Renormalization on Moyalspace

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Setting

Perturbative scalar quantum field theory

- of hyperbolic signature
- on a model of a noncommutative spacetime (Moyalspace) with noncommuting time

Problems/Difficulties

- No (straightforward) Wick rotation. Analytic continuation of Wightman functions yield Schwinger functions which are not those of the Euclidean approach and which seem to be useless to built a perturbative field theory [B10].
- Contrary to QFT on a vector space: divergences very different in hyerbolic theory, although mixing of ultraviolet/infrared divergences present in both settings. No renormalizable model (Grosse-Wulkenhaar?) with hyperbolic signature so far.

Please note however, that unitarity is not a problem.

Setting: Moyal space

Motivation. DFR: uncertainty relations (operational definition of an event in spacetime puts restrictions on localizability). Canonical commutation relations $[\hat{x}^\mu, \hat{x}^\nu] = i\lambda_P^2 Q^{\mu\nu}$.

Symbol calculus. For any $\theta \in \sigma(Q)$, θ antisymmetric $d \times d$-matrix of rank d (d even): Twisted convolution product, e.g. for $f, g \in \mathcal{S}(\mathbb{R}^d)$,

$$f * g(x) := \int \tilde{f}(p-k)\tilde{g}(k)\ e^{-\frac{i}{2}\langle p, \theta k\rangle}\ e^{-i\langle p, x\rangle}\ dkdp$$

Nonlocal: for $f, g \in D$, $f * g \in \mathcal{S}$, not compactly supported.

Setting: QFT on Moyal space

Effective Theory: Free theory (linear) remains unchanged (same hyperbolic PDOp), use symbol calculus to define interaction term (products) for fields on ordinary $\mathbb{R}^d$ with twisted convolution products. Alternative definition [BDFP04] not discussed today.

Nonlocal theory, so: What is renormalization?

Hyperbolic (scalar) quantum field theory – position space formulation

A quantum field Φ on d-dimensional Minkowski space is an operator-valued distribution, i.e. a map from $\mathcal{D}(\mathbb{R}^d)$ to (unbounded) operators on a dense subspace D in a Fock space $\mathcal{F}$ over a Hilbert space such that for any $\psi_1, \psi_2 \in D$, the map

$$\mathcal{D}(\mathbb{R}^d) \ni g \mapsto \langle \psi_1, \Phi(g)\psi_2 \rangle \in \mathbb{C}$$

is a distribution in $\mathcal{D}'(\mathbb{R}^d)$.

We say: Φ is an operator-valued distribution on $\mathcal{D}(\mathbb{R}^d)$ with dense domain $D \subset \mathcal{F}$.

The free field is an operator-valued distribution that solves the linear field equation, e.g.

$$(\partial_t^2 - \Delta_{\mathbf{x}} + m^2)\,\phi = 0$$

for the massive ($m > 0$) scalar field.

Here, $\mathcal{F}$ is the symmetric (bosonic) Fock space of the Hilbert space of L^2-functions on the positive mass shell H_+ with Lorentz-invariant measure μ.

$H_+ := \{(p_0, \mathbf{p}) \in \mathbb{R}_{>0} \times \mathbb{R}^{d-1} \mid p_0 = \sqrt{\mathbf{p}^2 + m^2}\}$,
$d\mu(p) = d\mathbf{p}/2p_0$.

Products of free fields are ill-defined as distributions – need for renormalization. By an inductive subtraction procedure (counterterms) define Wick ordered tensor products

$$:\phi \otimes \cdots \otimes \phi:$$

"Theorem 0" of Epstein and Glaser (1972):

> The product of a k-fold Wick (tensor) product of free fields $:\phi \otimes \cdots \otimes \phi:$ and any translation-invariant distribution $u \in \mathcal{D}'(\mathbb{R}^{kd})$ exists as an operator valued distribution with a dense invariant domain D in $\mathcal{F}$ (wavefunctions from $\mathcal{S}$).

In particular, Wick products $:\phi^k:$ (Wick product in coinciding points) are well-defined with dense invariant domain D.

Significance:

Perturbation theory produces translation invariant distributions (before renormalization in general from $\mathcal{D}'(\mathbb{R}^{kd} \setminus D)$) which are multiplied with Wick (tensor) products of fields.

Therefore, renormalization only concerns the singularities of ordinary distributions. We only have to guarantee that after renormalization they are still translation invariant.

In scalar theory, in the setup of the Dyson series/time ordered products: the translation invariant distributions which appear are convolutions and products of the Feynman propagator G_F (a fundamental solution of the free (=linear) field equation).
Note: not so simple on Moyal space!

Good news: $\mathrm{singsupp}\, G_F$ is the boundary of the *lightcone*, but from Hörmander's criterion (wavefront sets) we find that products $(G_F)^k$ can be defined as the pullback of the k-fold tensor product along the diagonal map as distributions in $\mathcal{D}'(\mathbb{R}^d \setminus \{0\})$. Only 0 is (potentially) a problem, 'ultraviolet divergence'. Details

Renormalization in position space: extension of distributions.

Steinman's singular order measures how bad the singularity in 0 is (depends on the dimension d): If sing ord ≥ 0, renormalization is needed, otherwise not. Power counting of divergences.

Ex: $(G_F)^2 \in \mathcal{D}'(\mathbb{R}^4 \setminus \{0\})$ needs renormalization, sing ord $= 0$ 'logarithmic divergence'.

Freedom to renormalize in different ways, given as a linear combination of the δ-distribution and its derivatives up to the distribution's scaling degree. Locality of counterterms.

Renormalizability:

Power counting for all contributions in perturbation theory.

If singular order is bounded by a universal constant C for all orders n of perturbation theory, the theory is renormalizable. Finite number of types of counterterms.

Otherwise, we need infinitely many physical conditions to fix renormalization: theory is not renormalizable.

Note: theory is superrenormalizable if singular order is ≥ 0 only for finite number $n \leq N$.

Main tool:

Renormalization is an inductive procedure. Renormalization at lower orders (extension of distributions on $\mathbb{R}^{kd}$) is used for renormalization at higher orders (distributions on $\mathbb{R}^{nd}$ with $n > k$)

Graphical language (k, n number of vertices): insert renormalized subgraphs in larger graphs. Combinatorics controlled by Zimmerman's forest formula (Connes-Kreimer Hopf-Algebra).

On Moyal space:

- We are very far from an abstract notion of 'renormalizability' in the sense explained above.
- Locality of counterterms? May have to be given up.
- Minimal requirement: finite number of types of counterterms.
- However, it sems that already main tool of renormalization breaks down.

QFT on Moyal space: Euclidean setting

No details. Theory's building block: Unique fundamental solution G_E of the underlying elliptic (linear) PDE. Twistings appear.

Seiberg+Raamsdonck: Break-down of main tool of renormalization theory shown in an example: Perturbation theory produces at low order (2 vertices) a distribution u_E with Fourier transform

$$\tilde{u}_E(p) \; = \; \tilde{G}_E(p) \; (\theta^* G_E)(p)$$

If θ is nondegenerate, the pullback $\theta^* G_E$, symbolically, $G_E \circ \theta$, w.r.t. multiplication with θ is defined (in the sense of Hörmander, Details). No need for renormalization of the corresponding position space distribution. Contrary to $\theta = 0$ case, where this is an ill-defined (tadpole) contribution.

Observe: in $\tilde{u}_E$, both position and momentum space distributions appear. Absolutely uncommon in QFT on vectorspaces.

Euclidean setting: The mixing

First note: wavefront set $(\theta^* G_E)$ = wavefront set (G_E)

There are higer order contributions in perturbation theory in which products of $\tilde{u}_E(p) = \tilde{G}_E(p)\ (\theta^* G_E)(p)$ appear.

As we have seen: Products $(\theta^* G_E)^k$ are defined only on $\mathbb{R}^d \setminus 0$ (for k large enough) e.g. for $k \geq 2$ if $d = 4$ they need renormalization.

Problem: The divergence appears at $p = 0$, in the infrared. 'Mixing of UV and IR'.

Similar problems for many graphs. Arbitrarily many different types of counterterms needed.

Solution: get rid of such divergences altogether.
Grosse- Wulkenhaar model where free theory (propagator) is modified. Lead to constructive QFT for $d = 2$ (2011).

Hyperbolic setting? No Wick rotation for noncommuting time, so we have to study it independently.

Moyal space: Hyperbolic setting

The examples which caused these mixing problems in the Euclidean setting were shown to be well-defined in the hyperbolic setting [B] (Reason: different distributions appear there, not in general twisted products of Feynman propagators, have different properties).

Study question of finite number of types of counterterms systematically [B, Doplicher, Fredenhagen, Piacitelli].

First step: Modification of Wick products

- Twisted tensor products of free fields defined as op-valued distribution. Subtelty: no compactly supported testfunction (nonlocality of the twisted product).
- Ordinary subtraction procedure which leads to Wick products: arbitrarily many different types of counterterms needed.
- Modified subtraction procedure in which only a certain type of subtraction terms are allowed was given, combinatorics understood long ago [BDFP 03], leads to modified Wick products ('quasiplanar Wick products')
- Functional analysis almost completed: well-definedness in coinciding points, existence of invariant domain etc. [BDFP]

Moyal space: Mixing problem

At the same time, ongoing investigation of particular graphs in the framework of the Dyson series. Result [B10]: There is a mixing problem (in 1-P-non-irreducible graphs)!

Complicted graphical language to keep track of the combinatorics. Vertices have sub-structure (rows of dots to encode the order of edges entering and leaving the vertex). Consider, e.g.

 in φ^3-theory, and

 in φ^4-theory.

Corresponding distributions (formal integral kernels)

$$\tau(\zeta_0 - (\theta q)_0)\, \Delta_+^k(\zeta)$$

$k = 2, 3$, where Δ_+ is the 2-point function (a tempered distribution), τ is the Heaviside step function, and where $\zeta = x - y$ (with x, y the vertices), and q is the momentum assigned to the free dot: action of the free field that corresponds to that dot on Fockspace yields an integration against a wavefunction $\psi(q)$. Again, we see position space distributions τ, Δ_+ involving also momenta.

Lemma: By Hörmander's criterion is a well-defined product of distribution on $\mathbb{R}^{2d} \times \mathbb{R}$.

Now consider the 1-P non-irreducible graphs

 and 

In the adiabatic limit, we have momentum conservation. Using this to simplify the expressions, we find that the two convolutions appear with opposite signs,

$$\theta(\zeta_0 - (\theta q)_0)\, \Delta_+^k(\zeta)\ \theta(\eta_0 + (\theta q)_0)\, \Delta_+^k(\eta)$$

Lemma By Hörmander's criterion, their product is a well-defined distribution outside 0. Steinman's scaling degree methods tell us that requires renormalization for $d \geq 6$ ($k = 2$ case) and for $d \geq 4$ ($k = 3$ case).

Divergence occurs for $(\theta q)_0 = 0$: partly an infrared divergence: Renormalization in the sense of extension of distirbutions would include terms acting on the wavefunctions.

Conclusion

- Need for strange counterterms found in perturbation theory (Dyson setup) on Moyal space.
- Conjecture: Arbitrarily many types of such counterterms needed.
- Very far from systematic understanding of renormalization (in hyperbolic) theories on Moyal space.
- Model too crude (constant commutators) ...

Extra: Wavefront sets

Let u be a distribution with compact support. Define $\Sigma(u)$= cone $\setminus\{0\}$ $\subset \mathbb{R}^{n*}$ of directions in which $\tilde{u}$ does not decrease rapidly (global smoothness).

Local version: Let u be a distribution on $\mathcal{D}(\mathbb{R}^n)$, let g be a bump function around $x \in \mathbb{R}^n$, define $\Sigma_x(u) := \bigcap_g \Sigma(gu)$. Wavefront set

$$WF(u) = \{(x,p) \in \mathbb{R}^n \times \dot{\mathbb{R}}^n \mid p \in \Sigma_x(u)\}$$

Thm [Hörmander]: Let $\varphi : M \to N$ be smooth, u a distribution on $\mathcal{D}(N)$. If there are no points $(x,p) \in WF(u)$ s.t. p is normal to $d\varphi(T_x M)$, then the pullback $\varphi^*(u)$ can be defined.

Cor (sufficient condition for existence of the product of distributions) Pullback along the diagonal map of the tensor product $u \otimes v$ of two distributions yields the product of the two distributions and is defined provided that there are no points $(x,p) \in WF(u)$ s.t. $(x,-p) \in WF(v)$.

Extra: Wavefront sets

Wavefront set WF of the Feynman propagator G_F

singsupp $G_F =$ boundary of the lightcone.

- for $x = 0$: $(x, p) \in WF(G_F)$ for all $p \in \mathbb{R}^n \setminus \{0\}$.
- for $x \in$ forward lightcone $x^2 = 0, x_0 > 0$: $(x, p) \in WF(G_F)$ iff $p \in$ forward lightcone, i.e. $p^2 = 0, p_0 > 0$.
- backward lightcone $x^2 = 0, x_0 < 0$: $(x, p) \in WF(G_F)$ iff $p \in$ backward lightcone, i.e. $p^2 = 0, p_0 < 0$.

so everywhere but in 0, G_F satisfies Hörmander's sufficient condition for the existence of the product of distributions $\Rightarrow$ G_F^k well-defined distribution in $\mathcal{D}'(\mathbb{R}^n \setminus \{0\})$.

back

Extra: Wavefront sets, $G_E \circ \theta$

Thm [Hörmander]: Let $\varphi : M \to N$ be smooth, u a distribution on $\mathcal{D}(N)$. If there are no points $(x, p) \in WF(u)$ s.t. p is normal to $d\varphi(T_x M)$, then the pullback $\varphi^*(u)$ can be defined.

Set of normals of $\varphi : \mathbb{R}^n \to \mathbb{R}^n$, $\varphi(x) = \theta x$ is

$$N = \{(\varphi(x), \xi) \in \mathbb{R}^n \times \mathbb{R}^n \mid {}^t\varphi'(x)\xi = 0\} = \{(\varphi(x), 0) \in \mathbb{R}^n \times \mathbb{R}^n\}$$

back

Twisted product

For $f, g \in \mathcal{S}(\mathbb{R}^d)$, d even,

$$f * g(x) := \int \tilde{f}(p-k)\tilde{g}(k)\ e^{-\frac{i}{2}\langle p,\theta k\rangle}\ e^{-ipx}\ dkdp$$

for an antisymmetric $d \times d$-matrix θ of rank d.
Associative, defines a continuous map $\mathcal{S} \times \mathcal{S} \to \mathcal{S}$.

Twisted tensor product

For $f, g \in \mathcal{S}(\mathbb{R}^d)$,

$$(fg)^{\otimes\theta}(x) := \int \tilde{f}(p)\tilde{g}(k)\ e^{-\frac{i}{2}\langle p,\theta k\rangle}\ e^{-ipx-iky}\ dkdp$$

Combinatorics of twisted tensor products

Recall: A contraction in a finite ordered set N is a pair (A, α) with $\emptyset \subseteq A \subseteq N$ and $\alpha : A \to N \setminus A$ injective such that $\alpha(a) > a$ for all $a \in A$ (w.r.t. the order of N).

Recall that before, we assigned to a contraction in $N := \{1, \dots, n\}$ a continuous map

$$\rho_C^0 : \mathcal{D}(\mathbb{R}^{nd}) \to \mathcal{D}(\mathbb{R}^{(n-2|A|)d})$$

such that, in particular, for the empty contraction $(A = \emptyset)$, we had $\rho_\emptyset = id$.

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Now, different assignment:

$$\rho_C : \mathcal{S}(\mathbb{R}^{nd}) \to \mathcal{S}(\mathbb{R}^{(n-2|A|)d})$$

1) For C the empty contraction, we have (Fourier transformed)

$$\widehat{\rho_\emptyset(g)}(k_1, \dots, k_n) = e^{-i \sum_{i<j} \langle k_i, \theta k_j \rangle} \hat{g}(k_1, \dots, k_n)$$

for $g \in \mathcal{S}(\mathbb{R}^{nd})$. So, for $g = h_1 \otimes \cdots \otimes h_n$, the function $\rho_\emptyset(g)$ is the twisted tensor product of the h_i's. If $\theta = 0$, we have $\rho_{\emptyset, \theta=0} = \rho^0_\emptyset$ (ordinary tensor product).

Twisted tensor product of distributions

By duality: For $u_1, \dots, u_n \in \mathcal{S}'(\mathbb{R}^d)$, $g \in \mathcal{S}(\mathbb{R}^{nd})$,

$$(u_1 \cdots u_n)^{\otimes\theta}(g) := u_1 \otimes \cdots \otimes u_n(\rho_\emptyset(g))$$

for the empty contraction in $\{1, \dots, n\}$.

Corresponds to the formulas used in the physics literature such as $u(x_1) * u(x_2)$.

Twisted tensor product of distributions

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2) General contractions:

Def'n Let C be a contraction in $\{1, \ldots, n\}$. Setting, for $g \in \mathcal{S}$,

$$\widehat{\rho_C(g)}(k_U) := \ldots \text{ see blackboard } \ldots$$

defines a continuous map

$$\rho_C : \mathcal{S}(\mathbb{R}^{nd}) \to \mathcal{S}(\mathbb{R}^{(n-2|A|)d})$$

Again, by duality, corresponding formulas for tempered distributions.

Observe: $\rho_{C,\theta=0} = \rho^0_C$. But for $\theta \neq 0$, ρ_C takes values in Schwartz functions and does not i.g. preserve supports (nonlocality).

Consider in this setting, the ordinary Wick poducts again – for now forgetting about the functional analysis and only looking at the combinatorics.

Recall first that the n-fold twisted tensor product is

$$\phi^{\otimes n,\theta}(g) = \phi^{\otimes n}(\rho_\emptyset(g))$$

with $\rho_\emptyset$ denoting ρ_C for the empty contraction. Now, we can write the recursive formula for the twisted Wick product as follows:

$$:\phi^{\otimes(n+1)}:(\rho_\emptyset(g)) = (\phi\otimes :\phi^{\otimes n}:)(\rho_\emptyset(g)) - \sum_C :\phi^{\otimes n-1}:(\rho_C(g))$$

with the sum over all contractions $C = (\{1\},\alpha)$ in $\{1,\ldots,n+1\}$.

Same formula as before, only with ρ_C instead of ρ_C^0. Also the case for the ordinary Wick theorem.

Now, look at subtraction terms in recursive definition of Wick products explicitly.

Problem: nonlocal subtractions occur. Attempt: avoid certain subtractions.

New combinatorics: quasiplanar Wick product:

$$\vdots \phi^{\otimes(n+1)} \vdots(\rho_\emptyset(g)) = (\phi \otimes \vdots \phi^{\otimes n} \vdots)(\rho_\emptyset(g)) - \sum_C \vdots \phi^{\otimes |U|} \vdots (\rho_C(g))$$

where the sum now runs over all contractions in $\{1, \ldots, n+1\}$ with $A \sqcup \alpha(A) < U$ (all edges on the left hand side of the dots) and all edges are in one connected component.

Why? Well, hope that this suffices (Example of 4 fields shows that more than one edge in the subtraction terms is necessary) to give well-defined products in the limit of coinciding points (more in a minute).

Quasiplanar Wick theorem

For $g \in \mathcal{S}(\mathbb{R}^{(n+m)d})$,

$$(\vdots \phi^{\otimes n} \vdots \otimes \vdots \phi^{\otimes m} \vdots)(\rho_\emptyset(g)) = \sum_{C \in \mathcal{C}_{qW}(N,M)} \vdots \phi^{\otimes |U|} \vdots (\rho_C(g))$$

where $\mathcal{C}_{qW}(N, M)$ is the set of all contractions in $N \sqcup M$ ($|N| = n, |M| = m$) such that every connected component has at least one edge $(a, \alpha(a))$ with $a \in N$ and $\alpha(a) \in M$ (including the empty contraction). Again, $U := (N \sqcup M) \setminus A \setminus \alpha(A)$.

Conj - Thm [proof in progress with DFP]

Gelfand-Shilov functions of type S (more specifically, $S^{1/2}_{1/2}$) provide a dense invariant domain in $\mathcal{F}$ on which quasiplanar Wick products are defined in coinciding points.

Not understood: Does the full Theorem 0 of Epstein and Glaser hold for quasiplanar Wick products on this domain?

In the adiabatic limit, problems occur [B10] in 1-P-reducible graphs! Ultraviolet-infrared mixing problem (different mechanism than in Euclidean).

The quasiplanar Wick products do not suffice to renormalize (if adiabatic limit is performed) ... New product admittting some nonlocal contractions (keeping the number of tapes of counterterms finite)? ... New interesting combinatorics!

Localized noncommutativity?

Main Tool of renormalization

Singularity in 0: unrenormalized distributions defined on $\mathcal{D}(\mathbb{R}^{nk} \setminus \{0\})$. Renormalization: extension of distributions. In momentum space (Fourier transform) corresponds to counterterm subtraction procedure.

Main tool: Start with simple graphs and renormalize them; for complicated graph:

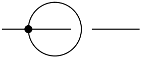
renormalize all subgraphs

take care of the remaining "overall divergence"

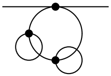
Formalism's combinatorics: Zimmermann's forest formula; in momentum space reformulated in the framework of the Hopf algebras by Connes+Kreimer.

What is UV-IR mixing?

Seiberg + Raamsdonck: In Euclidean framework,

 is ultraviolet well defined

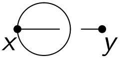
But when inserted into higher order graph, e.g. (in $d = 4$)



such graphs produce infrared singularities.

Only way out: special models [Grosse+Wulkenhaar, Rivasseau et al]. Interesting in their own right: Borel summable? [Rivasseau]

Example on E_θ



$\leftrightarrow \quad u_E(x-y)$ with Fourier transform

$$\tilde{u}_E(p) \;=\; \Delta_E(\theta p)\, \tilde{\Delta}_E(p)$$

$\tilde{u}_E$ contains both the fundamental solution G_E of the free field equation (recall: unique since elliptic PDOp) and its Fourier transform $\tilde{G}_E$.

If θ is nondegenerate, $G_E \circ \theta$ is defined as the pullback w.r.t. θ of the ordinary Euclidean propagator in the sense of Hörmander, and it is smooth outside $\{0\}$ (method: wavefront sets. Details). So, in itself,

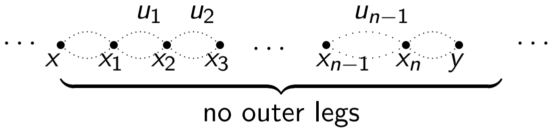
$$\tilde{u}_E = \tilde{G}_E \cdot G_E \circ \theta$$

is well-defined distribution on $\mathbb{R}^n$, even tempered.

Origin of UV-IR mixing

But: In d dimensions, products $(G_E \circ \theta)^k$ are ill-defined for $k \geq d - 2$ due to well-known ultraviolet singularity of G_E in 0. Since this term appears in the Fourier transform $\tilde{u}_E$ (i.e. in momentum space) this singularity occurs at small momenta ($p = 0$), hence as an infrared divergence.

Products of Fourier transforms $\tilde{u}$ appear in graphs of the form



in the adiabatic limit (testfunctions replaced by constants)

$$\rightarrow \quad \prod \tilde{u}_i$$

What is different on M_θ?

Recall: Perturbative setup for hyperbolic (Minkowskian) signature unrelated to Euclidean. In particular, no Feynman propagators in general.

Prop [B07] In 1-PI graphs, no mixing was found in φ^3, φ^4-theories of Minkowskian signature for subgraphs up to 2nd order perturbation theory.

Ex B10: A mixing of a different kind was found in the Hamiltonian framework in 1-P-reducible graphs... in the adiabatic limit.

Examples:

 in ϕ^3, and  in ϕ^4

Use adiabatic limit (testfunctions replaced by constants) to simplify the occuring twisting factors. Then the distributions corresponding to these graphs are (formal integral kernels):

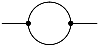
$$\theta(\zeta_0 - (\theta q)_0)\, \Delta_+^k(\zeta)$$

with $k = 2, 3$, resp., and $\zeta = x - y$ relative coordinate, q external momentum.

Well-defined distributions by Hörmander's criterion. Even tempered.

Extra: Renormalization in position space

Renormalization $\simeq$ extension of distributions:



For $g \in \mathcal{D}(\mathbb{R}^4)$,

$$\int dxdy\ \left(G_F(x-y)^2\right)_R\ :\phi(x)\phi(y)\,g(x)\,g(y)$$

$$= \int dudy\ G_F(u)^2 \Big(:\phi(u+y)\,\phi(y):\ g(u+y)\,g(y) \\ - w(u)\ :\phi(y)\,\phi(y):g(y)\,g(y) \Big)$$

w renormalization functions (counterterms) with $w(0) = 1$, fixed by renormalization conditions. In general, remaining freedom: finite renormalizations.

Adiabatic limit: $g = 1$ in the end.

Extra: Why no Feynman propagators?

Rules in S-matrix approach very complicated [B, Piacitelli, Sibold, Denk+Schweda]. Still not many calculations done so far.

I will only tell you what happens in principle:

Extra: Why no Feynman propagators?

In ordinary field theory, time ordering τ and contraction of fields $\Delta_\pm$ conspire to yield Feynman propagator:

$$\Delta_F = \tau\Delta_+ + (1-\tau)\Delta_-$$

τ Heaviside function,
and 2-point function $\Delta_+(x-y) = \langle\Omega|\varphi(x)\varphi(y)|\Omega\rangle$.
In particular, $\Delta_F^2 = \tau\Delta_+^2 + (1-\tau)\Delta_-^2$.

No longer true on M_θ: Typically, time ordering separate from contrations of fields. Twistings only between contractions, e.g.

$$\tau\ \Delta_+ \star \Delta_+ + (1-\tau)\ \Delta_- \star \Delta_-$$

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$$\tau \; \Delta_+ \star \Delta_+ + (1-\tau) \; \Delta_- \star \Delta_- \neq \Delta_F \star \Delta_F$$

for nondegenerate noncommutativity matrix θ.

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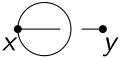
back

Extra: Graphs on M_θ

Time ordering and 2-point-function come separately:



$$\leftrightarrow \quad u(x-y)$$

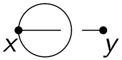
with Fourier transform

$$\tilde{u}(p) = \int dk_0\ \tilde{\tau}(p_0 - k_0)\ \tilde{\Delta}_+(k_0, \mathbf{p})\ \Delta_+(\theta(\omega_{\mathbf{p}}, \mathbf{p}))\ + \ldots$$

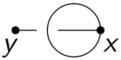
with Heaviside step "function" τ,
and 2-point "function" $\Delta_+(x-y) = \langle \Omega | \varphi(x) \varphi(y) | \Omega \rangle$.

Time order τ only affects $\tilde{\Delta}_+$, not tadpole part.

To see well-definedness easier to consider



$+$



$\rightarrow \quad u(x-y)$

with

$$\tilde{u}(p) = \frac{1}{p_0^2 - \mathbf{p}^2 - m^2 + i\epsilon}\,\Delta_+(\theta(\omega_{\mathbf{p}}, \mathbf{p})) = \tilde{\Delta}_F(p)\;\Delta_+(\theta(\omega_{\mathbf{p}}, \mathbf{p}))$$

Products of tadpole part $\Delta_+ \circ \theta$ are well-defined distributions (method: wavefront sets) $\Rightarrow$ Products of $\tilde{u}$ are well-defined.
Argument in $\Delta_+ \circ \theta$ is on-shell $\Rightarrow$ also $\tilde{u}(p)\tilde{u}(-p)$ is well-defined!

Extra: Graphs on M_θ (ctd.)

To see well-definedness easier to consider

$\rightarrow \quad u(x-y)$

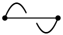
with

$$\tilde{u}(p) = \frac{1}{p_0^2 - \mathbf{p}^2 - m^2 + i\epsilon} \; \Delta_+(\theta(\omega_{\mathbf{p}}, \mathbf{p})) = \tilde{\Delta}_F(p) \; \Delta_+(\theta(\omega_{\mathbf{p}}, \mathbf{p}))$$

Products of tadpole part $\Delta_+ \circ \theta$ are well-defined distributions (method: wavefront sets) $\Rightarrow$ Products of $\tilde{u}$ are well-defined.
Argument in $\Delta_+ \circ \theta$ is on-shell $\Rightarrow$ also $\tilde{u}(p)\tilde{u}(-p)$ is well-defined!

Lemma UV-IR mixing from tadpole-like graphs is absent on M_θ for [B07] for φ^k with $k = 3, 4, 5, 6$ in any dimension d.

More graphs:

Fish graph:  $\to$ $\tilde{u}_E$ and $\tilde{u}_M$ (Euclidean and Minkowskian).

$$\tilde{u}_E(p) = \int \frac{1}{(k-p)^2 + m^2} \, \frac{1}{k^2 + m^2} \, e^{-ik\theta p} \, d^n k$$

is oscillatory integral, well-defined in itself. Products are defined only on testfunctions not having 0 in the support $\Rightarrow$ $p = 0$ problem $\Rightarrow$ UV/IR mixing.

Expression for $\tilde{u}_M$ complicated. Crucial difference: twisting on-shell such that $\tilde{u}_M$ is smooth function. In fact,

$$\tilde{u}_M(0) = \int \frac{1}{(2\omega_{\mathbf{k}})^3} \, e^{+2i\omega_{\mathbf{k}}\theta^{0i}k_i} \, d^{n-1}\mathbf{k}$$

can be solved explicitly, for $n = 4$: [Denk+Schweda03,B04].

Extra: Analytic continuation of the (free) n-point-function

- Usually, the n point function of free fields

$$W(x_1 - x_2, \ldots, x_{n-1} - x_n) = \langle \Omega | \varphi(x_1) \cdots \varphi(x_n) | \Omega \rangle$$

(built from contractions, no time ordering!) is the boundary value of a certain analytic function (Euclidean theory).
- Fourier transform of this analytic function $\Rightarrow$ Schwinger functions.
- An analytic continuation in momentum space "$p_4 \to ip_0$" leads to Feynman propagators.

Proposition [B08]: This can also be done on M_θ. On the Euclidean side one finds twisted products of ordinary Euclidean propagators, but with on-shell twistings (loss of $O(n)$-invariance).

Example

Contribution to 4-point function (free fields, cf. [B06]) in $n = 4$

$$\Delta_+^{(\star 2)}(x, y) \propto \int \frac{1}{\omega_{\mathbf{k}}} \frac{1}{\omega_{\mathbf{p}}} \; e^{-i\tilde{k}x - i\tilde{k}y} \; e^{-i\tilde{p}\theta\tilde{k}} \; d^3\mathbf{k} d^3\mathbf{p}$$

with $\tilde{k} = (\omega_{\mathbf{k}}, \mathbf{k})$, and $\tilde{p} = (\omega_{\mathbf{p}}, \mathbf{p})$, is boundary value of a function f_2^θ which is analytic in a certain region. Explicitly, for $z = (x_0, \mathbf{x}, y_0, \mathbf{y})$ and $\eta = (x_4, \mathbf{0}, y_4, \mathbf{0})$ with $x_4, y_4 > 0$ and $s_2^\theta(\mathbf{x}, x_4 + ix_0, \mathbf{y}, y_4 + iy_0) := f_2^\theta(z - i\eta)$:

$$s_2^\theta(x, y) \; = \; \frac{1}{(2\pi)^8} \int \int \frac{1}{k^2 + m^2} \; \frac{1}{p^2 + m^2} \; e^{+ikx} \; e^{+ipy} \; e^{-i\tilde{p}\theta\tilde{k}} \; d^4 k d^4 p$$

with on-shell momenta in the twisting!!

So, for the Schwinger function we have found

$$s_2^{\theta}(x,y) = \frac{1}{(2\pi)^8} \int\int \frac{1}{k^2+m^2} \; \frac{1}{p^2+m^2} \; e^{+ikx} \; e^{+ipy} \; e^{-i\tilde{p}\theta\tilde{k}} \; d^4k d^4p$$

with on-shell momenta in the twisting.
So, the twisting is independent of the components k_4 and p_4.
Therefore, analytic continuation $k_4 \to k_4 - ik_0$ (and likewise for p) can be performed as usual and yields Fourier transforms of Feynman propagators with on-shell twistings:

$$\tilde{\Delta}_F(k_0, \mathbf{k}) \; \tilde{\Delta}_F(p_0, \mathbf{p}) \; e^{-i\tilde{p}\theta\tilde{k}}$$

Not clear so far whether this is useful. Naively: twisted convolution with on-shell twistings not associative. However, shows that relation Euclidean/Minkowskian subtle on Moyal space.