Wick Rotation on Degenerate Moyal Space

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Joint work with H. Grosse, G. Lechner and R. Verch

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Introduction I

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- Wightman functions $\leftrightarrow$ Schwinger functions [OS 1973, OS2 1975]
- Euclidean net of algebras $\rightarrow$ Haag-Kastler net in Minkowski space-time [Schl 1999]

$$\begin{array}{ccc} \mathcal{E} & \longleftrightarrow & \mathcal{E}_\theta \\ \updownarrow & & ? \\ \mathcal{M} & \longleftrightarrow & \mathcal{M}_\theta \end{array}$$

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- At present no exact connection to noncommutative Lorentzian theories is known

Algebraic Approach: Spaces of Restricted Symmetry I

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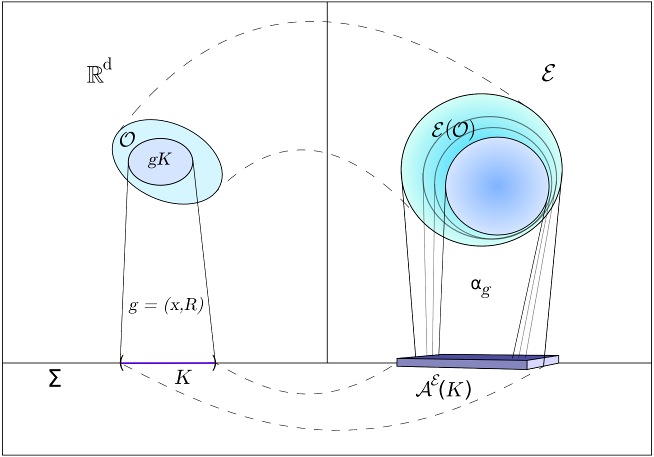

Schematic picture of the time-zero condition.

Algebraic Approach: Spaces of Restricted Symmetry II

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- $\exists\, e \in \mathbb{R}^4, |e| = 1$ such that
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r_e denotes the e-reflection, $r_e : x \mapsto x - 2(e, x)e$.

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$$
\begin{aligned}
V(t)[A]_\sigma &:= \left[\alpha_{((t,0,0,0),1)}(A)\right]_\sigma, \text{for } t \geq 0 \\
\tilde{V}(\beta)[A]_\sigma &:= \left[\alpha_{(0,R(\beta,0))}(A)\right]_\sigma\ ,\ R(\beta, 0) := R(\beta) \oplus 1
\end{aligned}
$$

Algebraic Approach: Spaces of Restricted Symmetry V

Some remarkable works concern continuation of such representations. Most notably: Fröhlich, Osterwalder & Seiler [FOS 1983], Fröhlich [F 1980] ; Klein and Landau [KL 1981, KL2 1983]

Shown there: operators generalizing $V(t)$ and $\tilde{V}(\beta)$ are generated by densely def. symm. operators which continue to self-adjoint operators on $\mathcal{H}^{\mathcal{M}}$.

Jorgensen and Ólafsson [JO 1999] gave a general treatment.

Algebraic Approach: Spaces of Restricted Symmetry VI

In this way:

Imaginary time-translations $V(t)$ and x_0, x_1-rotations $\tilde{V}(\beta)$

$\overset{\text{analytically}}{\longrightarrow}$ real time-translations e^{itH} and x_0, x_1-boosts $\mathrm{e}^{i\beta L}$:

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$\rightarrow$ It results a well-defined $\mathcal{P}_\theta(4)$-action, denote by $\alpha_g^{\mathcal{M}}$.

$\mathcal{P}_\theta(4) := (O(1,1) \times SO(2)) \ltimes \mathbb{R}^4$.

Algebraic Approach: Main Proposition

Proposition

- The operators $U(x, \Lambda(\beta, \alpha))$ form a unitary, weakly continuous representation of the reduced Poincaré group $\mathcal{P}_\theta(4)$ on $\mathcal{H}^{\mathcal{M}}$.

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- The vacuum vector $\Omega := [1]_\sigma$ is invariant under $U(g)$ for all $g \in \mathcal{P}_\theta(4)$.
- The joint spectrum of the generators H, P_1, P_2, P_3 of the translations lies in the closed lightwedge

$$Y := \{p \in \mathbb{R}^4 : p_0 \geq |p_1|\}.$$

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$U(y) := \mathrm{e}^{iPy} = \int \mathrm{d}E(p) \, \mathrm{e}^{ipy}$

Warped Convolution II

The symmetry group $E_\theta(4) = (O(2) \times SO(2)) \ltimes \mathbb{R}^4$ was chosen such that

$$R\theta = \theta R \quad \text{for all} \quad R \in E_\theta(4).$$

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Thus, the (space-space) noncommutative deformation of $\mathcal{E}(\mathcal{O})$ defined by

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Remark: Remains true in case of full rank noncommutativity Q for $(SO(2) \times SO(2)) \ltimes \mathbb{R}^4$.

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- Instead, consider nets indexed by better suitable regions in $\mathbb{R}^4$.

For now, we follow the second option.

Warped Convolution IV

Therefore, we define the cylindrical subsets

$$\mathrm{C} := \{\mathcal{O} \subsetneq \mathbb{R}^4 \mid \mathcal{O}_2 \text{ bounded}, \ \mathcal{O} + x = \mathcal{O} \ \forall \, x \in 0 \times \mathbb{R}^2\} \ ,$$

where $\mathcal{O}_2$ denotes the projection of $\mathcal{O}$ onto commutative $\mathbb{R}^2 \times 0$,

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where $\mathcal{O}_2$ denotes the projection of $\mathcal{O}$ onto commutative $\mathbb{R}^2 \times 0$, as well as the time-zero stripes:

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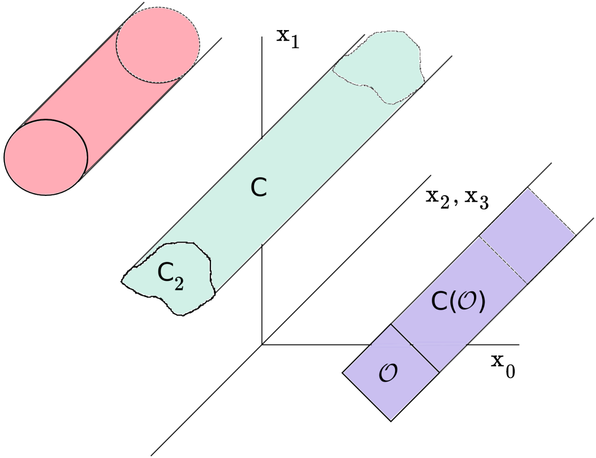
Warning: Does not mean **localization** in C !

Cylindrical Subsets



Warped Convolution: (TZ)

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Lemma: $E_\theta(4)$ enough for $\mathcal{C}$

1. $\forall\, C \in \mathcal{C}, \forall\, S \in \mathcal{S}\ \exists g \in E_\theta(4)$:

$$gS \subset C$$

2. If $g \in E(4)$ s.t. for $S \in \mathcal{S}$ we have $gS \subset C$ for a $C \in \mathcal{C}$

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Corollary

If $\mathcal{E}(C)$ is $E(4)$-cov. and satisfies (TZ)
$\Rightarrow \mathcal{E}_\theta(C)$ is $E_\theta(4)$-cov. and satisfies $(\mathrm{TZ})_\theta$

Warped Convolution: $(TZ)_\theta$

Proof (Cor., sketched).

Indeed, contemplate such $\mathcal{E}(C)$. From the lemma we have

$$\begin{aligned} & \{\alpha_g \mathcal{A}_0(S) \mid S \in \mathbb{S}\,,\, g \in E(d), gS \subset C\} \\ = & \{\alpha_g \mathcal{A}_0(S) \mid S \in \mathbb{S}\,,\, g \in E_\theta(d), gS \subset C\} \end{aligned}$$

$\mathcal{E}_\theta(C)$ is well-defined and $E_\theta(4)$-covariant $\Rightarrow$

$$\mathcal{E}_\theta(C) = \left(\bigcup_{S \in \mathbb{S}} \{\alpha_g \mathcal{A}_\theta(S) \mid S \in \mathbb{S}\,,\, g \in E_\theta(d)\,,\, gS \subset C\} \right)''$$

□

Warped Convolution: Minkowskian Net

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Minkowskian Net

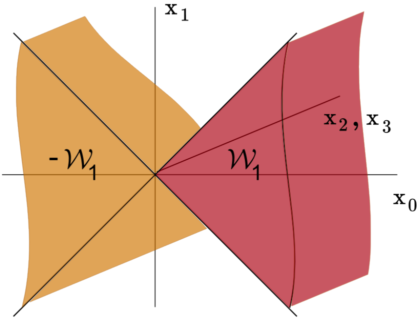
$$\mathcal{M}_\theta(C) := \left(\bigcup_{S \subset \mathbb{S}} \{ \alpha_g^{\mathcal{M}}(\pi_\sigma(A)) \mid g \in \mathcal{P}_\theta(4),\ gS \subset C, A \in \mathcal{A}_\theta^{\mathcal{E}}(K) \} \right)''$$

defines a Haag-Kastler net with modified locality (wedge locality).

π_σ . . . repr. of $\mathcal{E}_\theta$ on $\mathcal{H}^{\mathcal{M}}$.

Wedge Locality

$$\mathcal{W}_1 := \{x \in \mathbb{R}^4 \mid x_1 > |x_0|\}$$



$$A \in \mathcal{M}(\mathcal{W}_1) \quad , \quad B \in \mathcal{M}(-\mathcal{W}_1)$$
$$\Rightarrow [A_\theta, B_{-\theta}] \quad = \quad 0$$

Summary

- Input data: $E(4)$-covariant Euclidean net $\mathcal{E}(\mathcal{O})$ of v.N. algebras, regular reflection-positive Euclidean functional σ and the (restrictive!) time-zero condition

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- Input data: $E(4)$-covariant Euclidean net $\mathcal{E}(\mathcal{O})$ of v.N. algebras, regular reflection-positive Euclidean functional σ and the (restrictive!) time-zero condition
- Deformation: Build the algebra $\mathcal{E}_\theta(C(\mathcal{O}))$ in terms of warped convolutions.
  $\rightarrow \mathcal{E}_\theta$ is $E_\theta(4)$-cov. & well-def. on C.

Commuting Diagram

$$\begin{array}{ccc} \mathcal{E} & \longleftrightarrow & \mathcal{E}_\theta \\ \updownarrow & & \downarrow \\ \mathcal{M} & \longleftrightarrow & \mathcal{M}_\theta \end{array}$$

$\updownarrow$	...	known Wick rotation
$\longrightarrow$	...	warped convolution
$\longleftarrow$	...	commutative limit, i.e. $\vartheta \to 0$
$\downarrow$	...	this talk

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- "Covariantize" to have full symmetry group $E(4)$ at hand.

THANK YOU FOR YOUR ATTENTION !

K. Osterwalder, R. Schrader.
Axioms for Eucliden Green's Functions,
Comm.Math.Phys. 31; 83–112, 1973.

K. Osterwalder, R. Schrader.
Axioms for Eucliden Green's Functions II,
Comm.Math.Phys. 42; 281, 1975.

J. Fröhlich.
Unbounded, Symmetric Semigroups on a Separable Hilbert Space Are Essentially Selfadjoint,
Adv. Appl. Math. 1, 237-256, 1980

A. Klein, Landau.
Construction Of A Unique Self-Adjoint Generator for a Symmetric Local Semigroup,
J.Funct.Anal. 44; 121, 1981.

A. Klein, Landau.
From the Euclidean group to the Poincaré group via Osterwalder-Schrader positivity,
Comm.Math.Phys. 87; 469–484, 1983.

J. Fröhlich and K. Osterwalder and E. Seiler.
On Virtual representations of symmetric spaces and their analytic continuation

Annals Math. 118; 461–489, 1983.

D. Schlingemann.
From Euclidean Field Theory to Quantum Field Theory,
Rev. Math. Phys. 11; 1151–1178, 1999.

P. Jorgensen, G. Ólafsson.
Unitary Representations and Osterwalder-Schrader Duality,
Preprint: http://arxiv.org/abs/math/9908031, 1999.

H. Grosse, R. Wulkenhaar.
Renormalization of ϕ^4 theory on noncommutative $\mathbb{R}^4$ in the matrix base,
Comm.Math.Phys. 256; 305-374, 2004.

V. Rivasseau, M. Disertori, R. Gurau, J. Magnen.
Vanishing of Beta Function of Non Commutative ϕ^4_4 Theory to all orders,
Phys.Lett. B649; 95-102, 2006.

D. Buchholz, S.J. Summers.
Warped Convolutions: A Novel Tool in the Construction of Quantum Field Theories,
Preprint: http://arxiv.org/abs/0806.0349, 2008.

V. Rivasseau, R. Gurau, J. Magnen, A. Tanasa.
A Translation-invariant renormalizable non-commutative scalar model,
Comm.Math.Phys. 287; 275-290, 2009.

D. Buchholz, G. Lechner, S.J. Summers.
Warped Convolutions, Rieffel Deformations and the Construction of Quantum Field Theories,
Preprint: http://arxiv.org/abs/1005.2656, 2010.

Z. Wang,
Construction of 2-dimensional Grosse-Wulkenhaar Model,
Preprint: http://arxiv.org/abs/1104.3750v1, 2011.