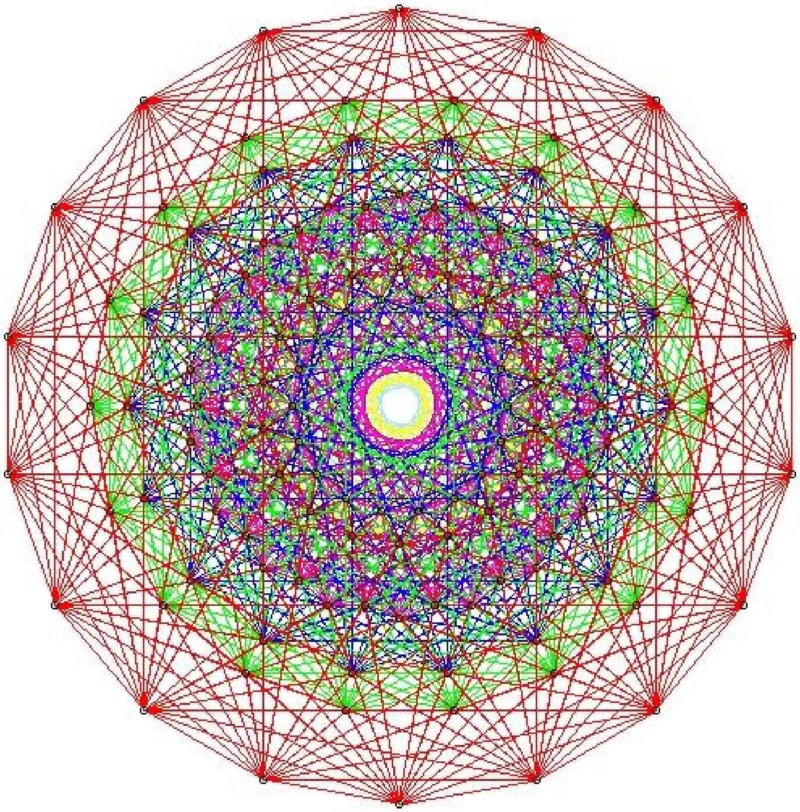
B. L. Cerchiai (Università di Milano) Bayrischzell, May 22, 2011

Exceptional Lie groups
from supergravity to quantum information theory

based on arXiv:1005.2231 with S. Cacciatori, A. Marrani,
arXiv:0902.3973, 1006.3101 with A. Marrani, S. Ferrara, B. Zumino,
and on arXiv:1003.4255, with B. Van Geemen



Abstract:

Here my results on the geometry of the exceptional Lie group E_7 are summarized and some of its applications to black holes in supergravity and quantum information theory are mentioned.

In particular, starting from a symplectic construction due to Adams, I have been able to compute a simple parametrization, the Iwasawa decomposition, of the coset $\frac{E_{7(7)}}{SU(8)}$, which describes the scalar manifold of the $\mathcal{N} = 8$, $d = 4$ supergravity. The explicit expression of the Lie algebra is used to study the origin as scalar configuration of the large $\frac{1}{8}$-BPS extremal black hole attractors. In such a framework it turns out that the $U(1)$ symmetry spanning such attractors is broken down to a discrete subgroup $\mathbb{Z}_4$. These results are compared with the ones obtained in other known bases.

Recently an intriguing relation between quantum information theory and supergravity has been discovered by Duff and Ferrara, linking entanglement measures for qubits to black hole entropy, in certain cases involving the quartic invariant of the 56-dimensional representation of the Lie group E_7. Here I recall the relatively straightforward manner in which three-qubits lead to E_7, or at least its Weyl group.

Plan

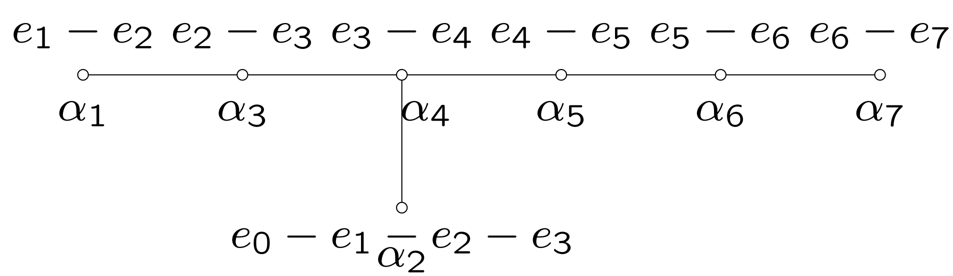
- The exceptional Lie group E_7 and its split form $E_{7(7)}$
- The Adams–Iwasawa parametrization
- Application to the large $\frac{1}{8}$-BPS black hole attractors in $\mathcal{N} = 8$ $d = 4$ supergravity
- Application to the classification of charge orbits
- From three-qubits to the Weyl group of E_7
- Conclusions and outlook

Main idea

Study algebraic structures, in this case exceptional Lie groups, acting as a symmetry of some physical system (in this case supergravity, black holes and quantum information theory) with the aim of getting information on the properties of these models from their geometry.

General properties of the exceptional Lie group E_7

- It is a simple exceptional Lie group of rank 7 and dimension 133.
- Dynkin diagram: (Here a system of 7 simple roots $\{\alpha_i\}$ is expressed in terms of a basis $\{e_i\}$ in $\mathbb{R}^8$)



$$\Downarrow$$

- The Cartan subalgebra, the maximal Abelian subalgebra, has dimension 7.
- The dimension of its fundamental representation is 56 and it admits a symplectic structure.

Properties of $E_{7(7)}$

- It is one of the four real forms of E_7: it is the split form (i.e. the maximally non compact form). The second subscript in the name $E_{7(7)}$ indicates the difference between the number of non compact (70) and of compact (63) generators.
- Its maximal compact subgroup is given by $SU(8)$.
- By exploiting its symplectic structure, following [Adams, *"Lectures on exceptional Lie groups"*, The University of Chicago Press, Chicago and London (1996)], its fundamental representation can be constructed as the

  algebra $\mathfrak{sl}(V) \oplus \Lambda^4 V^4$ acting on $\Lambda^2 V \oplus \Lambda^2 V^*$

  with V an 8-dimensional real vector space, e.g. $\mathbb{R}^8$,
  V^* its dual and $\wedge$ the wedge product.

The Iwasawa parametrization

[arXiv:1005.2231 with S. Cacciatori and A. Marrani]

It can be applied only to non compact forms of the group.

1) Choice of a suitable realization of the Lie algebra corresponding to the group, in this case the fundamental representation of E_7: 133 matrices of dimension 56.

2) Determination of a suitable fibration for the group with the maximal compact subgroup as a fiber $K = SU(8)$
$\Longrightarrow$ Since the Adams' construction is based on the subgroup $SL(8)$, it is necessary to perform a Cayley rotation to switch to a complex manifestly $SU(8)$-covariant basis.

3) A 7-dimensional non compact Cartan subalgebra (completely outside of $\mathfrak{su}(8)$) is chosen as a pivot and the corresponding system of positive roots is calculated. This is possible because it is the split form.

4) Computation of a realization of a generic element g of the group by using the above fibration to construct a well-defined parametrization of the group

$$g = KAN$$

with $K = SU(8)$ the fiber,
A=maximal Abelian subgroup generated by the Cartan subalgebra,
N=nilpotent subgroup generated by the eigenmatrices of the adjoint action of the maximal Abelian subgroup on the system of positive roots.

$$\Downarrow$$

This immediately yields an expression for the coset of the symmetric space $\frac{E_{7(7)}}{SU(8)}$, which is particularly manageable because it is expressed in terms of a nilpotent matrix.

5) It is possible to perform the calculation of
the vielbein: $J = g^{-1} dg$;
the corresponding metric: $ds^2 = -\frac{1}{2} Tr J \otimes J$
(in components: $g_{ij} = -\frac{1}{2} \delta_{lm} J^l{}_i J^m{}_j$);
and the Haar measure: $d\mu = det(J)$.
The range of the compact parameters can then be determined by calculating the volume and fixing the minimal range which makes the Haar measure non singular.

Application to the $\frac{1}{8}$-BPS black hole attractors in $\mathcal{N}=8$ $d=4$ supergravity

Motivation: Recently, there has been a rapid development in the study of $\mathcal{N}=8$ $d=4$ supergravity:

- From the study of the group cohomology and the corresponding Wess-Zumino consistency conditions, it is likely anomaly-free with respect to its chiral $SU(8)$ and continuous $E_{7(7)}$ symmetries [Bossard, Hillmann, Nicolai, arXiv:1007.5472].
- As a consequence, it is conjectured to be ultraviolet finite when doing perturbative quantum field theory computations [Kallosh, arXiv:1009.1135], because of possible $E_{7(7)}$-invariant counterterms being ruled out by different arguments [see e.g. Bern, Carrasco, Johansson, arXiv:0902.3765 for 4 loops computations, Beisert, Elvang, Freedman, Kiermaier, Morales, Stieberger, arXiv:1009.1643 for 7 loops].

The scalar sector of a non linear sigma model in supergravity theories is described by the symmetric Einstein manifold G/K with G the non compact group of global U-duality transformations, and K its maximal compact subgroup, describing the local $\mathcal{R}$ symmetry (for a reference see e.g. [Ferrara, Marrani, arXiv:0808.3567] or [Boya, arXiv:0811.0554]).

Start with the (bosonic) part of the Lagrangian for $\mathcal{N} = 8, d = 4$ pure ungauged supergravity.
In this case $G = E_{7(7)}$ and $K = SU(8)$.
$\Longrightarrow$ scalar manifold $\frac{G}{K} = \frac{E_{7(7)}}{SU(8)}$.
There are no matter multiplets.

- The knowledge of an explicit polynomial expression for the coset should allow a deeper understanding of the geometry of the $E_{7(7)}$ symmetry and provide a method to study the conjecture of the UV finiteness of the theory in a more abstract way, while the previous computations have mainly been done through an explicit calculation with different techniques of the possible counterterms by hand at higher and higher loops.

- It should also allow to analyze the orbits which do not pass through the origin and cannot be studied at the perturbative level of the Lie algebra, but need to be studied at the group level, e.g. the non-BPS large orbit.

The origin of $\frac{E_{7(7)}}{SU(8)}$ as $\frac{1}{8}$-BPS attractor in the Iwasawa parametrization

Main feature of the Iwasawa decomposition: choice of a completely non-compact 7-dim. Cartan subalgebra of $SL(8,\mathbb{R})$

$$\Downarrow$$

The maximal manifest covariance of the whole framework is $SL(7,\mathbb{R})$, breaking the maximal possible off-shell covariance $SL(8,\mathbb{R})$ according to the following embedding:

$$SU(7)\times U(1)_{\mathcal{E}} \subsetneq^{\max}_{symm} SU(8).$$

On the other hand:
Orbit of large $\frac{1}{8}$-BPS black holes:

$$\mathcal{O}_{\frac{1}{8}-BPS,\text{large}} = \frac{E_{7(7)}}{E_{6(2)}}$$

with moduli space:

$$\mathcal{M}_{\frac{1}{8}-BPS,\text{large}} = \frac{E_{6(2)}}{SU(6) \times SU(2)}.$$

The maximal compact symmetry exhibited by the central charge is then $SU(6) \times SU(2)$, which is the maximal compact subgroup of $E_{6(2)}$.
This corresponds to the embedding:

$$SU(2) \times SU(6) \times U(1)_{\mathcal{A}} \subsetneq^{\text{max}}_{symm} SU(8).$$

The subgroups $U(1)_{\mathcal{E}}$ from the Iwasawa construction and $U(1)_{\mathcal{A}}$ from the black hole moduli space do not coincide.
$\Longrightarrow$ The $U(1)_{\mathcal{A}}$ symmetry acting on the black hole attractors is broken down to the discrete subgroup

$$\mathbb{Z}_4 = U(1)_{\mathcal{A}} \cap U(1)_{\mathcal{E}}$$

and their dyonic nature is spoiled.

$$\Downarrow$$

The Adams–Iwasawa parametrization is suitable to single out the purely magnetic or purely electric component of the black hole attractor.

Comparison with other known approaches

- Sezgin-van Nieuwenhuizen

[Sezgin, Van Nieuwenhuizen, *Renormalizability Properties of Spontaneously Broken $\mathcal{N}=8$ Supergravity*, Nucl. Phys. **B195**, 325 (1982)]

It has a manifest covariance $USp(8) \subsetneq^{\text{max}}_{symm} SU(8)$, corresponding to the maximal compact subgroup of the $\mathcal{N}=8, d=5$ U-duality group $E_{6(6)}$: $USp(8) = mcs\left(E_{6(6)}\right)$.

By recalling the explicit form of "large" non-BPS charge orbit in $\mathcal{N}=8$, $d=4$ supergravity $\mathcal{O}_{nBPS} = \frac{E_{7(7)}}{E_{6(6)}}$ it is suitable for a study of "large" dyonic non-BPS $d=4$ extremal black holes.

- Cremmer-Julia /de Wit-Nicolai

[Cremmer, Julia, *The $SO(8)$ Supergravity*, Nucl. Phys. **B159**, 141 (1979), de Wit, Nicolai, *$\mathcal{N}=8$ Supergravity*, Nucl. Phys. **B208**, 323 (1982)]

This parametrization exhibits a manifest covariance $SO(8) = mcs\left(SL\left(8,\mathbb{R}\right)\right) \subsetneq^{\text{max}}_{symm} E_{7(7)}$, suitable for a study of $\frac{1}{8}$–BPS "large" dyonic extremal $d=4$ black holes.

Application to the classification of black holes orbits in 4 and 5-dimensional supergravity

based on arXiv:0902.3973, 1006.3101 with A. Marrani, S. Ferrara, B. Zumino

Black holes can be described in terms of an effective potential V_{BH}. It gives the Bekenstein-Hawking entropy when evaluated at its critical points:

$$\frac{S_{BH}}{\pi} = V_{BH}|_{\partial_\phi V_{BH}=0} = V_{BH}\left(\phi_H\left(\mathcal{P}\right), \mathcal{P}\right),$$

where it can be expressed in terms of the invariant of the duality group:

$$V_{BH}|_{\partial_\phi V_{BH}=0} = \left|\mathcal{I}\left(\mathcal{P}\right)\right|$$

Here $\phi \in \frac{G}{K}$ are the scalar fields,

$\mathcal{P} = \{p^\Lambda, q_\Lambda\}$ the black hole charges.

Central charges Z^{AB} (in the asymptotical Minkowski space-time):

$$\left\{Q_\alpha^A, Q_\beta^B\right\} = \epsilon_{\alpha\beta} Z^{AB}\left(\phi_\infty, \mathcal{P}\right)$$

where ϕ_∞ denotes the set of values taken by the scalar fields at radial infinity ($r \to \infty$) within the considered static, spherically symmetric and asymptotically flat dyonic extremal BH background and A, B are indices relative to the $\mathcal{R}$-symmetry.

Using the matrix describing the coset $S\left(\phi\right) = \sqrt{2}\begin{pmatrix} Ref & -Imf \\ Reh & -Imh \end{pmatrix}$ the central charges can be reexpressed as:

$$Z_{AB}\left(\phi, \mathcal{P}\right) \equiv f^{\Lambda}_{AB} q_\Lambda - h_{AB|\Lambda} p^{\Lambda}$$

This is the relation between the scalar dependent "dressed charges" Z and the "bare charges" p, q.

Application to 4-dimensional $\mathcal{N} = 8$ supergravity

The invariant [Cremmer Julia]

$$\begin{aligned}\mathcal{I}_{4,\mathcal{N}=8} &= \tfrac{1}{2^2}\left[2^2 Tr\left(\left(Z_{AC}\overline{Z}^{BC}\right)^2\right)\right.\\ &\left.-\left(Tr\left(Z_{AC}\overline{Z}^{BC}\right)\right)^2 + 2^5 Re\left(Pf\left(Z_{AB}\right)\right)\right]\end{aligned}$$

is defined as the unique combination of $Z_{AB}(\phi,\mathcal{P})$ satisfying the $E_{7(7)}$-invariant condition:

$$\partial_\phi \mathcal{I}_{4,\mathcal{N}=8}\left(Z_{AB}(\phi,\mathcal{P})\right) = 0, \quad \forall\phi \in \frac{E_{7(7)}}{SU(8)}$$

Orbits

It turns out that there are 5 charge orbits:

2 large: $\mathcal{I}_{4,\mathcal{N}=8} \neq 0$

3 small: $\mathcal{I}_{4,\mathcal{N}=8} = 0$.

The attractor mechanism holds for the large orbits. It does not hold for the small orbits.

A classification of the orbits of black strings in 5 dimensions is performed along similar lines in [arXiv:1006.3101].

From three-qubits to the Weyl group of E_7

[arXiv:1003.4255 with B. Van Geemen]

Motivation: Relation between quantum information theory and supergravity discovered by Duff and Ferrara [Duff, Ferrara, quant-ph/0609227, hep-th/0612036, arXiv:0704.0507], linking entanglement measures for qubits to black hole entropy.
Recently, by using this analogy, the classification of black hole orbits in the STU supergravity model in 3 dimensions has been applied to the classification of entaglement classes of 4-qubits [Borsten, Dahanayake, Duff, Marrani, Rubens, arXiv:1005.4915], which correspond to nilpotent orbits of $SL(2,\mathbb{C})^4$.

Qubits in quantum information theory

k-qubits: non-zero elements of the finite abelian group $L_k = \mathbb{Z}_k$

State space $\mathcal{H}_k$: 2^k-dimensional complex vector space of $\mathbb{C}$-valued maps $L_k \to \mathbb{C}$

The action of the qubits on this state space can be extended to an action of the group generated by the generalized Pauli matrices, denoted as
Heisenberg group $H_k = \mu_4 \times L_k \times L_k^*$
with L_k^* the dual of L_k, $\mu_4 = \{\pm i, \pm 1\}$ and group operation:

for $s, t \in \mu_4$, $x, y \in L_k$, $x^*, y^* \in L_k^*$
$(s, x, x^*)(t, y, y^*) := (st(-1)^{y^*(x)}, x + y, x^* + y^*)$.

E.g. for $k = 1$ it is generated by the Pauli matrices:

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad Y = i \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

The Heisenberg group is non-abelian and has a quotient $V_k = H_k/\mathbb{C}^\times$ which is in a natural way a symplectic vector space of dimension $2k$ over the finite field $F_2 = \mathbb{Z}_2$.

$E : (L_k \times L_k^*) \times (L_k \times L_k^*) \to F_2$; $E((x, x^*), (y, y^*)) = y^*(x) - x^*(y)$.

The normalizer $N_k := \{ M \in GL(\mathcal{H}_k) : M H_k M^{-1} = H_k \}$ of H_k, viewed as a subgroup of $GL(\mathcal{H}_k)$, is generated by transvections, which can be explicitly given.

Observation: The quotient of N_k by the subgroup $\mathbb{C}^\times \cdot H_k$, consisting of scalar multiples of the identity and the Heisenberg group, is the finite symplectic group $Sp(2k, F_2)$ (acting naturally on V_k).

Qutrits and the appearance of E_7

In the case three-qubits $k = 3$, there is a surjective homomorphism from the Weyl group $W(E_7)$ onto $Sp(6, F_2)$, with kernel just $\pm I$.

$$W(E_7)/ < -I > \cong Sp(6, F_2)(\cong N_3)$$

This homomorphism can be seen in two ways:

1) Coxeter relations

2) a surjective homomorphism $\pi : Q(E_7) \to V_3$, where $Q(E_7)$ is the root lattice of E_7. It is compatible with the scalar product on $Q(E_7)$ and the symplectic form on V_3.

Properties of the homomorphism π

- It maps the 63 pairs of roots $\pm\alpha$ of E_7 to the $2^6 - 1 = 63$ non-zero elements of V_3.

- The reflections in $W(E_7)$ defined by the roots of E_7 correspond to the transvections $Sp(6, F_2)$.

- Lie subgroups isomorphic to $SL(2, \mathbb{C})^7$ in E_7, corresponding to entanglement of 7-qubits, can be determined by the choice of seven perpendicular roots of E_7. The homomorphism, in turn, allows to identify such a choice with the choice of a Lagrangian (i.e. maximally isotropic) subspace in V_3.
  $\Longrightarrow$ These embeddings can be studied through the Fano plane by means of del Pezzo surfaces, allowing e.g. to count them (there are 135), in view of their classification.

Conclusions and outlook

- I have constructed the Iwasawa decomposition for the coset $\frac{E_{7(7)}}{SU(8)}$. It is particularly simple because of the nilpotency of matrix involved, which allows us to explicitly study the $\frac{1}{8}$-BPS black hole attractors in $\mathcal{N}=8$ $d=4$ supergravity .

- I have compared it to other known parametrizations for supergravity, showing how different approaches are useful to highlight different facets of the theory.

  In the Adams–Iwasawa basis, due to the choice of a particular Cartan subalgebra, the $U(1)$ symmetry spanning the attractors is broken down to a discrete subgroup $\mathbb{Z}_4$: purely electric and magnetic attractors are singled out.

- It provides an interesting arena, to discuss the the conjecture of perturbative ultraviolet finiteness of $\mathcal{N}=8$, $d=4$ supergravity.

- The Iwasawa parametrization can be applied for the analysis other noncompact cosets relevant for supergravity. Moreover, the exponentiation of the nilpotent matrix on which it is based could be performed explicitly, allowing the study of such symmetric spaces not only at the perturbative level of the Lie algebra, but also globally at the group level.

- I have reviewed some of the geometry behind the appearance of the exceptional Lie group E_7 starting from three-qubits in quantum information theory, which should shed some light on the relation with black holes.

- The methods used to show this link also provide a very natural interpretation for the emergence of the Fano plane when one restricts the 56-dimensional representation of the complex Lie group E_7 to seven (commuting) copies of $SL(2)$.