Poisson algebraic geometry of Kähler submanifolds

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References

The results in this talk are based on the following papers:

- *On the classical geometry of embedded surfaces in terms of Poisson brackets.* J. Arnlind, J. Hoppe and G. Huisken. arXiv:1001.1604
- *Discrete curvature and the Gauss-Bonnet theorem.* J. Arnlind, J. Hoppe and G. Huisken. arXiv:1001.2223
- *On the classical geometry of embedded surfaces in terms of Nambu brackets.* J. Arnlind, J. Hoppe and G. Huisken. arXiv:1003.5981
- *Multi linear formulation of differential geometry and matrix regularizations.* J. Arnlind, J. Hoppe and G. Huisken. arXiv:1009.4779
- *On the geometry of Kähler–Poisson structures.* J. Arnlind and G. Huisken. arXiv:1103.5862

The work we've done over the past year originated in the following concrete question

Question

Which geometric quantities can be written as algebraic expressions in the Poisson algebra of functions on an embedded surface?

Answer: Almost everything; Curvature, Gauss' equations, Codazzi-Mainardi equations, complex structure, etc. Moreover, everything is expressed in terms of the embedding coordinates.

Similar work

Over the last year we have learned that related formulas have been worked out in different contexts; in particular, let me mention the work of H. Steinacker et al. as well as J. Madore et al.

For instance, the Gaussian curvature of a surface embedded in $\mathbb{R}^m$ can be computed as

$$K = \frac{1}{\gamma^4} \sum_{j,k,l=1}^{m} \bigg(\frac{1}{2} \{\{x^j, x^k\}, x^k\}\{\{x^j, x^l\}, x^l\} - \frac{1}{4} \{\{x^j, x^k\}, x^l\}\{\{x^j, x^k\}, x^l\} \bigg),$$

where

$$\gamma^2 = \frac{1}{2} \sum_{i,k=1}^{m} \{x^i, x^k\}^2.$$

Here, x^i are the embedding coordinates of the surface in $\mathbb{R}^m$.

Why?

I believe the above question (and its generalization to arbitrary manifolds) is interesting in itself, in particular in relation to non-commutative geometry, but what lead us to it?

We were interested in matrix regularizations used to regularize Membrane Theory. In this context, functions on a manifold are mapped to hermitian matrices of increasing dimension such that the Poisson bracket is approximated by the matrix commutator.

In this case, matrices corresponding to the embedding coordinates of the surface are given as solutions of the classical equations of motion. The theory is expected to contain surfaces of arbitrary topology.

Why?

Thus, given a particular solution to the equations of motion, how can one identify its topology? In differential geometry, one can of course compute the genus starting from the embedding coordinates. In the matrix language, there is however no natural analog of the partial derivatives, but the particular combination of derivatives found in the Poisson bracket can be approximated.

Given a Poisson bracket expression for the genus in terms of the embedding coordinates, we can readily write down the corresponding matrix formula and simply compute the regularized genus (with convergence when choosing appropriate approximation properties).

Higher dimensional manifolds

First idea

Try to express the geometry of a n-dimensional submanifold in terms of a n-ary algebraic structure, a "Nambu bracket".

Result of first idea

On a n-dimensional submanifold, the differential geometry can be expressed in terms of a n-ary Nambu bracket.

(I won't say more about this in the following.)

Higher dimensional manifolds

Second idea

Find a particular class of manifolds for which one can express the geometry in terms of Poisson brackets.

Result of second idea

The results for surfaces can be extended to almost Kähler manifolds.

Why can we write objects in terms of the Poisson bracket?

A Poisson bivector θ is such that $\{f, h\} \equiv \theta^{ab}(\partial_a f)(\partial_a h)$ defines a Poisson bracket.

Definition

Let (Σ, g) be a Riemannian manifold. A *Kähler–Poisson structure* on (Σ, g) is a Poisson bivector θ such that

$$\gamma^2 g^{ab} = \theta^{ap}\theta^{bq} g_{pq}.$$

for some $\gamma \in C^\infty(\Sigma)$.

Proposition

Let (Σ, g) be a Riemannian manifold. A Kähler–Poisson structure exists on (Σ, g) if and only if it is an almost Kähler manifold.

Kähler–Poisson structures on submanifolds

Assume Σ is a submanifold of the Riemannian manifold M, embedded via coordinates $x^1, \dots, x^m$, and assume that there exists a Kähler–Poisson structure on Σ. Define

$$\mathcal{D}^{ij} = \frac{1}{\gamma^2}\{x^i, x^k\}\{x^j, x^l\}\eta_{kl},$$

where η denotes the metric on M.

Proposition

The map $\mathcal{D} : TM \to TM$ is the projection onto $T\Sigma$.

Proof

Let X be a vector in $T\Sigma$ and write $X = X^i\partial_i = X^a\partial_a$. One then computes

$$\begin{aligned}
\mathcal{D}^{ij}X_j &= \frac{1}{\gamma^2}\theta^{ab}(\partial_a x^i)(\partial_b x^k)X_j\theta^{pq}(\partial_p x^j)(\partial_q x^l)\eta_{kl}\\
&= \frac{1}{\gamma^2}\theta^{ab}\theta^{pq}g_{bq}(\partial_a x^i)(\partial_p x^j)X_j\\
&= g^{ap}(\partial_a x^i)(\partial_p x^j)X^c(\partial_c x^k)\eta_{jk}\\
&= g^{ap}g_{pc}X^c(\partial_a x^i) = X^a(\partial_a x^i) = X^i.
\end{aligned}$$

Since $(\partial_a x^i)N_i = 0$ for any vector normal to the submanifold, it follows that $\mathcal{D}^{ij}N_j = 0$.

Covariant derivatives

Let $\bar{\nabla}$ denote the covariant derivative on M. In local coordinates one writes

$$\bar{\nabla}_X Y^i = X^k \partial_k Y^i + \bar{\Gamma}^i_{jk} X^j Y^k.$$

Assuming $X, Y \in T\Sigma$ one can write $X^i = \mathcal{D}^{ij} X_j$ which gives

$$\begin{aligned}\bar{\nabla}_X Y^i &= \mathcal{D}^{kl} X_l \partial_k Y^i + \bar{\Gamma}^i_{jk} X^j Y^k \\ &= \frac{1}{\gamma^2}\{Y^i, x^j\}\{x^l, x^m\}\eta_{jm} X_l + \bar{\Gamma}^i_{jk} X^j Y^k,\end{aligned}$$

where all derivatives reside in Poisson brackets. Hence, the covariant derivative on Σ can be expressed in terms of Poisson brackets as

$$\nabla_X Y = \mathcal{D}\big(\bar{\nabla}_X Y\big)$$

for all $X, Y \in T\Sigma$.

Curvature

One can proceed to explore Gauss' and Weingarten's equations. For instance, denoting

$$\hat{\nabla}_i = \mathcal{D}_i^k \bar{\nabla}_k$$
$$\Pi^{ij} = \eta^{ij} - \mathcal{D}^{ij}$$

one can compute the curvature of Σ as

$$X^i Y^j Z^k V^l \Big[\bar{R}_{ijkl} + \big(\hat{\nabla}_k \Pi_{im}\big)\big(\hat{\nabla}_l \Pi_j^m\big) - \big(\hat{\nabla}_l \Pi_{im}\big)\big(\hat{\nabla}_k \Pi_j^m\big) \Big]$$

for $X, Y, Z, V \in T\Sigma$, where $\bar{R}_{ijkl}$ is the curvature tensor of M.

Some more formulas

For simplicity, let us consider the case of $M = \mathbb{R}^m$.

$$(\nabla u)^i = \hat{\nabla}^i(u) = \frac{1}{\gamma^2}\{u, x^k\}\{x^i, x_k\}$$

$$\Delta(u) = \hat{\nabla}_i\hat{\nabla}^i(u) = \frac{1}{\gamma^2}\{\frac{1}{\gamma^2}\{u, x^k\}\{x^i, x_k\}, x^j\}\{x_i, x_j\}$$

$$\text{div}(Y) = \hat{\nabla}_i Y^i = \frac{1}{\gamma^2}\{Y^i, x^k\}\{x_i, x_k\}$$

Complex structure

The basic quantity $\mathcal{P}^{ij} = \{x^i, x^j\}$ is related to the almost complex structure, namely

$$\frac{1}{\gamma}\mathcal{P}^i{}_j X^j = \mathcal{J}(X)^i$$

for all $X \in T\Sigma$. What if the complex structure is integrable? In particular, the complex structure is parallel with respect to the Riemannian connection. This can be formulated as

$$X_j Y_k \big(\widetilde{\nabla}^i \widetilde{\mathcal{D}}^{jk}\big) = 0,$$

where

$$\widetilde{\nabla}^i = \frac{1}{\gamma}\{x^i, x^k\}\bar{\nabla}_k \equiv \widetilde{\mathcal{D}}^{ik}\bar{\nabla}_k.$$

The operators $\widetilde{\mathcal{D}}$ and $\widetilde{\nabla}$ should be compared with $\mathcal{D}$ and $\hat{\nabla}$

$$\widetilde{\mathcal{D}}^{ik} = \frac{1}{\gamma}\{x^i, x^k\} \qquad \mathcal{D}^{ik} = \frac{1}{\gamma^2}\{x^i, x^j\}\{x^k, x^l\}\eta_{jl}$$

$$\widetilde{\nabla}^i = \widetilde{\mathcal{D}}^{ik}\bar{\nabla}_k \qquad \hat{\nabla}^i = \mathcal{D}^{ik}\bar{\nabla}_k$$

These new operators contain "half" the number of Poisson brackets. The formula $X_j Y_k\big(\widetilde{\nabla}^i\widetilde{\mathcal{D}}^{jk}\big) = 0$ can be used to reduce many formulas; e.g. in $\mathbb{R}^m$

$$\begin{aligned}\Delta(u) &= \frac{1}{\gamma^2}\{\frac{1}{\gamma^2}\{u, x^k\}\{x^i, x_k\}, x^j\}\{x_i, x_j\} = \hat{\nabla}^i\hat{\nabla}_i(u)\\ &= \widetilde{\mathcal{D}}^{ik}\widetilde{\nabla}_k\big(\widetilde{\mathcal{D}}_{il}\widetilde{\nabla}^l(u)\big) = \mathcal{D}^{kl}\widetilde{\nabla}_k\widetilde{\nabla}_l(u) + \widetilde{\mathcal{D}}^{ik}\big(\widetilde{\nabla}_k\widetilde{\mathcal{D}}_{il}\big)\widetilde{\nabla}^l(u)\\ &= \widetilde{\nabla}^l\widetilde{\nabla}_l(u) = \frac{1}{\gamma}\{\frac{1}{\gamma}\{u, x^i\}, x_i\}\end{aligned}$$

Discretized / Non-commutative geometrical concepts

Since we have formulated differential geometry in terms of Poisson brackets, it is suggestive to introduce discretized geometrical concepts by simply replacing Poisson brackets by commutators. In a matrix regularization, at least you now that they will converge in the norm sense (although there are some interesting subtleties here).

Theorems in differential geometry that can be formulated in terms of Poisson brackets have their analogues in matrix regularizations, by simply mapping all quantities to matrices. Is it possible to prove some theorems without reference to the underlying manifold? Solely in terms of matrix manipulations?

An example

Let us consider a particular example. On a compact closed manifold, a bound on the Ricci curvature induces a bound on the eigenvalues of the Laplace operator:

$$R_{ab} \geq \kappa g_{ab} \quad \Rightarrow \quad \lambda \geq \frac{n\kappa}{n-1},$$

where n is the dimension of the manifold, and λ is an eigenvalue of the Laplace operator, i.e. $\Delta(u) = -\lambda u$.

Differential geometric proof

Let us recall the proof. One rewrites

$$\int_\Sigma (\Delta u)^2 = -\lambda \int_\Sigma u\Delta u = \lambda \int_\Sigma |\nabla u|^2$$

On the other hand

$$\begin{aligned}\int_\Sigma (\Delta u)^2 &= \int_\Sigma \nabla_i\nabla^i(u)\nabla_k\nabla^k(u) = -\int_\Sigma \nabla^i(u)\nabla_i\nabla_k\nabla^k(u)\\ &= -\int_\Sigma \Big(\nabla^i(u)\nabla_k\nabla_i\nabla^k(u) - R_{ik}\nabla^i(u)\nabla^k(u)\Big)\\ &\geq \frac{1}{n}\int_\Sigma (\Delta u)^2 + \kappa\int_\Sigma |\nabla u|^2 = \Big(\frac{\lambda}{n}+\kappa\Big)\int_\Sigma |\nabla u|^2\end{aligned}$$

Comparing the two calculations gives $\lambda \geq n\kappa/(n-1)$.

What do we actually use?

- Relation between covariant derivatives and curvature
- Partial integration
- Cauchy-Schwartz inequality.

Can we do this with matrices? Of course, since there exists a map from the manifold to matrices, and the above concepts can be expressed in terms of matrix algebra, the result must hold. But, can one do it in terms of *pure* matrix manipulations?

- Laplace operator $\Delta(A) = -\frac{1}{\hbar_\alpha^2}\big[[A, X^i], X_i\big]$.
- Partial integration:
  $\operatorname{Tr}[X, Y]Z = \operatorname{Tr}[X, YZ] - \operatorname{Tr} Y[X, Z] = -\operatorname{Tr} Y[X, Z]$.

A lot of (but not all, so far) manipulations can be done on the matrix side.

Proposition

Let $(T^\alpha, \hbar_\alpha)$ be a C^2-convergent matrix regularization of (Σ, ω) and let $\{\hat{u}_\alpha\}$ be a C^2-convergent eigenmatrix sequence of $\hat{\Delta}_\alpha$ with eigenvalues $\{-\lambda_\alpha\}$. If $\hat{K}_\alpha \geq \kappa \mathbb{1}_{N_\alpha}$ for some $\kappa \in \mathbb{R}$ and all $\alpha > \alpha_0$, then $\lim_{\alpha\to\infty} \lambda_\alpha \geq 2\kappa$.

The result depends on that the matrix algebras has an underlying manifold structure. Can one prove this for sequences of matrix algebras (satisfying some matrix conditions) without reference to any manifold? We believe it might be possible.

However, let us start by examining the case of a commutative algebra with a Poisson structure, instead of noncommutative Poisson algebras (as in the case of matrices).

Algebraic abstraction

Having expressed differential geometry in terms of Poisson algebras, one may wonder if standard geometrical results (now written as Poisson algebra statements) hold for general Poisson algebras?

As expected, the class of Poisson algebras is too large, and one has to find a suitable subclass that mimics a function algebra on a manifold.

Thus, we need to encode the condition $\gamma^2 g^{ab} = \theta^{ap}\theta^{bq} g_{pq}$ in the ambient space function algebra.

Kähler–Poisson algebras (with $M = \mathbb{R}^m$)

As we have seen, a simple consequence of the Kähler–Poisson structure condition is that $\mathcal{D}^{ij}$ is a projection. By denoting $\mathcal{P}^{ij} = \{x^i, x^j\}$ we formulate it in the following way:

Definition (Kähler–Poisson algebra)

Let $(\mathcal{A}, \{\cdot,\cdot\}$ be the field of fractions of the polynomial algebra $\mathbb{C}[x^1, \ldots, x^m]$ together with a Poisson structure $\{\cdot,\cdot\}$. The pair $(\mathcal{A}, \{\cdot,\cdot\})$ is called an *almost Kähler–Poisson algebra* if there exists $\gamma^2 \in \mathcal{A}$ such that

$$\mathcal{P}^i{}_j \mathcal{P}^j{}_k \mathcal{P}^k{}_l = -\gamma^2 \mathcal{P}^i{}_l \qquad (*)$$

where repeated indices are summed over from 1 to m. (Note that there is no difference between upper and lower indices.)

Tangent space and normal space

Let us consider the space of derivations $\mathrm{Der}(\mathcal{A})$, spanned by ∂_i, and let $\mathcal{P}$ act on $X = X^i\partial_i$ as

$$\mathcal{P}(X) = \mathcal{P}^i{}_j X^j \partial_i.$$

Condition $(*)$ implies that $\mathcal{D}^{ij} = \gamma^{-2}\mathcal{P}^i{}_k\mathcal{P}^{jk}$ is a projector, i.e. $\mathcal{D}^2 = \mathcal{D}$, which allows for a very natural definition of the tangent space of the "submanifold" as a projective module.

$$\mathcal{X}(\mathcal{A}) = \{\mathcal{D}(X) : X \in \mathrm{Der}(\mathcal{A})\}$$

The dimension of $\mathcal{X}(\mathcal{A})$ is called the *geometric dimension of* $\mathcal{A}$. By writing $\Pi = \mathbb{1} - \mathcal{D}$ we also obtain the normal space as

$$\mathcal{N}(\mathcal{A}) = \{\Pi(X) : X \in \mathrm{Der}(\mathcal{A})\}.$$

We also set $(X, Y) = X^i Y_i$.

Covariant derivative

We learned from differential geometry that the covariant derivative on the submanifold can be written as (in the case of $M = \mathbb{R}^m$)

$$\nabla_X Y^i = \mathcal{D}\big(\hat{\nabla}_X Y\big)^i = \mathcal{D}^{ij} X^k \mathcal{D}_k(Y_j),$$

where $\mathcal{D}_k(u) = \mathcal{D}^l_k \partial_l u = \gamma^{-2}\{u, x^l\}\mathcal{P}_{kl}$. Let us take this as a definition for derivations $X, Y \in \mathcal{X}(\mathcal{A})$.

Affine connection

Proposition

Let $\mathcal{A}$ be an almost Kähler–Poisson algebra. For all $X, Y, Z \in \mathcal{X}(\mathcal{A})$ and $u \in \mathcal{A}$, the covariant derivative has the following properties

1. $\nabla_X(Y + Z) = \nabla_X Y + \nabla_X Z$,
2. $\nabla_{(X+Y)} Z = \nabla_X Z + \nabla_Y Z$,
3. $\nabla_{(uX)} Y = u\nabla_X Y$,
4. $\nabla_X(uY) = \nabla_X(u)Y + u\nabla_X Y$,

where $\nabla_X(u) = X^k \mathcal{D}_k(u)$.

Torsion-free metric connection

Proposition

The covariant derivative in an almost Kähler–Poisson algebra has no torsion, i.e. $\nabla_X Y - \nabla_Y X - [X, Y] = 0$ *for all* $X, Y \in \mathcal{X}(\mathcal{A})$.

Proposition

In an almost Kähler–Poisson algebra it holds that $(\nabla_X \mathcal{D})(Y, Z) = 0$ *for all* $X, Y, Z \in \mathcal{X}(\mathcal{A})$.

Bianchi identities

By introducing

$$R(X,Y,Z) \equiv R(X,Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z$$

one can prove the Bianchi identities.

Proposition

Let $\mathcal{A}$ be an almost Kähler–Poisson algebra and let R be the curvature tensor of $\mathcal{A}$. For all $X,Y,Z,V \in \mathcal{X}(\mathcal{A})$ it holds that

$$R(X,Y,Z) + R(Z,X,Y) + R(Y,Z,X) = 0$$
$$(\nabla_X R)(Y,Z,V) + (\nabla_Y R)(Z,X,V) + (\nabla_Z R)(X,Y,V) = 0.$$

Sectional curvature

Introduce the sectional curvature with respect to $X, Y \in \mathcal{X}(\mathcal{A})$

$$K(X, Y) = \frac{R(X, Y, X, Y)}{\mathcal{D}(X, X)\mathcal{D}(Y, Y) - \mathcal{D}(X, Y)^2}.$$

Proposition

Let $\mathcal{A}$ be an almost Kähler–Poisson algebra with curvature tensor R and geometric dimension $n \geq 3$. If $K(X, Y) = k \in \mathcal{A}$ for all $X, Y \in \mathcal{X}(\mathcal{A})$ then $\{k, u\} = 0$ for all $u \in \mathcal{A}$.

Eigenvalues of the Laplacian

Since we have Poisson algebraic expressions for the Laplace operator and the curvature, one can ask the question: Does a bound on the (algebraic) Ricci curvature induce a bound on the eigenvalues of the (algebraic) Laplacian?

To prove this we need to introduce some more concepts in the algebra.

$*$-algebras and states

We make the Kähler–Poisson algebra into a $*$-algebra by setting $(x^i)^* = x^i$. A *state* on a $*$-algebra is a $\mathbb{C}$-linear functional such that

$$\int_{\mathcal{A}} a^* = \overline{\int_{\mathcal{A}} a} \quad \text{and} \quad \int_{\mathcal{A}} a^* a \geq 0$$

for all $a \in \mathcal{A}$. A state is called *tracial* if in addition

$$\int_{\mathcal{A}} \nabla_i X^i = 0$$

for all $X \in \mathcal{X}(\mathcal{A})$. An element $a \in \mathcal{A}$ is called *positive* if it can be written as $a = \sum a_i^* a_i$ for some $a_i \in \mathcal{A}$.

Ready to go!

Now we have all the ingredients to prove the desired theorem:

- Covariant derivatives, and their relation to curvature.
- Partial integration (tracial state).
- Cauchy-Schwartz inequality since $(X^*, X) \geq 0$.

Theorem

Let $\mathcal{A}$ be an almost Kähler–Poisson algebra, of geometric dimension n, with a tracial state, and let $-\lambda$ be an eigenvalue of the Laplace operator corresponding to an eigenvector u such that $\langle u, u\rangle > 0$. If there exists $\kappa \in \mathbb{R}$ such that $R(X^*, X) \geq \kappa(X^*, X)$ for all $X \in \mathcal{X}(\mathcal{A})$ then $\lambda \geq n\kappa/(n-1)$.

Note that the proof is now purely algebraic!

Our belief is that one can continue and prove many classical theorems in the context of almost Kähler–Poisson algebras.

Example

Let $\mathcal{A} = \mathbb{C}[x^1, x^2, x^3]$ be the polynomial algebra in three variables together with the Poisson structure

$$\{x^i, x^j\} = \varepsilon^{ijk}\partial_k C$$

where C is an arbitrary (hermitian) element of $\mathcal{A}$, and ε^{ijk} is the totally anti-symmetric Levi-Civita symbol. It is easy to check that $\mathcal{A}$ is an almost Kähler–Poisson algebra with

$$\gamma^2 = (\partial_1 C)^2 + (\partial_2 C)^2 + (\partial_3 C)^2,$$

and that $\{\gamma^2, x^i\} = 0$ for $i = 1, 2, 3$.

The projection operator $\mathcal{D}_{ik}$ is computed to be

$$\mathcal{D}_{ik} = \delta_{ik} - \frac{1}{\gamma^2}\big(\partial_i C\big)\big(\partial_k C\big),$$

which gives $\Pi_{ik} = (\partial_i C)(\partial_k C)/\gamma^2$. Hence, the geometric dimension of $\mathcal{A}$ is 2, and a basis for $\mathcal{N}(\mathcal{A})$ (which is then one-dimensional) is given by $\sum_{i=1}^{3}(\partial_i C)\partial_i$. By using Gauss formula, one computes the curvature to be

$$R(X,Y,Z,V) = \frac{1}{\gamma^2}\Big(\big(\partial^2_{ik}C\big)\big(\partial^2_{jl}C\big) - \big(\partial^2_{il}C\big)\big(\partial^2_{jk}C\big)\Big)X^iY^jZ^kV^l.$$

Summary

- We have shown that the differential geometry of an almost Kähler submanifold can be expressed as Poisson brackets of the embedding coordinates.
- Consequently, we defined almost Kähler–Poisson algebras, as algebraic analogues of function algebras.
- Almost Kähler–Poisson algebras have natural concepts of tangent and normal space, as well as a nice theory of curvature.
- The connection has all the properties one wants, like being torsion free and metric as well as satisfying the Bianchi identities.
- We have illustrated the usefulness of these algebras by proving algebraic counterparts of several classical theorems in differential geometry.

Outlook

- How far can one push the analogy with differential geometry? Can we prove more theorems? A theory of Chern classes?
- One needs to fully understand the isomorphisms between almost Kähler–Poisson algebras (which is the equivalent of coordinate transformations).
- What is the natural algebraic generalization of submanifolds of curved spaces?
- Choosing more general types of algebras in the definition of almost Kähler–Poisson algebras.
- Non-commutative Kähler–Poisson algebras?