From Quantum Gravity to Quantum Field Theory via Noncommutative Geometry

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Bayrischzell 21.05.2011

Outline of Talk

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- General relativity from a Hamilton operator.

Noncommutative Geometry

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- **A Spectral Triple** is a collection (B, H, D):
  a $*$-algebra B represented as operators in the Hilbert space H; a self-adjoint, unbounded operator D, acting in H such that:
  1. The resolvent of D, $(1 + D^2)^{-1}$, is compact.
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- Example from physics: *the standard model coupled to gravity*

[Chamseddine, Connes, Dubois-Violette, Lizzi, Lott, Marcolli, ...]

$$B = C^\infty(M) \otimes B_F \qquad B_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$$

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Our goal

To construct a framework which combines noncommutative geometry with elements of quantum gravity/quantum field theory.

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$$\{A^i_j(x), E^k_l(y)\} = \delta^i_l \delta^k_j \delta(x-y)$$

- The Hamiltonian involves two constraints

$$H = \int N\epsilon^{ab}_c E^i_a E^j_b F^c_{ij} + N^i E^j_b F^c_{ij}$$

(Hamilton, spatial diffeomorphism)

- Shift focus from connections to holonomy and flux variables

$$h_L(A) = \mathrm{Hol}(L, A)$$

L loop on Σ

$$F_S^a(E) = \int_S \epsilon^i{}_{jk} E_i^a dx^j dx^k$$

S surface in Σ.

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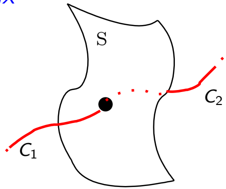

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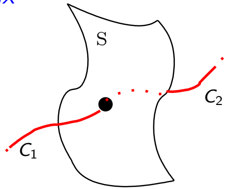

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- These are the variables used in Loop Quantum Gravity.

Our Project

- **Aim:** To construct a spectral triple that involves an algebra of holonomy loops, i.e. functions on a space $\mathcal{A}$ of connections:

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- Such a spectral triple will be a geometrical construction *over* the configuration space $\mathcal{A}$ (i.e. 'quantum')
- It turns out that an algebra generated by holonomy loops is naturally noncommutative. Thus, we are immediately within the realm of noncommutative geometry.

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  3. take the limit (projective, inductive) over graphs to obtain a spectral triple over the space of connections $\mathcal{A}$.

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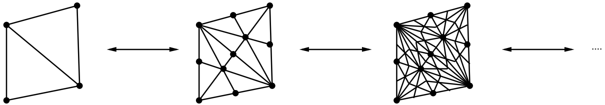

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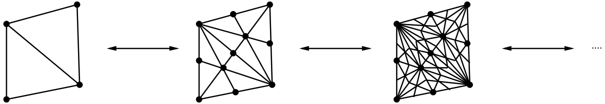

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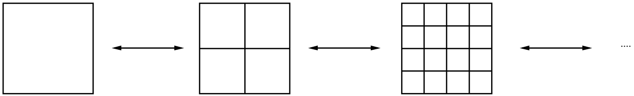

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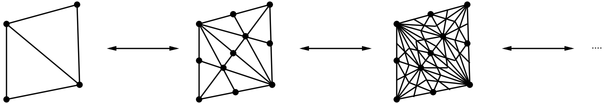

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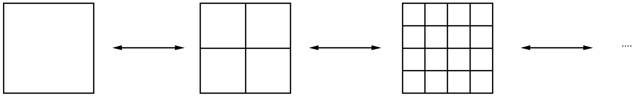

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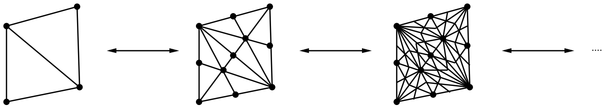

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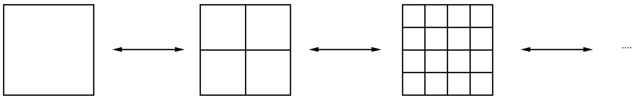

- Both systems of graphs (and many more) permit a spectral triple construction.
  - But the cubic lattices turn out to be natural (classical limit).

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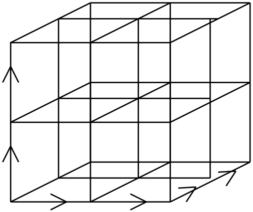
$$\nabla : \epsilon_i \to g_i$$

where G is a compact Lie-group.



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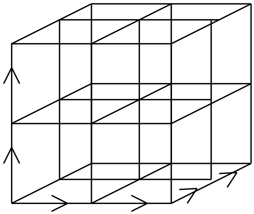

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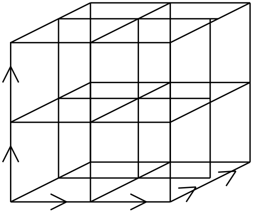

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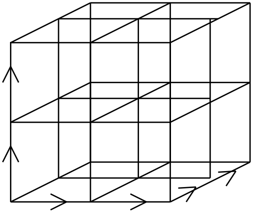

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- Think of the space $\mathcal{A}_\Gamma$ as a coarse-grained approximation of the space $\mathcal{A}$ of smooth connections.

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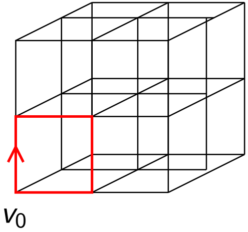

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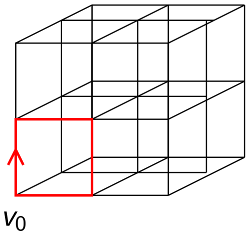

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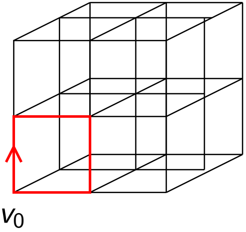

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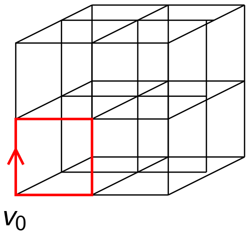

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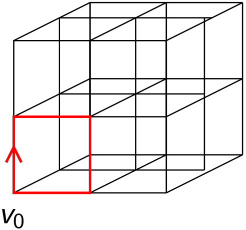

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  - These elements have a natural norm

$$\|a\| = \sup_{\nabla \in \mathcal{A}_\Gamma} \| \sum a_i \nabla(L_i) \|_G$$

where the norm on the rhs is the matrix norm in G.

- **Hilbert space:** There is a natural Hilbert space

$$\mathcal{H}_\Gamma = L^2(G^n, Cl(T^*G^n) \otimes M_l(\mathbb{C}))$$

involving a matrix factor $M_l(\mathbb{C})$ (l size of rep. of G). L^2 is with respect to the Haar measure on G^n. $Cl(T^*G^n)$ is the Clifford bundle over G^n.

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- **Dirac operator:** at the level of a single graph Γ we can just pick any Dirac operator D on G^n (restrictions on D show up later)

A family of lattices

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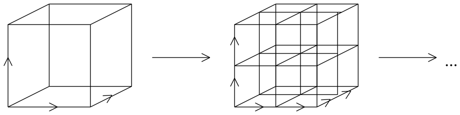
- Consider an infinite system of nested, 3-dimensional lattices

$$\Gamma_0 \to \Gamma_1 \to \Gamma_2 \to \dots$$

with Γ_i a subdivision of Γ_{i-1}

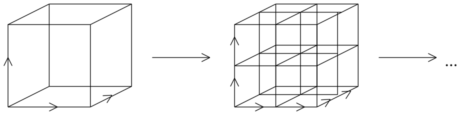


A family of lattices

- Consider an infinite system of nested, 3-dimensional lattices

$$\Gamma_0 \to \Gamma_1 \to \Gamma_2 \to \dots$$

with Γ_i a subdivision of Γ_{i-1}



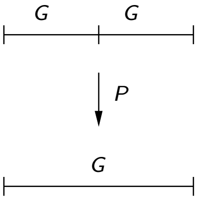
On the level of the associated manifolds $\mathcal{A}_{\Gamma_i}$ this gives rise to a projective system

$$\mathcal{A}_{\Gamma_0} \xleftarrow{P_{10}} \mathcal{A}_{\Gamma_1} \xleftarrow{P_{21}} \mathcal{A}_{\Gamma_2} \xleftarrow{P_{32}} \mathcal{A}_{\Gamma_3} \xleftarrow{P_{43}} \dots$$

- The projections involve:
  1. products

$$P : G^2 \to G \,,$$
$$(g_1, g_2) \to g_1 \cdot g_2$$

for edges which are subdivided in two,



because

$$Hol(\nabla, \epsilon_1) \cdot Hol(\nabla, \epsilon_2) = Hol(\nabla, \epsilon_1 \cdot \epsilon_2)$$

  2. omission of variables for new edges.

- Consider next a corresponding system of spectral triples

$$(\mathcal{B},\mathcal{H},D)_{\Gamma_0} \leftrightarrow (\mathcal{B},\mathcal{H},D)_{\Gamma_1} \leftrightarrow (\mathcal{B},\mathcal{H},D)_{\Gamma_2} \leftrightarrow \dots$$

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  which are compatible with the maps between graphs.
- This requirement restricts the choice of D.
- At the level of a graph Γ, a compatible operator has the form

$$D = \sum_k a_k D_k$$

  where the sum runs over different copies of G and where

$$D_k(\xi) = \sum_a \mathbf{e}_k^a \cdot d_{e_k^a}(\xi) \qquad \xi \in L^2(G, Cl(TG))$$

  where $d_{e_k^a}$ are left-translated vectorfields on the k'th copy of G and $\mathbf{e}_k^a$ are elements in the Clifford algebra. The a_n's are free parameters related to the level of refinement (the sum over copies of G is wrt a change of variables).

The limit

- In the limit of repeated subdivisions, this gives us a candidate for a spectral triple (inductive limits)

$$(\mathcal{B}, \mathcal{H}, D)_{\Gamma_i} \longrightarrow (\mathcal{B}, \mathcal{H}, D)_{\infty}$$

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- Result: For a compact Lie-group G the triple $(\mathcal{B}, \mathcal{H}, D)_\infty$ is a semi-finite* spectral triple:
  - D's resolvent $(1 + D^2)^{-1}$ is compact (wrt. trace) and
  - the commutator $[D, b]$ is bounded

  provided the sequence $\{a_i\}$ approaches ∞.

**semi-finite: everything works up to a symmetry group with a trace (CAR algebra)* [Carey, Phillips, Sukochev].

What physical interpretation does this spectral triple construction have?

Space of connections

- *The spectral triple is a geometrical construction over a space $\mathcal{A}$ of connections.*

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- To see this take the limit of intermediate spaces $\mathcal{A}_\Gamma$

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There is a natural map

$$\chi : \mathcal{A} \to \overline{\mathcal{A}}\,, \quad \chi(\nabla)(\epsilon_i) = Hol(\nabla, \epsilon_i)$$

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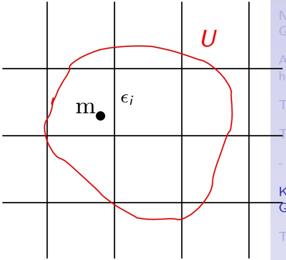
- **Result:** χ is a dense embedding

$$\mathcal{A} \hookrightarrow \overline{\mathcal{A}}$$

- **Argument:** given $\nabla_1, \nabla_2 \in \mathcal{A}$ they will differ in a point $m \in \Sigma$ and in a neighborhood U of m. Choose a small edge ϵ_i in a graphs Γ_i so that $\epsilon_i \in U$. Thus

$$Hol(\nabla_1, \epsilon_i) \neq Hol(\nabla_2, \epsilon_i)$$



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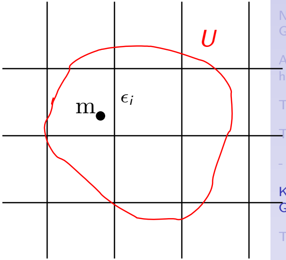

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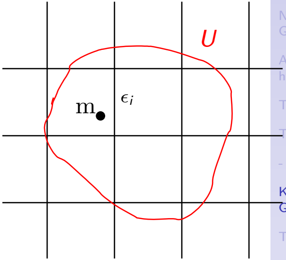

- This result mirrors a result in LQG based on piece-wise analytic graphs.
- This result holds for many different systems of ordered graphs. Fx triangulations with barycentric subdivisions.

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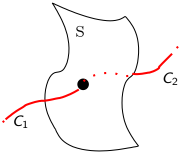

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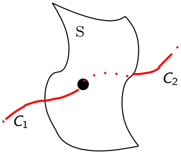

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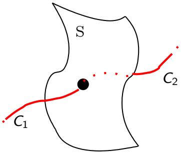

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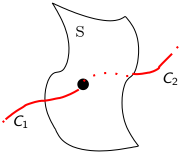

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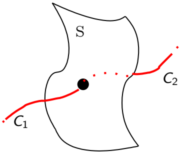

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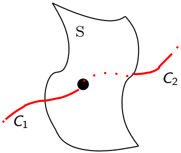

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  - the flux operators are stored in the Dirac type operator.

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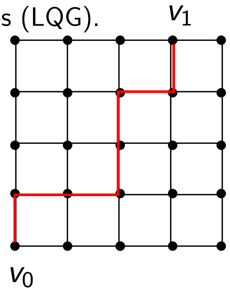

$$L' = \mathcal{U}_p(v_0, v_1) L \mathcal{U}_p^*(v_0, v_1)$$

where $p = \{\epsilon_{i_1}, \epsilon_{i_2}, \ldots, \epsilon_{i_n}\}$ is a path from v_0 to v_i and $\mathcal{U}_p$ the corresponding parallel transport along p

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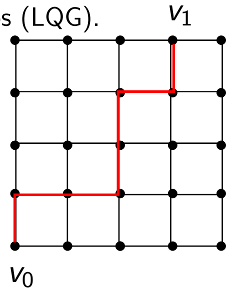

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- **Aim**: to use this deficit to identify natural states which exhibit an independency on the choice of basepoint.

- Introduce the operators

$$\tilde{\mathcal{U}}_p = \tilde{\mathcal{U}}_{i_1}\tilde{\mathcal{U}}_{i_2} \cdot \ldots \cdot \tilde{\mathcal{U}}_{i_n}$$

with

$$\tilde{\mathcal{U}}_i = \frac{\mathrm{i}}{2}\left(\mathbf{e}_i^a\sigma^a + \mathrm{i}\mathbf{e}_i^1\mathbf{e}_i^2\mathbf{e}_i^3\right)\nabla(\epsilon_i)$$

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- These operators are unitary and mutually orthogonal

$$\langle\tilde{\mathcal{U}}_p|\tilde{\mathcal{U}}_{p'}\rangle = \begin{cases} 1 & \text{if} \quad p = p' \\ 0 & \text{if} \quad p \neq p' \end{cases}$$

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- We find that

$$\langle\tilde{\mathcal{U}}_i|L|\tilde{\mathcal{U}}_i\rangle = \langle\tilde{\mathcal{U}}_i|\mathrm{Tr}(L)|\tilde{\mathcal{U}}_i\rangle$$

which shows that these operators remove the dependency on the basepoint.

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$$\xi_k(\psi) = \frac{1}{N}\sum_i \tilde{\mathcal{U}}_{p_i}\psi(v_i)\mathcal{U}_{p_i}^{-1}$$

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$$[D, \nabla(\epsilon_i)] = a_n \mathbf{e}_i^a \nabla(\epsilon_i)\sigma^a \sim a_n \tilde{\mathcal{U}}_i$$

$\tilde{\mathcal{U}}_p$ is something like an n-form.

Semi-Classical States

- Pick a point (A, E) in phase-space (Ashtekar variables). Coherent states $\phi^{t,k}_{(E,A)}$ in $L^2(\mathcal{A}_{\Gamma_k})$ are given by $(t \sim l_P^2)$

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- Consider now states

$$\Psi^t_k(\psi, E, A) = \xi_k(\psi) \Phi^{t,k}_{(A,E)}$$

- This is a natural sequence of states $\{\Psi^t_k\}$ assigned to each level of subdivision of lattices.

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$$\langle \tilde{\mathcal{U}}_{i_1}\tilde{\mathcal{U}}_{i_2}\dots\tilde{\mathcal{U}}_{i_n}\psi(v_i)\dots|\mathbf{e}^a_{i_{n+1}} d_{\mathbf{e}^a_{i_{n+1}}}|\tilde{\mathcal{U}}_{i_1}\tilde{\mathcal{U}}_{i_2}\dots\tilde{\mathcal{U}}_{i_{n+1}}\psi(v_{i+1})\dots\rangle$$

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$$\lim_{k\rightarrow\infty}\lim_{t\rightarrow 0}\langle\Psi_k^t|tD|\Psi_k^t\rangle = \frac{1}{2}\int_\Sigma d^3x\psi^*(x)\left(\sigma^a E^m_a\nabla_m + \nabla_m\sigma^a E^m_a\right)\psi(x)$$

provided we set $a_n = 2^{3n}$ and write $\nabla(\epsilon_i) \simeq 1 + A_i$ and $\nabla_i = \partial_i + [A_i, \cdot]$.

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provided we set $a_n = 2^{3n}$ and write $\nabla(\epsilon_i)\simeq 1 + A_i$ and $\nabla_i = \partial_i + [A_i,\cdot]$.

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The Dirac operator in 3 dimensions

- The expectation value of D on the states Ψ_k^t will only involve terms of the form (due to Clifford elements)

$$\langle \tilde{\mathcal{U}}_{i_1}\tilde{\mathcal{U}}_{i_2}\dots\tilde{\mathcal{U}}_{i_n}\psi(v_i)\dots|\mathbf{e}^a_{i_{n+1}} d_{\mathbf{e}^a_{i_{n+1}}}|\tilde{\mathcal{U}}_{i_1}\tilde{\mathcal{U}}_{i_2}\dots\tilde{\mathcal{U}}_{i_{n+1}}\psi(v_{i+1})\dots\rangle$$

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- Important: the gravitational variables emerges from our loop/flux operators.

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- The lapse and shift fields come as $M_i = N(x) 1_2 + N^a(x)\sigma^a$.

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  Q: can we change the order of this double limit? (!)
- These computations are very sign-sensitive. This indicates that we are missing some grading.

Many particle states

- Consider states of the form:

$$\Psi_k^t(\psi_1, \ldots, \psi_n, E, A) := \xi_k(\psi_1) \ldots \xi_k(\psi_n) \Phi^{t,k}_{(A,E)}$$

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- When we compute the expectation value of the Dirac type operator D_M on these states we obtain, in the semi-classical limit, a system of fermions coupled to the gravitational field, with an additional "interaction" (here, $n=2$, $M=1_2$)

$$\stackrel{\text{cl + cont.}}{\longrightarrow} \int_\Sigma dx \int_\Sigma dy \, \mathrm{Tr}\left(\mathcal{U}(y,x)\left(\nabla\psi_2^*(x)\right)\psi_1(x)\mathcal{U}^{-1}(y,x)\psi_1^*(y)\psi_2(y)\right)$$

$$+ \text{'symmetric terms'}$$

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- **Q:** what interactions (local, non-local) emerge through perturbation around this flat limit?
- **Q:** what about the symmetric sector? Bosons?

Pure gravity

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[work in progress]

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- What about the pure gravity sector? The operator

$$H_M = \sum_i M_i[D^2, [D^2, L_i - L_i^{-1}]]$$

where L_k, $k \in \{1, 2, 3\}$, are loops in a plaquet in $\Gamma_k \backslash \Gamma_{k-1}$, will descent to the Hamilton

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- Contact to general relativity in a semi-classical continuum limit.
- The fermionic degrees of freedom cancel out.

- Consider the operator

$$D_M + H_M$$

and its expectation value on the states $\Psi_k^t(\psi_1, \ldots, \psi_n, E, A)$

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- **Q**: does the constraint algebra close (semi-classically)?

Spectral action functional

- The spectral action (trace of heat-kernel) resembles a partition function

$$Tr \exp(-s(D)^2) \sim \int_{\overline{\mathcal{A}}} [d\nabla] \exp\left(-s(D)^2\right) \delta_{\nabla}(\nabla)$$

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- This object is finite.
- It is not clear to us what role this object should play in our approach.

Connes Distance Formula

- Given a spectral triple $(\mathcal{B}, \mathcal{H}, D)$ over a manifold $\mathcal{M}$ the distance formula reads

$$d(\xi_x, \xi_y) = \sup_{b \in \mathcal{B}} \left\{ |\xi_x(b) - \xi_y(b)| \big| |[D, b]| \leq 1 \right\}$$

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- If they differ only on short scales, then the distance will be 'small' (difference weighted with large a's - small distance).
- The spectral triple construction is a metric structure on a configuration space of connections. This idea goes back to Feynman, Singer ...

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- An operator H_M can be constructed which gives the pure gravity Hamiltonian in the semi-classical limit - contact to classical general relativity.

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Outlook (wishful thinking):

- Apply Tomita-Takesaki theory.
- Contact to Connes work on the Standard Model: analyze the algebra in the semi-classical limit.