Dirac Field on Moyal Minkowski Spacetime, NC Potential Scattering

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QFT on NC Spacetimes – “Conceptualities”

- ⋆ The operational significance of NC spacetime in its relation to the QFT on it is often not very clear.
- ⋆ What replaces the locality concept which is central to QFT in Minkowski spacetime on an NC spacetime?
- ⋆ There are (more or less) good arguments for all of the various models of NC spaces (spacetimes). Which is the most appropriate (if any)? What conceptual and mathematical framework is needed to stage a systematic discussion of this question?

QFT on NC Spacetimes – “Conceptualities”

- ⋆ What about general covariance? General relativity is one of the main motivations for considering NC spacetime. In QFT on classical spacetime, one can formulate general covariance for QFTs. This requires to consider not just a few particular spacetime models, but a whole class of spacetimes (abstractly characterized — “model independent”).
- ⋆ Actually, what is a QFT on an NC spacetime? What are its characterizing properties (needed for a sound physical interpretation)? Is there a model-independent framework — model-independent both on the NC geometry side and on the QFT side?

Model-independent NC spacetimes

There is a model-independent approach to Riemannian NC geometry — the spectral geometry approach developed by Alain Connes.

Some (many? all?) of the examples of NC spaces usually considered (when they correspond to NC generalizations of Riemannian geometries) fulfill the conditions of spectral geometry.

The strength of the spectral geometry approach is based on

- “Naturality” of the axioms
- Structural theorems, including “reconstruction” of a Riemannian manifold with spin structure in the “classical case”

Axioms for LOSTs = LOrentzian Spectral Triples

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A LOST is a collection of objects as follows:

$$(\mathcal{A}_0 \subset \mathcal{A}_2 \subset \mathcal{A}_b, \mathcal{H}, D, \beta, \gamma, J)$$

with the properties:

general (NC)

$\mathcal{H}$ is a Hilbert space

$\mathcal{A}_0$ is a pre-C^*-algebra of bounded linear operators on $\mathcal{H}$, $\mathcal{A}_b$ is a preferred unitalization

D is a lin op with dense C^∞ domain $\mathcal{H}^\infty = \mathcal{A}_2\mathcal{E}$ with a finitely generated $\mathcal{A}_b$-module $\mathcal{E}$

classical

$\mathcal{H} = L^2$ spinors on a Lorentzian manifold M with spin structure

$\mathcal{A}_0 = C_0^\infty(M)$, $\mathcal{A}_b = C_b^\infty(M)$

$D =$ D = Dirac-operator, with C^∞ domain of smooth sections in the spinor bundle where all D-derivatives are L^2

Axioms for LOSTs = LOrentzian Spectral Triples

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general (NC)

β, γ are bounded operators on $\mathcal{H}$, J is anti-unitary.
On $\mathcal{H}^\infty$, have
$\beta^* = -\beta$, $\beta^2 = -1$, $D^* = \beta D \beta$,
$[JaJ, b] = 0$ $(a, b \in \mathcal{A}_0)$
and several other relations

Let $\langle D \rangle = \sqrt{(DD^* + D^*D)}$.
Then all finite commutators $[X, [X, \ldots, [X, a]]]$, are bounded operators, where $a \in \mathcal{A}_0$,
$X = D,\ D^*,\ \langle D \rangle$

classical

$\beta = n^a \Gamma_a{}^A_B$,

n^a a timelike vectorfield on M,

$\Gamma_a{}^A_B$ is the spin connection,

$\gamma = \Gamma_0 \cdot \Gamma_1 \cdots \Gamma_3$,

J corresponds to charge conjugation

D is a first order PDO

Axioms for LOSTs = LOrentzian Spectral Triples

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general (NC)	**classical**
$a(1 - \langle D \rangle)^{-1}$ is compact $(a \in \mathcal{A}_0)$.	$\langle$D$\rangle$ is elliptic, n is the (spectral) dimension of M.
There is a minimal $n \in \mathbb{N}$ so that the Dixmier-trace of $a\langle D \rangle^{-n}$ is finite and non-vanishing for all $a \in \mathcal{A}_0$.	
plus 3 more conditions: γ is the image of a Hochschild cycle, β belongs to the 1-forms of $\mathcal{A}_b$, and Poincaré duality (alternatively, closedness and Morita-equivalence of $\mathcal{A}_b$ via $\mathcal{E}$)	essentially orientability and Hodge-duality

Tentative Results for LOSTs

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Let $(\mathcal{A}, \mathcal{H}, D, \beta, \gamma, J)$ and $(\tilde{\mathcal{A}}, \tilde{\mathcal{H}}, \tilde{D}, \tilde{\beta}, \tilde{\gamma}, \tilde{J})$ be two LOSTs.

They are called equivalent if there is $U : \mathcal{H} \to \tilde{\mathcal{H}}$ so that

$$U \mathrm{X} U^{-1} = \tilde{\mathrm{X}} \quad \text{for} \quad \mathrm{X} = \mathcal{A},\ \beta, \gamma,\ J$$

and

$$[\tilde{D}, U \,.\, U^{-1}] = U[D,\ .\,]U^{-1}$$

Thm (caveat: still has status of conjecture – no complete proof yet)

To each Lorentzian manifold with spin structure M there corresponds a LOST with commutative $\mathcal{A}$ and D = Dirac operator. The LOSTs corresponding to isometric Lorentzian manifolds with equivalent spin structures are equivalent.

Conversely, if a LOST has a commutative $\mathcal{A}$, then it derives from a Lorentzian manifold with spin structure. The Lorentzian manifolds deriving from equivalent LOSTs are isometric and have equivalent spin structures.

QFT on a LOST

One can obtain a QFT on a LOST essentially by an abstract form of 2nd quantization.

To a LOST $\mathbf{L} = (\mathcal{A}, \mathcal{H}, D, \beta, \gamma, J)$ one can associate a CAR-$*$-algebra $\mathcal{F}(\mathbf{L})$ generated by symbols $\Phi(f)$, $f \in \mathcal{H}^\infty$ with the relations:

- $f \mapsto \Phi(f)$ is linear
- $\Phi(f)^* = \Phi(Jf)$
- $\Phi(Df) = 0$
- $\{\Phi(f)^*, \Phi(h)\}_+ = (f, Rh)_{\mathcal{H}}$

Here, R would in the classical case correspond to the difference between the advanced and retarded fundamental solutions of the Dirac operator:

$$R = R_{\mathrm{adv}} - R_{\mathrm{ret}}\,, \quad DR_{\mathrm{adv}}f = f = DR_{\mathrm{ret}}f$$

Existence and uniqueness of R_{adv} and R_{ret} is an extra assumption for a LOST. It is viewed as an NC generalization of globally hyperbolic spacetimes
→ globally hyperbolic spectral triples (GHYSTs).

GHYST QFT

Existence and uniqueness of $\boldsymbol{R}_{\text{adv}}$ and $\boldsymbol{R}_{\text{ret}}$ is (still) difficult to characterize in the LOST setting. (Is related to the question of localizability in NC geometry, which is difficult.)

Guided by the classical case, it should be possible to define GHYSTs in a covariant manner using an equivalence concept for LOSTs. Then, if two GHYSTs

$$\mathbf{L} \xrightarrow{\boldsymbol{U}} \widetilde{\mathbf{L}} \quad \text{are equivalent by a unitary } \boldsymbol{U},$$

there should be a $(C)^*$ algebraic equivalence

$$\mathcal{F}(\mathbf{L}) \xrightarrow{\alpha_U} \mathcal{F}(\widetilde{\mathbf{L}}) \,, \quad \alpha_{\boldsymbol{U}}(\boldsymbol{\Phi}(\boldsymbol{f})) = \widetilde{\boldsymbol{\Phi}}(\boldsymbol{U}\boldsymbol{f})$$

This is the starting point for a concept of covariant QFT over Lorentzian spectral geometries.

NC Spacetime QFT

What are the observables in $\mathcal{F}(\mathbf{L})$, and how are they related to elements in $\mathcal{A}_0$?

One idea is to follow the action of $a \in \mathcal{A}_0$ on elements in $\mathcal{H}^\infty$ through the process of abstract 2nd quantization: We consider

$$w_a : \Phi(f) \mapsto \Phi(aRf)$$

In a particular example, this corresponds (essentially) to the derivation of an operation — a scattering morphism — on $\mathcal{F}(\mathbf{L})$.

As example, we take 3-dim Moyal-deformed Minkowski spacetime where

$$\theta = (\vartheta^{\mu\nu}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \vartheta \\ 0 & -\vartheta & 0 \end{pmatrix}$$

Dirac field on Moyal-Minkowski space / commutative time case

- $\mathcal{A}_0 = \mathcal{S}(\mathbb{R}^3)_\star$ with the Moyal-product

$$f \star h = (2\pi)^{-3} \int\int f(x - \theta u)h(x + v)\mathrm{e}^{-iu\cdot v}\, du\, dv$$

- $\mathcal{H} = L^2(\mathbb{R}^3, \mathbb{C}^2)$
- D is the usual (massless) Dirac-operator on 3-dim Minkowski space
- $\beta = \Gamma_0$, $\gamma = \Gamma_0 \cdot \Gamma_1 \cdot \Gamma_2$, J = charge conjugation
- In this case, R_{adv} and R_{ret} exist uniquely, and coincide with the "classical" objects
- Let $c(x^0, x^1, x^2) = a(x^0)b(x^1, x^2)$ with $C_0^\infty(\mathbb{R})$ and $b \in \mathcal{S}(\mathbb{R}^2)$, both real. Define the "non-commutative (time-dependent) potential"

$$V(c)f = c \star f + f \star c\,, \quad f \in \mathcal{H}$$

Dirac field scattering by NC potential / commutative time case

- Write the field equation

$$(D + \lambda V(c))\varphi = 0$$

  in Hamiltonian form.
  With $\varphi_t = \varphi(t, \,.\,)$, the field eqn becomes

$$\begin{aligned} H(t)\varphi_t = (H_0 + \lambda H_c(t))\varphi_t &= 0 \\ H_0 &= (\Gamma_0)^{-1} i \Gamma_k \partial^k \\ H_c(t)\varphi_t &= (\Gamma_0)^{-1} a(t)(b \star \varphi_t + \varphi_t \star b) \end{aligned}$$

  where $b \star \varphi_t$ is the 2-dim Moyal product (w.r.t. (x^1, x^2)), i.e. the NC product of the 2-dim Moyal plane
- One can show ess. selfadjointness of the $H(t)$ on a suitable domain.

Dirac field scattering by NC potential / commutative time case

- The solution of the eqn is given by a 2-parametric family of unitaries:

$$\varphi_t = U_\lambda(t, t_0)\varphi_{t_0}$$

- Note that $a(t)$ and hence $H_c(t)$ have compact time-support

  $\Longrightarrow s_\lambda = \mathrm{e}^{it_+ H_0} U_\lambda(t_+, t_-)\mathrm{e}^{-it_- H_0}$
  is constant if $t_+, -t_- > \tau$ for some finite τ
  s_λ is the 1-particle scattering operator
- Define for solutions χ of the free (potential = 0) Dirac eqn

$$\Psi(\chi) = \Phi(Rf)\,, \quad \chi = Rf$$

  s_λ induces a Bogoliubov-transformation on the CAR-algebra $\mathcal{F}(\mathbf{L})$ generated by the $\Psi(\chi)$,

$$\alpha_\lambda(\Psi(\chi)) = \Psi(s_\lambda \chi)$$

- Let $(\pi_0, \mathcal{H}_0, \Omega_0)$ be the GNS representation of the canonical vacuum state (ground state for the canonical ground state on $\mathcal{F}(\mathbf{L})$ w.r.t. the potential-free time evolution) and set $\Psi_0(\chi) = \pi_0(\Psi(\chi))$

Theorem (M. Borris, R.V., CMP 293)

α_λ is implementable in the vacuum representation π_0, i.e. there is a unitary S_λ on $\mathcal{H}_0$ so that

$$\pi_0(\alpha_\lambda(\Psi(\chi))) = S_\lambda \Psi_0(\chi) S_\lambda^{-1}$$

Moreover, it holds that

$$\left.\frac{d}{d\lambda}\right|_{\lambda=0} S_\lambda \Psi_0(\chi) S_\lambda^{-1} = i\Psi_0(R(c \star \chi + \chi \star c))$$

Dirac field scattering by NC potential / commutative time case

Hence, one can take

$$X_\lambda(c) = i(\frac{d}{d\lambda}S_\lambda)S_\lambda^{-1}$$

as **observables associated with elements c in the algebra of NC "spacetime coordinates" $\mathcal{A} = \mathcal{S}(\mathbb{R}^3)_\star$.**

This is very reminiscent of Bogoliubov's formula.

The $X_\lambda(c)$ are generators of transformations of quantum fields associated with "probing" the quantum field by coupling it to a non-commutative (scalar) potential.

(Actually, in order to respect gauge covariance, one should use elements c in the differential algebra of $\mathcal{A}$ instead of elements of the algebra itself.)

Dirac field scattering by NC potential / commutative time case

In Minkowski spacetime with $\mathcal{A} = \mathcal{S}(\mathbb{R}^3)$:

$$[:\Phi_0^+\Phi_0:(c), \Psi_0(\chi)] = 4i\Psi(Rc\chi)$$

With the Rieffel-product

$$A\ {}_{\theta}\!\star B = \frac{1}{(2\pi)^n}\int\int \alpha_{\theta u}(A)\alpha_v(B)e^{iu\cdot v}\,du\,dv$$

for operators A, B on $\mathcal{H}_0$,
α_y = action of translations, one obtains

$$\begin{aligned}[X_0(c), \Psi_0(\chi)] &= \Psi_0(R(c\star\chi + \chi\star c)) \\ &= \frac{1}{4}([:\Phi_0^+\Phi_0:(c)\ {}^{\theta\star}_{\ ,}\ \Psi_0(\chi)] + [:\Phi_0^+\Phi_0:(c)\ {}^{\star\theta}_{\ ,}\ \Psi_0(\chi)])\end{aligned}$$

Dirac field on Moyal-Minkowski space / NC time case

NC function algebra $\mathcal{A}_0 = \mathcal{S}(\mathbb{R}^4)_\star$ with the Moyal-product

$$f \star h = (2\pi)^{-4} \int\int f(x - \theta u) h(x + v) \mathrm{e}^{-iu \cdot v}\, du\, dv$$

now with

$$\theta = \vartheta \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

NC potential

$$V_\xi(c)\varphi = \xi(c \star (\xi\varphi) \star c)$$

where $\xi \geq 0$ is a C_0^∞ cut-off function

Dirac field on Moyal-Minkowski space / NC time case

Fix $\tau > 0$ so that ξ is compactly supported in

$$\mathcal{M}_\tau = \{(x^0, x^1, x^2, x^3) : |x^0| < \tau\}$$

For $\lambda > 0$ small enough there exist advanced and retarded fundamental solutions

$$\mathcal{R}^{\pm}_{\lambda V}$$

for the Dirac operator

$$D + \lambda V_\xi(c) \qquad \text{on} \qquad \mathcal{M}_\tau$$

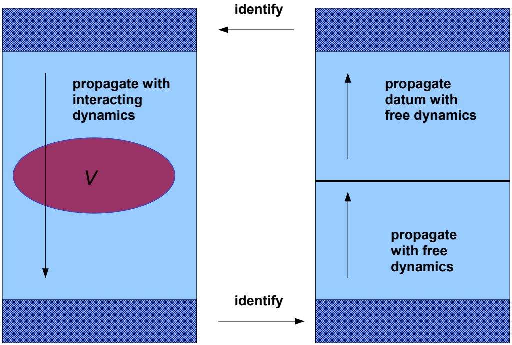
$$\mathcal{R}^{\pm}_{\lambda V} = R^{\pm}(1 + \lambda V_\xi(c) R^{\pm})^{-1} = R^{\pm}\left(\sum_{j=0}^{\infty}(-1)^j(\lambda V_\xi(c) R^{\pm})^j\right)$$

Dirac field on Moyal-Minkowski space / NC time case

With the help of $\mathcal{R}^{\pm}_{\lambda V}$ one can define an isometric 1-particle scattering operator

$$s_\lambda = s_\lambda(c, \xi, \tau)$$

on the space of solutions χ of the free Dirac equation $D\chi = 0$ on $\mathcal{M}_\tau$



Dirac field on Moyal-Minkowski space / NC time case

Then s_λ induces an endomorphism

$$\alpha_\lambda(\Psi(\chi)) = \Psi(s_\lambda \chi)$$

on the CAR-algebra of the quantized free Dirac field on $\mathcal{M}_\tau$

On this CAR-algebra one obtains the derivation

$$\delta_{\tau,\xi,c}(A) = \left.\frac{d}{d\lambda}\right|_{\lambda=0} \alpha_\lambda(A)$$

Upon taking away the cut-offs ($\tau \to \infty$, $\xi \to 1$) one obtains a derivation

$$\delta_c(\Psi(\chi)) = \Psi(R(c \star \chi \star c))$$

on the CAR-algebra of the free Dirac field on all of (Moyal-) Minkowski spacetime.

Dirac field on Moyal-Minkowski space / NC time case / Summary

Expected: In the vacuum representation, $\delta_c(\Psi(\chi))$ is induced by a symmetric operator

$$[X_0(c), \Psi_0(\chi)] = \Psi_0(R(c \star \chi \star c))$$

Concluding:

- Moyal-Minkowski spacetime can be seen as a model for a LOST
- Certain combinations of Moyal-Rieffel products between QFT operators can be interpreted operationally in terms of NC potential scattering + Bogoliubov's formula
- This renders a correspondence

  $$\mathcal{A}(\mathbf{L}) \to \mathcal{F}(\mathbf{L})\,, \quad c \mapsto X_0(c)$$

  which may be seen as generalization of an "observable quantum field" on an NC spacetime
- Starting point for generalization of local algebras of observables over NC spacetimes?