Non-commutative closed string geometry from flux compactifications

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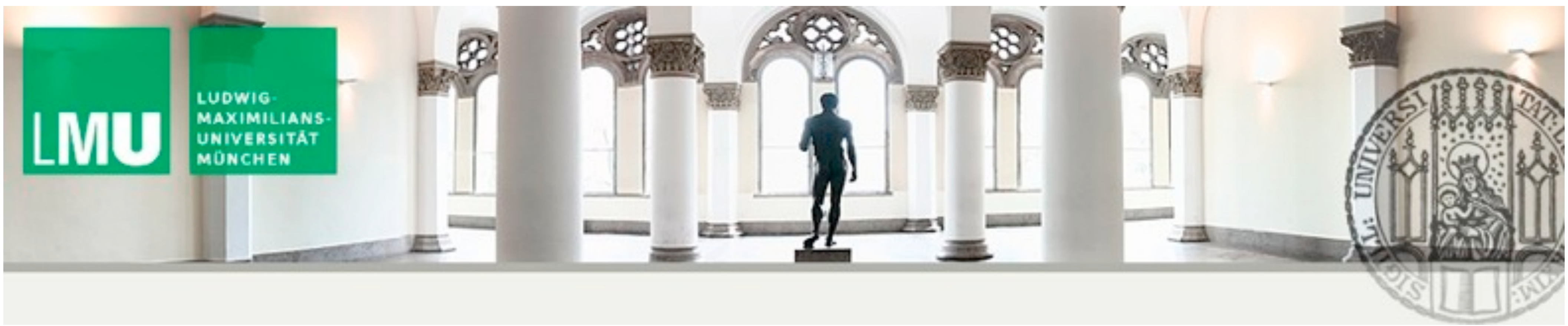
and MPI München



Bayrischzell, 23. May 2011

I) Introduction

Closed string flux compactifications:

- Moduli stabilization ➡ string landscape

- AdS/CFT correspondence

- Generalized geometries

- Here: closed string non-commutative (non-associative) geometry

Non-commutative geometry and string theory (a):

Open strings:

2-dimensional D-branes with 2-form F-flux $\Rightarrow$

coordinates of open string end points become non-commutative:

$$[X_i(\tau), X_j(\tau)] = \epsilon_{ij}\Theta\,, \quad \Theta = -\frac{2\pi i\alpha' F}{1+F^2}$$

(A. Abouelsaood, C. Callan, C. Nappi, S. Yost (1987);
J. Fröhlich, K. Gawedzki (1993); V. Schomerus (1999); ....)

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and even non-associative:

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$$[[X_i(\tau,\sigma), X_j(\tau,\sigma)], X_k(\tau,\sigma)] + \mathrm{perm.} \simeq F^{(3)}_{ijk}$$

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Outline:

II) T-duality

III) Non-commutative geometry

IV) Algebraic structure and
new uncertainty relations

V) Outlook (non-associative gravity)

II) T-duality

How does a closed string see geometry?

Consider compactification on a circle with radius R:

$$X(\tau,\sigma) = X_L(\tau+\sigma) + X_R(\tau-\sigma)$$

$$\begin{aligned} X_L(\tau+\sigma) &= \frac{x}{2} + p_L(\tau+\sigma) + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\alpha_n e^{-in(\tau+\sigma)}\,, \\ X_R(\tau-\sigma) &= \frac{x}{2} + p_R(\tau-\sigma) + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\tilde{\alpha}_n e^{-in(\tau-\sigma)} \end{aligned}$$

(KK momenta)

$$\begin{aligned} p_L &= \frac{1}{2}\left(\frac{M}{R} + (\alpha')^{-1}NR\right), \qquad & p &= p_L + p_R = \frac{M}{R} \\ p_R &= \frac{1}{2}\left(\frac{M}{R} - (\alpha')^{-1}NR\right) & \tilde{p} &= p_L - p_R = (\alpha')^{-1}NR \end{aligned}$$

(dual momenta - winding modes)

T-duality: $T: \quad R \longleftrightarrow \frac{\alpha'}{R}, \quad M \longleftrightarrow N$

$$T: \quad p \longleftrightarrow \tilde{p}\,, \quad p_L \longleftrightarrow p_L\,, \quad p_R \longleftrightarrow -p_R\,.$$

- Dual space coordinates: $\tilde{X}(\tau,\sigma) = X_L - X_R$

$(X, \tilde{X})$: Doubled geometry: (O. Hohm, C. Hull, B. Zwiebach (2009/10))

T-duality is part of diffeomorphism group.

$$T: \quad X \longleftrightarrow \tilde{X}\,, \quad X_L \longleftrightarrow X_L\,, \quad X_R \longleftrightarrow -X_R$$

- Shortest possible radius: $R \geq R_c = \sqrt{\alpha'}$

Compactification on a 2-dimensional torus:

Background: $R_1,\ R_2,\ e^{i\alpha},\ B$

2 complex background parameters:

$$\tau = \frac{e_2}{e_1} = \frac{R_2}{R_1} e^{i\alpha}\,,$$
$$\rho = B + iR_1 R_2 \sin\alpha\,.$$

T-duality transformations:

- $SL(2,\mathbb{Z})_\tau: \quad \tau \to \dfrac{a\tau + b}{c\tau + d}$
- $SL(2,\mathbb{Z})_\rho: \quad \rho \to \dfrac{a\rho + b}{c\rho + d}$

They act as shifts/rotations on doubled coordinates.

- T-duality in x_1 $\Leftrightarrow$ Mirror symmetry:

$$\tau \leftrightarrow \rho \iff B \leftrightarrow \Re\tau$$

Three-dimensional backgrounds $\Rightarrow$ twisted 3-tori:

(A. Dabholkar, C. Hull (2003) ; S. Hellerman, J. McGreevy, B. Williams (2004); J. Derendinger, C. Kounnas, P. Petropoulos, F. Zwirner (2004); J. Shelton, W. Taylor, B. Wecht (2005); G. Dall'Agata, S. Ferrara (2005)...)

Fibrations: 2-dim. torus that varies over a circle:

$$T^2_{x^1,x^2} \hookrightarrow M^3 \hookrightarrow S^1_{x^3}$$

The fibration is specified by its monodromy properties.

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Two (T-dual) cases:

(i) Geometric spaces (manifolds)

$$x^3 \to x^3 + 2\pi \;\Rightarrow\; \tau(x^3 + 2\pi) = \frac{a\tau(x^3) + b}{c\tau(x^3) + d}$$

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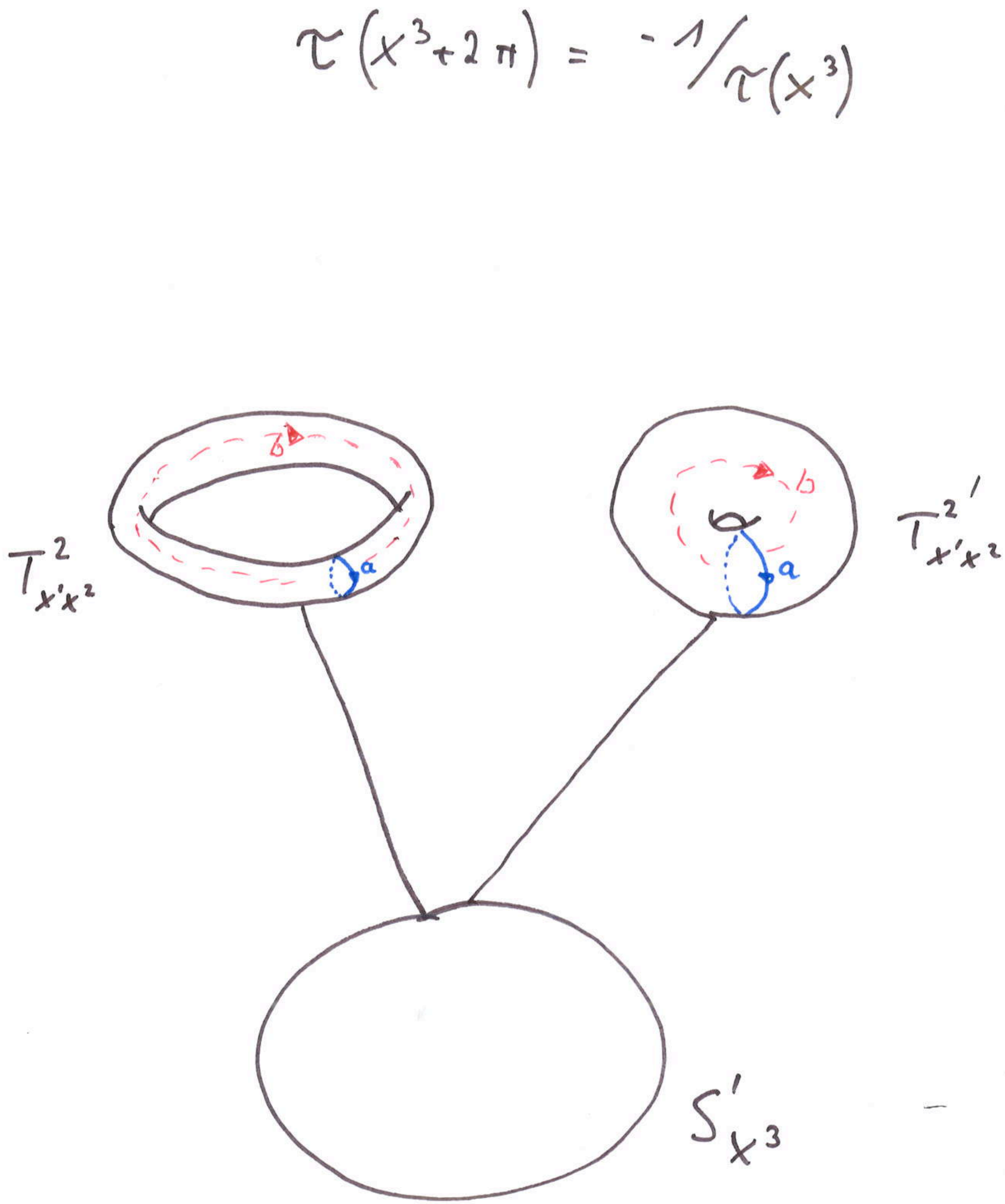

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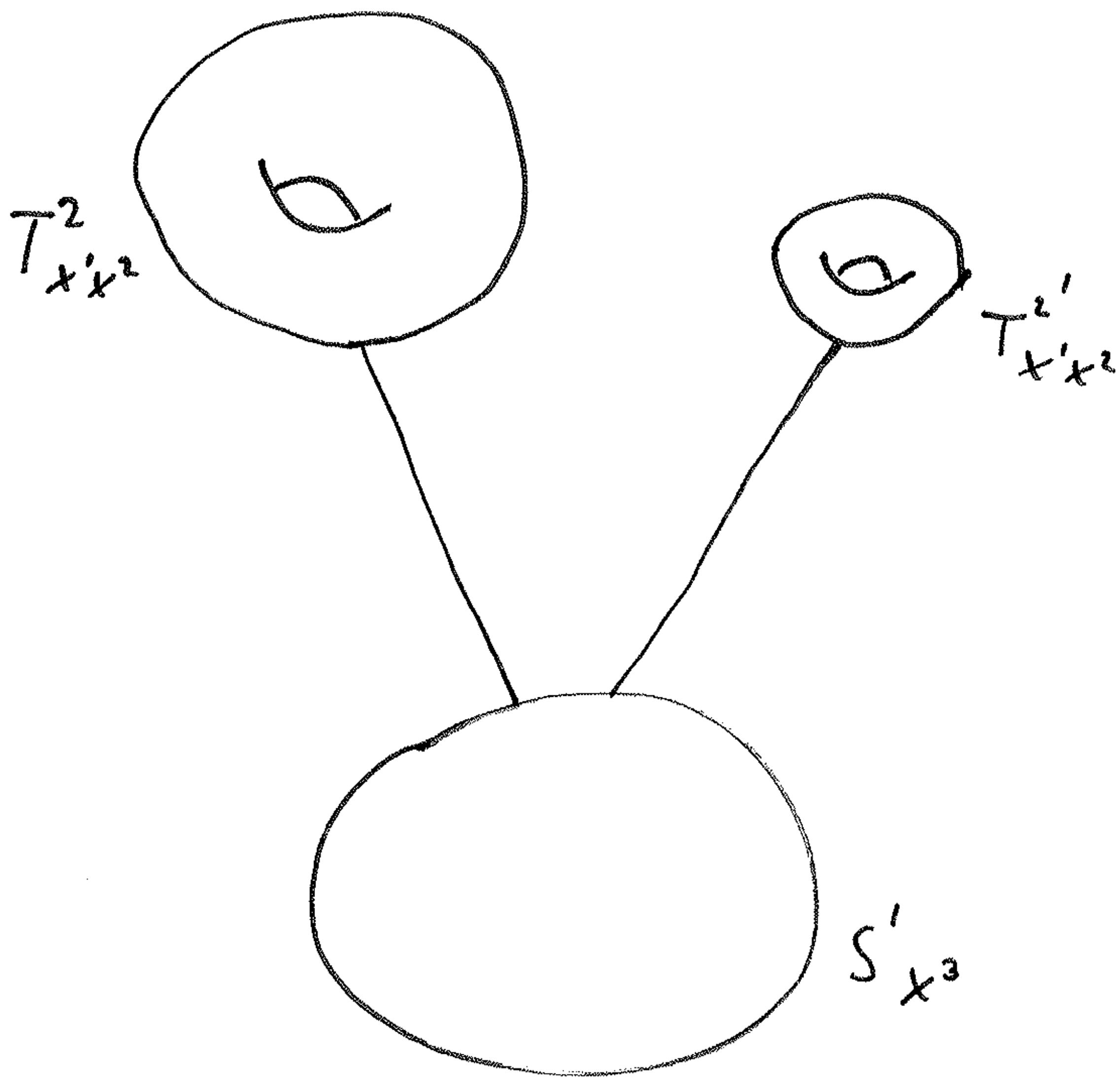

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Two different kind of monodromies for the fibrations:

(i) elliptic monodromies: finite order

$$SL(2,\mathbb{Z})_\tau, SL(2,\mathbb{Z})_\rho : \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$$

order 4 order 6

(ii) parabolic monodromies: infinite order

$$SL(2,\mathbb{Z})_\tau, SL(2,\mathbb{Z})_\rho : \quad \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \text{ or } \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix}$$

Both types in general contain geometric spaces as well as non-geometric backgrounds.

III) Non-commutative geometry

3.1) Open strings on D2-branes:

(i) D2-branes with gauge F-flux

Mixed D/N boundary conditions:

$$\begin{aligned} \partial_\sigma X_1 + F_{12}\,\partial_\tau X_2 &= 0, \\ \partial_\sigma X_2 - F_{12}\partial_\tau X_1 &= 0 \end{aligned}$$

$$[X_1(\tau,0), X_2(\tau,0)] = -\frac{2\pi i\alpha' F_{12}}{1+(F_{12})^2}$$

$\updownarrow$ T-duality (Seiberg-Witten map)

T-duality in X_1 :

(ii) D1-branes at angles

Boundary conditions:

$$\begin{aligned} N: \quad \partial_\sigma X_1 + F_{12}\,\partial_\sigma X_2 &= 0, \\ D: \quad \partial_\tau X_2 - F_{12}\,\partial_\tau X_1 &= 0. \end{aligned}$$

$$[X_1(\tau,0), X_2(\tau,0)] = 0$$

Geom. angle: $\nu = \dfrac{\operatorname{arccot} F_{12}}{\pi}$

Open string CFT with F-flux is exactly solvable $\Rightarrow$

Origin of open string non-commutativity:

a) shifted oscillator frequencies due to boundary conditions:

$$X_1 = x_1 - \sqrt{\alpha'} \sum_{n \in Z} \frac{\alpha_{n+\nu}}{n+\nu} e^{-i(n+\nu)\tau} \sin[(n+\nu)\sigma + \theta_1] - \sqrt{\alpha'} \sum_{m \in Z} \frac{\alpha_{m-\nu}}{m-\nu} e^{-i(m-\nu)\tau} \sin[(m-\nu)\sigma - \theta_1],$$

$$X_2 = x_2 + i\sqrt{\alpha'} \sum_{n \in Z} \frac{\alpha_{n+\nu}}{n+\nu} e^{-i(n+\nu)\tau} \sin[(n+\nu)\sigma + \theta_1] - i\sqrt{\alpha'} \sum_{m \in Z} \frac{\alpha_{m-\nu}}{m-\nu} e^{-i(m-\nu)\tau} \sin[(m-\nu)\sigma - \theta_1].$$

$$\nu = \frac{\mathrm{arccot}\, F_{12}}{\pi}$$

(A. Abouelsaood, C. Callan, C. Nappi, S. Yost (1987); C. Chu, P. Ho (1999))

b) Two-point function in open string CFT:

$$
\begin{aligned}
\langle X_1(z^1), X_2(z^2)\rangle &= -\frac{2\pi i\alpha' F_{12}}{1+(F_{12})^2}\log\left(\frac{z^1-\bar{z}_2}{\bar{z}_1-z_2}\right) \\
\Big|_{\sigma=\sigma'=0} &= -\frac{2\pi i\alpha' F_{12}}{1+(F_{12})^2}\epsilon(\tau_1-\tau_2)
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This function has a jump when changing the order of z^1 and z^2 on the real line.

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Non-commutative gauge theories - Moyal-Weyl $\star$- product:

$$f_1(x)\star f_2(x)\star\ldots\star f_N(x) :=$$

$$\exp\left[i\sum_{m<n}\Theta^{ab}\,\partial_a^{x_m}\,\partial_b^{x_n}\right] f_1(x_1)\,f_2(x_2)\ldots f_N(x_N)\Big|_{x_1=\ldots=x_N=x}$$

$$S\simeq\int d^n x\,\mathrm{Tr}\hat{F}_{ab}\star\hat{F}^{ab}$$

3.2) Closed strings on a 3-dim. space:

Can the closed string also see a non-commutative space?

What deformation is needed?

Yes: one needs 3-form flux: H/ω/Q/R

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(ii) Non-geometric spaces (T-folds) $\updownarrow$ T-duality

$$[X^1(\tau,\sigma), X^2(\tau,\sigma)] \neq 0$$

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More general:

Doubled geometry: Closed string non-commutativity in $(X, \tilde{X})$-space

Problem:

- Background is non-constant.
- CFT is in general not exactly solvable

Ways to handle:

- Study SU(2) WZW model with H-flux

(R. Blumenhagen, E. Plauschinn, arXiv:1010.1263)

- Consider sigma model perturbation theory for small H-field

(R. Blumenhagen, A. Deser, D.L., E. Plauschinn, F. Rennecke, work in progress)

- Consider monodromy properties and the corresponding closed string boundary conditions

$\Rightarrow$ Shifted closed string mode expansion

Specific example: elliptic monodromy

C. Hull, R. Reid-Edwards (2009))

(i) Geometric space (ω -flux) $(\omega_{123} \sim \partial_{x^3} g_{x^1 x^2} \sim \partial_{x^3} \Re\tau(x^3))$

$$\tau(x^3) = \frac{(1+i)\cos(Hx^3) + \sin(Hx^3)}{\cos(Hx^3) - (1+i)\sin(Hx^3)} \quad (H \in \frac{1}{4} + \mathbb{Z})$$

Monodromy: $\tau(x^3 + 2\pi) = -1/\tau(x^3)$

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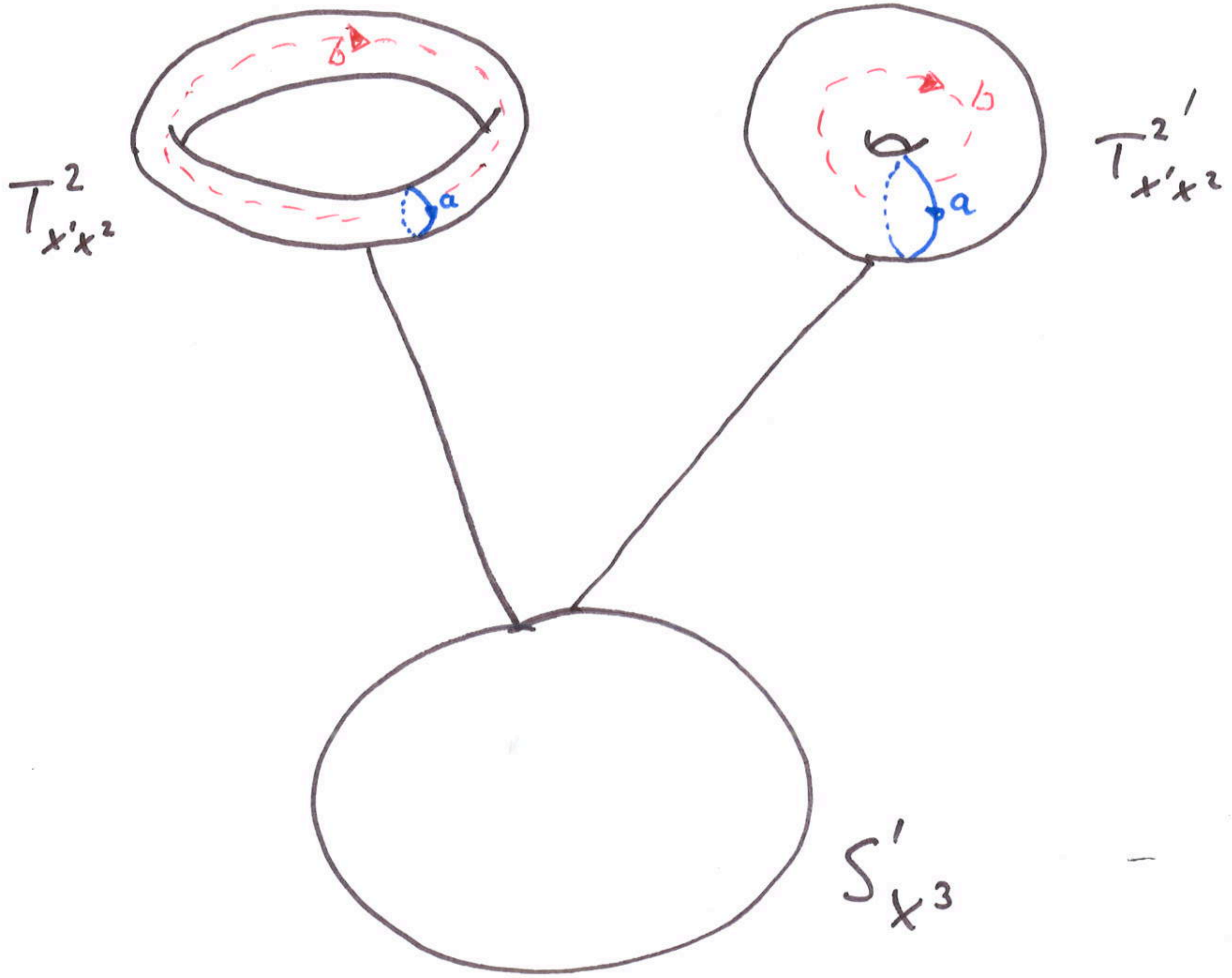

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This induces the following $\mathbb{Z}_4$ symmetric closed string boundary condition:

winding number

$$X^3(\tau, \sigma + 2\pi) = X^3(\tau, \sigma) + 2\pi N_3$$

$$X_L(\tau, \sigma + 2\pi) = e^{i\theta} X_L(\tau, \sigma), \quad \theta = -2\pi N_3 H,$$

$$X_R(\tau, \sigma + 2\pi) = e^{i\theta} X_R(\tau, \sigma).$$

L-R symmetric order 4 rotation

(Complex coordinates: $X_{L,R} = X^1_{L,R} + iX^2_{L,R}$)

Corresponding closed string mode expansion $\Rightarrow$

$$X_L(\tau+\sigma) = i\sqrt{\frac{\alpha'}{2}}\sum_{n\in\mathbb{Z}}\frac{1}{n-\nu}\alpha_{n-\nu}e^{-i(n-\nu)(\tau+\sigma)}\,, \qquad \nu=\frac{\theta}{2\pi}=-N_3H\,,$$

$$X_R(\tau-\sigma) = i\sqrt{\frac{\alpha'}{2}}\sum_{n\in\mathbb{Z}}\frac{1}{n+\nu}\tilde{\alpha}_{n+\nu}e^{-i(n+\nu)(\tau-\sigma)}$$

(shifted oscillators!)

Then one obtains:

$$[X_L(\tau,\sigma),\bar{X}_L(\tau,\sigma)] = -[X_R(\tau,\sigma),\bar{X}_R(\tau,\sigma)] = \tilde{\Theta}$$

$$\tilde{\Theta} = \alpha'\sum_{n\in\mathbb{Z}}\frac{1}{n-\nu} = -\alpha'\pi\cot(\pi N_3 H)$$

$$[X^1(\tau,\sigma),X^2(\tau,\sigma)] = [X_L^1+X_R^1,X_L^2+X_R^2] = 0$$

T-dual geometry (mirror symmetry): $\tau(x^3) \leftrightarrow \rho(x^3)$

(ii) Non-geometric space (Q-flux)

$$\rho(x^3) = \frac{(1+i)\cos(Hx^3) + \sin(Hx^3)}{\cos(Hx^3) - (1+i)\sin(Hx^3)} \qquad (H \in \frac{1}{4} + \mathbb{Z})$$

$\Rightarrow$ **H-field:** $H(x^3) = H\dfrac{10 - 12\sin(2Hx^3) - 6\cos(2Hx^3)}{(2\sin(2Hx^3) + \cos(2Hx^3) - 3)^2}$

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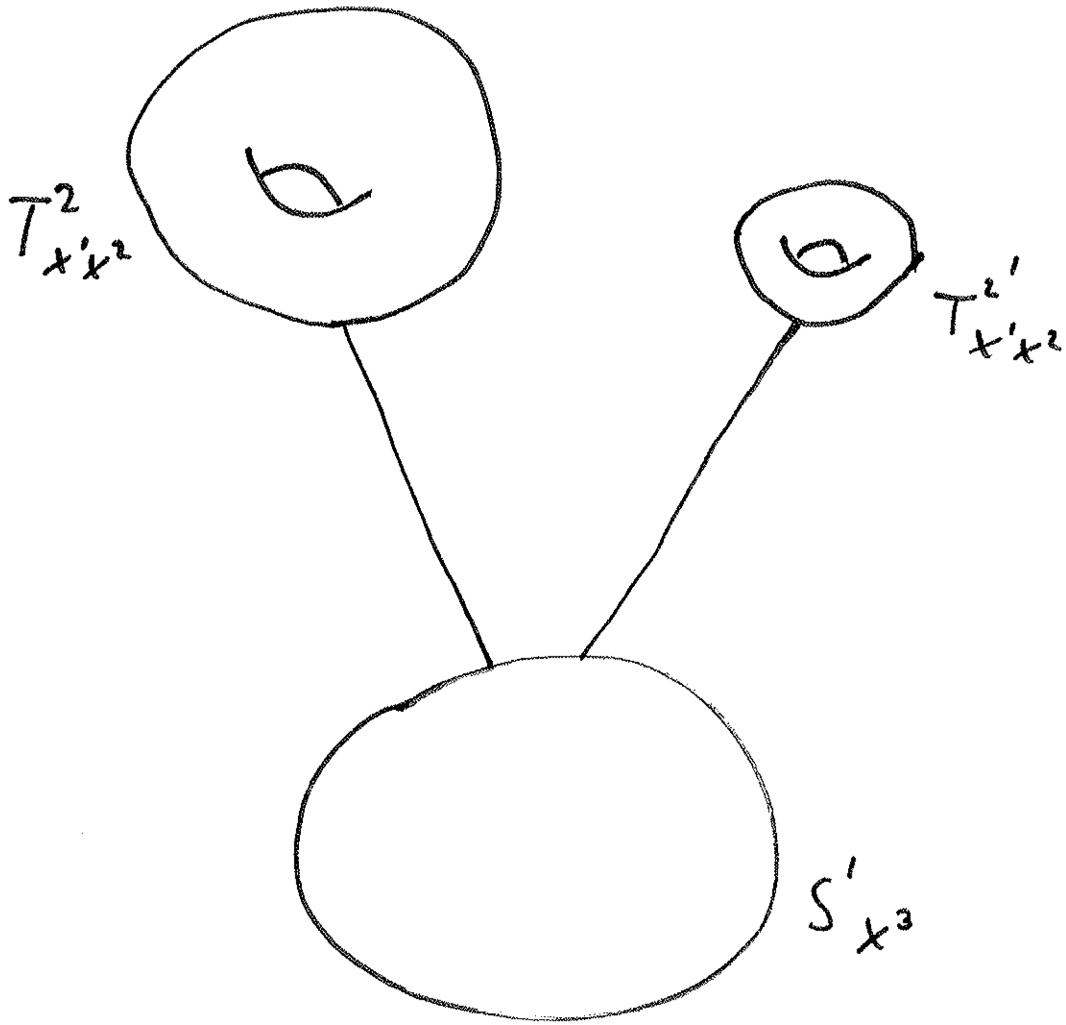

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Corresponding closed string mode expansion $\Rightarrow$

$$X_L(\tau+\sigma) = i\sqrt{\frac{\alpha'}{2}}\sum_{n\in\mathbb{Z}}\frac{1}{n-\nu}\alpha_{n-\nu}e^{-i(n-\nu)(\tau+\sigma)}\,, \qquad \nu = \frac{\theta}{2\pi} = -N_3 H\,;$$

$$X_R(\tau-\sigma) = i\sqrt{\frac{\alpha'}{2}}\sum_{n\in\mathbb{Z}}\frac{1}{n+\nu}\tilde{\alpha}_{n-\nu}e^{-i(n-\nu)(\tau-\sigma)}$$

Then one finally obtains:

$$[X_L(\tau,\sigma),\bar{X}_L(\tau,\sigma)] = [X_R(\tau,\sigma),\bar{X}_R(\tau,\sigma)] = \tilde{\Theta}$$

$$[X^1(\tau,\sigma),X^2(\tau,\sigma)] = [X_L^1+X_R^1,X_L^2+X_R^2] = i\tilde{\Theta}$$

T-duality in x^3 - direction $\Rightarrow$ R-flux

Winding no. $N_3 \Longleftrightarrow$ Momentum no. M_3

$$[X^1(\tau,\sigma), X^2(\tau,\sigma)] = i\Theta$$

$$\Theta = \alpha' \sum_{n\in\mathbb{Z}} \frac{1}{n-\nu} = -\alpha'\pi\cot(\pi M_3 H)$$

Chain of T-dualities:

geom. space: $[X^1(\tau,\sigma), \tilde{X}^2(\tau,\sigma)] = i\tilde{\Theta}$

$\Updownarrow \quad T_{x^2}$

T-fold: $[X^1(\tau,\sigma), X^2(\tau,\sigma)] = i\tilde{\Theta}$

$\Updownarrow \quad T_{x^3}$

R-background: $[X^1(\tau,\sigma), X^2(\tau,\sigma)] = i\Theta$

IV) Algebraic structure and new uncertainty relation

Act on wave functions $\Rightarrow$ replace momentum (winding)

numbers by (dual) momentum operator:

$$M_3 \equiv \sqrt{\alpha'}\, p^3 \,, \qquad N_3 \equiv \sqrt{\alpha'}\, \tilde{p}^3$$

Then one obtains the following non-commutative algebra:

$$[X^1, X^2] \simeq i l_s^3 F^{(3)}\, p^3 \quad ([X^i, X^j] \simeq i\epsilon^{ijk} F^{(3)} p^k)$$

Corresponding uncertainty relation:

$$(\Delta X^1)^2 (\Delta X^2)^2 \geq l_s^6 (F^{(3)})^2 \, \langle p^3 \rangle^2$$

Use $\quad [p^3, X^3] = -i$

$$\Longrightarrow \quad [[X^1, X^2], X^3] + \text{perm.} \simeq F^{(3)}\, l_s^3$$

Non-associative algebra!

This nicely agrees with the non-associative closed string structure found by Blumenhagen, Plauschinn in the SU(2) WZW model: arXiv:1010.1263

Origin of closed string non-associativity:

Three-point function in closed string CFT:

(WZW-model: R. Blumenhagen, E. Plauschinn, arXiv:1010.1263)

$$\big\langle X^a(z_1,\overline{z}_1)\; X^b(z_2,\overline{z}_2)\; X^c(z_3,\overline{z}_3)\big\rangle =$$

$$= F^{abc}\left\{\left[L\left(\tfrac{z_{12}}{z_{13}}\right) + L\left(\tfrac{z_{13}}{z_{23}}\right) + L\left(\tfrac{z_{32}}{z_{12}}\right)\right] - \left[L\left(\tfrac{\overline{z}_{12}}{\overline{z}_{13}}\right) + L\left(\tfrac{\overline{z}_{13}}{\overline{z}_{23}}\right) + L\left(\tfrac{\overline{z}_{32}}{\overline{z}_{12}}\right)\right]\right\},$$

$$(z_{ij} = z_i - z_j)$$

Rogers dilogarithm: $L(x) = \mathrm{Li}_2(x) + \dfrac{1}{2}\log(x)\log(1-x)$

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$$(z_{ij}=z_i-z_j)$$

Rogers dilogarithm: $L(x)=\mathrm{Li}_2(x)+\frac{1}{2}\log(x)\log(1-x)$

This function is discontinuous when $z_1\to z_2=1,\ z_1\to z_3=0$

It develops a jump when all three points approach each other, i.e. $z_1\to z_3=0,\ z_2\to z_3=0$

V) Summary & Outlook

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- What is the algebra of closed string states (functions) on this space? Is there something like a Moyal-Weyl $\star$- product?

Closed string correlation functions $\Rightarrow$

Non-associative $\triangle$- product:

$$f_1(y) \triangle f_2(y) \triangle \ldots \triangle f_N(y) :=$$

$$\exp\left[\sum_{m<n<r} F^{abc}\, \partial_a^{y_m}\, \partial_b^{y_n} \partial_c^{y_r}\right] f_1(y_1)\, f_2(y_2) \ldots f_N(y_N)\Big|_{y_1=\ldots=y_N=y}$$

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- Is there are non-commutative (non-associative) theory of gravity? Is there a map to commutative gravity (like SW-map for gauge theories)? (Non-commutative geometry & gravity: P. Aschieri, M. Dimitrijevic, F. Meyer, J. Wess (2005))

Comparison between open and closed strings:

Aspect	open string	closed string
$(n$-$1)$-probe	$S^1 = \partial D$	S^2
n-flux	F_{ij}	F_{ijk}
n-bracket	$[X^i, X^j] \simeq \alpha' \, \theta^{ij}$	$[X^i, X^j, X^k] \simeq (\alpha')^2 F^{ijk}$
n-point fct.	$\langle X^i, X^j \rangle \simeq \alpha' \, \theta^{ij} \times \log$	$\langle X^i, X^j, X^k \rangle \simeq (\alpha')^2 F^{ijk} \times (\mathrm{di}-\log)$
n-product	$f_1 \star f_2$	$f_1 \triangle f_2 \triangle f_3$
eff. action	$\mathcal{L} \sim F \star F$	?