DIRAC OPERATORS FROM PRINCIPAL CONNECTIONS.

Andrzej Sitarz

Jagiellonian University, Kraków

22.05.2011, Bayrischzell

1. MOTIVATION
2. THE CLASSICAL CASE
3. THE NONCOMMUTATIVE CASE
4. EXAMPLE: NC TORUS
5. OUTLOOK

THE LANDSCAPE:

- The language of spectral triples (might) describes the geometry of noncommutative manifolds.
- There are interesting examples of noncommutative manifolds: in particular tori and spheres (both θ-deformations and q-deformations)
- Some examples have nice symmetries and an interesting geometry (principal bundles in a general sense)
- The algebraic description of the quantum principal bundles is well known (we have the notion of connections, strong connections etc.)
- The spectral triples do exists for most of the objects mentioned above.

THE QUESTIONS:

- What is the relation between the Dirac operators on the total space of the principal bundle and the quotient space ?
- What is the connection formulated in the language of spectral geometry ?
- What are the conditions that allow us to give the answers ?
  Do we need to work with *real* spectral triples?
  Or just arbitrary spectral triples ?
- Which axioms of Connes' geometry are significant ?
- Is there any link between the spectral constructions and the algebraic ones ?

THE CLASSICAL GEOMETRY OF CIRCLE BUNDLES

We assume that M is a compact odd-dimensional Riemannian spin manifold, on which S^1 acts freely and isometrically. We can view M as total space of S^1 principal bundle over $N = M/S^1$. For simplicity we assume that the lengths of fibres is constant.

The S^1 principal bundle has a unique connection one-form ω such that $\ker \omega$ is orthogonal to the fibres.

The metric on M is completely characterized by the metric (g_N) on N, length of each fibre (l) and the connection ω.

Can we express the Dirac operator on M in terms of these data (g_N, l, ω) ?

Yes: Bernd Amman & Christian Bär (1998)

THE CLASSICAL GEOMETRY OF CIRCLE BUNDLES

The action of S^1 on M induces an action on the spinor bundle S_M, let us call the fundamental vertical vector field X and the associated action of the Lie derivative ∂_X. It leads to the decomposition of the spinor bundle into the eigenspaces of ∂_X:

$$L^2(S_M) = \bigoplus_k V_k.$$

Let $L = M \times_{S^1} \mathbb{C}$ be a complex line bundle over N. There exists a homothety of Hilbert spaces:

$$Q_k : L^2(S_N \otimes L^{-k}) \to V_k.$$

We take a natural connection on L given by ω.

THE CLASSICAL GEOMETRY OF CIRCLE BUNDLES

We define the horizontal Dirac operator on S_M as a unique closed linear operator D_h such that on each V_k is given by $Q_k D_k (Q_k)^{-1}$ where D_k is the twisted Dirac operator on $L^2(S_N \otimes L^{-k})$.
We define the vertical Dirac operator as:

$$D_v = \gamma(\frac{X}{l})\partial_X,$$

Then the Dirac operator on S_M is:

$$D = D_h + \frac{1}{l} D_v - l\frac{1}{4}\gamma(\frac{X}{l})\gamma(d\omega).$$

This is used a tool to obtain the eigenvalues (and their multiplicities) of the Dirac operator on some manifolds, like $\mathbb{C}P^m$, m odd.

THE CLASSICAL GEOMETRY OF CIRCLE BUNDLES

Where does the zero-order term Z comes from ? Omitting Z still provides a Dirac operator of M for the linear connection, which preserves the metric $\tilde{g}$ but has a nonvanishing (in general) torsion. This can be see easily by looking at the Christoffel symbols:

$$-\tilde{\Gamma}^0_{ij} = \tilde{\Gamma}^j_{i0} = \tilde{\Gamma}^j_{0i} = \frac{\ell}{2} d\omega(e_i, e_j),$$
$$\tilde{\Gamma}^0_{i0} = \tilde{\Gamma}^0_{0i} = \tilde{\Gamma}^i_{00} = \tilde{\Gamma}^0_{00} = 0.$$

If, in the latter formula we put $\tilde{\Gamma}^k_{ij} = 0$ whenever one or more of the indices i, j, k is zero, we get a linear connection, which is still compatible with the metric but the components

$$T^0_{ij} = e^0(\nabla_{e_i} e_j - \nabla_{e_j} e_i - [e_i, e_j]) = de^0(e_i, e_j) = \ell\, d\omega(e_i, e_j)$$

of the torsion tensor do not vanish (in general).

THE NONCOMMUTATIVE CASE: PRINCIPAL FIBRE BUNDLES.

DEFINITION

Let H be a unital Hopf algebra and $\mathcal{A}$ be a right H-comodule algebra. We denote by $\mathcal{B}$ the subalgebra of invariant elements of $\mathcal{A}$. We say the $\mathcal{B} \hookrightarrow \mathcal{A}$ is a Hopf-Galois extension iff the canonical map χ :

$$\mathcal{A} \otimes_{\mathcal{B}} \mathcal{A} \ni a' \otimes a \mapsto \chi(a' \otimes a) = a' a_{(0)} \otimes a_{(1)} \in \mathcal{A} \otimes H, \quad (1)$$

is an isomorphism.

In the purely algebraic settings the connections are defined as right-colinear maps from the Hopf algebra H to the first order universal differential calculus $\Omega^1_u(\mathcal{A})$ over $\mathcal{A}$.

THE NONCOMMUTATIVE CASE: STRONG CONNECTIONS.

DEFINITION

We say that a right H-colinear map $\omega : H \to \Omega^1_u(\mathcal{A})$ is *a strong* universal connection if the following conditions hold:

$$\omega(1) = 0, \quad \Delta_R \circ \omega = (\omega \otimes \mathrm{id}\,) \circ \mathrm{Ad}_R,$$

$$d_u(a) - a_{(0)}\omega(a_{(1)}) \in \left(\Omega^1_u(\mathcal{B})\right)\mathcal{A}, \quad \forall a \in \mathcal{A},$$

$$(m \otimes \mathrm{id}\,) \circ (\mathrm{id}\, \otimes \Delta_R) \circ \omega = 1 \otimes (\mathrm{id}\, - \varepsilon).$$

It is possible to extend this definition of connections for nonuniversal differential calculi, however only after requiring certain compatibility conditions between the differential calculus on $\mathcal{A}$ and a given calculus over the Hopf algebra H.

THE NONUNIVERSAL CALCULUS.

Choosing a subbimodule $\mathcal{N} \subset \mathcal{A} \otimes \mathcal{A}$ we have an associated first order differential calculus over $\mathcal{A}$. If the canonical map χ maps $\mathcal{N}$ to $\mathcal{A} \otimes Q$, where $Q \subset \ker\varepsilon \subset H$ is an *Ad*-invariant vector space then it is possible to use a calculus over H determined by Q using the Woronowicz construction of bicovariant calculi.

From coaction to action...

DEFINITION

For a $U(1)$ Hopf-Galois extension $\mathcal{B} \hookrightarrow \mathcal{A}$ we say that $\omega : \mathbb{Z} \to \Omega^1_u(\mathcal{A})$ is *a strong* universal connection iff:

$$\omega(0) = 0, \qquad g \triangleright \omega = \omega, \ \ \forall g \in U(1),$$

$$d_u(a) - a\omega(k) \in \Big(\Omega^1(\mathcal{B})\Big)\,\mathcal{A}, \ \ \forall a \in \mathcal{A}^{(k)},$$

$$m \circ (\mathrm{id} \otimes \pi_n)\omega(k) = \delta_{kn} - \delta_{n0}.$$

THE NONCOMMUTATIVE CASE: THE SETUP.

We assume that there exists a real spectral triple over $\mathcal{A}$, which is $U(1)$ equivariant, that is the action of $U(1)$ extends to the Hilbert space and the representation, the Dirac operator and the reality structure are $U(1)$ equivariant. We denote by π the representation of $\mathcal{A}$ on $\mathcal{H}$, D is the Dirac operator and J the reality structure. Let δ be the operator on $\mathcal{H}$ which generates the action of $U(1)$ on the Hilbert space. The $U(1)$ equivariance of the reality structure and D means that:

$$J\delta = -\delta J, \quad D\delta = \delta D,$$

whereas the equivariance of the representation is:

$$[\delta, \pi(a)] = \pi(\delta(a)), \;\; \forall a \in \mathcal{A},$$

where $\delta(a)$ is the derivation of a arising from the $U(1)$ action. For simplicity, we take the dimension of the spectral triple over $\mathcal{A}$ to be odd, then the dimension of spectral triple over $\mathcal{B}$ is even (specifically: top dimension 3 and the dimension of the quotient 2.)

PROJECTABLE TRIPLES.

We define the space $\mathcal{H}_k \subset \mathcal{H}$, $k \in \mathbb{Z}$, to be a subspace of vectors homogeneous of degree k in $\mathcal{H}$ that is, they are eigenvectors of δ of eigenvalue k.

DEFINITION

We say that the $U(1)$ equivariant spectral triple $(\mathcal{A}, D, J, \mathcal{H}, \delta)$ is projectable along the fibres if there exists an operator Γ, a $\mathbb{Z}_2$ grading of the Hilbert space $\mathcal{H}$, which satisfies the following conditions:

$$\forall a \in \mathcal{A} : [\Gamma, \pi(a)] = 0,$$
$$\Gamma J = -J\Gamma, \quad \Gamma\delta = \delta\Gamma, \quad \Gamma^* = -\Gamma,$$

and the *horizontal Dirac operator*: $D_h = \frac{1}{2}\Gamma[D, \Gamma]$, generates the same bimodule of one-forms over $\mathcal{B}$ as D:

PROJECTABLE TRIPLES.

The *horizontal Dirac operator* generates the same bimodule of one-forms over $\mathcal{B}$ as D:

$$[D_h, b] = [D, b], \;\; \forall b \in \mathcal{B}.$$

Let D_v denote the vertical part of the Dirac operator:

$$D_v = \frac{1}{\ell}\Gamma\delta.$$

DEFINITION

We say that the $U(1)$ bundle has fibre of constant length (taken to be $2\pi\ell$) if

$$Z = D - D_h - D_v$$

is an operator of zero order, which commutes with the elements from the commutant:

$$[Z, Ja^* J^{-1}] = 0, \;\; \forall a \in \mathcal{A}.$$

SPECTRAL TRIPLE(S) ON THE BASE SPACE.

THEOREM

The data $(\mathcal{B}, \mathcal{H}_0, D_0, \gamma_0, j_0)$ *gives a real spectral triple of KR-dimension* 2 *over* $\mathcal{B}$. *For* $k \neq 0$, $(\mathcal{B}, \mathcal{H}_k, D_k, \gamma_k)$ *are twisted spectral triples over* $\mathcal{B}$, *which are pairwise real:*

$$\gamma_k D_k = -D_k \gamma_k \quad j_k D_k = D_{-k} j_k,$$
$$j_k \gamma_k = -\gamma_{-k} j_k.$$

DEFINITION

We say that the first order differential calculus over $\mathcal{A}$ given by the Dirac operator D is compatible with the standard de Rham calculus over $U(1)$ if the following holds:

$$\forall p_i, q_i \in \mathcal{A} : \sum_i p_i [D, q_i] = 0 \Rightarrow \sum_i p_i \delta(q_i) = 0.$$

Spectral triple(s) on the base space.

Lemma

The image by the canonical map of the ideal defining the first order differential calculus is in $\mathcal{A} \otimes (\ker \varepsilon)^2$.

Definition

We say that $\omega \in \Omega^1_D(\mathcal{A})$ is a strong connection for the $U(1)$ bundle $\mathcal{B} \hookrightarrow \mathcal{A}$ if the following conditions hold:

$$[\delta, \omega] = 0, \quad \textit{(U(1) invariance of } \omega)$$

$$\text{if } \omega = \sum_i p_i [D, q_i] \text{ then } \sum_i p_i \delta(q_i) = 1, \quad \textit{(vertical field condition)},$$

$$\forall a \in \mathcal{A} : [D, a] - \delta(a)\omega \in \Omega^1_D(\mathcal{B})\mathcal{A}, \quad \textit{(strongness)}$$

STRONG CONNECTIONS AND TWISTING...

THEOREM

The map:

$$\nabla_\omega : \mathcal{A}^{(k)} \ni a \mapsto [D, a] - na\omega \in \Omega^1_D(\mathcal{B})\mathcal{A}^{(k)},$$

defines a $\Omega^1_D(\mathcal{B})$-valued connection (covariant derivative) over $\mathcal{A}^{(k)}$.

LEMMA

The following defines on a dense domain V_M in $\mathcal{H}_k$ an operator D_M:

$$D_M(h \otimes_{\mathcal{B}} m) = (D_0 h \otimes_{\mathcal{B}} m) + h\nabla(m),$$

where the last product is defined in the following way:

$$h(\rho \otimes_{\mathcal{B}} m) = (j_0 \overline{\rho} j_0^{-1}) h \otimes_{\mathcal{B}} m,$$

where $\overline{\rho}$ denotes the involution on the space of one forms.

COMPATIBILITY BETWEEN DIRAC AND CONNECTION

DEFINITION

We say that the connection ω is compatible with the Dirac operator D if both D_ω and D_h coincide on a dense subset of $\mathcal{H}$.

We have (almost) everything to go and look at examples.

THE EASY EXAMPLES

Apart from the classical case, there are some easy examples that work: basically all that arise from noncommutative tori (or θ-type) deformations. For example S^3_θ is a noncommutative S^1 bundle over the classical sphere.

The NC torus algebra:

$$U_1 e_{k,l,m} = e_{(k+1),l,m},$$
$$U_2 e_{k,l,m} = e^{2\pi k\theta_{21}} e_{k,(l+1),m},$$
$$U_3 e_{k,l,m} = e^{2\pi(k\theta_{31}+l\theta_{32})} e_{k,l,(m+1)},$$

EXAMPLE: NC TORUS.

LEMMA

The differential calculus generated by D_A satisfies the compatibility condition if $A_3 = 0$.

LEMMA

If $A_3 = 0$ and $\Gamma = \sigma^3$, then the projection of D_A onto $\mathbb{T}^2_\theta$ gives a real spectral triple over the two-dimensional torus and the differential calculi over $\mathbb{T}^2_\theta$ have the required property.

NC TORUS: CONNECTIONS AND DIRACS

LEMMA

A $U(1)$ connection over $\mathbb{T}^3_\theta$ (for a choice of the $U(1)$ action) is a one-form:

$$\omega = \sigma^3 + \sigma^2 \omega_2 + \sigma^1 \omega_1,$$

where $\omega^1, \omega^2 \in \mathbb{T}^2_\theta$ are $U(1)$ invariant elements of the algebra $\mathbb{T}^3_\theta$. Every such connection is strong.

LEMMA

For any antiselfadjoint connection ω the associated Dirac operator D_ω has the form:

$$D_\omega = D - (\sigma^2 J \omega_2 J^{-1} + \sigma^1 J \omega_1 J^{-1})\delta_3.$$

BABY GAUSS-BONNET

Actually one can take an easier example: 2-dimensional noncommutative torus, which can be treated as an S^1 bundle over S^1 (in a noncommutative sense). One can construct connections, Diracs - a Dirac compatible with a connection woul look like:

$$D = \sigma^1 f(U^o)\delta_U + \sigma^2\delta_V,$$

... and almost immediately you get the Gauss-Bonnet for this family of Diracs on the NC torus. Why? It is just the same as a Dirac on the *commutative* torus (say, generated by U^o, V.)

THE (OPEN) QUESTIONS

- Can one do it for the q-case ?
- Can one do it for principal fibre bundles with some other groups?
- Can one show a real Gauss-Bonnet theorem for the torus ?
- Where comes the torsion ?

THANK YOU!