κ-Minkowski: topology, symmetries and uncertainty relations

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Spectrum of volume operators for universal

differential calculus of DFR spacetime

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Bayrischzell, May 14-17, 2010

Outline

Canonical κ-Minkowski (with L. Dąbrowski)
- Relations
- Representations and Radial Quantisation
- Weyl Quantisation and C*-algebra
- Uncertainty Relations

Covariant κ-Minkowski (with L. Dąbrowski and M. Godłiński)
- The covariantised model

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Relations

κ-Minkowski relations for $d+1$-dimensional space:

$$[T, X_j] = iX_j, \quad j = 1, \ldots, d,$$
$$[X_i, X_j] = 0, \quad j, j = 1, \ldots, d$$

where $T = T^*, X_j = X_j^*$ on Hilbert space $\mathfrak{H}$.

Interpretation: T=time, $\boldsymbol{X} = (X_1, \ldots, X_d)$=space; generators of a localisation algebra. Not observables: ideally they are noncommutative analogues of classical localisation x of an observable field $A(x)$.

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Introduced in 90's by Lukierski, Ruegg, then Majid…
Mainly studied from the point of view of finitely generated algebras.

Here we take Weyl's point of view: the corresponding *-algebra = pre-C*-algebra.

C* = minimal requirement for: spectrum(selfadjoint)$\subset \mathbb{R}$, and existence of functional calculus with spectral mapping. Not a technicality, indispensible for a sound Quantum theory.

Representations

Set

$$R^2 = X_1^2 + \cdots + X_d^2.$$

Assume $\exists R^{-1}$; set $X_j = C_j R$; Of course $[R, C_j] = 0$; moreover

$iC_j R = [T, C_j R] = C_j[T, R] + [T, C_j]R = iC_j R + [T, C_j]R$

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Simpler proof: Given irrep (T, R), sign(R) =central= ± 1.

Case $R \neq 0$: Then setting $P = T$, $Q = \log(\pm R)$, we have $[P, Q] = -iI$ and we may use von Neumann uniqueness.

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Irreps

$$T = P, \qquad R = e^{-Q}, \qquad X_j = c_j R, \quad \boldsymbol{c} = (c_1, \ldots, c_d) \in \mathbb{R}^d, \quad \boldsymbol{c} \neq \mathbf{0}. \tag{1}$$

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Direct integration gives universal representation. The orthogonal projection

$$E = \sum_j C_j^2$$

is spectral for R, corresponding to continuous spectrum $(0, \infty)$; $I - E$ corresponds to discrete spectrum $\{0\}$.

Notations

Given a function $f(x_1, \ldots, x_n)$ and selfadjoint operators $A_1, \ldots, A_n$,

$$f(A_1, \ldots, A_n) = \int dk_1 \cdots dk_n \hat{f}(k_1, \ldots, k_n) e^{i \sum_j k_j A_j},$$

where

$$\hat{f}(k_1, \ldots, k_n) = \frac{1}{(2\pi)^n} \int dx_1 \cdots dx_n f(x_1, \ldots, x_n) e^{-i \sum_j k_j x_j}.$$

In particular we will consider cases where $n = d + 1$ and $n = 2$:

$$f(T, \boldsymbol{X}) = f(T, X_1, \ldots, X_d), \quad f(T, R).$$

Let $f = f(t, \boldsymbol{x})$ be classical fn (Weyl symbol); Weyl quantisation:

$$f(T, \boldsymbol{X}) = \frac{1}{(2\pi)^{d+1}} \int_{\mathbb{R}\times\mathbb{R}^d} d\alpha\, d\boldsymbol{\beta}\, \widehat{F}(\alpha, \beta) e^{i(\alpha T + \boldsymbol{\beta}\cdot\boldsymbol{X})}.$$

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$$f(T \quad, X \quad) = \left(\int_{S^{d-1}}^{\oplus} d\boldsymbol{c} \, f_{\boldsymbol{c}}(T^{(\boldsymbol{c})}, R^{(\boldsymbol{c})}) \right) \oplus f(Q, \mathbf{0})$$

$$\in \quad \mathcal{C}(S^{d-1}, \mathcal{K}) \oplus \mathcal{C}_\infty(\mathbb{R})$$

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4. the large scale limit of this component is the Minkowski spacetime with the time axis removed;
5. the time axis always remains classical and corresponds to the component $\mathcal{C}_\infty(\mathbb{R})$.

Let $T = P, R = e^{-Q}$ and $f = f(t, r)$; then

$$(f(T, R)\xi)(s) = \int du\, K_f(s, u)\xi(u),$$

$$K_f(s, u) = (\mathscr{F}_1 f)\left(u - s, \frac{e^{-s} - e^{-u}}{u - s}\right);$$

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3. twisted covariance also can be “pulled back” to κ-Minkowski.

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2. If $\Delta T, \Delta R$=classical period and radius of electron (Hydrogen atom), then $L \ll 10$ light years. There would be no atomic physic on α-Centauri.

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and such that there exists a unitary representation of the Lorentz group, such that

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In addition, we want it to be the smallest possible covariant central extension of κ-Minkowski; hence we require

$$V^\mu V_\nu = I.$$

Good news: it exists!

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Joint spectrum of the V^μ is the upper mass shell $H_m^+ = \{v \in \mathbb{R}^4 : v^\mu v_\mu = 1\}$, and

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1. Besides the DFR model, this is another model with two characteristic, dimensionful parametres, while the Lorentz group is kept undeformed.
2. Twisted covariance is equivalent to ordinary form-covariance, up to dismissing a huge non invariant set of otherwise admissible localisation states.

Outline

Canonical κ-Minkowski (with L. Dąbrowski)
- Relations
- Representations and Radial Quantisation
- Weyl Quantisation and C*-algebra
- Uncertainty Relations

Covariant κ-Minkowski (with L. Dąbrowski and M. Godłiński)
- The covariantised model

Spectrum of volume operators for universal differential calculus of DFR spacetime (with D. Bahns, S. Doplicher and K. Fredemhagen)

(Dubois-Violette) Given unital algebra $\mathcal{A}$, take

$$\Lambda(A) = \bigoplus_n \Lambda^n(A) = \bigoplus_n A^{n\otimes}$$

with product and differential

$$(a_1 \otimes \cdots a_n) \cdot (b_1 \otimes \ldots \otimes b_m) = a_1 \otimes \cdots \otimes a_{n-1} \otimes a_n b_1 \otimes b_2 \otimes \cdots \otimes b_m,$$
$$da = a \otimes I - I \otimes a,$$

(extended as a graded differential). Define $\Omega(A)$ as the d-stable subalgebra of $\Lambda^n(A)$, generated by A.

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DFR model:

$$[q^\mu, q^\nu] = iQ^{\mu\nu},$$
$$[q^\mu, Q^{\mu\nu}] = 0,$$
$$Q^{\mu\nu} Q_{\mu\nu} = 0,$$
$$Q^{\mu\nu} (*Q)_{\mu\nu} = \pm 4I.$$

Irreducibles are canonical quantum spacetimes.

The (unbounded) selfadjoint operators q^μ are uniquely affiliated to $\mathcal{E}$. If $\otimes$ is understood as tensor product of $Z(M(\mathcal{E}))$ moduli,

$$dq^\mu = q^\mu \otimes I - I \otimes q^\mu$$

is a well defined as a selfadjoint operator, interpreted as separation of independent events. It "lives" in $\mathcal{E} \otimes \mathcal{E}$.

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Can define the covariant volume operator: e.g.

$$V = dq^0 \wedge dq^1 \wedge dq^2 \wedge dq^3 = \epsilon_{\mu\nu\rho\sigma} dq^\mu dq^\nu dq^\rho dq^\sigma$$

(but also area operators $dq^\mu \wedge dq^\nu$, 3-volume operators,. . .).
In particular V “lives” in $\underbrace{\mathcal{E} \otimes \cdots \otimes \mathcal{E}}_{\text{5 factors}}$

Strength: use the abstract universal differential calculus to define them, but then can compute spectra as operators affiliated to C*-algebras.

V is a normal operator and has pure point spectrum

$$\mathrm{spec}_{pp}(V) = S = \pm 2 + \mathbb{Z}a_+a_- + i\,(\mathbb{Z}a_+ + \mathbb{Z}a_-)\,.$$

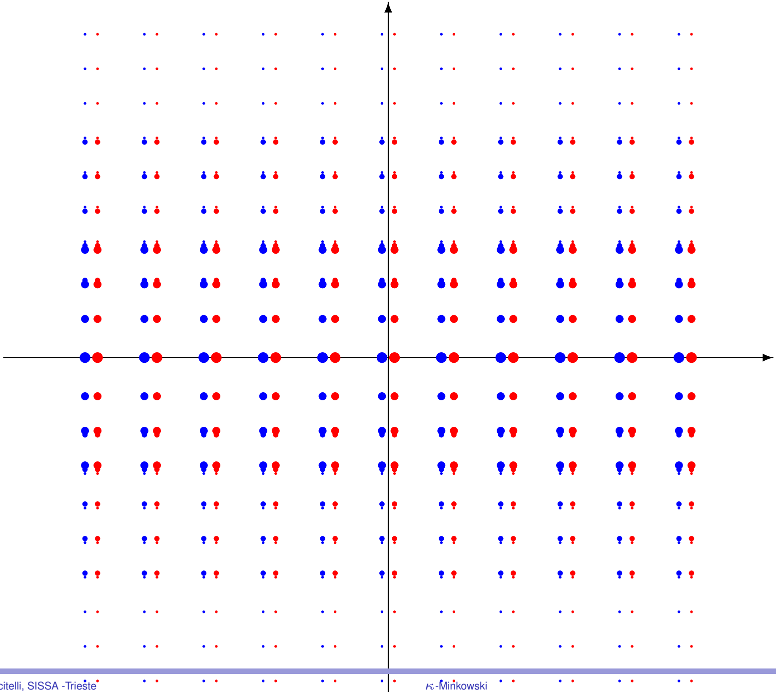
where

$$a_\pm = \sqrt{5 \pm 2\sqrt{5}}.$$

Then

$$\mathrm{spec}(V) = \overline{\mathrm{spec}_{pp}(V)} = \pm 2 + \mathbb{Z}\sqrt{5} + i\mathbb{R}.$$

Note that spec(V) stays away from zero by a constant of order of λ_p^4.



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3. Review on DFR model + general comment on Tw.Cov [arXiv:1004.5261]
4. Volume operators [arXiv:1005.2130] (here also: parallel transport and generalised covariant derivatives).