwise connected** if any two points in X can be connected by a path. All arcwise connected spaces are connected (take a path from $x_1 \in \mathcal{O}_1$ to $x_2 \in \mathcal{O}_2$; then $f^{-1}(f(I) \cap \mathcal{O}_1$ would be open and closed in I). As a counterexample for the inverse direction consider the graph A of the function $y = \sin 1/x$ from $(0, 1)$ to $[-1, 1]$. The closure $\overline{A} = A \cup \{(0, y) : -1 \leq y \leq 1\}$ of A in $\mathbb{R}^2$ is connected, but not arcwise connected. A closed path, i.e. the case where $a = b$, is called a **loop** with base point a in X.

1.2 Homotopy

Continuous images of (arcwise) connected spaces are (arcwise) connected. Connectedness is therefore a **topological invariant**, i.e. a property that is invariant under homemorphisms. One important task of topology is to find useful topological invariants that characterize the topological properties of manifolds. These invariants may themselves have an algebraic structure (for example a group or a ring structure). Then we are in the realm of **algebraic topology** [MA91,BO82,CR78]. As a first example in this direction we will now consider the homotopy groups.

We are mainly interested in manifolds, which locally look like $\mathbb{R}^n$. Hence, the local topology is fixed and the interesting things happen globally, and should therefore be independent of deformations. We call two continous maps $f : X \to Y$ and $g : X \to Y$ **homotopic** if there exists a continuous map $F : X \times I \to Y$ with $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$ for all $x \in X$. For a pair of topological spaces X and Y we can now consider homotopy classes of continuos maps. Of particular interest for the construction of topological invariants is the case where $X = S^n$ is a sphere of some dimension n.

We first consider the most important case $n = 1$: We denote by $\pi_1(X, x_0)$ the set of homotopy classes $[a]$ of loops $a : I \to X$ in X with base point $x_0 = a(0) = a(1)$. We can define a product on this space in the following way:

$$[a] * [b] := [a * b], \qquad (a * b)(t) := \begin{cases} a(2t) & \text{if } 0 \le t \le \frac{1}{2} \\ b(2t-1) & \text{if } \frac{1}{2} \le t \le 1 \end{cases}. \tag{1.4}$$

This operation is associative (modulo homotopy!), the class of contractible loops (the homotopy class of the constant loop) acts as unit element, and an inverse is given by $a^{-1}(t) := a(1-t)$. Hence $\pi_1(X, x_0)$ is a group, which we call the **fundamental group** of X (with base point x_0). This group is, in general, not commutative. If X is arcwise connected it is easy to see that π_1 is independent of the base point. A topological space is called **simply connected** if it is connected and the fundamental group is trival (i.e. all loops are contractible).

A covering space of a topological space $(X, \mathcal{T})$ is a connected and locally connected [2] space $(\tilde{X}, \tilde{\mathcal{T}})$ together with a surjective map $\pi : \tilde{X} \to X$ with the property that for every point $x \in X$ there exists a neighborhood $U(x)$ such that π is a homeomorphism from $\tilde{U}$ to $U(x)$ for every connected component $\tilde{U}$ of $\pi^{-1}U(x)$ (loosely speaking, $\tilde{X}$ locally looks like X). It is an important theorem that for every connected and locally connected topological space X there exists a covering space that is simply connected. This space is unique up to homeomorphisms and is therefore called **universal covering space**. It can be constructed as the space of homotopy classes of paths with a fixed starting point (we thus get $|\pi_1|$ copies of each point). For each curve $f : I \to X$ we can define a **lift** $\tilde{f}$ of f to the covering space $\tilde{X}$ with $f = \pi \circ \tilde{f}$, which is unique by continuity once we choose, for example, the image $\tilde{f}(0)$ of its starting point.

The higher homotopy groups are defined in a similar way: $\pi_n(X, x_0)$ denotes the space of homotopy classes of maps $f : S^n \to X$ with $x_0 \in f(S^n)$. Loosely speaking, multiplication is defined by moving some parts of the spheres together and then taking away some common, topologically trivial n-dimensional piece of surface that has the base point at its boundary to form a bigger sphere. It can be shown that this operation[3] defines a product on $\pi_n(X, x_0)$ that satisfies the group axioms and that is abelian if $n > 1$ [NA90] (as for the case $n = 1$ it is independent of x_0 if X is arcwise connected). These groups are useful to capture more of the topology of X. For a chunk of Emmentaler cheese, for example, the fundamental group would be trivial, whereas π_2 tells us something about the holes. A more important application is instanton physics [eg80,NA90]: Instantons are topologically non-trivial gauge field

[2] $(X, \mathcal{T})$ is called locally connected if every point has a basis of neighborhoods consisting of connected sets.

[3] For a precise definition we may use the fact that the sphere S^n is homeomorphic to $I^n/\partial I^n$, i.e. to a hypercube with the boundary identified with one point [since $\tan \pi(x - \frac{1}{2})$ is bi-continuous on the interior $(0, 1)$ of I, this open interval is homeomorphic to $\mathbb{R}$; the same is true for a product of n such factors, and we already know that the one point compactification of $\mathbb{R}^n$ is S^n]. Continuous functions from $I^n/\partial I^n$ to Y are called n-loops with base point equal to the image of the boundary. The product of two such objects can be defined by attaching two hypercubes along a face before identifying the boundary with the base point of the sphere.

configurations of minimal Euclidean action. They contribute to non-perturbative phenomena like tunneling and quantum mechanical violation of conservation laws.

Discrete groups can often be described by a presentation $\langle g_i\rangle/\langle R_I\rangle$ where g_i generate the group and R_I denotes relations (i.e. equations) emong the generators. By definition, a cyclic group has one generator. It can either be freely generate (i.e. there are no relations) and thus isomorphic to $\mathbb{Z}$, or there can be one relation $g^n = e \equiv 1$. Then the group is denoted by $\mathbb{Z}_n$. The order of a group is the number of its elements, the order of a group element h is the smallest positive integer l with $h^l = 1$, or infinity if such an l does not exist.

Theorem: A finitely generated abelian group is isomporhic to $\mathbb{Z}^r \times \mathrm{Tor}$, where r is called the *rank* and the *torsion part* Tor is the finite group that consists of all elements of finite order. The torsion Tor is isomorphic to $\mathbb{Z}_{n_1} \times \ldots \times \mathbb{Z}_{n_s}$, where we can either choose n_i to be a divisible by n_{i+1} (this yields the minimal set of factors) or n_i to be powers of prime numbers (yielding the maximal value of s).

Exercise 3: Show that $\mathbb{Z}_m \times \mathbb{Z}_n = \langle g,h\rangle/\langle g^m = h^n = 1\rangle$ is isomorphic to $\mathbb{Z}_\lambda \times \mathbb{Z}_\gamma = \langle G,H\rangle/\langle G^\lambda = H^\gamma = 1\rangle$ where $\lambda = \mathrm{lcm}(m,n)$ and $\gamma = \gcd(m,n)$. Concretely, it follows from the Euler algorithm that the greatest common divisor can be represented as $\gamma = am + bn$ with some integers a, b. Then we can choose $G = gh$ and $H = h/G^{\frac{am}{\gamma}} = h^{\frac{bn}{\gamma}} g^{-\frac{am}{\gamma}}$.

We conclude with a list of homotopy groups of spheres:

$$\pi_{k<n}(S^n) = 0, \qquad \pi_n(S^n) = \mathbb{Z} \;\; [n \ge 1], \qquad \pi_{n+1}(S^n) = \mathbb{Z}_2 \;\; [n \ge 3], \tag{1.5}$$

$$\pi_{n>1}(S^1) = 0, \qquad \pi_3(S^2) = \mathbb{Z}, \qquad \pi_4(S^2) = \mathbb{Z}_2, \qquad \pi_5(S^2) = \pi_5(S^3) = \mathbb{Z}_2. \tag{1.6}$$

Further results are $\pi_6(S^2) = \pi_6(S^3) = \mathbb{Z}_{12}$; a general formula for $\pi_k(S^n)$ is not known.

Since $S^0 = \{-1, 1\}$ the elements of π_0 correspond to the arcwise connected components of X (the continuous maps from $S^0 \to X$ cooorespond to the pairs of points (x_0, x_1) in X; keeping x_0 point fixed we get one homotopy class for each component of X). π_0 cannot be given a natural group structure.[4]

[4] Any group structure would require choices for the interpretation of connected components that cannot be made on the basis of purely topological data. The situation is different, for example, for topological groups: The Lorentz group has 4 connected components (which can be reached from the identity by parity and time reversal) and $\pi_0 = (\mathbb{Z}_2)^2$ would be a natural identification.

1.3 Manifolds

A (topological) **manifold** of dimension n is a Hausdorff space such that every point has an open neighborhood homeomorphic to an open subset of $\mathbb{R}^n$. An open subset U of some topological space M together with a homeomorphism φ from U to an open subset V of $\mathbb{R}^n$ is called a (coordinate) **chart** of M. The coordinates $(x^1, \ldots, x^n)$ of the image $\varphi(x) \in \mathbb{R}^n$ of a point $x \in U$ are called the **coordinates** of x in the chart (U, φ). Two charts (U_1, φ_1) and (U_2, φ_2) are called C^r **compatible** if $V = U_1 \cap U_2 = \emptyset$ or if the homeomorphisms $\varphi_2 \circ \varphi_1^{-1}$ is an r times continuously differentiable map from $\varphi_1(V) \subset \mathbb{R}^n$ onto $\varphi_2(V) \subset \mathbb{R}^n$. A C^r **atlas** on a manifold X is a set of C^r compatible coordinate charts such that the domains of the charts cover X. Two atlases are called equivalent iff their union is again a C^r atlas. A C^r manifold is a manifold together with an equivalence class of C^r atlases. A **differentiable manifold** is a C^∞ manifold. Similarly, we can define an **analytic manifold** and a **complex manifold** by requiring that all functions $\varphi_2 \circ \varphi_1^{-1}$ are analytic or holomophic, respectively. In the latter case the φ_i should be complex coordinates, i.e. maps from the neighborhoods U_i to subsets of $\mathbb{C}^n$ and n is the complex dimension of the manifold (its real dimension is $2n$). A subset S of a manifold M is called a **submanifold** of dimension s if for every point in $x \in S$ there exists a chart (U_x, φ) of M containing x such that $U_x \cap S$ is identical to the subset of $U_x \cap M$ for which the last $n - s$ coordinates vanish.

A **Lie group** G is a group that is also a differentiable manifold such that the operation $f : G \times G \to G$ with $f(x, y) = xy^{-1}$ is differentiable. A **left/right group action on a manifold** is a differentiable map $\sigma : G \times M \to M$ such that $\sigma_g \circ \sigma_h = \sigma_{gh}$ or $\sigma_g \circ \sigma_h = \sigma_{hg}$, respectively, where $\sigma_g(x) := \sigma(g, x)$. We say that the action of G on M is

- **effective** if $\sigma_g(x) = x \ \forall x \in M \Rightarrow g = e$ (i.e. only the identity e acts trivially),
- **free** if $g \neq e \Rightarrow \sigma_g(x) \neq x \ \forall x \in M$ (i.e. only σ_e has fixed points),
- **transitive** if $\forall x, y \in M$ there exists a $g \in G$ such that $\sigma_g(x) = y$.

The **isotropy group** (also called **little group** or **stabilizer**) of a point $x \in M$ is the subgroup $H(x) = \{g \in G | \sigma_g(x) = x\}$ of G consisting of the group elements that have x as a fixed point.

Examples: The following subgroups of the matrix groups $GL(n, \mathbb{R})$ and $GL(n, \mathbb{C})$ of invertible real/complex matrices (the name means 'general linear') are called the **classical Lie groups**:

$SL(n, \mathbb{R})$ is the group of 'special linear' matrices, i.e. matrices with determinant 1.

$SO(n, \mathbb{R})$ is the group of orthogonal matrices with det $= 1$ (i.e. the connected component of $O(n, \mathbb{R})$. Orthogonal matrices leave the metric $g_{mn} = \delta_{mn}$ of Euclidean space invariant. Consider an antisymmetric matrix $\omega_{mn} = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. The $2n \times 2n$ matrices

$Sp(2n, \mathbb{R})$ that leave the n-fold tensorproduct of ω_{mn} invariant are called symplectic matrices.

$SU(n)$ is the group of special unitary matrices, i.e. complex unitary matrices with det $= 1$.

Cartan showed that in addition to these inifinite series of *classical* Lie groups there is only a finite number of *exceptional Lie groups* with 'simple Lie Algebra'. They are called E_6, E_7, E_8, F_4 and G_2. In his classification $SL(n+1,\mathbb{R})$ and $SU(n+1)$, which are related by analytic continuation, are denoted by A_n, B_n corresponds to $SO(2n+1)$, C_n corresponds to $Sp(2n)$, and D_n corresponds to $SO(2n)$. Actually the truth is a little more complicated: For each of the groups $A_1,\ldots,G_2$ there is a unique choice of the real form of the Lie algebra such that the Lie Group becomes compact. And there is, in addition, a choice in the global structure, which corresponds to dividing by a subgroup of the center of the universal covering group.[5]

In quantum mechanics the groups SO_3 and SU_2 are of particular importance. The group manifold of SU_2 is homeomophic to S^3, since its elements are of the form $\left(\begin{smallmatrix}\alpha & \beta\\ -\beta^* & \alpha^*\end{smallmatrix}\right)$ with $\alpha^2+\beta^2=1$. It is the double covering of $SO_3 \cong \mathbb{RP}^3$ (this can be illustrated with the *belt trick:* A twisted belt with its ends kept parallel corresponds to a closed path in SO_3. A twist by 4π can be undone without twisting the ends). The correspondence between SO_3 and SU_2 can be constructed in the following way: Vectors in $\mathbb{R}^3$ can be identified with traceless Hermitian matrices via $H_x := \vec{x}\vec{\sigma}$, where σ_i are the Pauli matrices. Since $|\vec{x}|^2 = \operatorname{tr} H_x^2$ the 'adjoint' action $H \to A^{-1}HA$ of SU_2 matrices A on Hermitian matrices H_x leaves lengths invariant and hence defines an SO_3 action on $\mathbb{R}^3$. This map from SU_2 to SO_3 is two-to-one since A and $-A$ define the same transformation.[6]

Returning to general manifolds, we will next consider homology, which can be thought of as the study of topological properties of integration domains. Here we use approximations of manifolds by simplices, the higher dimensional analogues of triangles. But before entering the general definition of homology groups we first consider the use of triangulations in the classification of surfaces (i.e. 2-dimensional manifolds).

1.4 Surfaces

A subset of a surface X is called a (topological) triangle if it is homeomorphic to some triangle in $\mathbb{R}^2$. A finite collection of triangles T_i is called a **triangulation** of X if $X = \bigcup T_i$ and if any non-empty intersection $T_i \cap T_j$ is either a common vertex or a common edge of T_i and T_j for $i \neq j$. We can give an orientation to a triangle by choosing an order for its vertices up to cyclic permutations. This induces a direction for the edges of the triangle. Let a be the edge from P to Q. Then we denote the edge from Q to P by a^{-1}. We say that a triangulation is oriented

[5] The center of a group is the (abelian) subgroup that consists of all elements that commute with all other group elements.

[6] $\pm\mathbf{1}$ are the only SU_2 transformations that are mapped to the identity, as can be shown using Schur's lemma.

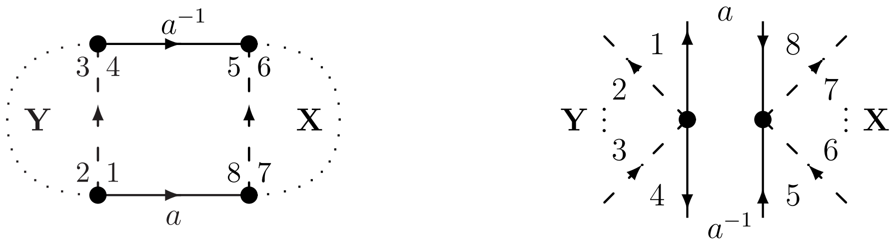

Fig. 1: Trying to paste a neighborhood of P for a non-linked edge a.

if we assign orientations in such a way that common edges of two triangles are always oriented in reverse directions. A surface is orientable if it admits an oriented triangulation.

Let us denote a triangle by the symbol abc if it has vertices PQR and is bounded by the oriented edges $P \xrightarrow{a} Q \xrightarrow{b} R \xrightarrow{c} P$. To any triangulation with n triangles we can assign a polygon with $n+2$ edges by joining the triangles along edges c_i and c_i^{-1} (in a first step we only join new triangles along 1 edge). We thus obtain a topological model of the surface by sewing the polygon along its bounding edges, i.e. by identifying the segments of the boundary according to their symbols. If the original surface X has no boundary then each edge occurs exactly twice, and the orientations must be inverse if X is orientable. By cutting and pasting such polygons in an appropriate way we can bring them into a normal form and thereby classify the topologies of triangulable surfaces.

Theorem: The normal form of a connected and oriented compact triangulable surface is either aa^{-1} or $\prod_{i=1}^{g}(a_i b_i a_i^{-1} b_i^{-1})$, where all vertices of the polygon correspond to a single point on the surface in the latter case. g is called the genus of the surface, with $g := 0$ if the normal form is aa^{-1}.

Proof: We proceed in three steps [FA80]. First we show that we can choose all vertices to correspond to the same point. Then we convince ourselves that each of the edges a of the resulting polygon is linked to some other edge b, i.e. that the symbol of the polygon is of the form $a \cdots b \cdots a^{-1} \cdots b^{-1} \cdots$. Eventually we bring all linked edges together.

1. We single out some vertex P which we want to become the base point of all edges. Assuming that there are more than 2 edges and that we remove any factor aa^{-1}, the symbol representing the polygon must have the form $P \xrightarrow{a} Q \xrightarrow{b} R \cdots R \xrightarrow{b^{-1}} Q \cdots$. If $Q \neq P$ we now cut the polygon from P to R along a line c and glue the resulting triangle abc^{-1} back to the polygon along b. Now the polygon has the form $P \xrightarrow{c} R \cdots R \xrightarrow{c^{-1}} P \xrightarrow{a} Q \cdots$, i.e. we removed one vertex Q and replaced it by a vertex P at a later position along the polygon. Since we did not change the number of edges, the iteration of this process must lead to a polygon with all vertices identical to P in a finite number of steps (unless we obtain aa^{-1}, the normal form for $g = 0$).

2. Next we show for a polygon with only one vertex P that every edge a must be linked with some other edge. Assuming that this is not the case we consider a polygon of the form $aXa^{-1}Y$ where the segments X and Y of the boundary have no common edges. Gluing together all

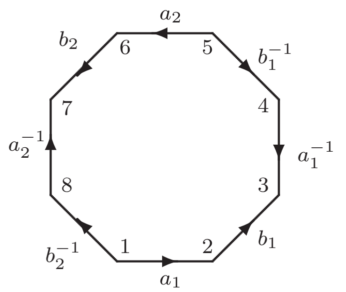

Fig. 2: Normal form for genus 2.

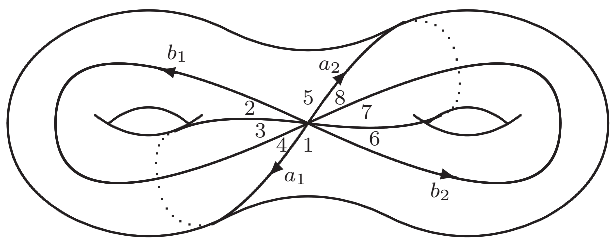

Fig. 3: Decomposition of a genus 2 surface.

edges in X and all edges in Y we obtain a surface which only has the two bounding edges, namely a and a^{-1} as shown in Fig. 1. Sewing along a would have to produce a surface without boundary. But this is not possible as we observe by checking the neighborhood of P, which ought to look like some small disk in $\mathbb{R}^2$.

3. Considering some fixed linked pair a, b of a polygon $aXbYa^{-1}Zb^{-1}W$. we now make a cut c from the end point of a to the beginning of a^{-1} and glue the resulting polygons along b. This gives us a polygon whose symbol is of the form $aca^{-1}ZYc^{-1}XW$ (with c homotopic to XbY on the original surface). Next we make a cut d from the beginning of c^{-1} to the beginning of a^{-1} and glue along a. This results in a symbol of the form $dcd^{-1}c^{-1}XWZY$ (with $[d^{-1}] = [a^{-1}ZY]$ or $[a] = [ZYd]$). Note that we never made any modifications within a segment of the boundary that is abbreviated by a capital letter so that linked edges that already are in normal form stay together. This completes the proof. $\square$

We have not yet shown that surfaces with different genus are topologically distinct. This can be achieved by finding a topological invariant that distinguishes between different genera. The **Euler number** of a 2-surface is defined by $\chi := v - e + f$, where v, e and f are the numbers of vertices, edges and faces of a triangulation, respectively.

Exercise 4: Show that the Euler number is independent of the triangulation and show that $\chi = v - e + f = 2 - 2g$ for compact orientable surfaces of genus g.
Hint: The formula for the Euler number can also be used for decompositions into arbitrary polygons. Any refinement of a polygon decomposition can be achieved with the following moves: a: split an edge into two edges with a new vertex, b: add a new vertex in the interior of a polygon and join it with a vertex, c: join two vertices with an edge. Any two triangulations admit (up to homotopy) a common refinement.

The normal form of a compact orientable surface suggests a presentaion of the **fundamental group:** π_1 of a genus g surface is generated by a_i and b_i with the single relation $\prod_{i=1}^{g} a_i b_i a_i^{-1} b_i^{-1} = 1$ (the boundary of the polygon model Fig. 2 of the surface is contractible and it can be shown that no other relations exist).

The **non-orientable case** can be treated in a similar way [`MA91`]: The normal form of a compact surface is $a_1a_1\cdots a_qa_q$ with all vertices identified, so that the Euler number is $\chi = 2-q$. Any non-orientable surface has an orientable double cover (take two copies of either orientation for each triangle of a triangulation and glue the copies along edges whose orientations match). Going from a non-orientable X_q to the orientible double cover X_g the Euler number doubles, thus the genus of X_g is $g = q-1$. The case $q = 1$ is the projective space $\mathbb{P}^2$, whose double cover is the sphere with $g = 0$. The double cover of the Klein bottle $q = 2$ is the torus $g = 1$.

The gluing of two surfaces S_i along some common triangle is called connected sum $S_1\#S_2$. It is easy to see that $\chi(S_1\#S_2) = \chi(S_1) + \chi(S_2) - 2$. All compact surfaces can be obtained as connected sums of tori and projective planes [MA91]. In other words, the orientable surfaces can be obtained from the sphere by attaching g handles, whereas the non-orientable ones are obtained by attaching q Möbius strips (whose boundaries only have one component). The connected sum with $\mathbb{RP}^2$ (or, equivalently, the attachment of a Möbius strip) is often drawn as a **crosscap** $\bigotimes$. It is easy to see that the Euler number for an arbitrary surface with boundaries is

$$\chi = 2 - 2h - b - c \tag{1.7}$$

where h is the number of handles, b is the number of components of the boundary and c is the number of crosscaps. Attaching a handle to an *unorientable* surface is equivalent to two crosscaps. The surface is thus characterized by its orientability, its Euler number and the number of components of the boundary.

Exercise 5:
• Show that $\mathbb{RP}^2 = D \cup M$ and that $M = D\#\mathbb{RP}^2$ can be viewed as an annulus (or cylinder) one of whose boundaries is closed by a crosscap (D is the disk and M is a Möbius strip).
• Show that a cylinder closed by two crosscaps (i.e. Möbius strips) gives a Klein bottle.
• Enumerate all surfaces of non-negative Euler number.

In higher dimensions the situation is more complicated: In 1958 A.A.Markov showed that there exists no algorithm for a classification of compact triangulable 4-manifolds; Poincaré's conjecture from the beginning of the 20th century that S^3 is the only simply connected compact 3-manifold was an open problem until recently when a proof sketched by Perelman in a series of papers 2002 / 2003 was checked and confirmed in 2006 (in 4-dimensions manifolds like $S^2 \times S^2$ have these properties but are different from S^4). Triangulability of surfaces has been shown in 1925 by T.Radó [AH60], who pointed out the necessity of assuming a countable basis for the topology. In 1952 E.Moise proved triangulability of 3-manifolds; recently A.Casson and M.Freedman showed that some 4-manifolds cannot be triangulated [MA91].

1.5 Homology

The fundamental group of a manifold is relevant, for example, for analytic continuation. If we are interested in integrals over closed curves (or higher-dimensional submanifolds), however, then the order in which the different parts of a closed path is passed through is irrelevant. This suggests to modify the concept of homotopy groups by allowing to split a loop into several elementary loops, which may be deformed individually and which may be added formally with integral (or real) coefficients. Such a formal sum is called a *cycle*. We then also need a new concept of equivalence: Since integrals of total derivatives are equal to integrals over the boundary of the integration domain (see below) the appropriate notion of equivalence for domains of dimension r is that a domain is in the class of 0 if it is the boundary of some $(r+1)$-dimensional domain. Integration domains that differ by boundaries are called **homologous**. The essential fact we will use is that a boundary has no boundary (see below). Therefore integrals of total derivatives over boundaries are zero.

A formal sum of r-dimensional (integration) domains is called r**-chain**. Since $\partial B = 0$ if $B = \partial M$ is the boundary of a chain M, we can write $\partial^2 = 0$ for the boundary operator ∂, which maps r-chains to $(r-1)$-chains. A chain is called a **cycle** Z if it satisfies $\partial Z = 0$. Of course it is not true in general that a cycle must be a boundary. The **homology group** $H_r(X) = Z_r/B_r$ is defined as the quotient of the group $Z_r(X)$ of r-cycles by the group $B_r(X)$ of r-boundaries.[7] The additive group structure is given by the formal sums of cycles, hence H_r is an abelian group. If we insist on integral or real coefficients then it is safer to write $H_r(X,\mathbb{Z})$ or $H_r(X,\mathbb{R})$. Being pedantic, we may insist that we have, in fact, different boundary operators $\partial_r : C_r \to B_{r-1} \subseteq C_{r-1}$ that act on r-dimensional chains. Thus we have a finite sequence[8]

$$0 \to C_n \xrightarrow{\partial_n} C_{n-1} \ldots C_r \xrightarrow{\partial_r} C_{r-1} \ldots C_0 \xrightarrow{\partial_0} 0 \tag{1.8}$$

of homomorphisms ∂_r of abelian groups C_n with $\partial_{r-1} \circ \partial_r = 0$. Such a structure is called a **complex** and the sequence is called **exact** if $\operatorname{Ker} \partial_{r-1} = \operatorname{Im} \partial_r$. The vector space dimensions $b_r(X) = \dim H_r(X,\mathbb{R})$ are called **Betti numbers** of the manifold.

To be more precise about the definition of the cycle groups C_r and their subgroups $Z_r \supseteq B_r$ we use (generalized) triangulations, i.e. decompositions of the manifold into simplexes: A **simplex** $\sigma_k = \langle p_0 p_1 \ldots p_k \rangle := \{x \in \mathbb{R}^n \mid x = \sum_{i=0}^{k} c_i p_i,\ c_i \geq 0,\ \sum_{i=0}^{k} c_i = 1\}$ in $\mathbb{R}^n$ is the **convex hull** of $k+1$ affine independent points (i.e. the points p_i should span a k-dimensional affine subspace of $\mathbb{R}^n$; the $k+1$ unique numbers c_i are called **barycentric coordinates** of $x \in \sigma_k$).

[7] Besides *homotopy* and *homology* groups there is also the **holonomy** group, which is the subgroup of linear transformations of tangent vectors that can be achieved by parallel transport along loops. This is, however, not a topological but a geometrical concept, since we need more structure (namely a connection) to define parallel transport on manifolds.

[8] The quotion group $\operatorname{CoKer}(\partial_r) = C_{r-1}/\partial_r(C_r)$ is called *cokernel* of ∂_r.

The convex hull of $q+1$ vertices of σ_k is called a face of σ_k. It is a q simplex which is called a proper face of σ_k if $q<k$ and a facet if $q=k-1$. A finite set K of simplexes σ_i in $\mathbb{R}^n$ is called a **simplicial complex** if each face of a simplex in K belongs to K and if the intersection $\sigma_i \cap \sigma_j$ of any two simplexes in K is either empty or a face of both simplexes, σ_i and σ_j. The union $|K| = \bigcup_{\sigma_i \in K} \sigma_i$ is called the polyhedron of K. A topological space is said to be triangulable if it is homeomorphic to $|K|$ for some simplicial complex K.

In order to define simplicial homology we also need to consider orientations. Let $\sigma = (p_0 \ldots p_r) = (-)^r (p_1 \ldots p_r p_0) = -(p_1 p_0 p_2 \ldots p_r)$ denote an **oriented simplex**, which is given by an ordered set of vertices up to even permutations; odd permutations reverse the orientation. The boundary of an oriented simplex σ_r is a formal sum of the oriented facets of the simplex: $\partial_r(p_0 \ldots p_r) = (p_1 p_2 \ldots p_r) - (p_0 p_2 \ldots p_r) + \ldots + (-1)^r (p_0 p_1 \ldots p_{r-1})$. In particular, the boundary of an oriented line is its end point minus its initial point. It is easy to check that the boundary of a boundary is zero $\partial_{r-1} \circ \partial_r = 0$. We can now go on to define chains and cycles as above. The r-**chain group** $C_r(K)$ of a simplicial complex is the free abelian group generated by the r-simplexes of K, i.e. it consists of formal sums of σ_r's with integral coefficients. The **cycle group** Z_r is the kernel of ∂_r and the **boundary group** B_r is the image of ∂_{r+1}. Hence Z_r consists of chains without boundaries and B_r consists of boundaries. Two cycles are homologous if their difference is a boundary.

Since $\partial^2 = 0$ all boundaries are cycles and we define the **simplicial homology groups** (with integral coefficients) $H_r(K) = Z_r(K)/B_r(K)$ as the group of homology classes of cycles. Note that the homology group is not always a free abelian group. $H_r(X) \cong H_r(K)$ is a finitely generated abelian group that can be shown to be independent of the triangulation. It's rank is equal to the Betti number b_r. An example of a homology group with torsion is $\mathbb{P}_2$, for which $H_1 = \mathbb{Z}_2$ (if we take twice the non-contractible loop, which generates H_1, then we obtain a boundary). The torsion subgroup of $H_r(K)$ can be thought of as the 'twisting' of the complex K [NA83]. We can get rid of the torsion subgroup by allowing real (or rational) coefficients of chains, because then we can divide the equation $na = \partial b$ for a generator of finite order n by n, i.e. $H_r(K, \mathbb{R}) \cong \mathbb{R}^{b_k}$.

Euler Poincaré theorem: Let K be an n-dimensional simplicial complex and let I_r be the number of r-simplexes in K. Then the Euler characteristic is related to the Betti numbers by:

$$\chi(K) := \sum_{r=0}^{n} (-1)^r I_r = \sum_{r=0}^{n} (-1)^r b_r. \tag{1.9}$$

Exercise 6: Show the Euler Poincaré theorem and compute the Betti numbers for a genus g surface.

Hint: Use that $b_r = \dim Z_r - \dim B_r$ and convince yourself that $I_r = \dim C_r = \dim(\ker \partial_r) +$

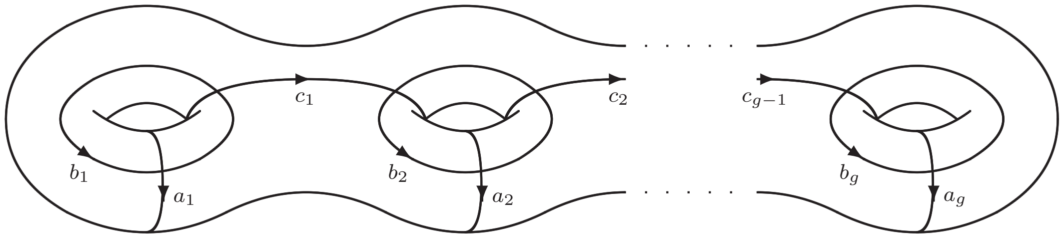

Fig. 4: a_i and b_i are a basis of homology cycles for a compact orientable genus g surface.

$\dim(\operatorname{im}\partial_r) = \dim Z_r + \dim B_{r-1}$. For an orientable connected manifold $b_0 = b_n = 1$.

For an arcwise connected topological space it can be shown that the first homotopy group is isomorphic to the abelianization of the fundamental group [MA91] (the abelianization can be obtained by adding commutativity to the other relations of a presentation of π_1).

A more general setting for the definition of the homology groups H_r is *singular homology*, where one admits arbitrary continous images of simplices for the decomposition of an arbitrary topological space X. The required conditions for the gluing of the faces can be defined via the parametrization. H_r can be computed efficiently using the Mayer-Vietoris exact sequence and CW-complexes [MA91].

It is often useful to consider the dual spaces C^r, called cochain spaces, and the dual complex with the coboundary operators $d_r : C^r \to C^{r+1}$ defined as the adjoint to ∂_r. Note the d_r increases the degree r. The quotients $H^r = \operatorname{Ker} d_r / \operatorname{Im} d_{r-1}$ are called cohomology groups. They have the advantage that there exists a ring structure, the cup product, which is compatible with the grading by the degree r. We will come back to this concept in the context of differential forms and de Rham cohomology, where the coboundary operator will be realized as a differential operator.

1.6 Intersection numbers and the mapping class group

Requiring transversal intersections of representatives of homology groups on smooth orientable manifolds we can define an intersection product, which maps $H_r \times H_s \to H_{r+s-n}$. For even dimensions this defines a scalar product on middle-dimensional cycles that only depends on the class in $H_{n/2}$. This product has an interesting application to the characterization of automorphisms of orientable surfaces.

There are homeomorphisms of surfaces onto themselves that cannot be continuously deformed to the identity. The group of homotopy classes of such homeomorphisms is called

mapping class group (MCG), which can be shown to be generated by **Dehn twists**, i.e. twists by 2π around non-contractible loops. The twists around a_i, b_i and c_i in Fig. 4 provide a non-minimal set of generators of a subgroup of the MCG, called the **(Siegel) modular group**.

The modular group (MG) is isomorphic to $Sp(2g,\mathbb{Z})$, which can be seen as follows: Modular transformations leave the intersection matrix of a canonical homology basis $a_i \cap b_j = -b_j \cap a_i = \delta_{ij}$ invariant. Since $D_{a_1} b_1 = a_1 + b_1$, $D_{c_1} b_1 = b_1 + c_1$ and $D_{c_1} b_2 = b_2 - c_1$ with $c_1 = a_1 a_2^{-1}$ their action on (a_1, b_1, a_2, b_2) is described by

$$D_{a_1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad D_{b_1} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad D_{c_1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 1 & 1 \end{pmatrix}. \tag{1.10}$$

The quotient MCG/MG is called Torelli group, which is non-trivial for $g > 1$ (it is generated by Dehn twists around homologically trivial cycles and thus leaves intersection numbers of homology cycles invariant).

Chapter 2

Differentiable manifolds

In differential geometry [CH77, eg80, GO62, KO63, NA90] we are interested in spaces that locally look like $\mathbb{R}^n$ for some dimension n: A **differentiable manifold** $\mathcal{M}$ is a (topological) space that can be covered by a collection of **local coordinate charts** $\mathcal{U}_I \subseteq \mathcal{M}$ with $\bigcup \mathcal{U}_I = \mathcal{M}$ and local coordinates defined on $\mathcal{U}_I$. These coordinates $x_i^{(I)}$ have to be compatible in the sense that $x_i^{(I)}$ and $x_j^{(J)}$ are related by differentiable transition functions on $\mathcal{U}_I \cap \mathcal{U}_J$ whenever that intersection is non-empty. To give a meaning to the word *local* we first need a **topology** on $\mathcal{M}$, which tells us about the neighborhoods of points. The charts $\mathcal{U}_I$ should be open sets with respect to this topology, and, in turn, the open sets of $\mathcal{U}_I$ (in the topology inherited from $\mathbb{R}^n$ via the local coordinates) should provide a basis for the topology of $\mathcal{M}$.

The local coordinates tell us how to differentiate functions and allow us to define the **tangent space** and infinitesimal operations, as well as cotangent vectors and general tensors (tangent vectors have upper/contravariant indices). Spaces of vector and tensor fields may themselves be considered as manifolds, which leads us to the notion of fiber **bundles**. The coordinate independent **exterior derivative** can be defined on differential forms, which correspond to anti-symmetric tensor fields with lower indices. These are useful in integration theory and in formulating structure equations and integrability conditions.

2.1 Tangent space and tensors

Differentiable (and topological) manifolds can always be embedded in Euclidean spaces of higher dimensions. We could therefore think of tangent vectors as vectors in that embedding space. It does not make much sense, however, to deal with all the ambiguous additional structure if we are only interested in intrisic geometrical properties of a manifold. It is thus more useful to work with a more abstract definition: Tangent vectors can be constructed as directional derivatives along smooth curves, which we identify by their action on $C^\infty(M)$ evaluated at a given point. Vector fields are, therefore, derivations on the algebra of smooth functions.

Smooth functions on a manifold form an algebra, because they form a ring (they can be added and multiplied) as well as a vector space over the field $\mathbb{R}$ (linear combinations again are smooth functions). A **tangent vector** $v_x : C^\infty(\mathcal{M}) \to \mathbb{R}$ at a point $x \in X$ is a linear map

$$v_x(\alpha f + \beta g) = \alpha v_x(f) + \beta v_x(g) \tag{2.1}$$

that satisfies the **Leibniz rule**

$$v_x(fg) = f(x)v_x(g) + g(x)v_x(f) \tag{2.2}$$

(here it would be sufficient to consider the algebra of smooth functions that are defined in some neighborhood of x, since v_x only depends on the functions in an arbitrarily small neighborhood). From this definition it follows that $v_x = v^m\partial_m$ is a linear combination of partial derivatives with respect to any coordinates x^m. This must be true for any choice of coordinates, so the chain rule $\frac{\partial}{\partial x^n} = \frac{\partial \hat{x}^m}{\partial x^n}\frac{\partial}{\partial \hat{x}^m}$ implies that the local coordinates (or components) of a vector v transform contravariantly $\hat{v}^m = \frac{\partial \hat{x}^m}{\partial x^n} v^n$ under a diffeomorphism $x \to \hat{x}(x)$. The tangent vectors v_x at a point x form a vector space of the dimension of the manifold, the tangent space $T_x(\mathcal{M})$ Taylor expansion can be used to show that the vectors ∂_n form a basis of $T_x(\mathcal{M})$.

More general tensors can be obtained as duals and tensor products of tangent vectors. The **dual** $V^* = \mathrm{Hom}(V, \mathbb{R})$ of a **vector space** is the linear space of *linear forms* (or functionals) on V, i.e. $w \in V^*$ is a linear map (or homomorphism) $w : V \to \mathbb{R}$. The value of this map evaluated on a vector v is denoted by the bracket $\langle w, v\rangle := w(v)$, which is also called duality **pairing**. If we have a basis E_i of vectors then there is a natural dual basis, the **co-basis** e^j that is defined by the following action of the co-vectors e^i on the basis of V:

$$\langle e^j, E_i\rangle = \delta^j{}_i \qquad \Rightarrow \qquad \langle w, v\rangle := w(v) = w_i v^i \quad \text{with} \quad w = w_j e^j,\ v = v^i E_i. \tag{2.3}$$

The dual of $T_x\mathcal{M}$ is the space $T^*_x\mathcal{M}$ of cotangent vectors at a point $x \in \mathcal{M}$. General **tensors** with m upper and n lower indices are elements of the tensor product[1] of m copies of $T_x\mathcal{M}$ and n copies of $T^*_x\mathcal{M}$. The coordinates (components) of such a tensor t are defined by $t = t^{i_1\ldots i_m}{}_{j_1\ldots j_n} E_{i_1} \otimes \ldots \otimes E_{i_m} \otimes e^{j_1} \otimes \ldots \otimes e^{j_n}$ (possibly with some other ordering of the indices).

The basis dual to the partial derivatives ∂_i that come with a coordinate system is denoted by dx^j. Since vectors $v \in T_x\mathcal{M}$ and $w \in T^*_x\mathcal{M}$, as well as the duality pairing $\langle w, v\rangle$ are independent of the choice of coordinates, we find the following transformations of indices under diffeomorphism $x \to \hat{x}(x)$:

$$\partial_i = (D\hat{x})_i{}^j \hat{\partial}_j, \qquad d\hat{x}^j = dx^i (D\hat{x})_i{}^j, \qquad \hat{v}^j = v^i (D\hat{x})_i{}^j, \qquad w_i = (D\hat{x})_i{}^j \hat{w}_j, \tag{2.4}$$

[1] The tensor product $V \otimes W$ of two vector spaces is given by the set of all linear combinations of tensor products $v \otimes w$ of vectors $v \in V,\ w \in W$. A basis of the product space is given by all tensor products $E_i \otimes F_j$ of basis elements $E_i \in V,\ F_j \in W$, so that its dimension is the product of the dimensions of the factors. The *direct sum* $V \oplus W$, on the other hand, consists of pairs (v, w) with componentwise vector operations, so that $(E_i, 0)$ and $(0, F_j)$ provide a basis and the dimensions are added.

with the **jacobian matrix** $(D\hat{x})_i{}^j := \frac{\partial \hat{x}^j(x)}{\partial x^i}$ (the determinant of $D\hat{x}_i{}^j$, which will be important in integration theory, is called *jacobian*). For general tensors we get a jacobian matrix or its inverse for each index: The upper/lower indices of contravariant/covariant tensors transform inverse/identical to the basis ∂_i of tangent vectors. (Tensors could also be defined via a collection of component functions that transforms in this way under diffeomorphism, i.e. "a tensor is an object that transforms like a tensor".)

The formulas (2.4), with $\hat{x}(x)$ replaced by $f(x)$, are also valid for a more general smooth map $f : X \to Y$ from a manifold X to a manifold Y. The mapping among (co)vectors is then well-defined as long as we do not have to invert the (in general rectangular) jacobian matrix Df. This suggests that there should be natural maps $f_* : T_xX \to T_yY$ and $f^* : T_y^*Y \to T_x^*X$ with $y = f(x)$. Indeed, this is easy to see in a coordinate independent way: For any function g of Y we can define the **pull back** $f^*g := g \circ f$, which is a function on X. But this map $f^* : C^\infty(Y) \to C^\infty(X)$ can be used to define for each tangent vector $v_x : C^\infty(X) \to \mathbb{R}$ a vector $v_y = f_*v_x$ by $(f_*v_x)(g) := v_x(f^*g)$. The operation f_* is called **push forward** or **differential map** and it is defined for any contravariant tensor in the obvious way. It maps, in particular, tangent vectors of a curve to the tangent vectors of the image curve. In turn, cotangent vectors (and covariant tensors) can again be pulled back in a natural way: For an element $w_y \in T_y^*Y$ the **pull back** $w_x = f^*w_y$ is defined by $f^*w_y(v_x) := w_y(f_*v_x)$. If f is a diffeomorphism then we can, of course, push and pull arbitrary tensors in either direction along f.

Exercise 7: Show that f_*v_x is an element of T_yY and write down the pull back (push forward) for covariant (contravariant) tensors in terms of coordinates.

In differential geometry we are interested in **tensor fields** on manifolds rather than in tensors at a single point. These can be defined via (smooth/continuous) coordinate dependent component functions. For vector fields it is easy to write down a coordinate independent version of this definition: A smooth vector field is a **derivation** on the algebra $C^\infty(\mathcal{M})$, i.e. a linear map $v : C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M})$ that satisfies the Leibniz rule. This is so because a vector v_x assigns a number to a function, so that a vector field (a vector at each point) assigns a number at each point (a function) to a function.

2.2 Lie derivatives

A vector field $\xi = \xi^m\partial_m$ defines a curve through any given point on the manifold: We may think of moving along the direction $\xi^m(x)$ at each point with coordinates x^m on a manifold with a velocity that is proportional to the length of ξ. This suggests to consider the system of first order differential equations $\frac{d}{dt}x^m(t) = \xi^m(x(t))$, whose solutions $x(t)$ we call **integral**

curves of the vector field ξ. The integral curves of the basis vector fields ∂_i are the coordinate lines themselves.

Integral curves exist at least locally and they can be used to define a t-dependent map of points on the manifold, i.e. a *diffeomorphism* of the manifold onto itself. Such a diffeomorphism is often called an *active* transformations, to which we can associate a *coordinate transformation* (or *passive* transformation) by assigning the old coordinates of the target points as new coordinates to the original points. For infinitesimal transformations, i.e. for small t, we obtain the map $\phi_t : x^m \to \tilde{x}^m = x^m + t\xi^m + O(t^2)$ in either case (only the interpretation of $\tilde{x}^m$ is different).

The **Lie derivative** $\mathcal{L}_\xi T$ of a tensor field T is its variation induced by the infinitesimal transformation that comes with the vector field ξ. It is thus defined to be the term of leading order for $t \to 0$ in the difference $\phi_t^* T - T$ between a tensor field and its pull back along ϕ_t,

$$\mathcal{L}_\xi T := \left(\tfrac{d}{dt}(\phi_t^* T)\right)_{\big|_{t=0}}, \qquad \phi_t(x) = x + t\xi + O(t^2). \tag{2.5}$$

This is easy to compute for a function: $\phi_t^* f = f + t\xi(f) + O(t^2)$ so that $\mathcal{L}_\xi f = \xi(f) = \xi^m \partial_m f$. For the component functions of tensor fields, however, we have additional terms from the jacobian matrices and their inverses [see eq. (2.4)]. The difference between tensors of the same type is again a tensor, and so is the Lie derivative of a tensor. Putting the pieces together we find its components

$$\mathcal{L}_\xi T_i{}^p = \xi^m \partial_m T_i{}^p + \partial_i \xi^m T_m{}^p - \partial_n \xi^p T_i{}^n \tag{2.6}$$

with additional positive/negative terms for additional lower/upper indices.

In computations with tensor fields whose rank we do not want to specify it is often useful to write the part of the Lie derivative that comes from the jacobian factor in a more symbolic form. For this purpose we introduce the symbol $\Delta_i{}^j$ for infinitesimal $GL(n)$ transformation, which acts on (co)vectors by $\Delta_i{}^j v^u = \delta_i{}^u v^j$ and $\Delta_i{}^j v_l = -\delta_l{}^j v_i$ and which is extended to general tensors by the Leibniz rule and by linearity (i.e. $\Delta_i{}^j$ is a derivation on the tensor algebra; on a tensor with m upper and n lower indices it acts by a collection of m positive and n negative terms containing $\delta_i{}^u$ and $\delta_l{}^j$, respectively). Then

$$\mathcal{L}_\xi = \xi^l \partial_l - \partial_i \xi^k \Delta_k{}^i, \qquad \Delta_k{}^i v^n = \delta_k{}^n v^i, \qquad \Delta_k{}^i v_m = -\delta_m{}^i v_k; \tag{2.7}$$

the first term $\xi^l \partial_l$ of $\mathcal{L}_\xi$ is called *shift term* for obvious reasons.
Exercise 8: Show that $[\Delta_i{}^j, \Delta_k{}^l] = \delta_i{}^l \Delta_k{}^j - \delta_k{}^j \Delta_i{}^l$ is the commutator of GL_n-transformations (since both sides are derivations it is sufficient to check this on covariant and contravariant vectors, which generate the tensor algebra).

Symmetry transformations act simultaneously on each factor of a tensor product. If we consider infinitesimal transformations, then we are collecting all terms that are linear in a small parameter. We thus get one term for each factor in a product, so that infinitesimal transformations act as derivations on algebras. Non-commutativities of finite symmetry transformations manifest themselves in non-vanishing commutators of infinitesimal transformations. It is easy to check that the commutator of two derivations is again a derivation, and thus corresponds to another infinitesimal transformation. In the case of diffeomorphisms inifinitesimal transformations correspond to (Lie derivatives along) vector fields. On functions $\mathcal{L}_v(f) = v(f)$. This is the motivation for defining the **Lie bracket** of two vector fields as

$$[v, w] := v \circ w - w \circ v = (v^i \partial_i w^l - w^i \partial_i v^l)\partial_l = \mathcal{L}_v(w) = -\mathcal{L}_w(v), \tag{2.8}$$

which again is a vector field.

It is easy to check that the Lie bracket is anti-symmetric and satisfies the Jacobi identity,

$$[v, w] = -[w, v], \qquad \sum_{i,j,k}\!\!\!\!\circlearrowright [v_i, [v_j, v_k]] := [v_i, [v_j, v_k]] + [v_j, [v_k, v_i]] + [v_k, [v_i, v_j]] = 0. \tag{2.9}$$

It commutes with the push forward $f_*[v, w] = [f_*v, f_*w]$ and satisfies the following identities

$$\mathcal{L}_{fv}(w) = f[v, w] - w(f)v, \quad \mathcal{L}_v(fw) = f[v, w] + v(f)w, \qquad [\mathcal{L}_v, \mathcal{L}_w] = \mathcal{L}_{[v,w]} \tag{2.10}$$

for $f \in C^\infty(\mathcal{M})$; the last identity gives the commutator of Lie derivatives in terms of the Lie brackets of vector fields.

Note that all Lie brackets among the natural basis vector fields ∂_i vanish (partial derivatives commute). A set of n pointwise linearly independent vector fields v_i defines a basis of tangent space. It can be shown that their integral curves define local coordinates (of a submanifold if n is smaller than the number of dimensions) iff $[v_i, v_j] = 0\ \forall i, j$ (such a basis v_i of TM is called holonomous). Geometrically a non-vanishing Lie bracket $[v, w] \neq 0$ means that the flows along the integrals curves of v and w do not commute. If the Lie brackets among $r < n$ linearly independent vector fields satisfy the **Frobenius integrability condition**

$$[v_i, v_j] = f_{ij}{}^k v_k, \qquad i = 1, \ldots, r < n \tag{2.11}$$

(the Lie bracket closes on v_i) then the integral curves locally span an r-dimensional submanifold.

2.3 Differential forms

The **exterior (Grassmann) algebra** $\Lambda(V)$ over a vector space V is the associative algebra spanned by V with the **wedge product** defined by the relation $v \wedge v = 0$, and hence $v \wedge w = -w \wedge v$. A graded algebra is an algebra with a decomposition into a direct sum $\mathcal{A} = \bigoplus_{p\in P} \mathcal{A}_p$ for an abelian group P that is compatible with the product, i.e. $\mathcal{A}_p \cdot \mathcal{A}_q \subseteq \mathcal{A}_{p+q}$. Typical examples are $\mathbb{Z}^n$ gradings, like multidegrees of polynomials, or $\mathbb{Z}_2$ gradings (superalgebras), which correspond to decompositions into even and odd elements. $\Lambda(V) = \bigoplus \Lambda^p(V)$ with $\Lambda^1 \cong V$ is $\mathbb{Z}$-graded and $\Lambda^i = \{0\}$ unless $0 \le i <\le n = \dim V$. $\Lambda^p(V)$ has dimension $\binom{n}{p}$ and can be identified with the *linear* subspace of the tensor algebra consisting of totally antisymmetric tensors of degree p because the alternating sum over permutations

$$v_1 \wedge \ldots \wedge v_p = \sum_p \operatorname{sign}(\pi)\, v_{\pi(1)} \otimes \ldots \otimes v_{\pi(p)} \tag{2.12}$$

defines (by linear extension) an associative and graded-commutative product:

$$(\alpha \wedge \beta) \wedge \gamma = \alpha \wedge (\beta \wedge \gamma), \qquad \alpha \wedge \beta = (-)^{\alpha\beta} \beta \wedge \alpha. \tag{2.13}$$

$\Lambda(V)$ thus can be considered a *linear subspace* (with dimension 2^n) of the tensor algebra. It is, however, not a subalgebra because the wedge product and the tensor product are different associative products and $\Lambda^p \otimes \Lambda^q$ is not contained in Λ^{p+q}.

For a $\mathbb{Z}_2$ graded (associative) algebra $\mathcal{A}$ we can define graded derviations $Der_\pm(\mathcal{A})$, which are maps $\delta : \mathcal{A}_p \to \mathcal{A}_{p+|\delta|}$ that satisfy the graded Leibniz rule

$$\delta(A \cdot B) = \delta A \cdot B + (-)^{\delta\, A} A \cdot \delta B. \tag{2.14}$$

(In the exponent of $(-) \equiv (-1)$ we identify graded maps and algebra elements with their grading.) Even elements $\delta \in Der_+$, with $|\delta| \equiv 0$, are called derivations. Odd elements $\delta \in Der_-$ are called antiderivations. It is straightforward to check that the graded commutator $[\delta, \varepsilon]_\pm = \delta \circ \varepsilon - (-)^{\delta\varepsilon} \varepsilon \circ \delta$ of two graded derivations is again a graded derivation, whose degree is $|\delta| + |\varepsilon|$. Due to the graded antisymmetry and the graded Jacobi identity of the graded commutator we thus obtain a graded Lie algebra.

An important example of a Grassmann algebra is the exterior algebra over the cotangent space $\Lambda_x\mathcal{M} = \Lambda(T_x^*\mathcal{M})$. The elements with pure degree of $\Lambda\mathcal{M}$ are antisymmetric covariant tensors with smooth component functions, called differential forms $\omega \in \Lambda^p$ or p-forms with

$$\omega = \tfrac{1}{p!}\, dx^{i_1} \wedge \ldots \wedge dx^{i_p}\, \omega_{i_1 \ldots i_p}. \tag{2.15}$$

The wedge product thus becomes

$$\alpha \wedge \beta := \frac{1}{p!q!}\, dx^{i_1} \wedge \ldots \wedge dx^{i_{p+q}}\, \alpha_{i_1 \ldots i_p} \beta_{i_{p+1} \ldots i_{p+q}} \;\in\; \Lambda^{p+q} \qquad \forall \alpha \in \Lambda^p,\ \beta \in \Lambda^q; \tag{2.16}$$

and $(\alpha \wedge \beta)_{i_1 \ldots i_p j_1 \ldots j_q} = \frac{(p+q)!}{p!q!} \alpha_{[i_1 \ldots i_p} \beta_{j_1 \ldots j_q]}$ with $t_{[i_1 \ldots i_p]} := \frac{1}{p!} \sum_\pi \operatorname{sign}(\pi) t_{i_{\pi(1)} \ldots i_{\pi(p)}}$.

A **differential** on a graded algebra is a nilpotent antiderivation

$$d^2 = 0, \qquad d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-)^{|\alpha|} \alpha \wedge d\beta. \tag{2.17}$$

The total differential $df = dx^i\partial_i f$ maps functions to their gradient, which is a cotangent vector (as can be checked by its transformation properties). It can be extended to a differential on the whole exterior algebra, which is called **exterior derivative**. On coordinate function we find $d(x^n) = dx^n$. Moreover, since $d(dx^i) = d^2x^i = 0$, the differential on 1-forms is $d(dx^i\omega_i) = -dx^i \wedge dx^j\partial_j\omega_i = \frac{1}{2}dx^i \wedge dx^j(\partial_i\omega_j - \partial_j\omega_i)$, or $(d\omega)_{ij} = \partial_i\omega_j - \partial_j\omega_i$. As d^2 is a derivation, d is consistently defined on all of $\Lambda\mathcal{M}$ by its antiderivation property. In components we find

$$(d\omega)_{i_0\ldots i_p} = (p+1)\partial_{[i_0}\omega_{i_1\ldots i_p]} = \sum\!\!\!\!\!\!\bigcirc_{i_0\ldots i_p} (\pm)^{p+1}\partial_{i_0}\omega_{i_1\ldots i_p} \tag{2.18}$$

where $\sum\!\!\!\!\!\!\bigcirc$ denotes the cyclic sum (which is this case has $p+1$ terms). It can be shown that

$$(d\omega)(v_0,\ldots,v_p) = \sum_0^p (-)^l v_l(\omega(v_0,\ldots,v_p)) + \sum_{i<j}(-)^{i+j}\omega([v_i,v_j],v_o,\ldots,v_p) \tag{2.19}$$

with the obvious omissions of the vectorfields v_l and v_i, v_j in the arguments of ω.

In physics we are used to anticommuting wave functions, which implement the Pauli principle. It would be possible to count signs from anticommuting fermions and anticommuting differentials seperately, but it is equivalent and much simpler to use an overall $\mathbb{Z}_2$ grading to account for all signs. In turn, we will often omit the $\wedge$ symbol and just regard the differentials dx^i as anticommuting objects. We can then write $d = dx^i\partial_i$.

Exercise 9: In $\mathbb{R}^3$ tangent and cotangent vectors can be identified by index shifts with the Kronecker δ and the Levi-Civita symbol ε can be used to identify functions f with 3-forms $\frac{1}{6}f\varepsilon_{ijk}dx^idx^jdx^k$ and the components v^i of vector fields with 2-forms $\frac{1}{2}v^i\varepsilon_{ijk}dx^jdx^k$. Show that the differential d thus corresponds to gradient, curl, and divergence on functions, 1-forms, and 2-forms, respectively. Moreover, $d^2 = 0$ comprises the identities curl $\circ$ grad $= 0$ and div $\circ$ curl $= 0$.

Remark: In special relativity electromagnetic potentials and field strengths can be combined to a gauge potential $A = A_m dx^m$ and its exterior derivative $F = dA = \frac{1}{2}dx^idx^j(\partial_iA_j - \partial_jA_i)$. The two homogeneous Maxwell equations thus become $dF = 0$ (which is a purely topological equation; the analogous form of the inhomogeneous equations will depend on the metric).

The **interior product** i_v is the anti-derivation of degree -1 on Λ that vanishes on functions and that inserts v into forms, $i_v\, w := \langle w, v\rangle = w(v)$. This implies

$$i_v^2 = 0, \qquad \{d, i_v\} = \mathcal{L}_v, \qquad [\mathcal{L}_v, i_w] = i_{[v,w]}, \qquad (i_v\omega)_{n_2\ldots n_p} = v^l\omega_{ln_2\ldots n_p} \tag{2.20}$$

on arbitrary p-forms. The algebra of d, i_v and $\mathcal{L}_v = \{d, i_v\}$ implies $[d, \mathcal{L}_v] = 0$ by the graded Jacobi identity. Because of the properties of graded derivations it is sufficient to check this algebra for functions and 1-forms. The interior product is sometimes also called *interior derivative.*[2]

[2] The Lie derivative of 'tensor-valued p forms' is $\mathcal{L}_\xi = \{d, i_\xi\} - \partial_l\xi^n\Delta_n{}^l$, where $\Delta_n{}^l$ acts on those indices that are not contracted with coordinate differentials.

One of the main applications of differential forms is integration theory: Under a change of coordinates the component functions of an n-form transform with an inverse Jacobian because we get an inverse Jacobi matrix for each lower index and the antisymmetrization in all indices generates the determinant. This just compensates the Jacobian that we get for a change of variables in an n-dimensional integral. Therefore, if a domain can be covered by a single coordinate patch then

$$\int_C \omega := \int_C dx^1 \dots dx^n \omega_{1\dots n} \tag{2.21}$$

is a geometrical construction that does not depend on the coordinates chosen for the evaluation of the integral, except possibly for a sign. More general integration domains have to be decomposed into patches. To take care of the sign ambiguity we require that the manifold is orientable, i.e. that we can choose an atlas in such a way that all Jacobians for changes of coordinates are positive on the overlap of the respective patches. Then we allow only coordinates with positive orientation and coordinate independence of (2.21) implies that the result is independent of the decomposition of the complete integration domain.[3]

The same construction can be used if ω is a p form and if $C \subset M$ is some oriented p-dimensional submanifold. Recall that we always can pull back covariant tensors along differentiable maps; we can thus regard the integral of a p form as being defined via the integral of the pull back of that form over a p-dimensional coordinate domain that parametrizes the (sub)manifold. By linearity we can extend the definition of integration to integrals of p forms over arbitrary p-cycles. Using, for example, a decomposition with rectanguar coordinate domains it can be shown that integrals of $d\omega$ over cycles are related to integrals over boundaries:

$$\int_{C_r} d\omega_{r-1} = \int_{\partial C_r} \omega_{r-1} \tag{2.22}$$

This is **Stokes' theorem**, which generalizes Gauß' and Stokes' theorems in $\mathbb{R}^3$.

Since $d^2 = 0$ we can define the **de Rham complex**

$$0 \to \Lambda^0 M \xrightarrow{d_0} \Lambda^1 M \xrightarrow{d_1} \dots \xrightarrow{d_{n-2}} \Lambda^{n-1} M \xrightarrow{d_{n-1}} \Lambda^n M \xrightarrow{\partial_n} 0 \tag{2.23}$$

in analogy to the chain complex for the bounday operator. A p form ω is called **closed**, or a **cocycle**, if $d\omega = 0$. It is called **exact**, or a **coboundary**, if $\omega = d\omega'$ for some $p-1$ form ω'. The **de Rham cohomology** groups $H^p = Z^p/B^p = \operatorname{Ker} d_p / \operatorname{Im} d_{p-1}$ are the additive groups of closed p forms modulo exact p forms, where Z^p and B^p are the cocycle and coboundary groups, respectively. As the names suggest, these groups are dual to the homology groups, at least under certain conditions. This is the content of the

[3]For rigorous definitions and proofs one often uses partitions of unity $1 = \sum_i f_i$ with $0 \le f_i \le 1$ that are compatible with a locally finite covering U_i (i.e. $f_i \in C^\infty(M)$ vanishes outside U_i). One can use simplicial or, equivalently but more convenient for integration, rectangular coordinate domains.

de Rham theorem: The bilinear map

$$([C_r],[\omega_r]) \to \int_{C_r} \omega_r \in \mathbb{R}, \qquad C_r \in Z_r,\ \omega_r \in Z^r \tag{2.24}$$

is non-degenerate. (It is easy to see that this map is independent of the representatives C_r, ω_r of the (co)homology classes $[C_r] \in H_r$, $[\omega_r] \in H^r$ because of Stokes's theorem.) If M is compact then H_r and H^r are finite-dimensional and the dimensions $b^r = \dim H^r$ coincide with the Betti numbers $b_r = \dim H_r$.

The integral $(C, \omega) := \int_C \omega$ is called the **period** of the closed r-form ω over the r-cycle C. De Rham's theorem implies that, given a basis C_i of $H_r(M, \mathbb{R})$, a closed r-form is exact iff its periods over all C_i vanish. In turn, for any set of r real numbers there exists a closed form whose periods are given by these numbers. The de Rham theorem thus provides an amazing link between topology (counting faces in triangulations) and analysis (global existence of solutions to differential equations). In particular we obtain the

Poincaré lemma: If M is contractible then $H^r = 0$ for $r > 0$, i.e. all closed forms are exact. This applies, in particular, to contractible coordinate neighborhoods.

Proof (for $\mathbb{R}^n$): We use the 'homotopy operator'

$$K : \Omega(x) \to K(\Omega(x)) = i_x \int_0^1 \frac{dt}{t}\, \Omega_t(x), \qquad \Omega_t(x) := \Omega(tx) = \frac{t^p}{p!} \Omega_{i_1 \ldots i_p}(tx) dx^{i_1} \ldots dx^{i_p}, \tag{2.25}$$

on Λ^p, where i_x is the interior product with the vector field x^m. K satisfies

$$Kd + dK = 1 \tag{2.26}$$

on p-forms, which can be seen as follows: With $f : x \to tx$ we observe $f_*(x^m) = tx^m$, $f^*(\Omega) = \Omega_t$ and $i_x f^* = f^* i_{tx} = t f^* i_x$, so that

$$K(\Omega) = \int_0^1 dt f^*(i_x \Omega), \qquad \{K, d\}\Omega = \int_0^1 dt\, f^* \{i_x, d\}\Omega = \int_0^1 dt\, f^*(\mathcal{L}_x \Omega). \tag{2.27}$$

The Lie derivative $\mathcal{L}_x \Omega$ can be written as $\mathcal{L}_x \Omega(x) = \frac{d}{d\varepsilon} \Omega_{(1+\varepsilon)}(x)_{|\varepsilon=0} = \frac{d}{dt} \Omega_t(x)_{|t=1}$ and we find

$$\{K, d\}\Omega(x) = \int_0^1 dt \frac{d}{dt} \Omega_t(x) = \Omega_1(x) - \Omega_0(x) = \Omega(x). \tag{2.28}$$

Putting the pieces together we find $\Omega = d\omega$ with $\omega = K\Omega$ for any closed p-form Ω because $d\omega = \{K, d\}\Omega = \Omega$. Note that our construction yields a 'potential' ω that satisfies the 'radial gauge' condition $i_x \omega = 0$ (which makes ω unique because $d(\delta\omega) = i_x(\delta\omega) = 0 \Rightarrow \mathcal{L}_x(\delta\omega) = 0$).

Remark: For electromagnetic field strenghts the absense of magnetic sources $dF = 0$ thus implies the local existence of a gauge potential 1-form A with $F = dA$.

Chapter 3

Riemannian geometry

So far we only considered differentiable manifolds without additional structure. Such structure is needed if we want to introduce geometrical concepts like parallel transport of vectors, distances and curvature. To this end Riemannian geometry introduces a metric and a connection, which are closely related by a compatibility condition. First we add a **connection**, which tells us how to relate tangent spaces at different points and how to do parallel transport of vectors along curves on the manifold. The result of parallel transport in general depends on the curve, which leads to the notion of (intrinsic) **curvature** (and torsion). In many applications it is also important to be able to measure **distances** and **angles** with the parallel transport being compatible in the sense that it conserves scalar products. In order to describe spinors in curved space it is necessary to introduce the vielbein and the **spin connection**. A metric on tangent space will allow the definition of the **Hodge dual** and of an inner product on the exterior algebra.

3.1 Covariant derivatives and connections

The simplest situation is the embedding of a manifold into some Euclidean space E of higher dimension with coordinates X^μ. Then tangent vectors can be identified with vectors in E, with a basis of T_xM provided by $\partial_m X^\mu(x)$. The components of the induced metric are $g_{mn} = \partial_m X^\mu \partial_n X^\nu \delta_{\mu\nu}$. Partial differentition of tangent vectors $V^\mu = V^m \partial_m X^\mu$, however, in general takes us out of tangent space. In this situation we can define a *covariant derivative* of tangent vector fields by orthogonal projection $D_i V = \mathrm{tpr}(\partial_i V)$ of $\partial_i V$ to the tangent space at the appropriate point (the linear operator 'tpr' denotes this projection). We will see below that this leads to the unique metric-compatible torsion-free connection on the Riemannian manifold (M, g).

In general we would like to have a covariant notion of a derivative of a vector or tensor in some direction ξ, like, for example, a covariant derivative along the tangent vector to a coordinate line $\xi = \partial_i$. The problem is that the partial derivatives of vector components $\partial_i v^m$ compare components of vectors at different points. The corresponding tangent spaces are, however, different spaces that are not related in a coordinate independent way. We thus need to introduce a relation between tangent spaces at neighboring points.[1] (The notion of *parallel transport* is closely related to this: A vector field along a curve will be called parallel iff its covariant derivative in the direction of the tangent vector to the curve vanishes at each point). Such a relation is provided by an **affine connection**

$$D : TM \times TM \to TM, \qquad (\xi, v) \to D_\xi v \tag{3.1}$$

which by definition is bilinear and satisfies

$$D_{f\xi} v = f D_\xi v, \qquad D_\xi (fv) = \xi(f) v + f D_\xi v. \tag{3.2}$$

It can be thought of as a GL_n-valued 1-form, since we have a linear map $TM \to TM$ for each ξ. (The first of these equation states that $D_.()$ is a 1-form and the second that D_ξ acts as a derivation on products; by demanding that D_ξ is a derivation on arbitrary tensor products we can extend its definition to the tensor algebra.)

To understand the content of this defintion in terms of coordinates we evaluate the **covariant derivative** $D_i v^n := (D_{\partial_i} v)^n$ using the **connection coefficients** ${\Gamma_{ij}}^k$ that are defined by ${\Gamma_{ij}}^k \partial_k := D_{\partial_i} \partial_j$. Evaluating the components of $D_i(v) = D_i(v^n \partial_n)$ we obtain

$$D_i v^n = \partial_i v^n + {\Gamma_{ij}}^n v^j, \qquad D_i = \partial_i + {\Gamma_{ij}}^n {\Delta_n}^j. \tag{3.3}$$

The second formula, which involves the GL_n symbol, extends the definition of the covariant derivative to a derivation on the tensor algebra.

Exercise 10: Show that the connection coefficients transform as

$$ {\Gamma_{ij}}^k(y) = \left(\frac{\partial^2 x^l}{\partial y^i \partial y^j} + \frac{\partial x^m}{\partial y^i} \frac{\partial x^n}{\partial y^j} {\Gamma_{mn}}^l(x) \right) \frac{\partial y^k}{\partial x^l} \tag{3.4}$$

under coordinate transformations (use, for example, the definition ${\Gamma_{ij}}^k(y) \partial_{y^k} = D_{\partial_{y^i}}(\partial_{y^j})$, the equations (3.2) and $\partial_{y^l} = \frac{\partial x^n}{\partial y^l} \partial_{x^n}$). Inserting $x = y + \xi + O(\xi^2)$ this implies the infinitesimal transformation law

$$\delta_\xi {\Gamma_{ij}}^n = \partial_i \partial_j \xi^n + \mathcal{L}_\xi {\Gamma_{ij}}^n, \tag{3.5}$$

where $\mathcal{L}_\xi {\Gamma_{ij}}^n$ denotes the terms that would arise if ${\Gamma_{ij}}^n$ were a tensor.[2]

[1] The Lie derivative $\mathcal{L}_\xi$ of a vector depends on $\partial_i \xi$, so it is not a (coordinate independent) directional derivative.

[2] Note that ${\Gamma_{ij}}^k dx_j = D_i dx^k$ can be interpreted as a 'tensor' is some sense if we regard it as the difference to the flat connection on a chart $\mathcal{U}_x$; this is, however, not coordinate independent.

3.2 Curvature and torsion

The transformation law (3.4) implies that the anti-symmetric parts of the connection coefficients

$$T_{ij}{}^k := \Gamma_{ij}{}^k - \Gamma_{ji}{}^k \tag{3.6}$$

are the components of a tensor, which is called the **torsion tensor**; by an appropiate choice of local coodinates we may, on the other hand, make the symmetric part $\Gamma_{(ij)}{}^k$ vanish at any given point. Furthermore, the difference of two connections and variations of a connection are tensors.

Since the commutator $[D_m, D_n]$ is again a derivation it must be proportional to D_l and $\Delta_i{}^k$. We find the following algebra of derivations:

$$[D_i, D_j] = -T_{ij}{}^l D_l + R_{ijk}{}^l \Delta_l{}^k, \tag{3.7}$$

$[\Delta_i{}^k, D_j] = -\delta_j^k D_i$ and $[\Delta_i{}^k, \Delta_j{}^l] = \delta_i^l \Delta_j{}^k - \delta_j^k \Delta_i{}^l$, with the **curvature tensor**

$$R_{ijk}{}^l = \partial_i \Gamma_{jk}{}^l - \partial_j \Gamma_{ik}{}^l + \Gamma_{in}{}^l \Gamma_{jk}{}^n - \Gamma_{jn}{}^l \Gamma_{ik}{}^n. \tag{3.8}$$

Exercise 11: Check this results for the curvature $R_{ijk}{}^l$ by direct evaluation of $[D_i, D_j]$ or by application of (3.7) to a vector field. (Curvature and torsion are coefficients of independent covariant derivations in a tensorial equation and hence must themselves transform as tensors.)

Exercise 12: Show that the Jacobi identity $\circlearrowleft\!\!\!\!\!\sum_{ijk}[D_i, [D_j, D_k]]$ implies the **Bianchi identities**

$$1^{\text{st}}\ \text{BI}: \qquad \circlearrowleft\!\!\!\!\!\!\sum_{ijk} \left(R_{ijk}{}^l - D_i T_{jk}{}^l + T_{jk}{}^n T_{in}{}^l \right) = 0, \tag{3.9}$$

$$2^{\text{nd}}\ \text{BI}: \qquad \circlearrowleft\!\!\!\!\!\!\sum_{ijk} \left(D_i R_{jkl}{}^m - T_{jk}{}^n R_{inl}{}^m \right) = 0, \tag{3.10}$$

for curvature and torsion, where the *cyclic sum* $\circlearrowleft\!\!\!\!\!\sum_{ijk}$ denotes the sum over cyclic permutations of the ordered set (i, j, k).

Interpreting the torsion as vector valued two form $T = \frac{1}{2} dx^i dx^j T_{ij}{}^l \partial_l$ and the curvature as a GL_n-valued two form $R = \frac{1}{2} dx^i dx^j R_{ijk}{}^l \Delta_l{}^k$ it is straightforward to check the coordinate independent formulas

$$T(X,Y) = D_X Y - D_Y X - [X,Y], \qquad R(X,Y)Z = D_X D_Y Z - D_Y D_X Z - D_{[X,Y]} Z \tag{3.11}$$

for arbitrary vector fields X, Y, Z, which can be used as a coordinate independent definition of the tensors T and R. The geometrical meaning of curvature and torsion can be clarified by inserting $X = \partial_i$ and $Y = \partial_j$: Then the Lie bracket $[X, Y]$ does not contribute and nonvanishing torsion implies that 'parallelograms do not close'. Nonvanishing curvature implies that a vector changes when transported along an infinitesimal closed loop (like, for example, along a parallelogram of coordinate lines). The subgroup of GL_n that is generated by parallel transport of vectors along arbitrary closed loops on a manifold is called **holonomy group**.

A **Riemannian manifold** is a manifold with a **metric** $g = dx^m \otimes dx^n g_{mn}$, which by definition is a positive definite[3] symmetric bilinear form on tangent space

$$g(v,w) = v^m g_{mn} w^n, \qquad g_{mn} = g_{nm}, \qquad g := \det g_{mn} \neq 0. \tag{3.12}$$

A metric defines lengths and angles among tangent vectors at a given point and it is natural to demand that parallel transport does not change these quantities. This is equivalent to covariant constance $D_m g_{ij} = 0$ of the metric tensor. This property is called **metric compatibility**

$$D_m g_{ij} = \partial_m g_{ij} - \Gamma_{mij} - \Gamma_{mji} = 0, \qquad \text{with} \quad \Gamma_{ijk} := g_{kl}\Gamma_{ij}{}^l. \tag{3.13}$$

Exercise 13: Show that metric compatibility implies that the connection coefficients are

$$\Gamma_{ijn} = \hat\Gamma_{ijn} + \tfrac{1}{2}(T_{ijn} + T_{nij} - T_{jni}), \qquad \hat\Gamma_{ijn} := \tfrac{1}{2}(\partial_i g_{jn} + \partial_j g_{in} - \partial_n g_{ij}). \tag{3.14}$$

$\hat\Gamma_{ijn} = g_{nl}\hat\Gamma_{ij}{}^l$ is called **Christoffel symbol** or **Levi-Civita connection**. Thus the tensors metric and torsion fix a unique metric-compatible connection.
The torsion dependent contribution $K_{ijn} = K_{i[jn]} = \frac{1}{2}(T_{ijn} + T_{nij} - T_{jni})$ is called **contorsion**. Note that $\Gamma_{in}{}^n = \hat\Gamma_{in}{}^n$ but $\Gamma_{in}{}^i = \hat\Gamma_{in}{}^i + T_{in}{}^i$ and $g^{ij}\Gamma_{ij}{}^n = g^{ij}\hat\Gamma_{ij}{}^n + g^{nl}T_{li}{}^i$.

The metric provides a natural n form, the **volume form** $\sqrt{g}\, d^n x$ so that any scalar ϕ can be used to build an n form $\sqrt{g}\, d^n x \phi$ whose integral over the manifold is coordinate independent. It also can be used to define curvature tensors with fewer indices: A single contraction gives the **Ricci tensor** $\mathcal{R}_{ik} := R_{ijk}{}^j$ and a second contraction gives the **curvature scalar** $\mathcal{R} = g^{ik}R_{ijk}{}^j$, which is used to build the Einstein Hilbert action in general relativity

$$S_{EH} = \frac{1}{16\pi G_N}\int d^n x \sqrt{-g}\,\mathcal{R} \tag{3.15}$$

The following are some variational formulas that are useful for the derivation of the field equations:

$$\delta g^{ij} = -g^{im}g^{jn}\delta g_{mn}, \qquad \delta g = g\, g^{mn}\delta g_{mn}, \tag{3.16}$$

$$\delta\mathcal{R}_{mn} = D_m\delta\Gamma_{jn}{}^j - D_j\delta\Gamma_{mn}{}^j + T_{mj}{}^l\delta\Gamma_{ln}{}^j \qquad \textbf{Palatini identity} \tag{3.17}$$

$$\tfrac{\delta}{\delta g^{mn}}\left(\sqrt{g}e^\phi\hat{\mathcal{R}}\right) = \sqrt{g}\left(e^\phi\hat G_{mn} + \hat D_m\hat D_n e^\phi - g_{mn}g^{kl}\hat D_k\hat D_l e^\phi\right) \tag{3.18}$$

where $G_{mn} = \mathcal{R}_{mn} - \frac{1}{2}g_{mn}\mathcal{R}$ is the **Einstein tensor** and ϕ an arbitrary scalar field. The Palatini identity is independent of the metric and holds for arbitrary affine connections.[4] The formula for

[3] If g is non-degenerate but not positive definite then the manifold is called pseudo-Riemannian; the signature of g cannot change in a connected component of the manifold.

[4] It can be used to simplify the computation of the Einstein equations by treating $\Gamma_{ij}{}^l$ as an independent field whose usual dependence on the metric follows from its variational equation.

the veriation of the determinant of the metric is easily obtained by using $\ln\det X_{ij} = \operatorname{tr}\ln X_{ij}$, so that $\delta\ln g = \operatorname{tr}(g_{..})^{-1}\delta g_{..} = g^{mn}\delta g_{mn}$ with $g^{mn} := (g_{mn})^{-1}$.

For derivatives of the determinant $g = \det g_{mn}$ of the metric we find

$$\partial_p \ln\sqrt{g} = \hat\Gamma_{pm}{}^m = \Gamma_{pm}{}^m, \quad \partial_p(\sqrt{g}v^p) = \sqrt{g}(D_p + T_{pl}{}^l)v^p, \quad \mathcal{L}_\xi\sqrt{g} = \partial_p(\xi^p\sqrt{g}). \tag{3.19}$$

Therefore $\sqrt{g}\,(D_n X_{i\ldots j}Y^{ni\ldots j} + X_{i\ldots j}D_n Y^{ni\ldots j} + T_{nl}{}^l X_{i\ldots j}Y^{ni\ldots j}) = \partial_n(\sqrt{g}\,X_{i\ldots j}Y^{ni\ldots j})$ is a total derivative, which implies a covariant partial integration rule. These formulas also confirm that $\sqrt{g}\,\phi$ gives coordinate independ integrals: Under an infinitesimal coordinate transformation a *scalar density* $\sqrt{g}\,\phi$ transforms into the total derivative $\mathcal{L}_\xi(\sqrt{g}\,\phi) = \partial_m(\xi^m\sqrt{g}\,\phi)$, whose integral gives a surface term that just compensates the shift in the integration domain. (For a tensor $T^{\cdots}_{\cdots}$ the product $\sqrt{g}\,T^{\cdots}_{\cdots}$ is called a **tensor density**.)

Exercise 14: Show that the 'orthogonal projection to tangent space' definition of the covariant derivative that can be used if a Riemannian manifold is realized by an embedding into Euclidean space is equal to the Levi-Civita connection. Hint: Use the definition $\Gamma_{mn}{}^l\partial_l X^\mu := D_m(\partial_n X^\mu) := \operatorname{tpr}(\partial_m(\partial_n X^\mu))$ and relate Γ_{mnl} to the partial derivatives of the induced metric $g_{mn} = \partial_m X^\mu \partial_n X^\nu \delta_{\mu\nu}$. (Note that $V_\mu \operatorname{tpr}(W^\mu) = V_\mu W^\mu$ if V^μ is tangential.)

The length of a curve $x(t)$ with curve parameter t on a Riemannian manifold is given by the integral $S = \int ds = \int dt L(t)$ with $L = \sqrt{\dot x^m \dot x^n g_{mn}(x)}$. A curve of extremal length is called a **geodesic**. Variation of $x(t)$ implies

$$\frac{\delta S}{\delta x^p} = \frac{1}{2L}\dot x^m \dot x^n \partial_p g_{mn} - \frac{d}{dt}\left(\frac{2\dot x^n g_{np}}{2L}\right) = g_{pn}\dot x^n \frac{\dot L}{L^2} - \frac{g_{pn}}{L}\left(\ddot x^n + \dot x^i \dot x^j \Gamma_{ij}{}^n\right) = 0 \tag{3.20}$$

This equation does not fix $x(t)$ as a function of t because the 'action' S is reparametrization invariant. We may thus choose an **affine parametrization** of the curve, with the curve parameter proportional to the length (i.e. we impose $\dot L = 0$), to simplify the equation. Then only the last term remains and we obtain the geodesic equation

$$\ddot x^n + \dot x^i \dot x^j \hat\Gamma_{ij}{}^n. \tag{3.21}$$

This equation can be obtained directly using the action $S = \int dt L^2$. (In general relativity the action of a structureless free particle is proportional to the proper time; such particles therefore move on geodesics.)

There is an alternative definition of a geodesic as a curve whose tangent vectors are parallel and – with affine parametrization – of constant length along the curve (the curve is *autoparallel*). This means that the covariant derivative of $\dot x$ along $\dot x^m\partial_m$ vanishes, i.e. $D_{\dot x}\dot x^m = \ddot x^n + \dot x^i \dot x^j \Gamma_{ij}{}^n$. The two definitions agree for the Levi-Civita connection $\Gamma = \hat\Gamma$.

Since R and $\hat{R}$ both are tensors the difference has to be a tensor as well. It is, indeed, straightforward to check that

$$R_{ijk}{}^l = \hat{R}_{ijk}{}^l + \hat{D}_i K_{jk}{}^l - \hat{D}_j K_{ik}{}^l + K_{in}{}^l K_{jk}{}^n - K_{jn}{}^l K_{ik}{}^n, \tag{3.22}$$

where $K_{ijn} = \Gamma_{ijn} - \hat{\Gamma}_{ijn} = \frac{1}{2}(T_{ijn} + T_{nij} - T_{jni})$ is the contorsion. There is thus no loss in generality if we use the Levi-Civita connection and consider the torsion as an independent tensor field (the same is true for non-metricity components of the connection, as soon as a metric tensor if available).

The curvature tensor $\hat{R}$ of the Levi-Civita connection has additional symmetries: Since

$$\hat{R}_{ijkl} = \tfrac{1}{2}(\partial_i\partial_k g_{jl} - \partial_i\partial_l g_{jk} - \partial_j\partial_k g_{il} + \partial_j\partial_l g_{ik}) + \hat{\Gamma}_{ik}{}^n\hat{\Gamma}_{jln} - \hat{\Gamma}_{jk}{}^n\hat{\Gamma}_{iln} \tag{3.23}$$

symmetry of $\hat{\Gamma}_{ij}{}^k$ implies $\hat{R}_{ijkl} = \hat{R}_{klij} = -\hat{R}_{jikl}$; the first Bianchi identity reads $\sum_{ijk}\hat{R}_{ijk}{}^l = 0$.

The following formulas for a **Weyl rescaling** of the metric are valid for the Levi-Civita connection: Let $g^*_{ij} := e^{-2\sigma} g_{ij}$ and define $\sigma_l := \partial_l\sigma$, $\sigma_{mn} := D_m D_n\sigma + \sigma_m\sigma_n$. Then

$$e^{2\sigma} R^*_{ijkl} = R_{ijkl} + g_{il}\sigma_{jk} - g_{ik}\sigma_{jl} - g_{jl}\sigma_{ik} + g_{jk}\sigma_{il} + \sigma^n\sigma_n(g_{ik}g_{jl} - g_{jk}g_{il}) \tag{3.24}$$

$$R^*_{ij} = R_{ij} + (n-2)(g_{ij}\sigma^n\sigma_n - \sigma_{ij}) - g_{ij}\Delta\sigma, \quad e^{-2\sigma}R^* = R + (n-1)\left((n-2)\sigma^n\sigma_n - 2\Delta\sigma\right), \tag{3.25}$$

$$\Delta f := D^n D_n f = (g^{mn}\partial_m\partial_n - g^{ij}\Gamma_{ij}{}^n\partial_n)f, \tag{3.26}$$

$$\overline{\Gamma}^l := g^{mn}\Gamma_{mn}{}^l - \tfrac{2}{n+1}g^{ml}\Gamma_{mn}{}^n \;\Rightarrow\; \overline{\Gamma}^*_l = \overline{\Gamma}_l + \tfrac{(n+2)(n-1)}{n+1}\sigma_l \tag{3.27}$$

$$e^{-2\sigma}\Delta^* = \Delta - (n-2)\sigma^n\partial_n, \quad (\Delta^* + \tfrac{1}{4}\tfrac{n-2}{n-1}R^*)\phi^* = e^{\frac{n+2}{2}\sigma}(\Delta + \tfrac{1}{4}\tfrac{n-2}{n-1}R)\phi \quad \text{with} \quad \phi^* = e^{\frac{n-2}{2}\sigma}\phi \tag{3.28}$$

The conformal (Weyl) curvature tensor $C_{ijk}{}^l = C^*_{ijk}{}^l$ is given by

$$C_{ijkl} := R_{ijkl} + \frac{g_{il}R_{jk} - g_{ik}R_{jl} - g_{jl}R_{ik} - g_{jk}R_{il}}{n-2} + \frac{R(g_{ik}g_{jl} - g_{il}g_{jk})}{(n-2)(n-1)}, \tag{3.29}$$

$$C_{ijkl}C^{ijkl} = R_{ijkl}R^{ijkl} - \frac{4R_{ik}R^{ik}}{n-2} + \frac{2R^2}{(n-1)(n-2)}, \qquad \sqrt{g}C^2 = \left(\sqrt{g}C^2\right)^* \Leftrightarrow n = 4 \tag{3.30}$$

3.3 The Killing equation and the conformal group

A Riemannian manifold has a (continuous) symmetry if there is a family of coordinate transformations that leaves a fixed metric invariant. The vector field ξ that corresponds to an infinitesimal symmetry thus has to satisfy the **Killing equation**

$$\mathcal{L}_\xi g_{mn} = D_m\xi_n + D_n\xi_m = 0 \tag{3.31}$$

A **conformal transformation** is a coordinate transformation $x \to x'(x)$ that amounts to Weyl rescaling of the metric, i.e $g'_{mn}(x') = e^{-2\sigma(x)} g_{mn}(x)$. The existence of an infinitesimal conformal transformation thus requires the existence of a solution to the *conformal Killing equation*

$$\mathcal{L}_\xi g_{mn} = D_m \xi_n + D_n \xi_m = -2\sigma(x) g_{mn} \tag{3.32}$$

($\sigma(x) = -\frac{1}{n} D_l \xi^l$ follows from taking the trace). Such solutions exist, for example, in flat space.

The conformal group $SO(p+1, q+1)$ of a flat space with signature (p, q) can be obtained by solving the equation

$$h_{mn} := \partial_m \xi_n + \partial_n \xi_m + 2\sigma(x)\eta_{mn} = 0, \tag{3.33}$$

where $\eta_{mn} = \mathrm{diag}(1, \ldots, 1, -1, \ldots, -1)$ with $q = n - p$ negative entries. Taking the trace (i.e. contracting with the inverse metric) and the double divergence (i.e. contracting with $\partial_m \partial_n$) we obtain

$$\eta^{mn} h_{mn} = 2(\partial\xi + n\sigma) = 0 \qquad \Rightarrow \qquad \partial^m \partial^n h_{mn} = \left(2 - \tfrac{2}{n}\right) \Box \partial\xi = 0. \tag{3.34}$$

For $n \neq 1$ this implies $\Box\Lambda = \Box\partial\xi = 0$. (In one dimension there is, of course, no restriction on ξ.) Now we compute the symmetrized derivative of the divergence of h_{mn},

$$\partial_l \partial^m h_{mn} + \partial_n \partial^m h_{ml} = \Box(\partial_l \xi_n + \partial_n \xi_l) + 2\partial_l \partial_n \partial\xi + 4\partial_l \partial_n \Lambda = 2\left(1 - \tfrac{2}{n}\right) \partial_l \partial_n \partial\xi = 0, \tag{3.35}$$

where we used $\Box(\partial_l \xi_n + \partial_n \xi_l) = -2\eta_{ln} \Box\Lambda = 0$. In more than two dimensions this implies that all second derivatives of Λ vanish, i.e.

$$n > 2 \qquad \Rightarrow \qquad \Lambda = -\tfrac{1}{n} \partial\xi = 2\, bx - \lambda \tag{3.36}$$

for some constants λ and b_m. In order to solve for ξ we still need the antisymmetric part of $\partial_m \xi_n$, whose derivative is

$$\partial_l(\partial_m \xi_n - \partial_n \xi_m) = \partial_m \partial_l \xi_n - \partial_n \partial_l \xi_m = 2(\eta_{lm} \partial_n \Lambda - \eta_{ln} \partial_m \Lambda) = 4(\eta_{lm} b_n - \eta_{ln} b_m). \tag{3.37}$$

Integrating this equation we find

$$\tfrac{1}{2}(\partial_m \xi_n - \partial_n \xi_m) = \omega_{mn} + 2x_m b_n - 2x_n b_m \tag{3.38}$$

with an antisymmetric integration constant $\omega_{mn} = -\omega_{nm}$. Putting the pieces together

$$\partial_m \xi_n = \omega_{mn} + 2x_m b_n - 2x_n b_m + \eta_{mn}(\lambda - 2bx) \tag{3.39}$$

and thus

$$\xi^n = a^n + x^m \omega_m{}^n + \lambda\, x^n + x^2 b^n - 2bx\, x^n. \tag{3.40}$$

a, ω, λ and b generate translations, Lorentz transformations, dilatations and 'special conformal transformations', respectively.

Computing the Lie brackets of these vector fields it can be shown that the conformal group is isomorphic to $SO(p+1, q+1)$ for a space with signature (p, q). The finite form of the special conformal transformations is $x^n \to y^n = (x^n + x^2\, b^n)/(1 + 2bx + b^2 x^2)$. They form a subgroup, as can be seen by writing the transformation as a combination of two inversions and a translation $\vec{y}/y^2 = \vec{x}/x^2 + \vec{b}$ (note that $x^2/y^2 = 1 + 2bx + b^2 x^2$; the inversion $\vec{x} \to \vec{x}/x^2$ itself is also a conformal map, but it has negative functional determinant – the radial direction is reversed – and hence is not continuously connected to the identity). The functional determinant is $|\frac{\partial y}{\partial x}| = (\frac{y^2}{x^2})^n = (1 + 2bx + b^2 x^2)^{-n}$ and $\eta^{mn} \frac{\partial y^i}{\partial x^m} \frac{\partial y^j}{\partial x^n} = \eta^{ij}/(1 + 2bx + b^2 x^2)^2$.

3.4 Hodge duality and inner products

Given a metric with $\mathrm{sign}(g) = (-)^s$ the natural **volume form** is $\varepsilon = \sqrt{|g|}dx^1 \ldots dx^n$ and we can define the **inner product** for p forms $(\alpha|\beta) := \frac{1}{p!}\alpha_{i_1\ldots i_p}\beta^{i_1\ldots i_p}$, where

$$(dx^{i_1}\ldots dx^{i_p}|dx^{j_1}\ldots dx^{j_p}) = \det g^{i_a j_b} \tag{3.41}$$

The **Hodge star** operation $* : \Lambda^p \to \Lambda^{n-p}$ is defined by $(*\omega)_{i_{p+1}\ldots i_n} = 1/p!\, \varepsilon_{i_1\ldots i_n}\omega^{i_1\ldots i_p}$ with

$$\varepsilon(\alpha|\beta) = \alpha \wedge *\beta, \quad *^2 = (-)^{p(n-p)+s}, \quad (*\alpha| *\beta) = (-)^s(\alpha|\beta), \tag{3.42}$$

as well as

$$*1 = \varepsilon, \qquad \alpha \wedge *\beta = \beta \wedge *\alpha, \qquad \alpha \wedge *\alpha = 0 \Leftrightarrow \alpha = 0. \tag{3.43}$$

The **codifferential** δ, with

$$\delta\omega := (-)^p *^{-1} d * \omega = (-)^{pD+D+s+1} * d * \omega, \quad * \delta = (-)^p d*, \quad * d = (-)^{p+1}\delta*, \tag{3.44}$$

is, up to surface terms, the adjoint of the differential d w.r.t. the **scalar product**

$$[\alpha|\beta] := \int \alpha \wedge *\beta = [\beta|\alpha], \quad [d\alpha|\beta] = [\alpha|\delta\beta] + \int d(\alpha \wedge *\beta). \tag{3.45}$$

It allows us to define the **Laplace–Beltrami** operator $\Delta := d\delta + \delta d$ on p-forms in curved space, for which we find

$$(\Delta\omega)_{i_1\ldots i_p} = -g^{mn}D_mD_n\omega_{i_1\ldots i_p} - \sum_{1\le a\le p}\omega_{i_1\ldots j\ldots i_p}R_{i_a}{}^j + \sum_{0\le a<b\le p}\omega_{i_1\ldots j\ldots k\ldots i_p}R_{i_a i_b}{}^{jk}. \tag{3.46}$$

(In calculations it is sometimes useful to have, instead of an equivalent factor $1/p!$, angles $\langle\ldots\rangle$ around indices that enforce their ordering. Then $(\delta\omega)_{i_2\ldots i_p} = -\delta^{\langle j_1\ldots j_p\rangle}_{i_1\ldots i_p}D^{i_1}\omega_{\langle j_1\ldots j_p\rangle}$ with the symbol $\delta^{j_1\ldots j_p}_{i_1\ldots i_p} := \det \delta^{j_b}_{i_a}$ for $1 \le a, b \le p$.)

On a compact Riemannian manifold the Laplacian is a positive operator because

$$[\omega|\Delta\omega] = [d\omega|d\omega] + [\delta\omega|\delta\omega] \ge 0. \tag{3.47}$$

A p-form ω is called *harmonic* if $\Delta\omega = 0$. On a compact orientable Riemannian manifold ω is thus harmonic iff it is closed and co-closed [eg80, NA90]. The **Hodge decomposition** theorem states that there is a unique decomposition of a p-form into an exact, a co-exact and a harmonic piece:

$$\omega_p = d\omega_{p-1} + \delta\omega_{p+1} + \omega_{harm.} \tag{3.48}$$

Each cohomology class therefore has a unique harmonic representative, which implies the Hodge duality $b^r = b^{n-r}$ of Betti numbers. (The linear space of harmonic forms has the same dimension as the de Rham cohomology group, but no ring structure because δ is not an anti-derivation on the exterior algebra and the wedge product of harmonic forms need not be harmonic.)

In physics an important application are the inhomogeneous Maxwell equations $\delta F = j$, or, equivalently $d * F = *j$. The action can be written as $\int F \wedge *F$ and $F \to *F$ corresponds the electric-magnetic duality $E \to B$ and $B \to -E$ of the source-free equations.