The two-point function of noncommutative ϕ^4_4-theory

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(based on arxiv:0909.1389 with Harald Grosse
and work in progress with Harald Grosse and Vincent Rivasseau)

Introduction

- The Standard Model is a perturbatively renormalisable quantum field theory.
- Scattering amplitudes can be computed as formal power series in coupling constants such as $e^2 \approx \frac{1}{137}$.
  The first terms agree to high precision with experiment.
- The radius of convergence in e^2 is zero!
  We are far away from understanding the Standard Model (see e.g. confinement).

- Refined summation techniques (e.g. Borel) may establish reasonable domains of analyticity.

- Unfortunately, this also fails for QED due to the Landau ghost problem.

  It is expected to work for non-Abelian gauge theories because of asymptotic freedom.

  But these theories are too complicated.

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  But these theories are too complicated.

QFT's on noncommutative geometries may provide toy models for non-perturbative renormalisation in four dimensions.

ϕ^4_4-theory on Moyal space with oscillator potential

action functional

$$S[\phi] = \int d^4x \Big(\frac{1}{2}\phi \star \big(-\Delta + \Omega^2 \tilde{x}^2 + \mu^2\big)\phi + \frac{\lambda}{4}\phi \star \phi \star \phi \star \phi \Big)(x)$$

Moyal product $\star$ defined by Θ and $\tilde{x} := 2\Theta^{-1} \cdot x$

parameters: $\mu^2, \lambda \in \mathbb{R}_+$, $\Omega \in [0,1]$, redef'n $\phi \mapsto Z^{\frac{1}{2}}\phi$, $Z \in \mathbb{R}_+$

- renormalisable as formal power series in λ [Grosse-W.]
  means: well-defined perturbative quantum field theory
- β-function vanishes to all orders in λ for $\Omega = 1$
  [Disertori-Gurau-Magnen-Rivasseau]
  means: model is believed to exist non-perturbatively

Up to the sign of μ^2, this model arises from a spectral triple.

The matrix basis of noncommutative Moyal space

$$(f\star g)(x) = \int_{\mathbb{R}^d\times\mathbb{R}^d} \frac{dy\,dk}{(2\pi)^d}\, f(x+\tfrac{1}{2}\Theta\cdot k)\, g(x+y)\,\mathrm{e}^{iky}$$

- central observation (in 2D):
  $f_{00} := 2\mathrm{e}^{-\frac{1}{\theta}(x_1^2+x_2^2)} \quad\Rightarrow\quad f_{00}\star f_{00} = f_{00}$
- left and right creation operators applied to f_{00} lead to
  $f_{mn}(\rho,\varphi) = 2(-1)^m\sqrt{\frac{m!}{n!}}\mathrm{e}^{\mathrm{i}\varphi(n-m)}\Big(\sqrt{\frac{2}{\theta}}\rho\Big)^{n-m}\mathrm{e}^{-\frac{\rho^2}{\theta}}\, L_m^{n-m}(\tfrac{2}{\theta}\rho^2)$
- $\star$-product becomes simple matrix product:
  $f_{mn}\star f_{kl} = \delta_{nk} f_{ml}\ , \qquad \int d^2x\, f_{mn}(x) = \sqrt{\det(2\pi\Theta)}\,\delta_{mn}$
- $(-\Delta+\Omega^2\tilde{x}^2+\mu^2)f_{mn}$
  $= \big(\mu^2+\frac{2}{\theta}(1+\Omega^2)(m+n+1)\big) f_{mn}(x)$
  $-\frac{2}{\theta}(1-\Omega^2)\big(\sqrt{mn}\, f_{m-1,n-1} + \sqrt{(m+1)(n+1)}\, f_{m+1,n+1}\big)$

The 4D-action functional for $\Omega = 1$

expand $\phi(x) = \sum_{m_1,m_2,n_1,n_2} \phi_{mn} f_{m_1n_1}(x_1,x_2) f_{m_2n_2}(x_3,x_4)$

- matrices $(\phi_{mn})_{m,n\in\mathbb{N}^2_\Lambda} \in M_\Lambda$ with cut-off Λ in matrix size
- correlation functions generated by partition function

$$\mathcal{Z}[J] = N \int \Big(\prod_{m,n\in\mathbb{N}^2_\Lambda} d\phi_{mn} \Big) \exp\big(-S[\phi] + \mathrm{tr}(\phi J)\big)$$

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We are interested in $\mathbb{N}^2_\Lambda \to \mathbb{N}^2$. Correlation functions ill-defined unless $S[\phi]$ is a suitably divergent function of Λ:

$$S[\phi] = \sum_{m,n\in\mathbb{N}^2_\Lambda} \frac{1}{2}\phi_{mn} H_{mn} \phi_{nm} + V(\phi)$$

$$H_{mn} = Z\left(\mu^2_{bare} + |m| + |n|\right), \quad V(\phi) = \frac{Z^2\lambda}{4} \sum_{m,n,k,l\in\mathbb{N}^2_\Lambda} \phi_{mn}\phi_{nk}\phi_{kl}\phi_{lm}$$

with $|m| = m_1 + m_2$ and divergent $\mu_{bare}[\Lambda,\lambda], Z[\Lambda,\lambda]$.
There is no separate Λ-dependence in λ!

Ward identity

- inner automorphism $\phi \mapsto U\phi U^\dagger$ of M_Λ
  infinitesimally $\phi_{mn} \mapsto \phi_{mn} + \mathrm{i}\sum_{k\in\mathbb{N}^2_\Lambda}(B_{mk}\phi_{kn} - \phi_{mk}B_{kn})$
- not a symmetry of the action, but translation invariance of the measure $\mathcal{D}\phi = \prod_{m,n\in\mathbb{N}^2_\Lambda} d\phi_{mn}$ gives

$$0 = \frac{\delta W}{\mathrm{i}\delta B_{ab}} = \frac{1}{\mathcal{Z}}\int \mathcal{D}\phi\,\Big(-\frac{\delta S}{\mathrm{i}\delta B_{ab}} + \frac{\delta}{\mathrm{i}\delta B_{ab}}\big(\mathrm{tr}(\phi J)\big)\Big)e^{-S+\mathrm{tr}(\phi J)}$$

$$= \frac{1}{\mathcal{Z}}\int \mathcal{D}\phi\,\sum_n\Big((H_{nb}-H_{an})\phi_{bn}\phi_{na} + (\phi_{bn}J_{na} - J_{bn}\phi_{na})\Big)e^{-S+\mathrm{tr}(\phi J)}$$

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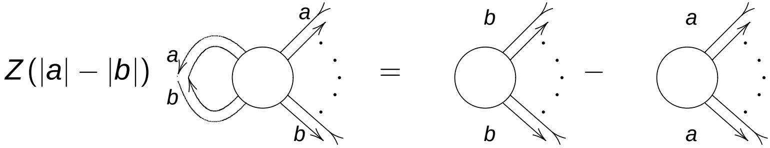
perturbation trick $\phi_{mn} \mapsto \frac{\delta}{\delta J_{nm}}$

$$0 = \Big\{\sum_n \Big((H_{nb}-H_{an})\frac{\delta^2}{\delta J_{nb}\,\delta J_{an}} + \Big(J_{na}\frac{\delta}{\delta J_{nb}} - J_{bn}\frac{\delta}{\delta J_{an}}\Big)\Big)$$
$$\times \exp\Big(-V\big(\tfrac{\delta}{\delta J}\big)\Big)e^{\frac{1}{2}\sum_{p,q}J_{pq}H^{-1}_{pq}J_{qp}}\Big\}_c$$

Interpretation

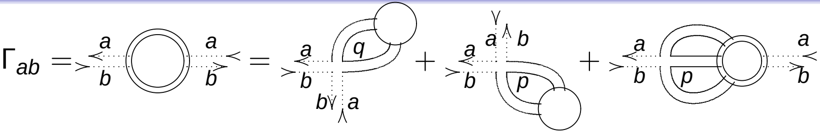
The insertion of a special vertex $V^{ins}_{ab} := \sum_n (H_{an} - H_{nb})\phi_{bn}\phi_{na}$

into an external face of a ribbon graph is the same as the difference between the exchanges of external sources $J_{nb} \mapsto J_{na}$ and $J_{an} \mapsto J_{bn}$



$$Z(|a|-|b|)G^{ins}_{[ab]\dots} = G_{b\dots} - G_{a\dots}$$

Two-point Schwinger-Dyson equation



- vertex is $Z^2\lambda$, connected two-point function is G_{ab}: first graph equals $Z^2\lambda \sum_q G_{aq}$

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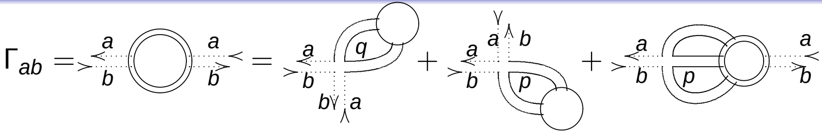
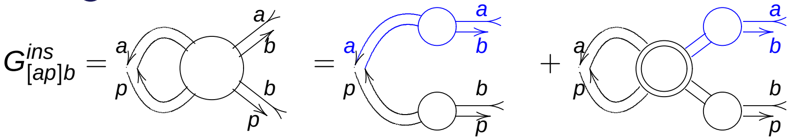


- vertex is $Z^2\lambda$, connected two-point function is G_{ab}:
  first graph equals $Z^2\lambda \sum_q G_{aq}$
- in other two graphs we open the p-face and compare with insertion into connected two-point function; it inserts
  1. either into one-particle reducible line
  2. or into 1PI function:



- amputation of G_{ab}:
  last two graphs together equal $Z^2\lambda \sum_p G^{-1}_{ab} G^{ins}_{[ap]b}$

Result (using $G^{-1}_{ab} = H_{ab} - \Gamma_{ab}$):

$$\Gamma_{ab} = Z^2\lambda \sum_p \Big(G_{ap} + G^{-1}_{ab} G^{ins}_{[ap]b} \Big) = Z^2\lambda \sum_p \Big(G_{ap} - G^{-1}_{ab} \frac{G_{bp} - G_{ba}}{Z(|p| - |a|)} \Big)$$

$$= Z^2\lambda \sum_p \Big(\frac{1}{H_{bp} - \Gamma_{bp}} + \frac{1}{H_{ap} - \Gamma_{ap}} - \frac{1}{H_{bp} - \Gamma_{bp}} \frac{(\Gamma_{bp} - \Gamma_{ab})}{Z(|p| - |a|)} \Big)$$

- This is a self-consistent functional equation for Γ_{ab}.
  It is non-linear and singular. Its singular part at $(a, b = 0)$ already appeared in [Disertori-Gurau-Magnen-Rivasseau].

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- This is a self-consistent functional equation for Γ_{ab}. It is non-linear and singular. Its singular part at $(a, b = 0)$ already appeared in [Disertori-Gurau-Magnen-Rivasseau].
- We perform the renormalisation directly in the SD-equation for Γ_{ab}. The Z-factors are essential for that.
- Taylor: $\Gamma_{ab} = Z\mu^2_{bare} - \mu^2 + (Z-1)(|a|+|b|) + \Gamma^{ren}_{ab}$
  $\Rightarrow \quad G^{-1}_{ab} = H_{ab} - \Gamma_{ab} = |a| + |b| + \mu^2 - \Gamma^{ren}_{ab}$
- $\Gamma^{ren}_{00} = 0$ and $(\partial\Gamma^{ren})_{00} = 0$ determine μ^2_{bare} and Z.

Integral representation

- We replace discrete indices $a, b, p \in \mathbb{N}^2$ by continuous indices $a, b, p \in (\mathbb{R}_+)^2$, and sums by integrals.
- This captures the $\Lambda \to \infty$ behaviour of the discrete version (or defines another interesting field theory).
- The mass-renormalised Schwinger-Dyson equation depends only on the length $|a| = a_1 + a_2$. Partial derivatives $\frac{\partial}{\partial a_i}$ needed to extract Z are equal. Therefore, Γ^{ren}_{ab} depends only on $|a|$ and $|b|$.
- Hence, $\displaystyle\int_{(\mathbb{R}_+)^2}^{(\Lambda)} dp_1\, dp_2\, f(|p|) = \int_0^{\Lambda} |p|\, d|p|\, f(|p|)$

Mass renormalisation = subtraction at 0:

$$(Z-1)(|a|+|b|)+\Gamma^{ren}_{ab}$$
$$=\lambda\int_0^\Lambda |p|\,d|p|\Big(\frac{Z}{|b|+|p|+\mu^2-\Gamma^{ren}_{bp}}+\frac{Z^2}{|a|+|p|+\mu^2-\Gamma^{ren}_{ap}}-\frac{Z^2+Z}{|p|+\mu^2-\Gamma^{ren}_{0p}}$$
$$-\frac{Z}{|b|+|p|+\mu^2-\Gamma^{ren}_{bp}}\frac{\Gamma^{ren}_{bp}-\Gamma^{ren}_{ab}}{|p|-|a|}+\frac{Z}{p+\mu^2-\Gamma^{ren}_{0p}}\frac{\Gamma^{ren}_{0p}}{|p|}\Big)$$

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- perturbative solution depends on combination $\frac{a}{1+a}$ and $\frac{\Lambda}{1+\Lambda}$
- change of variables

$$|a|=:\mu^2\frac{\alpha}{1-\alpha}\,,\quad |b|=:\mu^2\frac{\beta}{1-\beta}\,,\quad |p|=:\mu^2\frac{\rho}{1-\rho}\,,$$
$$\Gamma^{ren}_{ab}=:\mu^2\frac{1-\alpha\beta}{(1-\alpha)(1-\beta)}\Big(1-\frac{1}{G_{\alpha\beta}}\Big)\,,\quad \Lambda=:\mu^2\frac{\xi}{1-\xi}$$

- $\frac{\partial}{\partial a_i}\Big|_{a=0}=\frac{\partial}{\partial\alpha}\Big|_{\alpha=0}$ to extract Z

Theorem [Grosse-W., 2009]

The renormalised planar connected two-point function $G_{\alpha\beta}$ of self-dual n.c. ϕ_4^4-theory satisfies (and is determined by)

$$
\begin{aligned}
G_{\alpha\beta} = 1 + \lambda\Big(& \frac{1-\alpha}{1-\alpha\beta}(\mathcal{M}_\beta - \mathcal{L}_\beta - \beta\mathcal{Y}) + \frac{1-\beta}{1-\alpha\beta}(\mathcal{M}_\alpha - \mathcal{L}_\alpha - \alpha\mathcal{Y}) \\
& + \frac{(1-\alpha)(1-\beta)}{1-\alpha\beta}(G_{\alpha\beta}-1)\mathcal{Y} - \frac{\alpha(1-\beta)}{1-\alpha\beta}(\mathcal{L}_\beta + \mathcal{N}_{\alpha\beta} - \mathcal{N}_{\alpha 0}) \\
& + \frac{1-\beta}{1-\alpha\beta}\Big(\frac{G_{\alpha\beta}}{G_{0\alpha}} - 1\Big)(\mathcal{M}_\alpha - \mathcal{L}_\alpha + \alpha\mathcal{N}_{\alpha 0})\Big)
\end{aligned}
$$

with $\alpha, \beta \in [0,1)$ and

$$
\mathcal{L}_\alpha := \int_0^1 d\rho\, \frac{G_{\alpha\rho} - G_{0\rho}}{1-\rho} \qquad \mathcal{M}_\alpha := \int_0^1 d\rho\, \frac{\alpha\, G_{\alpha\rho}}{1-\alpha\rho}
$$

$$
\mathcal{N}_{\alpha\beta} := \int_0^1 d\rho\, \frac{G_{\rho\beta} - G_{\alpha\beta}}{\rho - \alpha} \qquad \mathcal{Y} = \lim_{\alpha\to 0} \frac{\mathcal{M}_\alpha - \mathcal{L}_\alpha}{\alpha}
$$

Discussion

- The previous integral equation for $G_{\alpha\beta}$ is non-perturbatively defined. It is non-linear and singular (at 1).
- Nonlinearity and singularity can be resolved in perturbation theory. Then: Is the perturbation series analytic at $\lambda = 0$?

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- Non-perturbative approach:
  - There are methods for singular but linear integral equations (Riemann-Hilbert problem).
  - Non-linearity treatable by implicit function theorem (or Nash-Moser theorem), but singularity is problematic.

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If we could solve the equation for $G_{\alpha\beta}$, then all other n-point function should result from a hierarchy of Ward-identities and Schwinger-Dyson equations which are linear (and inhomogeneous) in the highest-order function.

Theorem

The renormalised planar 1PI four-point function $\Gamma_{\alpha\beta\gamma\delta}$ of self-dual n.c. ϕ^4_4-theory satisfies (and is determined by)

$$\Gamma_{\alpha\beta\gamma\delta} = \lambda \cdot \frac{\displaystyle \Big(1 - \frac{(1-\alpha)(1-\gamma\delta)(G_{\alpha\delta}-G_{\gamma\delta})}{G_{\gamma\delta}(1-\delta)(\alpha-\gamma)} + \int_0^1 \rho\, d\rho \frac{(1-\beta)(1-\alpha\delta)G_{\beta\rho}G_{\delta\rho}}{(1-\beta\rho)(1-\delta\rho)} \frac{\Gamma_{\rho\beta\gamma\delta}-\Gamma_{\alpha\beta\gamma\delta}}{\rho-\alpha}\Big)}{\displaystyle G_{\alpha\delta}+\lambda\Big((\mathcal{M}_\beta-\mathcal{L}_\beta-\mathcal{Y})G_{\alpha\delta}+\int_0^1 d\rho \frac{G_{\alpha\delta}G_{\beta\rho}(1-\beta)}{(1-\delta\rho)(1-\beta\rho)} + \int_0^1 \rho\, d\rho \frac{(1-\beta)(1-\alpha\delta)G_{\beta\rho}}{(1-\beta\rho)(1-\delta\rho)} \frac{(G_{\rho\delta}-G_{\alpha\delta})}{(\rho-\alpha)}\Big)}$$

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Corollary

$\Gamma_{\alpha\beta\gamma\delta} = 0$ is not a solution!

We have a non-trivial (interacting) QFT in four dimensions!

Perturbative solution

- We look for an iterative solution $G_{\alpha\beta} = \sum_{n=0}^{\infty} \lambda^n G_{\alpha\beta}^{(n)}$.
- This involves iterated integrals labelled by rooted trees.

Up to $\mathcal{O}(\lambda^3)$ we need

$$I_\alpha := \int_0^1 dx\, \frac{\alpha}{1-\alpha x} = -\ln(1-\alpha)\,,$$

$$I_{\substack{\alpha\\ \bullet}} := \int_0^1 dx\, \frac{\alpha\, I_x}{1-\alpha x} = \mathrm{Li}_2(\alpha) + \frac{1}{2}\big(\ln(1-\alpha)\big)^2$$

$$I_{\substack{\alpha\\ \bullet\;\bullet}} = \int_0^1 dx\, \frac{\alpha\, I_x \cdot I_x}{1-\alpha x} = -2\,\mathrm{Li}_3\Big(-\frac{\alpha}{1-\alpha}\Big)$$

$$I_{\substack{\alpha\\ \bullet\\ \bullet}} = \int_0^1 dx\, \frac{\alpha\, I_{\substack{x\\ \bullet}}}{1-\alpha x} = -2\mathrm{Li}_3\Big(-\frac{\alpha}{1-\alpha}\Big) - 2\mathrm{Li}_3(\alpha) - \ln(1-\alpha)\zeta(2) + \ln(1-\alpha)\mathrm{Li}_2(\alpha) + \frac{1}{6}\big(\ln(1-\alpha)\big)^3$$

In terms of I_t and $A := \frac{1-\alpha}{1-\alpha\beta}$, $B := \frac{1-\beta}{1-\alpha\beta}$:

$$
\begin{aligned}
G_{\alpha\beta} = 1 &+ \lambda\Big\{A(I_\beta - \beta) + B(I_\alpha - \alpha)\Big\} \\
&+ \lambda^2\Big\{A\big(\beta I_{\underset{\bullet}{\beta}} - \beta I_\beta\big) - \alpha AB\big((I_\beta)^2 - 2\beta I_\beta + I_\beta\big) \\
&\quad + B\big(\alpha I_{\underset{\bullet}{\alpha}} - \alpha I_\alpha\big) - \beta BA\big((I_\alpha)^2 - 2\alpha I_\alpha + I_\alpha\big) \\
&\quad + AB\big((I_{\underset{\bullet}{\alpha}} - \alpha) + (I_{\underset{\bullet}{\beta}} - \beta) + (I_\alpha - \alpha)(I_\beta - \beta) + \alpha\beta(\zeta(2)+1)\big)\Big\} \\
&+ \lambda^3\Big\{A\mathcal{W}_\beta + \alpha AB\big(-\mathcal{U}_\beta + I_\alpha I_\beta + I_{\underset{\bullet}{\alpha}} I_\beta\big) + \alpha A^2 B\mathcal{V}_\beta \\
&\quad + B\mathcal{W}_\alpha + \beta BA\big(-\mathcal{U}_\alpha + I_\beta I_\alpha + I_{\underset{\bullet}{\beta}} I_\alpha\big) + \beta B^2 A\mathcal{V}_\alpha \\
&\quad + AB\big(\mathcal{T}_\beta + \mathcal{T}_\alpha - I_\beta(I_\alpha)^2 - I_\alpha(I_\beta)^2 - 6 I_\alpha I_\beta\big) \\
&\quad + AB^2\big((1-\alpha)(I_{\underset{\bullet}{\alpha}} - \alpha) + 3I_\alpha I_\beta + I_{\underset{\bullet}{\beta}} I_\alpha + I_\beta(I_\alpha)^2\big) \\
&\quad + BA^2\big((1-\beta)(I_{\underset{\bullet}{\beta}} - \beta) + 3I_\alpha I_\beta + I_{\underset{\bullet}{\alpha}} I_\beta + I_\alpha(I_\beta)^2\big)\Big\} + \mathcal{O}(\lambda^4)
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{T}_\beta &:= \beta I_{\underset{\bullet}{\beta}} - \beta I_\beta + (I_\beta - \beta)\;, \\
\mathcal{U}_\beta &:= -\beta I_{\underset{\bullet\;\bullet}{\beta}} - (I_\beta)^3 + \beta I_{\underset{\bullet}{\beta}} I_\beta + 2 I_{\underset{\bullet}{\beta}} I_\beta + \beta\zeta(2) I_\beta - 2\beta\zeta(3) \\
&\quad - 2(I_\beta)^2 + \beta(I_\beta)^2 + I_{\underset{\bullet}{\beta}} + \beta I_{\underset{\bullet}{\beta}} + 2 I_\beta - \beta^2\;, \\
\mathcal{V}_\beta &:= \beta I_{\underset{\underset{\bullet}{\bullet}}{\beta}} - \beta^2 I_{\underset{\bullet\;\bullet}{\beta}} - 2\beta^2\zeta(3) + 2\beta I_{\underset{\bullet}{\beta}} I_\beta - I_\beta^3 + 2\beta I_\beta \zeta(2) - 3\beta^2\zeta(2) \\
&\quad + (1-\beta)\big(2\beta I_{\underset{\bullet}{\beta}} - 3 I_\beta^2 + 3\beta I_\beta - 3 I_\beta + \beta\big)\;, \\
\mathcal{W}_\beta &:= (I_{\underset{\underset{\bullet}{\bullet}}{\beta}} - \beta\zeta(2)) - \frac{1}{2} I_\beta \frac{I_\beta - \beta}{\beta} + \frac{1}{2}(I_\beta)^2 - (I_{\underset{\bullet}{\beta}} - \beta) - \frac{1}{2}(I_\beta - \beta) - \frac{1}{2}\beta^2
\end{aligned}
$$

Remark: $\frac{I_\beta - \beta}{\beta} = \int_0^1 dx\; \frac{\beta x}{1 - \beta x}$

(optimal family of iterated integrals not yet determined)

Beyond perturbation

Ansatz (suggested by perturbation, but consistent in general)

$$G_{\alpha\beta} = 1 + \Big(\frac{1-\alpha}{1-\alpha\beta}\Big)\beta^2\mathcal{G}_\beta + \Big(\frac{1-\beta}{1-\alpha\beta}\Big)\alpha^2\mathcal{G}_\alpha + \Big(\frac{1-\alpha}{1-\alpha\beta}\Big)\Big(\frac{1-\beta}{1-\alpha\beta}\Big)\alpha\beta\mathcal{G}_{\alpha\beta}$$

- coupled system of integral equations for $\mathcal{G}_\alpha, \mathcal{G}_{\alpha\beta}$

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- coupled system of integral equations for $\mathcal{G}_\alpha, \mathcal{G}_{\alpha\beta}$
- The 1 inserted into $\mathcal{M}_\alpha$ produces $\lambda \ln(1-\alpha)$ in $\mathcal{G}_\alpha$ which spreads everywhere
- $\frac{1}{G_{0\alpha}} = \frac{1}{1+\alpha^2\mathcal{G}_\alpha}$ becomes singular at some $0 < \alpha(\lambda) < 1$ for any $\lambda < 0$.

  This could be cancelled by a common zero of $\mathcal{M}_\alpha - \mathcal{L}_\alpha + \alpha\mathcal{N}_{\alpha 0}$, which is hard to control.

New strategy

- To avoid $\ln(1-\alpha)$ we need $\mathcal{G}_\alpha = -1+\mathcal{S}_\alpha$ with $\lim_{\alpha\to 1} \mathcal{S}_\alpha = 0$
- This additional condition distinguishes one special value of $\lambda = \lambda_0$ at which we want to prove existence of the theory.

New strategy

- To avoid $\ln(1-\alpha)$ we need $\mathcal{G}_\alpha = -1+\mathcal{S}_\alpha$ with $\lim_{\alpha\to 1} \mathcal{S}_\alpha = 0$
- This additional condition distinguishes one special value of $\lambda = \lambda_0$ at which we want to prove existence of the theory.

ansatz $\mathcal{G}_{\alpha\beta} = -2 - \alpha\mathcal{G}_\alpha - \beta\mathcal{G}_\beta + \mathcal{T}_{\alpha\beta}$ with $\lim_{\alpha\to 1} \mathcal{T}_{\alpha\beta} = 0$

$\lim_{\alpha\to 1} \mathcal{S}_\alpha = 0$ equivalent to

$$-1 = \frac{\lambda}{1+\frac{\lambda}{2}}\Big(\int_0^1 d\rho\, \frac{\rho^2\mathcal{S}_\rho}{1-\rho} - 3\int_0^1 d\rho\, \rho^2\mathcal{S}_\rho + \int_0^1 d\rho\, \rho\mathcal{T}_{0\rho}\Big) \quad (*)$$

- (*) is intrinsically non-perturbative
- insert (*) back into equation for $\mathcal{G}_\alpha = -1+\mathcal{S}_\alpha$:

$$\mathcal{S}_\alpha + \frac{\lambda}{1+\frac{\lambda}{2}} \int_0^1 d\rho\, K(\alpha,\rho)\mathcal{S}_\rho$$
$$= \frac{\lambda}{1+\frac{\lambda}{2}} \Big(-(1-\alpha)\mathcal{Y} + (1-\alpha)\mathcal{S}_\alpha \mathcal{Y}$$
$$- \int_0^1 d\rho\, \frac{1}{\alpha}\Big(\frac{(1-\alpha)^2}{(1-\alpha\rho)^3}\rho\mathcal{T}_{\alpha\rho} - \rho(1-\alpha)\mathcal{T}_{0\rho}\Big)\Big)$$

with

$$K(\alpha,\rho) = \frac{1}{1-\rho}\frac{\rho^5(1-\alpha)^3}{(1-\alpha\rho)^3} + \frac{(1-\alpha)(1-\rho)^2}{(1-\alpha\rho)^3}\rho^2 - 3\frac{(1-\alpha)}{1-\alpha\rho}\rho^2$$
$$\mathcal{Y} = 1 + 3\int_0^1 d\rho\, \rho^2 \mathcal{S}_\rho - \int_0^1 d\rho\, \rho\mathcal{T}_{0\rho}$$

- integral operator K is unbounded, rhs non-linear

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- integral operator K is unbounded, rhs non-linear
- but K is bounded on functions vanishing polynomially at 1:

$$\Big|\int_0^1 d\rho\, K(\alpha,\rho)(1-\rho)^\nu\Big| \leq K_\nu(1-\alpha)^\nu\,, \qquad 0<\nu\leq 1$$

We put $\mathcal{S}_\alpha = (1-\alpha)g(\alpha)$ and $\tilde{K}(\alpha,\rho) = \frac{(1-\rho)}{(1-\alpha)}K(\alpha,\rho)$

Lemma

$(\mathrm{id} + \frac{\lambda}{1+\frac{\lambda}{2}}\tilde{K}) : C([0,1]) \to C([0,1])$ is invertible for $|\lambda| < \frac{3}{7}$, with

$$\Big\| \frac{\lambda}{1+\frac{\lambda}{2}} \Big(\mathrm{id} + \frac{\lambda}{1+\frac{\lambda}{2}}\tilde{K}\Big)^{-1} \Big\| \leq \frac{|\lambda|}{1-\frac{7}{3}|\lambda|}$$

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We put $\mathcal{T}_{\alpha\rho} = 0$ and define recursively $g_0 = 0$

$$g_{n+1} = \frac{\lambda}{1+\frac{\lambda}{2}}\left(\mathrm{id} + \frac{\lambda\tilde{K}}{1+\frac{\lambda}{2}}\right)^{-1}(\ell g_n - 1)\left(1 + 3\int_0^1 d\rho\, \rho^2(1-\rho)g_n(\rho)\right)$$

with $\ell(\alpha) = 1-\alpha$

Proposition

Let $\mathcal{T}_{\alpha\rho} = 0$ and $|\lambda| < \frac{12}{55} = 0.2\overline{18}$. Then:

1. The sequence (g_n) is uniformly convergent to $g = \lim_{n\to\infty} g_n \in C([0,1])$.
2. $\mathcal{S}_\alpha = (1-\alpha)g(\alpha) \in C_0([0,1])$ is the unique solution of our integral equation, with

$$|\mathcal{S}_\alpha| \leq \frac{12-43|\lambda|-\sqrt{(12-55|\lambda|)(12-31|\lambda|)}}{6|\lambda|}(1-\alpha) \leq \frac{55}{6}|\lambda|(1-\alpha)$$

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3. $G_{0\alpha} \geq (1-\alpha)(1+\alpha-\frac{55}{6}|\lambda|\alpha^2) > 0$ for all $\alpha \in [0,1[$.

Equation for $\mathcal{T}_{\alpha\beta}$ is regular in first approximation!

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With careful discussion of signs, this extends to $-\frac{6}{17} < \lambda \leq \frac{6}{5}$.

The next steps

1. establish differentiability of $\mathcal{S}_\alpha$ to control $\displaystyle\int_0^1 d\rho\, \frac{\mathcal{S}_\rho - \mathcal{S}_\alpha}{\rho - \alpha}$
2. interpret equation for $\mathcal{T}$ as recursion $\mathcal{T}^{n+1}(\mathcal{T}^n, \mathcal{S}^n)$ with $\mathcal{T}^0 = 0$ and $\mathcal{S}^0$ from Proposition
3. compute $\mathcal{T}^1$ and re-iterate (g_n) for smaller $|\lambda|$
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3. compute $\mathcal{T}^1$ and re-iterate (g_n) for smaller $|\lambda|$
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Vision

The resulting function $G_{\alpha\beta}$ solves the original problem only for

$$\frac{1}{\lambda} = -\frac{1}{2} - \int_0^1 d\rho\, \frac{\rho^2 \mathcal{S}_\rho}{1-\rho} + 3\int_0^1 d\rho\, \rho^2 \mathcal{S}_\rho - \int_0^1 d\rho\, \rho \mathcal{T}_{0\rho}$$

This equation defines the value of λ at which there is a realistic chance to prove non-perturbative existence of the theory.