The spectral action for Dirac operators with skew-symmetric torsion

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joint work with

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set-up

Consider 4-dim. Riemannian manifolds M, closed and spin.

General connection for vector fields has the form

$$\nabla_X Y = \nabla^{LC}_X Y + A(X,Y)$$

where ∇^{LC} is Levi-Civita connection
and A is a $(2,1)$-tensor field.

Torsion 3-form of ∇ is

$$T(X,Y,Z) = \langle \nabla_X Y - \nabla_Y X - [X,Y], Z \rangle$$

∇ is compatible with the Riemannian metric $\langle \cdot, \cdot \rangle$ and has same geodesics as ∇^{LC}
if and only if $A(X, Y, Z) = \langle A(X, Y), Z \rangle$ is totally anti-symmetric.
("Torsion is skew-symmetric.")

In this case: $T = 2A$, and hence

$$\nabla_X Y = \nabla_X^{LC} Y + \tfrac{1}{2} T(X, Y, \cdot)^{\#}$$

Induced connection for spinor fields is then

$$\nabla_X \psi = \nabla_X^{LC} \psi + \tfrac{1}{4} (X \lrcorner T) \cdot \psi,$$

where $(X \lrcorner T) \cdot$ is Clifford multiplication by the 2-form $T(X, \cdot, \cdot)$.

Dirac operator D is given by $D\psi = \sum_i e_i \cdot \nabla_{e_i}\psi$, for any orthonormal frame e_i.

Bochner formula (Agricola-Friedrich)

$$D^2 = \Delta + \frac{3}{4}dT + \frac{1}{4}R - \frac{9}{8}T_0^2,$$

where $\Delta = \widetilde{\nabla}^*\widetilde{\nabla}$ is the Laplacian associated to spin connection

$$\widetilde{\nabla}_X\psi = \nabla^{LC}_X\psi + \frac{3}{4}(X \lrcorner T)\cdot\psi,$$

dT is the exterior differential of the 3-form T,
R is the scalar curvature and
$T_0^2 = \frac{1}{6}\sum_{i,j=1}^{n} \|T(e_i, e_j, \cdot)^{\#}\|^2$.

Chamseddine-Connes:
Spectral action (bosonic part) of Dirac operator D is the number of eigenvalues of D in the interval $[-\Lambda, \Lambda]$ (with $\Lambda \in \mathbb{R}^+$):

$$I = \mathrm{Tr}\, f\left(\frac{D^2}{\Lambda^2}\right),$$

where Tr is the L^2-trace over the space of spinor fields, and f is cut-off function with support in $[-1, +1]$ which is constant near 0.

From the heat trace asymptotics for $t \to 0$

$$\mathrm{Tr}\left(e^{-t\,D^2}\right) \sim \sum_{n\geq 0} t^{n-2} a_{2n}(D^2)$$

(with Seeley-deWitt coefficients $a_{2n}(D^2)$)
one gets (by $t = \Lambda^{-2}$) an asymptotics for the spectral action

$$I = \mathrm{Tr}\, f\left(\frac{D^2}{\Lambda^2}\right) \sim \Lambda^4 f_4\, a_0(D^2) + \Lambda^2 f_2\, a_2(D^2) + \Lambda^0 f_0\, a_4(D^2)$$

as $\Lambda \to \infty$.
Here f_4, f_2, f_0 are moments of the cut-off function f.

Bochner formula is $D^2 = \Delta - E$ with $E = -\frac{3}{4}dT - \frac{1}{4}R + \frac{9}{8}T_0^2$.

Insert this into formulas for Seeley-deWitt coefficients:

$$\begin{aligned}
a_0(D^2) &= \frac{1}{4\pi^2}\int_M dvol \\
a_2(D^2) &= \frac{1}{96\pi^2}\int_M (6\,\mathrm{tr}(E) + 4R)\,dvol \\
a_4(D^2) &= \frac{1}{5760\pi^2}\int_M \Big(\mathrm{tr}\Big(60\,\Delta E + 60RE + 180E^2 + 30\,\Omega_{ij}\Omega_{ij}\Big) \\
&\qquad + 48\Delta^{LC}R + 20R^2 - 8\|Ric\|^2 + 8\|Riem\|^2\Big)\,dvol
\end{aligned}$$

where Ric and $Riem$ are Ricci/Riemannian curvature tensors of the metric,
and $\Omega_{ij} = \widetilde{\nabla}_{e_i}\widetilde{\nabla}_{e_j} - \widetilde{\nabla}_{e_j}\widetilde{\nabla}_{e_i} - \widetilde{\nabla}_{[e_i,e_j]}$ is the curvature of $\widetilde{\nabla}$.

Clifford relations and cylicity of trace $\Longrightarrow \mathrm{tr}(dT) = 0$
Therefore

$$a_2(D^2) = \frac{1}{16\pi^2} \int_M \left(\frac{9}{2} T_0^2 - \frac{1}{3} R \right) dvol.$$

Neglecting the a_4-term in the spectral action, one obtains the classical Einstein-Cartan-action: Only torsion free critical points upon variation of metric and torsion 3-forms.

We compute for the curvature Ω_{ij} of the connection $\widetilde{\nabla}$:

$$\Omega_{ij} = \sum_{a,b} \Big(\tfrac{1}{4}\langle R(e_i,e_j)e_a,e_b\rangle + \tfrac{3}{8}a(\nabla T)(e_i,e_j,e_a,e_b) + \tfrac{9}{16}\sum_c \big(T(e_i,e_c,e_a)T(e_j,e_c,e_b) - T(e_j,e_c,e_a)T(e_i,e_c,e_b)\big)\Big)e_a e_b,$$

where $a(\nabla T)(e_i,e_j,e_a,e_b) = \nabla_{e_i}T(e_j,e_a,e_b) - \nabla_{e_j}T(e_i,e_a,e_b)$.
Then using $\mathrm{tr}(e_k e_\ell e_s e_t) = 4(\delta_{\ell s}\delta_{kt} - \delta_{\ell t}\delta_{ks})$:

$$\sum_{i,j}\mathrm{tr}(\Omega_{ij}\Omega_{ij}) = -8\sum_{\substack{i\neq j\\ a,b}} \Big(\tfrac{1}{4}\langle R(e_i,e_j)e_a,e_b\rangle + \tfrac{3}{8}a(\nabla T)(e_i,e_j,e_a,e_b) + \tfrac{9}{8}c(T)(e_i,e_j,e_a,e_b)\Big)^2,$$

where $c(T)(e_i,e_j,e_a,e_b) = \sum_c T(e_i,e_c,e_a)T(e_j,e_c,e_b)$.

Representation theory of $O(4) \rightsquigarrow$ evaluation of $\sum_{i,j} \mathrm{tr}(\Omega_{ij}\Omega_{ij})$.

The forth Seeley-deWitt coefficient is

$$a_4(D^2) = \frac{1}{16\pi^2}\int_M \Big(\frac{1}{72}R^2 - \frac{1}{45}\|Ric\|^2 - \frac{7}{360}\|Riem\|^2 + \frac{81}{32}(T_0^2)^2 - \frac{27}{32}\|c(T)\|^2 + \frac{9}{16}\|dT\|^2 - \frac{3}{8}\|d^*T\|^2 - \frac{3}{8}\|sym_0^2(\nabla T)\|^2 - \frac{3}{8}R(T)\Big)\, dvol$$

with

$$R(T) = -\sum_{\substack{i\neq j\\ a,c}} Ric(e_j,e_a)T(e_i,e_c,e_a)T(e_j,e_c,e_i) + \sum_{\substack{i\neq j\\ a,b,c}} \langle W(e_i,e_j)e_a,e_b\rangle T(e_i,e_c,e_a)T(e_j,e_c,e_b)$$

Observations

- $R(T)$ couples torsion and the trace free component of the curvature tensor. $\rightsquigarrow$ Expect critical points of spectral action with non-zero torsion.
- Existence of the derivative terms of T. $\rightsquigarrow$ Torsion gets dynamical

Specific particle model

Ingredients of the particle model:

- internal Hilbert space: $\mathcal{H}_f$ with connection $\nabla^{\mathcal{H}_f}$
- full Hilbert space: $\mathcal{H}_{SM} = L^2(M,S) \otimes \mathcal{H}_f \ni \psi \otimes \chi$
- twisted connection (with torsion):
  $\widehat{\nabla}^{SM} = \nabla \otimes \mathrm{id}_{\mathcal{H}_f} + \mathrm{id}_S \otimes \nabla^{\mathcal{H}_f}$
- associated Dirac operator: $D^{\widehat{\nabla}^{SM}}$
- field Φ of endomorphisms of $\mathcal{H}_f$
  (encodes Higgs boson, Yukawa couplings, etc.)

The Chamseddine-Connes Dirac operator:

$$D_\Phi(\psi \otimes \chi) = D^{\widehat{\nabla}^{SM}}(\psi \otimes \chi) + \gamma_5 \psi \otimes \Phi\,\chi$$

Bochner formula for $D_\Phi(\psi \otimes \chi) = D^{\widehat{\nabla}^{SM}}(\psi \otimes \chi) + \gamma_5 \psi \otimes \Phi\, \chi$:

$$(D_\Phi)^2(\psi \otimes \chi) = \Delta^{\overline{\nabla}}(\psi \otimes \chi) - E_\Phi(\psi \otimes \chi)$$

with

$$\begin{aligned} E_\Phi(\psi \otimes \chi) &= \left((-\tfrac{3}{4} dT - \tfrac{1}{4} R + \tfrac{9}{8} T_0^2)\, \psi \right) \otimes \chi \\ &+ \tfrac{1}{2} \cdot \sum_{i \neq j} (e_i \cdot e_j \cdot \psi) \otimes \left(\Omega_{ij}^{\mathcal{H}}\, \chi \right) \\ &+ \sum_{i=1}^{n} \gamma_5 e_i \cdot \psi \otimes [\nabla_{e_i}^{\mathcal{H}_f}, \Phi]\, \chi - \psi \otimes (\Phi^2) \chi \end{aligned}$$

Bosonic Lagrangian of the SM coupled to gravity and torsion:

$$I = \frac{24\Lambda^4 f_4}{\pi^2}\int_M dvol$$
$$+ \frac{\Lambda^2 f_2}{\pi^2}\int_M \left\{27\, T_0^2 - 2R - a|\varphi|^2 - \tfrac{1}{2}c\right\} dvol$$
$$+ \frac{f_0}{2\pi^2}\int_M \Big\{\tfrac{1}{6}R^2 - \tfrac{4}{15}\|Ric\|^2 - \tfrac{7}{30}\|Riem\|^2 + \tfrac{243}{8}(T_0^2)^2 - \tfrac{27}{4}\|c(T)\|^2$$
$$+ \tfrac{27}{4}\|dT\|^2 - \tfrac{9}{2}\|d^*T\|^2 - \tfrac{9}{2}\|sym_0^2(\nabla T)\|^2 - \tfrac{9}{2}R(T)$$
$$+ g_3^2\|G\|^2 + g_2^2\|F\|^2 + \tfrac{5}{3}g_1^2\|B\|^2$$
$$+ a|D_\nu\varphi|^2 + b|\varphi|^4 + 2e|\varphi|^2 + \tfrac{1}{2}d + \tfrac{1}{6}R\left(a|\varphi|^2 + \tfrac{1}{2}c\right)$$
$$- \tfrac{9}{4}T_0^2\left(a|\varphi|^2 + \tfrac{1}{2}c\right)\Big\} dvol.$$

Observations

- Torsion couples to the Higgs field φ.
- $R(T)$ couples torsion and the trace free component of the curvature tensor. $\rightsquigarrow$ Expect critical points of spectral action with non-zero torsion.
- Existence of the derivative terms of T. $\rightsquigarrow$ Torsion gets dynamical

The full Standard Model action is given by

$$I_{SM} = \mathrm{Tr}\, f\left(\frac{D_\Phi^2}{\Lambda^2}\right) + \langle \Psi, D_\Phi \Psi \rangle \quad \text{with} \quad \Psi \in \mathcal{H}_{SM}$$

Some Questions

Experimental signatures:

- Torsion field of the Earth (Weyl tensor $\neq 0$)?
- Effects on freely falling particles with different spin?

The cosmological constant:

- Spectral action "predicts" huge cosm. constant: $\Lambda \sim 10^{17}\text{GeV}$
- Torsion might induce Fermi condensate (Perez,Rovelli)
- Dynamical cancelation of cosm. constant + inflation?

Further implications:

- Connections to LQG (Barbero-Immirzi parameter)?
- Torsion in QFT?