Differential structures on κ-Minkowski space, and Lorentz algebra

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Outline

1. Definition and basic properties of realizations of NC spaces
2. Examples of realizations
3. Similarity transformations
4. Differential structures on κ-Minkowski space

NC spaces of Lie algebra type

- Let $\mathfrak{g}$ be a Lie algebra generated by $X_1, X_2, \ldots, X_n$:

$$[X_\mu, X_\nu] = i \sum_{\lambda=1}^{n} a_{\mu\nu}^{\lambda} X_\lambda, \quad a_{\mu\nu}^{\lambda} \text{ deformation parameters}$$

Classical limit: $\quad a_{\mu\nu}^{\lambda} \to 0 \quad \Rightarrow \quad X_\mu \to x_\mu$ commutative coordinates

- Lie algebra type NC spaces
  - κ-deformed space

$$[X_\mu, X_\nu] = i(a_\mu X_\nu - a_\nu X_\mu), \quad a_\mu \in \mathbb{R},$$

  - generalized κ-deformed space

$$[X_\mu, X_\nu] = i\theta_{\mu\nu} X_0 + i(a_\mu X_\nu - a_\nu X_\mu), \quad \theta_{\mu\nu} = -\theta_{\nu\mu}, \quad X_0 \text{ central element}$$

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Realizations of NC spaces of Lie algebra type

- **Problem:** find realizations of X_k as elements of the Weyl algebra $\mathcal{A}_n$,

$$[x_\mu, x_\nu] = [\partial_\mu, \partial_\nu] = 0, \quad [\partial_\mu, x_\nu] = \delta_{\mu\nu}.$$

Lie algebra homomorphism $\varphi \colon \mathfrak{g} \to \bar{\mathcal{A}}_n$ (formal completion of $\mathcal{A}_n$)

$$\varphi([X_\mu, X_\nu]) = \hat{x}_\mu \hat{x}_\nu - \hat{x}_\nu \hat{x}_\mu, \quad \hat{x}_\mu = \varphi(X_\mu)$$

- We seek realizations of X_μ of the form

$$\hat{x}_\mu = \sum_{\alpha=1}^{n} x_\alpha \, \varphi_{\alpha\mu}(\partial), \quad \varphi_{\alpha\mu} \in D[[a]] \quad \text{formal power series in } a$$

Classical limit

$$\lim_{a\to 0} \varphi_{\alpha\mu} = \delta_{\alpha\mu} \quad \Rightarrow \quad \lim_{a\to 0} \hat{x}_\mu = x_\mu \quad \text{commutative coordinates}$$

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Realizations of NC spaces of Lie algebra type

- $\hat{x}_\mu = x_\mu + \text{deformation}(a, x, \partial)$

- Commutation relations imply

$$[\hat{x}_\mu, \hat{x}_\nu] = i(a_\mu \hat{x}_\nu - a_\nu \hat{x}_\mu) \quad \Leftrightarrow \quad \varphi_{\mu\nu} \text{ satisfy a system of nonlinear PDE's}$$

$$\sum_{\beta=1}^{n} \left(\frac{\partial \varphi_{\alpha\mu}}{\partial \partial_\beta} \varphi_{\beta\nu} - \frac{\partial \varphi_{\alpha\nu}}{\partial \partial_\beta} \varphi_{\beta\mu} \right) = i a_\mu \varphi_{\alpha\nu} - i a_\nu \varphi_{\alpha\mu}$$

- Simplifications needed

$$\varphi_{\mu\nu} = \varphi_{\mu\nu}(A, B), \quad A = ia\partial, \quad B = a^2 \partial^2$$

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Noncovariant realizations of κ-deformed space

- Assume $a_\mu = a_n \delta_{\mu n}$

$$[\hat{x}_i, \hat{x}_j] = 0, \quad [\hat{x}_n, \hat{x}_j] = ia\hat{x}_j, \quad i, j = 1, 2, \ldots, n-1. \tag{1}$$

- Ansatz

$$\hat{x}_i = x_i \varphi(A), \quad A = ia\partial_n,$$

$$\hat{x}_n = x_n \psi(A) + ia \sum_{k=1}^{n-1} x_k \partial_k \, \gamma(A), \quad \varphi(0) = \psi(0) = 1$$

Commutation relations (1) imply

$$\gamma = \frac{\varphi'}{\varphi}\psi + 1$$

- There is an infinite family of noncovariant realizations parameterized by real-analytic functions φ and ψ.

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Natural realization of κ-deformed space

- Assume commutation relations

$$[\hat{x}_\mu, \hat{x}_\nu] = i(a_\mu \hat{x}_\nu - a_\nu \hat{x}_\mu), \quad \mu, \nu = 1, 2, \ldots, n. \tag{2}$$

- Ansatz

$$\hat{x}_\mu = x_\mu \varphi(A, B) + i \left(\sum_{k=1}^{n} a_k x_k \right) \partial_\mu, \quad \varphi(0, 0) = 1$$

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Commutation relations (2) imply

$$\frac{\partial \varphi}{\partial A} + 1 = 0, \quad \frac{\partial \varphi}{\partial A} - 2(\varphi + A) \frac{\partial \varphi}{\partial B} = 0.$$

Solution

$$\varphi(A, B) = -A + \sqrt{1 - B}$$

- The natural realization belongs to an infinite family of covariant realizations.

Universal realization for a general Lie algebra type space

- Consider Lie algebra $\mathfrak{g}$ defined by

$$[\hat{x}_\mu, \hat{x}_\nu] = i \sum_{\lambda=1}^{n} a_{\mu\nu}^{\lambda}\, \hat{x}_\lambda .$$

- Associate to $\mathfrak{g}$ $n \times n$ matrix of differential operators

$$M_{\mu\nu} = i \sum_{\lambda=1}^{n} a_{\mu\nu}^{\lambda}\, \partial_\lambda .$$

- Let f be the generating function for the Bernoulli numbers B_n,

$$f(t) = \frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}, \quad \text{and define matrix} \quad f(M).$$

- The **universal realization** is defined by

$$\hat{x}_\mu = \sum_{\alpha=1}^{n} x_\alpha\, f(M)_{\alpha\mu}$$

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Similarity transformations

- Fix a φ-realization of a Lie algebra $\mathfrak{g}$:

$$\hat{x}_\mu = \sum_{\alpha=1}^{n} x_\alpha\, \varphi_{\alpha\mu}(\partial), \quad \mu = 1, 2, \ldots, n$$

- Define similarity transformations $T_P \colon \bar{\mathcal{A}}_n \to \bar{\mathcal{A}}_n$

$$T_P(u) = Ad(e^P)u, \quad P = \sum_{\alpha=1}^{n} x_\alpha A_\alpha(\partial), \quad A_\alpha \in D[[a]],\ A_\alpha(0) = 0$$

BCH $\Rightarrow$ $\{T_P\}$ is a group under composition

- Similarity transformations act covariantly on φ-realizations.

$$\mathfrak{g} \xrightarrow{\varphi} \bar{\mathcal{A}}_n \xrightarrow{T_P} \bar{\mathcal{A}}_n, \quad \tilde{\varphi} = T_P \circ \varphi \quad \text{new realization}$$

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Similarity transformations (cont.)

- Using $Ad(e^P)(u) = e^{\mathrm{ad}(P)}u$ one finds:

$$T_P(x_\mu) = \sum_{\alpha=1}^{n} x_\alpha \psi_{\alpha\mu}(\partial) \quad \text{for some } \psi_{\alpha\mu} \in D[[a]],$$

$$T_P(\partial_\mu) = \partial_\mu + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} Q^{k-1}(A_\mu) = \Lambda_\mu(\partial), \quad Q = \sum_{\alpha=1}^{n} A_\alpha \frac{\partial}{\partial\partial_\alpha}$$

- The relation $[T_P(\partial_\mu), T_P(x_\nu)] = \delta_{\mu\nu}$ implies:

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Differential algebra on κ-Minkowski space

We want to construct a differential algebra on κ-Minkowski space which is compatible with an action of the Lorentz algebra, and the number of one-forms **equals** the number of NC coordinates.

Some earlier work:

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Desired properties of differential algebra

Consider κ-Minkowski space with coordinates $\hat{x}_\mu$, $\mu = 0, 1, \ldots, n-1$.

Desired properties of the differential algebra $(\Omega, \hat{d})$:

1. exterior derivative maps k-forms to $(k+1)$-forms, $\hat{d} : \Omega^k \to \Omega^{k+1}$,
2. $\hat{d}$ is nilpotent, $\hat{d}^2 = 0$,
3. one-forms $\xi = \hat{d} \cdot \hat{x}_\mu$ anti-commute, $\{\xi_\mu, \xi_\nu\} = 0$,
4. $\hat{d}$ satisfies undeformed Leibniz rule

$$\hat{d} \cdot [\hat{f}(\hat{x})\hat{g}(\hat{x})] = [\hat{d} \cdot \hat{f}(\hat{x})]\hat{g}(\hat{x}) + \hat{f}(\hat{x})[\hat{d} \cdot \hat{g}(\hat{x})],$$

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Construction of differential forms

- In order to construct differential forms on κ-Minkowski space we extend the Weyl Algebra $\mathcal{A}_n$ by one-forms dx_μ:

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  where $0 \leq i_1 < i_2 < \ldots < i_p \leq n-1, \quad p = 0, 1, \ldots, n-1.$

- $\mathbb{Z}_2$-grading $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$

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Construction of differential forms (cont.)

- Define exterior derivative and one-forms by

$$\hat{d} = \sum_{\alpha,\beta=0}^{n-1} dx^{\alpha}\,\partial_{\beta}\,k_{\alpha\beta}(\partial), \quad \xi_{\mu} = \sum_{\alpha=0}^{n-1} dx^{\alpha}\,h_{\alpha\mu}(\partial), \quad k_{\mu\nu}, h_{\mu\nu} \in \mathbb{C}[[\partial]]$$

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$$\xi = \hat{d}\cdot\hat{x}_{\mu} \equiv [\hat{d}, \hat{x}_{\mu}].$$

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$$\hat{d} \to \sum_{\alpha=0}^{n-1} dx^{\alpha}\,\partial_{\alpha}, \quad \xi \to dx_{\mu}, \quad dx_{\mu} = [d, x_{\mu}].$$

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- Additional constraints come from the requirement

$$[\xi_\mu, \hat{x}_\nu] = \sum_{\lambda=0}^{n-1} K^\lambda_{\mu\nu} \xi_\lambda, \quad K^\lambda_{\mu\nu} \in \mathbb{C}.$$

  In this case any k-form can be written uniquely as

$$\sum_k p_k(\hat{x}) \xi_{i_1} \xi_{i_2} \dots \xi_{i_k}, \quad i_1 < i_2 < \dots < i_k.$$

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- $\hat{d} \cdot (\xi_{i_1} \xi_{i_2} \dots \xi_{i_k}) = 0,$
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κ-Minkowski space and Lorentz algebra

- Let $\mathfrak{M}$ be the κ-Minkowski space

$$[\hat{x}_i, \hat{x}_j] = 0, \quad [\hat{x}_0, \hat{x}_j] = ia_0\, \hat{x}_j, \quad i, j = 1, 2, \dots, n-1.$$

- Let $\mathfrak{L}$ be the Lorentz algebra

$$[M_{\mu\nu}, M_{\lambda\rho}] = \eta_{\nu\lambda} M_{\mu\rho} - \eta_{\mu\lambda} M_{\nu\rho} - \eta_{\nu\rho} M_{\mu\lambda} + \eta_{\mu\rho} M_{\nu\lambda}, \quad \mu, \nu = 0, 1, \dots, n-1,$$

$\eta = \mathrm{diag}(-1, 1, \dots, 1)$ Lorentz metric.

- If we demand that $[M_{\mu\nu}, \hat{x}_\lambda]$ is linear in $M_{\mu\nu}$ and $\hat{x}_\mu$, then

$$[M_{i0}, \hat{x}_0] = -\hat{x}_i + ia_0 M_{i0}, \quad [M_{i0}, \hat{x}_k] = -\delta_{ik}\hat{x}_0 + ia_0 M_{ik},$$
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Realizations of κ-Minkowski space

- **Problem:** find relizations of $\mathfrak{g}$ in $\mathcal{A}_n$,

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which posses the shift operator Z defined by

$$[Z, \hat{x}_\mu] = ia_\mu Z, \quad [Z, \partial_\mu] = 0.$$

The shift operator is essential in the construction of differnetial forms.

- Realization of $\mathfrak{M}$

$$\hat{x}_0 = x_0\varphi(A) + ia_0\left(\sum_{k=1}^{n-1} x_k\partial_k\right)\gamma(A), \quad A = -ia_0\partial_0,$$

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Realizations of Lorentz algebra

- To find realizations of the Lorentz algebra, use the natural realization

$$M_{\mu\nu} = X_\mu D_\nu - X_\nu D_\mu,$$

and the transformation of variables $(X_\mu, D_\mu) \mapsto (x_\mu, \partial_\mu)$:

$$\begin{aligned} M_{i0} &= x_i \varphi(A) \left(\frac{ia_0}{2} e^{\Psi(A)} \Box - \frac{e^{\Psi(A)} - 1}{ia_0} \right) \\ &\quad - \left[x_0 \Psi(A) + ia_0 \left(\sum_{k=1}^{n-1} x_k \partial_k \right) \gamma(A) \right] \frac{\partial_i}{\varphi(A)}, \\ M_{ij} &= x_i \partial_j - x_j \partial_i. \end{aligned}$$

- $\Box$ is a deformed Laplace operator

$$\Box = \triangle \frac{e^{-\Psi(A)}}{\varphi^2(A)} - \left(\frac{2}{ia_0} \right)^2 \sinh^2(\Psi(A)), \quad \triangle = \sum_{k=1}^{n-1} \partial_k^2$$

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- Ansatz for $\hat{d}$

$$\hat{d} = -dx_0 \partial_0 K_1(A) + \Big(\sum_{k=1}^{n-1} dx_k \partial_k \Big) K_2(A), \quad A = -ia_0 \partial_0 .$$

- Find K_1 and K_2 such that

$$\xi_0 = [\hat{d}, \hat{x}_0] = dx_0 Z^{-s}, \quad \xi_i = [\hat{d}, \hat{x}_i] = dx_i Z^{-t}, \quad s, t \in \mathbb{R},$$
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- Solutions

$$K_1(A) = \frac{1 - Z^{-s}}{sA}, \quad K_2(A) = \frac{Z^{-1}}{\varphi(A)}, \quad s \neq 0, t = 1$$

- Commutators $[\xi_\mu, \hat{x}_\nu]$ are closed:

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Action of the Lorentz algebra

- Define action of Lorentz generators by

$$M_{\mu\nu} \triangleright \hat{f}(\hat{x}, \xi) = [M_{\mu\nu}, \hat{f}(\hat{x}, \xi)] \cdot \hat{1}$$

$$\hat{1} \in U(\mathfrak{M}) \ , \ \hat{x}_\mu \cdot \hat{1} = \hat{x}_\mu \ , \ \partial_\mu \cdot \hat{1} = 0$$

- Restriction to $\hat{\Omega}^0$ agrees with the action found by A. Sitarz (1995):

$$M_{i0} \triangleright \hat{x}_0 = -\hat{x}_i, \qquad M_{ij} \triangleright \hat{x}_0 = 0,$$
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- Advantages
  - the action is consistent with Jacobi identities for $\hat{x}_\mu$ and ξ_μ,
  - there are exactly n one-forms ξ_ν.
- Drawbacks
  - $M_{\mu\nu} \triangleright \xi_\lambda = 0 \quad \Rightarrow \quad$ one-forms do not transform vector-like,
  - $[M_{\mu\nu} \triangleright \hat{d}] \neq 0 \quad \Rightarrow \quad \hat{d}$ is not Lorentz-invariant.

Action of the Lorentz algebra

- Define action of Lorentz generators by

$$M_{\mu\nu} \triangleright \hat{f}(\hat{x},\xi) = [M_{\mu\nu}, \hat{f}(\hat{x},\xi)] \cdot \hat{1}$$

$\hat{1} \in U(\mathfrak{M})$, $\hat{x}_\mu \cdot \hat{1} = \hat{x}_\mu$, $\partial_\mu \cdot \hat{1} = 0$

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Alternative approach

- Expand the algebra generated by x_μ, ∂_μ and dx_μ by Grassman variables q_μ:

$$\{q_\mu, q_\nu\} = 0, \qquad \{dx_\mu, q_\nu\} = \eta_{\mu\nu},$$
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- Consider new realizations of Lorentz generators

$$\widetilde{M}_{\mu\nu} = (\hat{x}_\mu \tilde{\partial}_\nu - \hat{x}_\nu \tilde{\partial}_\mu) Z + dx_\mu\, q_\nu - dx_\nu\, q_\mu,$$
$$\tilde{\partial}_0 = \partial_0 \frac{1 - Z^{-1}}{A}, \quad \tilde{\partial}_i = \partial_i \frac{Z^{-1}}{\varphi(A)}.$$

- $\widetilde{M}_{\mu\nu}$ and $\tilde{\partial}_\mu$ generate Poincaré algebra

$$[\tilde{\partial}_\mu, \tilde{\partial}_\nu] = 0, \quad [\widetilde{M}_{\mu\nu}, \tilde{\partial}_\lambda] = \eta_{\mu\lambda} \tilde{\partial}_\nu - \eta_{\nu\lambda} \tilde{\partial}_\mu,$$
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Alternative approach (cont.)

- Drawback

$$[\widetilde{M}_{\mu\nu}, \hat{x}_\lambda] \quad \textbf{is not linear in} \quad \widetilde{M}_{\mu\nu},\ \hat{x}_\mu!$$

- Exterior derivative in this realization is Lorentz-invariant

$$\hat{d} = \sum_{\alpha=0}^{n-1} dx^\alpha \tilde{\partial}_\alpha, \quad [\widetilde{M}_{\mu\nu}, \hat{d}] = 0.$$

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Alternative approach (cont.)

Special case $\varphi = \psi = 1 - A$:

$$\widetilde{M}_{\mu\nu} = x_\mu \partial_\nu - x_\nu \partial_\mu + dx_\mu q_\nu - dx_\nu q_\mu, \quad \hat{d} = \sum_{\alpha=0}^{n-1} dx_\alpha \, \partial^\alpha.$$

$$[\widetilde{M}_{\mu\nu}, \hat{x}_\lambda] = \hat{x}_\mu \eta_{\nu\lambda} - \hat{x}_\nu \eta_{\mu\lambda} + i\hat{x}_\lambda (a_\mu \partial_\nu - a_\nu \partial_\mu) Z,$$
$$[\widetilde{M}_{\mu\nu}, \xi_\lambda] = \xi_\mu \eta_{\nu\lambda} - \xi_\nu \eta_{\mu\lambda} + i\xi_\lambda (a_\mu \partial_\nu - a_\nu \partial_\mu) Z.$$

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