Deformation of quantum field theories and curved backgrounds

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Outline of the Talk

- Motivations
- Wedge regions in curved backgrounds
- Warped convolutions and curved backgrounds
- Conclusions

Based on

- C. D., Gandalf Lechner and Eric Morfa-Morales: Comm. Math. Phys. **305** (2011), 99 – ArXiv:1006.3548 [math-ph].

Motivations - Part I

Deformations of QFT have been thoroughly studied:

- as giving rise to quantum field theories on non-commutative spacetimes
- as a tool to construct new (interacting) models on commutative spacetimes

Yet, a closer look unveils that

- most of these models have been built on the Euclidean or on Minkowski space
- often a choice of a preferred coordinate system is employed
- sharp point-like localization is weakened to localization in wedge-shaped regions

Motivations - Part II

Hence a few natural questions

- what is the interplay between deformations such as warped convolutions and non-trivial geometries?
- does any notion of covariance and locality survive?

To this avail,

- we want consider curved backgrounds where warped convolutions can be applied
- we look for a suitable notion of wedge-regions in this framework
- we want to work out explicit examples

Wedges in Minkowski

In Minkowski spacetime $(\mathbb{R}^4, \eta)$, we call right wedge

$$W_R := \left\{(x_0, x_1, x_2, x_3) \in \mathbb{R}^4, \mid x_1 > |x_0|\right\}.$$

Let us notice that

- every other wedge W can be constructed from W_R acting with a suitable Poincaré-isometry
- Each $(W, \eta|_W)$ is a glob.-hyp. spacetime embedded in $(\mathbb{R}^4, \eta)$
- W_R possesses an edge defined as

$$E(W_R) := \left\{(0, 0, x_2, x_3) \in \mathbb{R}^4\right\}.$$

  and W_R is bounded by two non-parallel characteristic $3D$ planes whose intersection is $E(W_R)$

More on wedges in Minkowski

Further properties include:

- Each wedge is the causal completion of the world line of a uniformly accelerated observer.
- Each wedge is the union of a family of double cones whose tips lie on two fixed lightrays.
- The Poincaré group acts transitively on the family of all possible wedges.
- The family of wedges in Minkowski is causally separating; for any two spacelike separated double cones $O_1, O_2 \in \mathbb{R}^4$, there exists a wedge W such that $O_1 \subset W \subset O_2'$

The role of the edge

For a generalization to curved backgrounds we notice that

$$W_R \cup W_L = \mathbb{R}^4 \setminus \left(\overline{J^+(E(W_R))} \cup \overline{J^-(E(W_R))} \right),$$

where $W_L \doteq \left\{ (x_0, x_1, x_2, x_3) \in \mathbb{R}^4, \mid -x_1 > |x_0| \right\}$ is the left-wedge.

Notice:

- the edge admits a covariant characterization as the plane spanned by the flow of two spacelike Killing fields of η,
- consequently also the above region has been covariantly characterized.

Strategy: Can we transfer this characterization to curved backgrounds?

Admissible spacetimes

We shall only consider manifolds (M, g) which

1. are globally hyperbolic spacetimes
2. admit two *complete, commuting and smooth* Killing fields ξ_1, ξ_2 (why?)
3. M is diffeomorphic to $\mathbb{R} \times I \times E$, $I \subseteq \mathbb{R}$ while E is the 2D submanifold identified by the flow of ξ_1 and ξ_2 - Frobenius' theorem (why?)

We call $\Xi(M, g)$ the set of all ordered pairs $\xi \doteq (\xi_1, \xi_2)$ with ξ_1, ξ_2 as above.

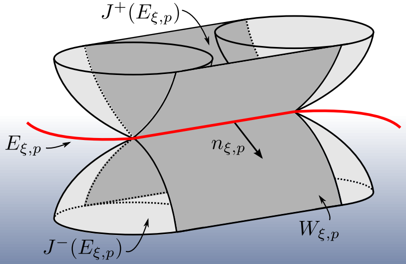
Edges in admissible spacetimes

Following Minkowski spacetime:

We call edge the submanifold of (M, g)

$$E_{\xi,p} \doteq \left\{ \varphi_{\xi,s}(p) \in M \ : \ s = (s_1, s_2) \in \mathbb{R}^2, \ p \in M \right\},$$

where $\xi \in \Xi(M, g)$ and $\varphi_{\xi,s} = \varphi_{\xi_1,s_1} \circ \varphi_{\xi_2,s_2}$ is the flow of the Killing pair.



Properties of the edges

Since (M, g) is globally hyperbolic, M is isometric to $\mathbb{R} \times \Sigma$ and $\exists \mathcal{T} : \mathbb{R} \times \Sigma \to \mathbb{R}$ such that

$$g = -\beta d\mathcal{T}^2 + h \quad \beta \in C^\infty(\mathbb{R} \times \Sigma, (0, \infty)) \text{ and } h \in Riem(\Sigma)$$

Hence:

- at each $p \in M$ we can assign an oriented basis of T_pM

$$(\nabla \mathcal{T}(p), \xi_1(p), \xi_2(p), n_{\xi,p}).$$

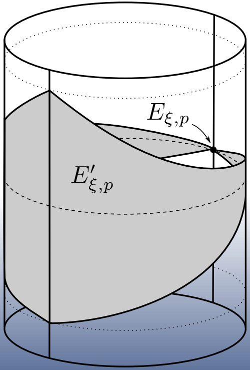
Lemma:

The causal complement $E'_{\xi,p}$ of an edge $E_{\xi,p}$ is the disjoint union of two connected components, which are the causal complements of each other

The reason for $M \sim \mathbb{R} \times I \times E$

The above lemma would not hold without the assumption on the topology of M!

Suppose (M, g) is such that $M \equiv \mathbb{R} \times \mathbb{S}^1 \times \mathbb{R}^2$, where $\mathbb{R} \times \mathbb{S}^1$ is the *Lorentz cylinder*. Then an edge looks like:



Wedges in curved backgrounds

Definition [Wedge]:

A wedge is a subset of an admissible spacetime (M, g) which is a connected component of the causal complement of an edge. Hence, for given $\xi \in \Xi(M, g)$ and $p \in M$, we call $W_{\xi,p}$ the component of $E'_{\xi,p}$ which intersects the curve $\gamma(t) \doteq \exp_p(tn_{\xi,p})$, $t > 0$.

Each wedge $W = W_{\xi,p}$

- is causally complete, *i.e.* $W'' = W$, hence globally hyp.,
- has a casual complement $W' = W_{\xi',p}$ where $\xi' = (\xi_2, \xi_1)$,
- is invariant under the Killing flow generating its edge.

Families of wedges and their properties

Let us now introduce the family of all wedges

$$\mathcal{W} \doteq \{W_{\xi,p} \ : \ \xi \in \Xi(M,g), \ p \in M\}.$$

The set $\mathcal{W}$

- is invariant under the action of the isometry group of (M, g) and under taking causal complements
- is such that two elements $W_{\xi,p}$ and $W_{\tilde{\xi},\tilde{p}}$ form an inclusion $W_{\xi,p} \subset W_{\tilde{\xi},\tilde{p}}$ if and only if $p \in \overline{W_{\tilde{\xi},\tilde{p}}}$ and $\exists N \in GL(2,\mathbb{R})$ with $\det N > 0$ such that $\tilde{\xi} = N\xi$.

Are there spacetimes which are admissible?

Examples of admissible spacetime

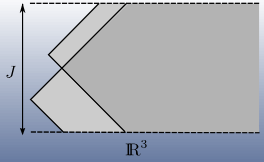
There are several interesting admissible spacetimes:

- every warped product of a globally hyperbolic manifold $X \sim \mathbb{R} \times I$ with a $2D$ Riemannian manifold E endowed with two complete spacelike commuting Killing fields
- Kasner spacetimes, *a.k.a.* Bianchi I models, that is $M \sim J \times \mathbb{R}^3$ with $J \subseteq \mathbb{R}$ and

$$ds^2 = dt^2 - e^{2f_1}dx^2 - e^{2f_2}dy^2 - e^{2f_3}dz^2. \quad f_i = f_i(t)$$

- as a particular case the FRW spacetime with flat spatial section

$$ds^2 = dt^2 - a^2(t)[dx^2 + dy^2 + dz^2] = a^2(\tau)[d\tau^2 - dx^2 - dy^2 - dz^2].$$



Basic Ingredients

We consider a QFT in the framework of Haag-Kastler axioms:

- We take a C^*-algebra $\mathcal{F}$. The elements are bounded functions of quantum fields on (M, g)
- A local structure exists: To each wedge W, we associate the C^*-subalgebra $\mathcal{F}(W) \subset \mathcal{F}$
- $\exists$ a strongly-continuous action α of $ISO(M, g)$ on $\mathcal{F}$
- $\exists$ a Bose/Fermi automorphism γ of $\mathcal{F}$ such that $\gamma^2 = 1$, $[\alpha, \gamma] = 0$,
- We assume that $\mathcal{F}$ is concretely realized on a separable Hilbert space $\mathcal{H}$
- $\mathcal{H}$ must carry a unitary rep. U of $ISO(M, g)$ and V of γ

First consequences

The triple of data $(\{\mathcal{F}(W)\}_{W\in\mathcal{W}},\alpha,\gamma)$ has the *structural properties* of a QFT

- *Isotony)* $\mathcal{F}(W)\subset\mathcal{F}(\widetilde{W})$ if $W\subset\widetilde{W}$
- *Covariance)* under action of $ISO(M,g)$, that is $\alpha_h(\mathcal{F}(W))=\mathcal{F}(hW)$ for all $h\in ISO(M,g)$ and for all $W\in\mathcal{W}$
- *Twisted Locality)* If we introduce the unitary operator $Z\doteq\frac{1}{\sqrt{2}}(1-iV)$

$$[ZFZ^*,G]=0,\quad \forall F\in\mathcal{F}(W),\ G\in\mathcal{F}(W'),\ W\in\mathcal{W}.$$

Note that covariance implies that $\forall\xi\in\Xi$, $\mathcal{F}$ carries an $\mathbb{R}^2$-action τ_ξ

$$\tau_{\xi,s}\doteq\alpha_{\varphi_{\xi,s}}=adU_\xi(s),\quad s\in\mathbb{R}^2$$

Note that isotony implies

$$\tau_{N\xi,s}(\mathcal{F}(W_{\xi,p}))=\mathcal{F}(W_{\xi,p}),\quad N\in GL(2,\mathbb{R}),\ s\in\mathbb{R}^2$$

Warped Convolutions

We have all the ingredients to define a deformed net $W \to \mathcal{F}(W)_\lambda$.

To cope with the non trivial geometry we call $F \in \mathcal{F}$ ξ-smooth if

$$\mathbb{R}^2 \ni s \mapsto \tau_{\xi,s}(F) \in \mathcal{F},$$

is smooth in the norm-topology of $\mathcal{F}$.

We call deformed operator (warped convolution) of a ξ-smooth $F \in \mathcal{F}$

$$\mathcal{F}_{\xi,\lambda} \doteq \frac{1}{4\pi^2} \lim_{\epsilon \to 0} \int dsds' e^{-iss'} \chi(\epsilon s, \epsilon s') U_\xi(\lambda Q s) F U_\xi(s' - \lambda Q s),$$

- $\lambda \in \mathbb{R}$, while $s, s' \in \mathbb{R}^2$
- $\chi \in C_0^\infty(\mathbb{R}^2 \times \mathbb{R}^2)$ and $\chi(0,0) = 1$
- Q is the standard antisymmetric 2×2 matrix.

Properties of the warped convolution

Lemma:

If $\xi \in \Xi$ and if $F, G \in \mathcal{F}$ are ξ-smooth, then

1. $F^*_{\xi,\lambda} = (F^*)_{\xi,\lambda}$
2. $F_{\xi,\lambda} G_{\xi,\lambda} = (F \times_{\xi,\lambda} G)_{\xi,\lambda}$ where
$$F \times_{\xi,\lambda} G \doteq \frac{1}{4\pi^2} \lim_{\epsilon \to 0} \int ds ds' e^{-iss'} \chi(\epsilon s, \epsilon s') \tau_{\xi,\lambda Qs}(F) \tau_{\xi,s'}(G).$$
3. If $[\tau_{\xi,s}(F), G] = 0$ for all $s \in \mathbb{R}^2$, then $[F_{\xi,\lambda}, G_{\xi,-\lambda}] = 0$
4. If a unitary $Y \in \mathcal{B}(\mathcal{H})$ commutes with $U_\xi(s)$, $s \in \mathbb{R}^2$, then $Y F_{\xi,\lambda} Y^{-1} = (Y F Y^{-1})_{\xi,\lambda}$ and the latter is ξ-smooth.

Note that the third property entails:

$$[Z\tau_{\xi,\lambda}(F)Z^*, G] = 0 \Longrightarrow [Z F_{\xi,\lambda} Z^*, G_{\xi,-\lambda}] = 0.$$

The deformed net

Let us consider the following data

- the net based on wedges $W \mapsto \mathcal{F}(W)$,
- the equivalence classes $[\xi]$ where $\xi \sim \xi'$, $\xi, \xi' \in \Xi$ iff $\exists h \in ISO(M, g)$ and $N \in GL(2, \mathbb{R})$ with $\xi' = N h_* \xi$.
- the decomposition of $\mathcal{W}$ as $\bigsqcup_{[\xi]} \mathcal{W}_{[\xi]}$

Fix a representative ξ for all $[\xi]$ and for $p \in M$

$$\mathcal{F}(W_{\xi,p})_\lambda \doteq \{F_{\xi,\lambda} \ : \ F \in \mathcal{F}(W_{\xi,p}),\ \xi\text{-smooth}\}^{\|\cdot\|},$$

$$\mathcal{F}(W_{\xi',p})_\lambda \doteq \left\{F_{\xi',\lambda} \ : \ F \in \mathcal{F}(W'_{\xi,p}),\ \xi'\text{-smooth}\right\}^{\|\cdot\|},$$

where $\|\cdot\|$ stands for norm closure in $\mathcal{B}(\mathcal{H})$.

Properties of the deformed net

Note that the def. above are extended to arbitrary wedges via

$$\mathcal{F}(hW_{\xi,p})_\lambda \doteq \alpha_h(\mathcal{F}(W_{\xi,p})_\lambda),$$

where $\alpha_h(F_{\xi,\lambda}) = \alpha_h(F)_{h_*\xi,\lambda}$ for all $h \in ISO(M,g)$.

Theorem:

The map $\lambda \mapsto \mathcal{F}(W)_\lambda$ identifies a well-defined isotonous, twisted wedge-local, ISO-covariant net of C*-algebras on $\mathcal{H}$, that is $\forall W, \widetilde{W} \in \mathcal{W}$

1. $\mathcal{F}(W)_\lambda \subset \mathcal{F}(\widetilde{W})_\lambda$ if $W \subset \widetilde{W}$,
2. $[ZF_\lambda, Z^*, G_\lambda] = 0$ for $F_\lambda \in \mathcal{F}(W)_\lambda$ and $G_\lambda \in \mathcal{F}(W')_\lambda$
3. $\alpha_h(\mathcal{F}(W)_\lambda) = \mathcal{F}(hW)_\lambda$ for all $h \in ISO(M,g)$
4. if $\lambda = 0$ then $\mathcal{F}(W)_0 = \mathcal{F}(W)$.

Outlook and Perspectives

We have

- identified a notion of wedges in a large class of curved spacetimes
- applied warped convolution deformation to QFT on these spacetimes
- proven that the deformed net preserves basic covariance and wedge-localization

We want to

- extend the construction to a larger class of manifolds
- extend the framework to non-Abelian isometries
- better understand the structure of the new models