Symmetry of universal deformation formulae

Pierre Bieliavsky

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2.The Moyal twist (Rieffel's version 1994)

Assume: $\mathbb{A}$ comes with an **Abelian symmetry**

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Then ($\mathbb{A}$: $C^\star$-algebra, α: strongly continuous isometrical), $a, b \in \mathbb{A}^\infty$ (C^∞ vectors), $\theta \in \mathfrak{so}(d)$:

$$a \star_\theta b \ := \int_{\mathbb{R}^d \times \mathbb{R}^d} e^{ix.y} \, \alpha_x(a) \, . \, \alpha_{\theta y}(b) \, \mathrm{d}x \, \mathrm{d}y$$

defines a pre-$C^\star$-**associative algebra** structure on $\mathbb{A}^\infty$.

3. Formal Drinfel'd theory

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Fact 1: Every $\mathcal{U}(\mathfrak{b})$-module algebra $(\mathbb{A}, \mu_0)$ yields a deformed **associative** algebra: $\mathbb{A}_F \; := \; (\, \mathbb{A}[[h]] \, , \, \mu_F \; := \; \mu_0 \, F \,) \, .$

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Fact 2: $\mathcal{U}_h(\mathfrak{b}) \;:=\; (\mathcal{U}(\mathfrak{b})[[h]]\,,\, \mu_{\mathcal{U}}\,,\, \Delta_F \;:=\; \Delta\, F_L\, F_R^{-1}\,)$ underlies a Hopf-algebra (Drinfeld non-standard quantum group).

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$\mathbb{A}_F$ is a $\mathcal{U}_h(\mathfrak{b})$-module algebra (**external symmetry**).

Symplectic Lie groups

Observation: $[F_1]$ is a left-invariant Poisson structure on $C^\infty(\mathbb{B})$ whose symplectic leaf $\mathbb{S}$ through unit e in $\mathbb{B}$ constitutes a immersed **Lie sub-group** of $\mathbb{B}$.

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Definition [Lichnerowicz]: A **symplectic Lie group** is a pair $(\mathbb{S}, \omega^{\mathbb{S}})$ where $\mathbb{S}$ is a connected Lie group and ω is a *left-invariant symplectic structure* on $\mathbb{S}$. This leads to the notion of **symplectic Lie algebra**.

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Structure theory for **Kahler** Lie groups [Piatetskii-Shapiro]: building blocks are **elementary normal J-groups** i.e. Iwasawa factors of $SU(1, n)$:

$$\mathbb{S} \;:=:\; AN \;\simeq\; \mathbb{R} \ltimes_{\text{solvable}} (\text{Heisenberg}) \,.$$

Proposition [Bieliavsky, Voglaire] Every *elementary normal J-group* $(\mathbb{S}, \omega)$ admits a unique structure of *Ricci type solvable symplectic symmetric space* i.e. admits a complete symplectic affine connection ∇ such that:

1. centered geodesic symmetries extends as global symplectic affine transformations:

$$s_x : \mathbb{S} \to \mathbb{S} : y \mapsto s_x y \quad (x \in \mathbb{S})$$

2. $\mathbb{S} \ < \ \mathrm{Aff}(\nabla) \cap \mathrm{Symp}(\omega)$
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Global diffeomorphism:

$$\Phi : \mathbb{S}^3 \to \mathbb{S}^3 : (x, y, z) \mapsto (t, s_z t, s_y s_z t) \quad t \; = \; s_x s_y s_z t$$

$$S(x, y, z) \; := \; \mathrm{Symplectic\ Area}\Phi(x, y, z) \; .$$

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Theorem [Bieliavsky, J. Sympl. Geom. 2002] There exists a map $\delta : \mathcal{O}_M(\mathbb{R})^\times \to C^\infty(\mathbb{S}^3)^{\mathrm{inv.}}$ such that to every $\mathbf{m} \in C^\infty(\mathbb{R}, \mathcal{O}_M(\mathbb{R}))$ satisfying $\mathbf{m}_0 \equiv 1$ is associated an associative Hilbert algebra $(\mathcal{H}_{\theta,\mathbf{m}}, \star_{\theta,\mathbf{m}})$ such that

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3. $$\text{left reg. } \mathbb{S} \;\subset\; \mathrm{Aff}(\nabla) \cap \mathrm{Symp}(\omega) \;\subset \mathtt{Aut}(\mathcal{H}_{\theta,\mathbf{m}}, \star_{\theta,\mathbf{m}})\ .$$

3. Universal Deformation Formulae

$$K_{\theta,\mathbf{m}}(x,y) \;:=\; \frac{1}{\theta^{\dim \mathbb{S}}}\, \sqrt{\mathrm{Jac}_{\Phi}(x,y,e)}\, \delta\mathbf{m}_{\theta}(x,y,e)\, e^{\frac{i}{\theta}S(x,y,e)} \;.$$

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Theorem [Bieliavsky, Gayral, 2010] $\mathbb{A}$: Fréchet algebra; $\alpha : \mathbb{S} \times \mathbb{A} \to \mathbb{A}$ strongly continuous isometrical action by automorphisms. Then

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1. For all $a, b \in \mathbb{A}^\infty$ (C^∞-vectors), the oscillating integral:

$$a \star^{\mathbb{A}}_{\theta,\mathbf{m}} b \;:=\; \int_{\mathbb{S}\times\mathbb{S}} K_{\theta,\mathbf{m}}(x,y)\,\alpha_x(a)\,\alpha_y(b)\,\mathrm{d}y\,\mathrm{d}z$$

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2. The map $\mathbb{A}^{\infty} \times \mathbb{A}^{\infty} \to \mathbb{A}^{\infty} : (a,b) \mapsto a \star^{\mathbb{A}}_{\theta,\mathbf{m}} b$ is an associative Fréchet algebra structure on $\mathbb{A}^{\infty}$.

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3. When $\mathbb{A}$ is $C^\star$, $(\mathbb{A}^\infty, \star^{\mathbb{A}}_{\theta,\mathbf{m}})$ is pre-$C^\star$ (non-Abelian Calderon-Vaillancourt).

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3. When $\mathbb{A}$ is $C^\star$, $(\mathbb{A}^\infty, \star^{\mathbb{A}}_{\theta,\mathbf{m}})$ is pre-$C^\star$ (non-Abelian Calderon-Vaillancourt).

4. For fixed $\mathbf{m}$, setting $\mathbb{A}_\theta := \mathrm{completion}(\mathbb{A}^\infty, \star^{\mathbb{A}}_{\theta,\mathbf{m}})$ defines a continuous field of $C^\star$-algebras deforming $\mathbb{A} = \mathbb{A}_0$.

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1. Multiplier Hopf algebras

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Lemma: Group co-product map $\Delta : \tilde{\mathcal{S}} \to M(\tilde{\mathcal{S}}\hat{\otimes}\tilde{\mathcal{S}})$.

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$$F^{L}_{\mathbf{m}} := \int_{\mathbb{S}\times\mathbb{S}} K_{\theta,\mathbf{m}}(x,y)\, R^{\star}_{x} \otimes R^{\star}_{y}\, \mathrm{d}x\, \mathrm{d}y\ .$$

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Theorem [Bieliavsky, D'Andrea, Gayral] Set

$$F^L_{\mathbf{m}} \;:=\; \int_{\mathbb{S}\times\mathbb{S}} K_{\theta,\mathbf{m}}(x,y)\, R^\star_x \otimes R^\star_y \,\mathrm{d}x\,\mathrm{d}y\ .$$

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(4) Group co-product map $\Delta : \tilde{\mathcal{S}} \to M_\star(\tilde{\mathcal{S}}\hat{\otimes}\tilde{\mathcal{S}})$

Recall: Kac-Takesaki operator: $W : L^2(\mathbb{S}\times\mathbb{S}) \to L^2(\mathbb{S}\times\mathbb{S})$

$$W\Phi(x,y) \ := \ \Phi(xy,y) \qquad (W := (I\otimes\mu_0)\circ(\Delta\otimes I))$$

is **unitary** and satisfies **pentagonal equation**:
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(2) the formula

$$W_\star \ := \ (I \otimes \star) \circ (\Delta \otimes I)$$

defines a multiplicative unitary operator on $\mathcal{H}_\star \hat{\otimes} \mathcal{H}_\star$ where $\mathcal{H}_\star := \overline{(\tilde{\mathcal{S}}, <\,,\,>_\star)}$.