Berezin's coherent states, symbols and transform revisited

Martin Schlichenmaier

Mathematics Research Unit
University of Luxembourg

Bayrischzell, May 20-23, 2011

OUTLINE

1. review of coherent state techniques related to the quantization of Kähler manifolds
2. covariant symbols, Berezin transform, equivalent star products
3. More details: in *Berezin-Toeplitz quantization for Compact Kähler manifolds. A Review of results*, Advances in Math. Phys. 38 pages, doi:10.1155/2010/927280
4. mainly restrict the treatment to the compact Kähler case – but some of the constructions work also in the non-compact case.

Results:

partly joint with M. Bordemann, E. Meinrenken, and A. Karabegov

(M, ω) a (compact) Kähler manifold.
i.e. M a complex manifold, ω a Kähler form (closed $(1,1)$ form which is positive), $\quad d\omega = 0$
in local holomorphic coordinates $\{z_i\}_{i=1,\dots n}$

$$\omega = \mathrm{i} \sum_{i,j=1}^{n} g_{ij}(z) dz_i \wedge d\bar{z}_j,$$

$(g_{ij}(z))_{i,j=1,\dots,n}$ is hermitian and positive definite matrix

Examples

1. $\mathbb{C}^n, \quad \omega = \mathrm{i} \sum_{i=1}^{n} dz_i \wedge d\bar{z}_i$ 2. $\mathbb{P}^1, \quad \omega = \frac{\mathrm{i}}{(1+z\bar{z})^2} dz \wedge d\bar{z}$
3. every Riemann surface 4. every (complex) torus
5. every (quasi-)projective manifold 6. very often moduli spaces

Quantization condition: (M, ω) is called quantizable, if there exists an associated quantum line bundle (L, h, ∇)

L is a holomorphic line bundle over M,
h a hermitian metric on L,
∇ a compatible connection fullfilling additionally

$$curv_{(L,\nabla)} = -\mathrm{i}\,\omega$$

locally this means $\mathrm{i}\bar\partial\partial \log \hat{h} = \omega$.

Note: Not all Kähler manifolds are quantizable
e.g. the tori are only quantizable if they have enough theta functions, i.e. if they are abelian varieties

(M, ω) a (compact) Kähler manifold

Consider now $L^m := L^{\otimes m}$, with metric $h^{(m)}$.

$\Gamma_\infty(M, L^m)$ the space of smooth sections
$\Gamma_{hol}(M, L^m) = \mathrm{H}^0(M, L^m)$ the space of global holomorphic sections

scalar product

$$\langle \varphi, \psi \rangle := \int_M h^{(m)}(\varphi, \psi)\,\Omega, \qquad \Omega := \frac{1}{n!}\underbrace{\omega \wedge \omega \cdots \wedge \omega}_{n}$$

$$\Pi^{(m)} : L^2(M, L^m) \longrightarrow \Gamma_{hol}(M, L^m)$$

Berezin-Toeplitz operator quantization

Take $f \in C^\infty(M)$, and $s \in \Gamma_{hol}(M, L^m)$

$$s \quad \mapsto \quad \Pi^{(m)}(f \cdot s) =: T_f^{(m)}(s)$$

defines

$$T_f^{(m)} : \quad \Gamma_{hol}(M, L^m) \to \Gamma_{hol}(M, L^m)$$

the Toeplitz operator of level m.

The Berezin-Toeplitz operator quantization is the map

$$f \mapsto \left(T_f^{(m)}\right)_{m \in \mathbb{N}_0}.$$

The BT quantization has the correct semi-classical behavior

Semi-classical behaviour

Theorem (Bordemann, Meinrenken, and Schl.)

(a)

$$\lim_{m\to\infty} ||T_f^{(m)}|| = |f|_\infty$$

(b)

$$||m\mathrm{i}\,[T_f^{(m)}, T_g^{(m)}] - T_{\{f,g\}}^{(m)}|| = O(1/m)$$

(c)

$$||T_f^{(m)} T_g^{(m)} - T_{f\cdot g}^{(m)}|| = O(1/m)$$

Poisson bracket $\{.,.\}$ is given by

$$\{\, f, g\,\} := \omega(X_f, X_g) \quad \text{with} \quad \omega(X_f, \cdot) = df(\cdot).$$

Further result: The Toeplitz map

$$T_{(m)} : C^{\infty}(M) \to End(\Gamma_{hol}(M, L^m))$$

is surjective

This implies operator $Q_f^{(m)}$ of geometric quantization (with holomorphic polarization) can be written as Toeplitz operator of a function (different for every m)

Indeed Tuynman relation:

$$Q_f^{(m)} = \mathrm{i}\ T^{(m)}_{f - \frac{1}{2m}\Delta f}.$$

$u(N), \; N \to \infty$ LIMIT

If we choose basis in $\Gamma_{hol}(M, L^m)$ then $T_f^{(m)}$ can be represented as $N \times N$ matrices, $N = \dim \Gamma_{hol}(M, L^m)$.

$C^\infty(M) \to gl(N, \mathbb{C}), \; f \mapsto \mathrm{i}\, T_f^{(m)}$ is a surjective linear map and we obtain an infinite sequence of matrices.

Fact: $T_{\bar{f}}^{(m)} = (T_f^{(m)})^*$,
hence for real valued f the operator $T_f^{(m)}$ is selfadjoint.

$C^\infty(M, \mathbb{R}) \to u(n), \; f \mapsto \mathrm{i}\, T_f^{(m)}$ (again surjective)
gives a sequence of $u(N)$, $N \to \infty$ matrices.

BEREZIN-TOEPLITZ DEFORMATION QUANTIZATION

Theorem (BMS, Schl., Karabegov and Schl.)

$\exists$ a unique differential star product

$$f \star_{BT} g = \sum \nu^k C_k(f,g)$$

such that

$$T_f^{(m)} T_g^{(m)} \sim \sum_{k=0}^{\infty} \left(\frac{1}{m}\right)^k T_{C_k(f,g)}^{(m)}$$

Further properties: it is of separation of variables type, (also called of Wick type)

with classifying Deligne-Fedosov class $\frac{1}{\mathrm{i}}(\frac{1}{\nu}[\omega] - \frac{\epsilon}{2})$ and Karabegov form $\frac{-1}{\nu}\omega + \omega_{can}$ $\square$

equivalence of star products:

$\star$ and $\star'$ (for the same Poisson structure) are equivalent iff there exists a formal series of linear operators

$$B = \sum_{i=0}^{\infty} B_i \nu^i, \qquad B_i : C^\infty(M) \to C^\infty(M),$$

with $B_0 = id$ such that $B(f) \star' B(g) = B(f \star g)$

- BMS Theorem (using Tuynman relation) $\implies$ there exists a star product $\star_{GQ}$ given by asymptotic expansion of product of geometric quantisation operators
- $\star_{GQ}$ is equivalent to $\star_{BT}$, $\quad B(f) := (id - \nu \frac{\Delta}{2}) f$
- it is not of separation of variable type

THE DISC BUNDLE

- quantization condition says L is a positive line bundle, by Kodaira embedding theorem there exists $m_0 \in \mathbb{N}$, such that $L^{(m_0)}$ has enough global holomorphic sections which can be used to embed M into projective space (such $L^{(m_0)}$ is called very ample),
- assume that bundle L is already very ample,
- pass to its dual $(U, k) := (L^*, h^{-1})$ with dual metric k
- inside of the total space U, consider the circle bundle

$$Q := \{\lambda \in U \mid k(\lambda, \lambda) = 1\},$$

- $\tau : Q \to M$ (or $\tau : U \to M$) the projection,

- the bundle Q is a contact manifold, i.e. there is a 1-form ν $(= (\frac{1}{2\mathrm{i}}(\partial - \bar{\partial}) \log \hat{h})_Q)$ such that $\mu = \frac{1}{2\pi}\tau^*\Omega \wedge \nu$ is a volume form on Q
- $$\int_Q (\tau^* f)\mu = \int_M f\,\Omega, \qquad \forall f \in C^\infty(M).$$
- $\mathcal{H}^{(m)}$ space of m-homogenous functions on Q which can be extended to the disc bundle ("interior" of the circle bundle), homogenous means $\psi(c\lambda) = c^m\psi(\lambda)$
- $\mathcal{H}$ is the space of all extendable functions

- Q is a S^1-bundle, L^m are associated line bundles
- sections of $L^m = U^{-m}$ are identified with those functions ψ on Q which are homogeneous of degree m,
- identification given via the map

$$\gamma_m : \mathrm{L}^2(M, L^m) \to \mathrm{L}^2(Q, \mu), \quad s \mapsto \psi_s \quad \text{where}$$

$$\psi_s(\alpha) = \alpha^{\otimes m}(s(\tau(\alpha))),$$

- Restricted to the holomorphic sections we obtain the unitary isomorphism

$$\gamma_m : \Gamma_{hol}(M, L^m) \cong \mathcal{H}^{(m)}.$$

Recall

$$\psi_s(\alpha) = \alpha^{\otimes m}(s(\tau(\alpha))),$$

Now we fix $\alpha \in U \setminus 0$ and vary the sections s.

- *coherent vector (of level m)* associated to the point $\alpha \in U \setminus 0$ is the element $e_\alpha^{(m)}$ of $\Gamma_{hol}(M, L^m)$ with (for all $s \in \Gamma_{hol}(M, L^m)$)

$$\langle e_\alpha^{(m)}, s \rangle = \psi_s(\alpha) = \alpha^{\otimes m}(s(\tau(\alpha)))$$

  for all $s \in \Gamma_{hol}(M, L^m)$.
- check:

$$e_{c\alpha}^{(m)} = \bar{c}^m \cdot e_\alpha^{(m)}, \qquad c \in \mathbb{C}^* := \mathbb{C} \setminus \{0\} \ .$$

- *coherent state (of level m)* associated to $x \in M$ is the projective class

$$\mathrm{e}_x^{(m)} := [e_\alpha^{(m)}] \in \mathbb{P}(\Gamma_{hol}(M, L^m)), \qquad \alpha \in \tau^{-1}(x), \alpha \neq 0.$$

- The coherent state embedding is the antiholomorphic embedding

$$M \quad \to \quad \mathbb{P}(\Gamma_{hol}(M, L^m)) \;\cong\; \mathbb{P}^N(\mathbb{C}), \qquad x \mapsto [e_{\tau^{-1}(x)}^{(m)}].$$

COVARIANT BEREZIN SYMBOL

Covariant Berezin symbol $\sigma^{(m)}(A)$ (of level m) of an operator $A \in \mathrm{End}(\Gamma_{hol}(M, L^{(m)}))$ is defined as

$$\sigma^{(m)}(A) : M \to \mathbb{C}, \quad x \mapsto \sigma^{(m)}(A)(x) := \frac{\langle e_\alpha^{(m)}, Ae_\alpha^{(m)} \rangle}{\langle e_\alpha^{(m)}, e_\alpha^{(m)} \rangle}, \quad \alpha \in \tau^{-1}(x).$$

Can be rewritten as

$$\sigma^{(m)}(A) = \mathrm{Tr}(AP_x^{(m)}).$$

with the coherent projectors

$$P_x^{(m)} = \frac{|e_\alpha^{(m)}\rangle\langle e_\alpha^{(m)}|}{\langle e_\alpha^{(m)}, e_\alpha^{(m)} \rangle}, \qquad \alpha \in \tau^{-1}(x)$$

Importance of the covariant symbol

- Construction of the Berezin star product, under very restrictive conditions on the manifolds
- $\mathcal{A}^{(m)} \le C^\infty(M)$, of level m covariant symbols.
- the symbol map is injective (follows from Toeplitz map surjective)
- for $\sigma^{(m)}(A)$ and $\sigma^{(m)}(B)$ the operators A and B are uniquely fixed, and we set

$$\sigma^{(m)}(A) \star_{(m)} \sigma^{(m)}(B) := \sigma^{(m)}(A \cdot B)$$

- $\star_{(m)}$ on $\mathcal{A}^{(m)}$ is an associative and noncommutative product
- Crucial problem, how to obtain from $\star_{(m)}$ a star product for all functions (or symbols) independent from the level m ?
- in general not possible, only for limited classes of manifolds

- Also the notion of a contravariant symbol exists.
- for a Toeplitz operator $T_f^{(m)}$ a contravariant symbol is f itself
- General definition will be given maybe later.

Berezin transform

The map

$$I^{(m)} : C^{\infty}(M) \to C^{\infty}(M), \qquad f \mapsto I^{(m)}(f) := \sigma^{(m)}(T_f^{(m)})$$

is called the Berezin transform (of level m).

THEOREM

Given $x \in M$ then the Berezin transform $I^{(m)}(f)$ has a complete asymptotic expansion in powers of $1/m$ as $m \to \infty$

$$I^{(m)}(f)(x) \quad \sim \quad \sum_{i=0}^{\infty} I_i(f)(x) \frac{1}{m^i},$$

where $I_i : C^\infty(M) \to C^\infty(M)$ are maps with
$I_0(f) = f, \qquad I_1(f) = \Delta f.$

- Δ is the Laplacian with respect to the metric given by the Kähler form ω,
- Complete asymptotic expansion: Given $f \in C^\infty(M)$, $x \in M$ and an $r \in \mathbb{N}$ then there exists a positive constant A such that

$$\left| I^{(m)}(f)(x) - \sum_{i=0}^{r-1} I_i(f)(x) \frac{1}{m^i} \right|_\infty \quad \le \quad \frac{A}{m^r}\ .$$

APPLICATION 1: BEREZIN STAR PRODUCTS

- take from asymptotic expansion of the Berezin transform the formal expression

$$I = \sum_{i=0}^{\infty} I_i \, \nu^i, \quad I_i : C^\infty(M) \to C^\infty(M)$$

- set $f \star_B g := I^{-1}(I(f) \star_{BT} I(g))$
- as $I_0 = id$ this $\star_B$ is a star product, called the Berezin star product
- I gives the equivalence to $\star_{BT}$.
- if the definition with the covariant symbol works it will coincide with the star product defined there.

APPLICATION 2: NORM PRESERVATION OF BT QUANTUM OPERATORS

Statement:

$$|f|_\infty - \frac{C}{m} \quad \leq \quad ||T_f^{(m)}|| \quad \leq \quad |f|_\infty$$

First,

$$|I^{(m)}(f)|_\infty = |\sigma^{(m)}(T_f^{(m)})|_\infty \quad \leq \quad ||T_f^{(m)}|| \quad \leq \quad |f|_\infty \,.$$

Second,

- take $x_e \in M$ a point with $|f(x_e)| = |f|_\infty$
- asymptotic expansion of the Berezin transform yields $|(I^{(m)}f)(x_e) - f(x_e)| \le C/m$ with a constant C
- hence,
$$\Big| |f(x_e)| - |(I^{(m)}f)(x_e)| \Big| \le C/m$$
- and
$$|f|_\infty - \frac{C}{m} = |f(x_e)| - \frac{C}{m} \quad \le \quad |(I^{(m)}f)(x_e)| \quad \le \quad |I^{(m)}f|_\infty \, .$$
- This gives the statement

Bergman kernel

- Main tool: for the asymptotic expansion of the Berezin-transform is asymptotic expansion of the Bergman kernel function in the neighbourhood of the diagonal (joint work with A. Karabegov).
- Szegö projectors $\Pi : \mathrm{L}^2(Q,\mu) \to \mathcal{H}$, and its components $\hat{\Pi}^{(m)} : \mathrm{L}^2(Q,\mu) \to \mathcal{H}^{(m)}$, the Bergman projectors
- Bergman projectors have smooth integral kernels, the Bergman kernels $\mathcal{B}_m(\alpha,\beta)$ on $Q \times Q$, i.e.

$$\hat{\Pi}^{(m)}(\psi)(\alpha) = \int_Q \mathcal{B}_m(\alpha,\beta)\psi(\beta)\mu(\beta).$$

$$\mathcal{B}_m(\alpha,\beta) = \psi_{e_\beta^{(m)}}(\alpha) = \overline{\psi_{e_\alpha^{(m)}}(\beta)} = \langle e_\alpha^{(m)}, e_\beta^{(m)} \rangle.$$

- connected to Berezin transform via

$$\left(I^{(m)}(f)\right)(x) = \frac{1}{\mathcal{B}_m(\alpha,\alpha)} \int_Q \mathcal{B}_m(\alpha,\beta)\mathcal{B}_m(\beta,\alpha)\tau^* f(\beta)\mu(\beta)$$

We need: Rawnsley's epsilon function

$$\epsilon^{(m)} : M \to C^\infty(M), \quad x \mapsto \epsilon^{(m)}(x) := \frac{h^{(m)}(e_\alpha^{(m)}, e_\alpha^{(m)})(x)}{\langle e_\alpha^{(m)}, e_\alpha^{(m)} \rangle}, \; \alpha \in \tau^{-1}(x).$$

As $\epsilon^{(m)} > 0$ we introduce the modified measure

$$\Omega_\epsilon^{(m)}(x) := \epsilon^{(m)}(x)\Omega(x)$$

on the space of functions on M.

Given $A \in \mathrm{End}(\Gamma_{hol}(M, L^{(m)}))$ then a contravariant Berezin symbol $\check{\sigma}^{(m)}(A) \in C^{\infty}(M)$ of A is defined by the representation of the operator A as an integral

$$A = \int_M \check{\sigma}^{(m)}(A)(x) P_x^{(m)} \Omega_{\epsilon}^{(m)}(x),$$

if such a representation exists.

- The Toeplitz operator $T_f^{(m)}$ admits such a representation with $\check{\sigma}^{(m)}(T_f^{(m)}) = f$, i.e. the function f is a contravariant symbol of the Toeplitz operator $T_f^{(m)}$. It is not unique.

- The Toeplitz map is surjective $\implies$ every operator has a contravariant symbol,
- on $\mathrm{End}(\Gamma_{hol}(M, L^{(m)}))$ introduce the Hilbert-Schmidt norm

$$\langle A, C \rangle_{HS} = Tr(A^* \cdot C) \ ,$$

- the Toeplitz map $f \to T_f^{(m)}$ and the covariant symbol map $A \to \sigma^{(m)}(A)$ are adjoint:

$$\langle A, T_f^{(m)} \rangle_{HS} = \langle \sigma^{(m)}(A), f \rangle_{\epsilon}^{(m)} \ .$$

- Using this, from the surjectivity of the Toeplitz map the injectivity of the covariant symbol map follows.