I The geometry of the Higgs

Burić & Madore

Outline

1. Yang-Mills potentials ... good
2. Higgs mechanism .... better
3. Noncommutative geometry .... best

Yang-Mills potentials

As example we choose the chiral SU_2 potential

$$A = \rho + \gamma^5 A_1$$

and action

$$S = \frac{1}{4} \int \mathrm{Tr}\, F F^*$$

No mass term.

Higgs mechanism (almost)

Let ψ be a Dirac spinor with values in the fundamental representation of the chiral SU_2 symmetry group.

Let D be covariant with connection A.

The norm with an arbitrary hermitean h

$$\bar{\psi}\psi = \psi^* h \psi$$

The covariant derivative of h is given by

$$Dh = dh + [A, h]$$

If this vanishes the connection is metric.

An example is the σ-model metric

$$h = \gamma^0(\sigma + \gamma^5 \pi), \qquad \pi = -\tfrac{1}{4} i \pi_a \sigma^a, \quad \sigma^2 = 1 - \tfrac{1}{4}\pi^2$$

A solution to the condition $Dh = 0$ is given by

$$A = j + \gamma^5 j^5,$$

where

$$j = [\pi, d\pi], \qquad j^5 = \sigma d\pi - \pi d\sigma$$

are the pion vector and axial-vector currents.

Using these the action can be extended

$$S = \tfrac{1}{4} \int \mathrm{Tr}\, F F^* + \tfrac{1}{2} m^2 \mathrm{Tr}\,(A - j)^2$$

to include a mass term for the chiral potential.

There is no Higgs (the pion) potential-energy term to break the symmetry

Schwinger, Zumino, Weinberg and others

Noncommutative geometry

One would introduce the chiral algebra

$$\mathcal{A} = \mathcal{C}(\mathbb{R}^4) \otimes (M_2 \oplus \gamma^5 M_2)$$

and study electromagnetism using $\mathcal{A}$ instead of $\mathcal{C}(\mathbb{R}^4)$

There is now a potential-energy term and the current is replaced by the Connes-Dirac operator, a 1-form which transforms as a Yang-Mills potential does but is also in fact gauge-invariant.

II The Frame Formalism

or

How it Came

That we Tergiversate

no More

and

Love Noncommutativity

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Ingredients

The usual plus a frame

$$[x^\mu, \theta^\alpha] = 0,$$

and thereto dual momenta

$$[p_\alpha, x^\mu] = \hbar e^\mu_\alpha.$$

with a quadratic consistency condition.

For the rotation coefficients one has

$$C^\alpha{}_{\beta\gamma} = F^\alpha{}_{\beta\gamma} - 4i\epsilon p_\delta Q^{\alpha\delta}_{-}{}_{\beta\gamma} \qquad F, Q \in \mathbb{R}$$

It follows that

$$e_\alpha C^\alpha{}_{\beta\gamma} = 0$$

Phase space

There are in gereral in position space 4 inner derivations with momenta $-Z_\alpha(x^\mu)$ which are dual to the frame.

Phase space has also 4 extra generators, outer derivations which we make inner by adding 4 momenta p_α. This is quantum mechanics.

The 'covariant derivative'

$$P_\alpha = p_\alpha + Z_\alpha(x^\mu)$$

lies thus in the commutant of the algebra.

Write a point in phase space as $y^i = (x^\lambda,\ p_\alpha)$.

The Heisenberg commutation relations are

$$[y^i, y^j] = J^{ij}$$

with

$$J^{ij} = \begin{pmatrix} 0 & \delta^\mu_\beta \\ -\delta^\nu_\alpha & 0 \end{pmatrix}.$$

The diagonal elements consist of the six position commutators

$$[x^\mu, x^\nu] = i\hbar J^{\mu\nu}.$$

as well as of the 'dual' momentum commutators

$$[p_\alpha, p_\beta] = (i\hbar)^{-1} L_{\alpha\beta}$$

with

$$L_{\alpha\beta} = K_{\alpha\beta} + F^\gamma{}_{\alpha\beta} p_\gamma - 2i\epsilon Q^{\gamma\delta}{}_{\alpha\beta} p_\gamma p_\delta.$$

The general noncommutative phase space is given by

$$J^{ij} = \begin{pmatrix} i\hbar J^{\mu\nu} & e^\mu_\beta \\ -e^\nu_\alpha & (i\hbar)^{-1} L_{\alpha\beta} \end{pmatrix}$$

The Leibniz rule can be written

$$e_\alpha J^{\beta\gamma} + C^{[\beta}{}_{\alpha\delta} J^{\delta\gamma]} = 0.$$

or in terms of the inverse

$$F_{\alpha\beta} = (J^{-1})_{\alpha\beta}.$$

either as

$$e_\alpha F_{\beta\gamma} + F_{\alpha\delta} C^{\delta}{}_{\beta\gamma} = 0.$$

or as a 'cocycle condition'

$$dF = 0, \qquad F = \frac{1}{2} F_{\alpha\beta} \theta^\alpha \theta^\beta$$

The solution

$$C^{\alpha}{}_{\beta\gamma} = J^{\alpha\eta} e_\eta F_{\beta\gamma}.$$

yields an explicit map

$$J^{\alpha\eta} \to C^{\alpha}{}_{\beta\gamma}$$

from the algebra to the geometry