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QUANTUM POINCARÉ COVARIANCE OF NONCOMMUTATIVE QFT AND BRAIDING

1. Introduction
2. NC free quantum fields and braided products
3. Quantum Poincaré symmetries of NC free quantum fields
4. Braided field commutators and braided locality of NC quantum free fields
5. Braiding and κ-deformed NC free quantum fields
6. Final remarks

Based on: M. Woronowicz + J.L., J.Phys.A45, (2012), arXiv: 1105.3612v3 [hep-th]; JW 2011 Workshop, Proceedings,

MAIN QUESTION:

How to introduce the free quantum fields covariant under deformed quantum Poincaré symmetries?

How to obtain deformed quantum free fields with covariant c-number commutator?

MAIN TOOL:

Necessity of introducing the intertwiner $R_{21} = R_{(2)} \otimes R_{(1)}$ by using universal R-matrix R ($R \equiv R_{12} = R_{(1)} \otimes R_{(2)} \equiv \equiv \sum\limits_J R^J_{(1)} \otimes R^J_{(2)}$)

$$U \otimes V \longleftrightarrow R_{21} \circ (V \otimes U) = (R_{(2)}V) \otimes (R_{(1)}U)$$

Covariant braided commutator:

$$[A, B] \rightsquigarrow [A, B]^{BR} = AB - R_{21}(BA)$$

1. INTRODUCTION

Dopplicher - Fredenhagen - Roberts (1994-5)	Due to quantum effects the space-time can not be measurable in classical way - is becoming noncommutative
DFR: canonical space-time algebra	$[\widehat{x}_\mu, \widehat{x}_\nu] = \frac{i}{\kappa^2}\widehat{\Sigma}_{\mu\nu} \quad [\widehat{\Sigma}_{\mu\nu}, \widehat{x}_\rho] = 0$ Effectively one inserts $\widehat{\Sigma}_{\mu\nu} = \underset{\substack{\uparrow \\ \text{numerical tensor}}}{\theta^{(0)}_{\mu\nu}} \cdot \mathbb{1}$

In general case:

$$[x_\mu, x_\nu] = 0 \Rightarrow [\hat{x}_\mu, \hat{x}_\nu] = \frac{i}{\kappa^2}\,\theta^{(0)}_{\mu\nu}(\kappa\hat{x}) = \theta^{(0)}_{\mu\nu} + \frac{1}{\kappa}\theta^{(1)\ \rho}_{\mu\nu}\hat{x}_\rho + \cdots$$

κ-fundamental mass parameter; $\theta^{(0)}_{\mu\nu}, \theta^{(1)\ \rho}_{\mu\nu}$ - dimensionless.

Two approaches to symmetries of the theories with noncommutative space-time $\widehat{x}_\mu$:

a) Classical Poincaré symmetries preserved, and noncommutativity of $\widehat{x}_\mu$ introduces the breaking of classical Poincaré symmetries (DFG (1994-5). This approach is used in the majority of papers on NC QFT starting with Filk (1996), modification of Wightman framework (Grosse, Lechner (2008); Buchholz, Summers (2008) ...). In particular

- **for canonical deformation ($\theta_{\mu\nu}(\widehat{x}) = \theta^{(0)}_{\mu\nu} = const.$)** $O(3,1) \longrightarrow O(2) \otimes O(1,1)$ **(Lorentz symmetry breaking)**

- **for Lie-algebraic deformation ($\theta_{\mu\nu}(\widehat{x}) = \theta^{(1)\ \rho}_{\mu\nu}\widehat{x}_\rho$) both Lorentz and classical translation symmetry broken.**

The parameters $\theta^{(0)}_{\mu\nu}, \theta^{(1)\,\rho}_{\mu\nu}, \ldots$ appear as the Poincaré symmetry breaking parameters, reducing Poincaré symmetry to some unbroken algebra (in general case any Poincaré symmetry can be broken).

Remark: As remedy for regaining broken symmetries one considers $\theta^{(0)}_{\mu\nu}, \theta^{(1)\,\rho}_{\mu\nu}$ as new tensorial commutative coordinates

$$x_\mu \longrightarrow (\widehat{x}_\mu, \theta^{(0)}_{\mu\nu}, \theta^{(1)\,\rho}_{\mu\nu}, \ldots)$$

Then one extracts physically meaningful quantities by averaging over arbitrary values of $\theta^{(0)}_{\mu\nu}, \theta^{(1)\,\rho}_{\mu\nu}$ (DFR (1994-5), Carlson et all (2002), Okumura (2003), Dąbrowski, Piacitelli (2007-2011))

Problem: the meaning of additional coordinates.

b) One considers the space-time noncommutativity relations as being the same in all deformed Poincaré symmetry frames

Classical Poincaré Hopf algebra (with primitive, symmetric coproducts) $\xrightarrow[\text{deformed}]{}$ Quantum Poincaré Hopf algebra (with nonsymmetric coproduct described by universal R-matrix)

For some noncommutative space-time algebras one can find the deformation of Poincaré symmetries providing covariance

Canonical DSR deformation: NC relation (in 1994) $\longrightarrow$ canonically deformed Poincaré algebra (in 2004)

In opposite direction one can look for the NC space-time algebra which is covariant under given quantum Poincaré algebra

κ-deformation: κ-deformed Poincaré algebra (in 1991) $\longrightarrow$ κ-deformed Minkowski space (in 1993-94)

Advantage of the approach with quantum symmetries: the NC space-time structures covariant under quantum Hopf-algebraic symmetries are less arbitrary

$$\begin{tikzcd}
\text{restrictions on quantum deformations of Poincaré symmetries} \arrow[r] & \text{restrictions on possible deformed Minkowski space-times} \\
\text{Partial classifications: S.L. Woronowicz + S. Zakrzewski (1996)} \arrow[u, Rightarrow] \arrow[r] & \text{Partial results - for Lie-algebraic deformations e.g. M. Woronowicz, J.L. (2006)} \arrow[u, Rightarrow]
\end{tikzcd}$$

If we introduce NC quantum fields the noncommutativity of space-time ($e^{ipx} \to e^{ip\hat{x}} \in \widehat{\mathcal{M}}$) is not sufficient - to get fields covariant under quantum Poincaré symmetries one should deform as well the field oscillators algebra $\mathcal{H}$ and introduce braiding between $\widehat{\mathcal{M}}$ and $\widehat{\mathcal{H}}$. We present the scheme with complete deformed covariance.

2. NONCOMMUTATIVE FREE QUANTUM FIELDS AND BRAIDED PRODUCT

From classical to deformed free fields:

$$\varphi(x) = \frac{1}{(2\pi)^3} \int d^4p \, \delta(p^2 - m) e^{ipx} \cdot A(p) \Rightarrow$$

$$\Rightarrow \widehat{\phi}(\hat{x}) = \frac{1}{(2\pi)^3} \int d^4p \, \delta(\underset{\substack{\nearrow \\ \text{deformed mass} \\ \text{Casimir}}}{C_2(p)}) \underset{\substack{\uparrow \\ \text{noncommutative} \\ \text{plane waves}}}{e^{ip\hat{x}}} \underline{\otimes} \underset{\substack{\nwarrow \\ \text{deformed} \\ \text{oscillators}}}{\widehat{A}(p)}$$

For twisted deformations - mass Casimir not deformed. For general deformation - can be modified mass Casimir as well introduced quantum-covariant volume form.

Deformed NC quantum field $\widehat{\phi}$ belongs to the braided tensor product

$$\widehat{\phi} \in \widehat{\mathcal{M}} \underline{\otimes} \widehat{\mathcal{H}} \qquad \widehat{f} \in \widehat{\mathcal{M}} \quad \widehat{h} \in \widehat{\mathcal{H}}$$

$\widehat{\mathcal{M}}$ - algebra of functions on noncommutative space-time
$\widehat{\mathcal{H}}$ - algebra of deformed field oscillators

We have infinite sum (continuous integral) of tensor products of basis functions $e^{ip\widehat{x}}$ and oscillators $\widehat{A}(p)$; if we consider the product of two fields $\widehat{\phi}$ one arrives at need of a braid factor

$$e^{ip\widehat{x}} \underline{\otimes} \widehat{A}(q) = \Psi_{\mathcal{M},\mathcal{H}}(\widehat{A}(q) \underline{\otimes} e^{ip\hat{x}}) \neq \widehat{A}(q) \underline{\otimes} e^{ip\widehat{x}} \quad \Leftarrow \quad \text{braiding } (p \neq q)$$

where

$$\Psi_{\mathcal{M},\mathcal{H}}(\widehat{h} \underline{\otimes} \widehat{f}) = (R_{(2)} \widehat{h}) \underline{\otimes} (R_{(1)} \widehat{f})$$

We need braided product of NC quantum fields $\widehat{\phi}$

Three algebras providing three sources of deformations:

(a) Algebra $\widehat{\mathcal{M}}(e^{ip\widehat{x}}, \cdot)$ of basic functions on noncommutative space-time - can be represented homomorphically by so-called Weyl map $\widehat{\mathcal{M}}(e^{ip\widehat{x}}, \cdot) \xrightarrow{W} \mathcal{M}(e^{ipx}, \star)$

$$e^{ip\widehat{x}}\, e^{iq\widehat{x}} = m(e^{ip\widehat{x}} \otimes e^{iq\widehat{x}}) \xrightarrow{W} m_\star(e^{ipx} \otimes e^{iqx}) = e^{ipx} \star e^{iqx}$$

We get the Weyl map of noncommutative fields ($C_2(p) = p^2 - m^2$)

$$\widehat{\phi}(\widehat{x}) \xrightarrow{W} \widehat{\varphi}(x) = \frac{1}{(2\pi)^4} \int d^4p\, \delta(p^2 - m^2) e^{ipx}\, \underset{\substack{\uparrow \\ \text{deformed}}}{\widehat{A}(p)} \in \mathcal{H}$$

because

$$\widehat{\mathcal{M}} \underline{\otimes} \widehat{\mathcal{H}} \xrightarrow{W} \underset{\substack{\uparrow \\ \text{classical functions}}}{\mathbb{1}} \otimes \widehat{\mathcal{H}} \simeq \widehat{\mathcal{H}}$$

In order to consider multilocal products of fields $\widehat{\phi}(x)$ the Weyl map should be generalized to bilocal products

$$e^{ip\widehat{x}}\, e^{iq\widehat{y}} \xrightarrow{W} e^{ipx} \star e^{iqy}$$

(b) Algebra $\widehat{\mathcal{H}}(\widehat{A}(p),\cdot)$ of the deformed field oscillators.

Standard field oscillators algebra $\widehat{\mathcal{H}}^{(0)}(A(p),\cdot)$, which can be written in covariant form as follows

$$\delta(p^2-m^2)\delta(q^2-m^2)[A(p),A(q)]=\epsilon(p_0)\delta(p^2-m^2)\delta^{(4)}(p+q)$$

is not covariant under any quantum symmetry characterized by nonsymmetric coproduct $\Delta(\widehat{g})$ of the symmetry generators.

If deformed Poincaré symmetry is described by the universal R-matrix intertwining the coproduct $\Delta=\Delta_{(1)}\otimes\Delta_{(2)}$ with flipped coproduct $\Delta_{21}=\Delta_{(2)}\otimes\Delta_{(1)}$

$$\Delta_{21}=R\Delta R^{-1} \ \leftrightarrow \ \Delta_{21}R=R\Delta \qquad R=R_{(1)}\otimes R_{(2)}$$

the deformed algebra of oscillators $a(\vec{p})=\widehat{A}(\vec{p},\omega(\vec{p}))$, $a^{+}(\vec{p})=\widehat{A}(-\vec{p},-\omega(\vec{p}))$ has been given by the relations

$$a^{+}(\vec{p})a(\vec{q})-(R_{(2)}\blacktriangleright a(\vec{q}))(R_{(1)}\blacktriangleright a^{+}(\vec{p}))=\omega_q(\vec{p})\delta^3(\vec{p}-\vec{q})$$
$$a(\vec{p})a(\vec{q})-(R_{(2)}\blacktriangleright a(\vec{q}))(R_{(1)}\blacktriangleright a(\vec{p}))=0 \qquad +\,\mathrm{H.C.}$$

The action $\blacktriangleright$ of deformed generators $\widehat{g} \in U_q(\mathcal{P}_{3,1})$ on the deformed Poincaré algebra module $\widehat{h} \in \widehat{\mathcal{H}}$ is given by

$$\widehat{g} \blacktriangleright \widehat{h} = ad_g\, \widehat{h} = \Sigma g_{(1)} \widehat{h} S(g_{(2)}) \qquad \Delta(\widehat{g}) = g_{(1)} \otimes g_{(2)}$$

(generalization of commutator)

If written in fourdimensional covariant form it can be written in general case as follows

$$\delta(C_2(p))\delta(C_2(q))(A(p)A(q)) - R_{(2)} \blacktriangleright A(q)R_{(1)} \blacktriangleright A(p) =$$
$$= \epsilon(p_0)C_2(p)\delta^{(4)}(p\widetilde{+}q)$$

where $p\widetilde{+}q = \Delta_{(1)}(p)\Delta_{(2)}(q)$ from $\Delta(P) = \Delta_{(1)}(P) \otimes \Delta_{(2)}(P)$ (i.e. $P_\mu \otimes 1 \leftrightarrow p_\mu$, $1 \otimes P_\mu \leftrightarrow q_\mu$; summ of terms in the coproducts in our notation is supressed)

(c) Algebra $\Phi(\widehat{\phi},\bullet)$ of the NC quantum fields $\widehat{\phi}$ with braided multiplication rule - we multiply the NC fields $\widehat{\phi}$ with taking into account the braid $\Psi_{\mathcal{M},\mathcal{H}}$:

$$\widehat{\phi}(\widehat{x})\bullet\widehat{\phi}(\widehat{y}) \equiv m_{\mathcal{M}\underline{\otimes}\mathcal{H}}[\widehat{\phi}(\widehat{x})\underline{\otimes}\widehat{\phi}(\widehat{y})] =$$

$$= (m\otimes m)\circ(id\otimes\Psi_{\mathcal{M},\mathcal{H}}\otimes id)[\widehat{\phi}(\widehat{x})\underline{\otimes}\widehat{\phi}(\widehat{y})]$$

For basic vectors in the expansion of $\widehat{\phi}$ we get:

$$m_{\mathcal{M}\underline{\otimes}\mathcal{H}}[(e^{ip\hat{x}}\underline{\otimes}\underset{\Psi_{\mathcal{M},\mathcal{H}}}{\underleftrightarrow{\widehat{A}(p))\underline{\otimes}(e^{iq\hat{y}}}}\underline{\otimes}\widehat{A}(q))] = m(e^{ip\hat{x}}\otimes R_{(2)}\blacktriangleright e^{iq\hat{y}})$$

$$\underline{\otimes}m(R_{(1)}\blacktriangleright\widehat{A}(p))\otimes\widehat{A}(q)$$

Using the Weyl map the first factor becomes commutative function of space-time, and second belongs to algebra $\widehat{\mathcal{H}}$ ⇒ after Weyl map the braided products of fields belong to the algebra $\widehat{\mathcal{H}}$. Important: Weyl map should be applied after performing the braid $\Psi_{\mathcal{M},\mathcal{H}}$

There are two $U_q(\mathcal{P}_{3;1})$ modules $\widehat{\mathcal{M}}$ and $\widehat{\mathcal{H}}$ and two actions $\triangleright$ and $\blacktriangleright$ of generators $\widehat{g}$ on $\widehat{\mathcal{M}}$ and $\widehat{\mathcal{H}}$. On $\widehat{f} \in \widehat{\mathcal{M}}$ one gets the differential realization

noncommutative space-time

$$\widehat{g} \triangleright \widehat{f} = \widehat{D}(\widehat{g})\widehat{f} \qquad \widehat{g} \in U_q(\mathcal{P}_{3;1})$$

$$\Uparrow$$

noncommutative vector field

After Weyl map $\widehat{D}(\widehat{g})$ becomes a differential operator $D(\widehat{g})$ on commutative Minkowski space.

If we act on tensor product $\widehat{\mathcal{M}} \otimes \widehat{\mathcal{H}}$ we should assume trivial action on "wrong" part of the tensor product

$$\widehat{g} \blacktriangleright \widehat{f} = \varepsilon(\widehat{g})\widehat{f} \qquad \widehat{g} \triangleright \widehat{h} = \varepsilon(\widehat{g})\widehat{h} \quad \begin{array}{l} \widehat{f} \in \widehat{\mathcal{M}} \\ \widehat{h} \in \widehat{\mathcal{H}} \end{array}$$

where $\varepsilon(1) = 1$; otherwise zero. These properties are needed if we calculate $\widehat{g} \blacktriangleright \widehat{\phi}(\widehat{x})$ and $\widehat{g} \triangleright \widehat{\phi}(x)$ and use Hopf-algebraic actions

$$\widehat{g} \blacktriangleright (e^{ip\widehat{x}} \otimes a(\vec{q})) = (\widehat{g}_{(1)} \blacktriangleright e^{ip\widehat{x}})\underline{\otimes}(\widehat{g}_{(2)} \blacktriangleright a(\vec{q}))$$

But $\Delta(\widehat{g}) = g_{(1)} \otimes g_{(2)} = \Delta^{(0)}(\widehat{g})$ + terms not containing terms $(\widehat{g} \otimes \mathbb{1})$ and $(\mathbb{1} \otimes \widehat{g})$

so the only term contributing to $g_{(1)} \blacktriangleright e^{ipx}$ is $\mathbb{1} \otimes \widehat{g}$, i.e. $g_{(1)} = \mathbb{1}$

$$\widehat{g} \blacktriangleright e^{ip\widehat{x}} \underline{\otimes} a(\vec{q}) = (\mathbb{1} \blacktriangleright e^{ip\widehat{x}}) \underline{\otimes} (g \blacktriangleright a(\vec{q})) = e^{ip\widehat{x}} \underline{\otimes} (g \blacktriangleright a(\vec{q}))$$

i.e. functions $\widehat{f}$ behave as numbers (scalar spectators) under the action $\blacktriangleright$ of symmetry generators.
Similarly only $\widehat{g} \otimes \mathbb{1}$ $(g_{(2)} = \mathbb{1})$ contributes to

$$\widehat{g} \triangleright (e^{ip\widehat{x}} \underline{\otimes} a(\vec{q})) = (\widehat{g} \triangleright e^{ip\widehat{x}}) \underline{\otimes} a(\vec{q})$$

However both actions $\blacktriangleright$ and $\triangleright$ occur in the definition of covariant braid factor between $\widehat{\mathcal{M}}$ and $\widehat{\mathcal{H}}$

$$\Psi_{\mathcal{M},\mathcal{H}}(\widehat{h} \underline{\otimes} \widehat{f}) = (R_{(2)} \triangleright \widehat{f}) \underline{\otimes} R_{(1)} \blacktriangleright \widehat{h}$$

3. QUANTUM POINCARÉ SYMMETRIES OF NONCOMMUTATIVE FREE QUANTUM FIELDS

a) Classical quantum free fields (scalar case)

The Poincaré covariance is given by formula

$$U(\Lambda, a)\varphi(x)U^{-1}(\Lambda, a) = \varphi(\Lambda x + a)$$

In infinitesimal form it takes shape of generalized Heisenberg equations, with translational and Lorentz sector

$$g \underset{\uparrow}{\blacktriangleright} \varphi \;\equiv\; [g, \varphi] \;=\; -g \underset{\uparrow}{\triangleright} \varphi \;\equiv\; -D^{(0)}(g)\varphi \qquad (g = \underset{\uparrow}{P_\mu}, \underset{\uparrow}{M_{\mu\nu}})$$

QM action on
algebra $\widehat{\mathcal{H}}^{(0)}$ of
free field oscillators

classical action on functions in $\varphi(x)$
$D^{(0)}(P_\mu) = i\partial_\mu$
$D^{(0)}(M_{\mu\nu}) = i(x_\mu\partial_\nu - x_\nu\partial_\mu)$

undeformed
Noether charges
$(g \in \widehat{\mathcal{H}}^{(0)})$

The standard Leibniz rule describes the action on the product of fields corresponding to $\Delta^{(0)}(g) = g \otimes \mathbb{1} + \mathbb{1} \otimes g$

$$D^{(0)}(g) \triangleright (\varphi(x)\varphi(y)) = (D^{(0)}(g)\varphi(x))\varphi(y) + \varphi(x)(D^{(0)}(g)\varphi(y))$$

b) Noncommutative deformed quantum free field $\hat{\phi} \in \widehat{\mathcal{M}} \otimes \widehat{\mathcal{H}}$

Both actions are modified ($\widehat{g}$ - deformed generators)

in $\widehat{\mathcal{H}}$: $\quad \widehat{g} \blacktriangleright \hat{\phi}(\hat{x}) = (\mathbb{1} \otimes adj_{\widehat{g}})\widehat{\phi}(x) \qquad adj_{\widehat{g}}\hat{h} = \underset{\Uparrow}{\widehat{g}_{(1)}\hat{h}}S(\widehat{g}_{(2)})$

in $\widehat{\mathcal{M}}$: $\quad \widehat{g} \triangleright \widehat{\phi}(\hat{x}) = (\widehat{D}(\widehat{g}) \otimes \mathbb{1})\widehat{\phi}(\hat{x})$

quantum adjoint

The modification of the action in module $\widehat{\mathcal{H}}$ is determined by the deformed coproduct and deformed antipode.

The action $\widehat{D}(\widehat{g})$ is adjusted in such a way that the basic covariance relation is valid

$$\widehat{g} \triangleright \widehat{\phi}(\hat{x}) = S(\widehat{g}) \blacktriangleright \widehat{\phi}(\hat{x})$$

deformed generalized
Heisenberg equations

One can introduce as well the third action $\widetilde{\triangleright}$ of the deformed generators $\widehat{g}$ on NC quantum fields $\widehat{\phi}$ (Fiore 2008)

$$\widehat{g}\,\widetilde{\triangleright}(f \otimes h) = (\widehat{g}_{(1)} \triangleright f) \otimes (\widehat{g}_{(2)} \blacktriangleright h)$$

where $f \in \widehat{\mathcal{M}}$, $h \in \widehat{\mathcal{H}}$ (e.g. $f = e^{ip\hat{x}}, \quad h = a(p)$)

Applying covariance condition one gets

$$(\widehat{g}_{(1)} \triangleright f) \otimes h = f \otimes S(\widehat{g}_{(1)})h$$

Further

$$\widehat{g} \,\widetilde{\triangleright}(f \otimes h) = f \otimes S(\widehat{g}_{(1)})\, g_{(2)} h = \in (g) f \otimes h$$

and we get the formula expressing quantum covariance

$$\widehat{g} \;\widetilde{\triangleright}\; \widehat{\phi} = \in (g)\widehat{\phi}$$

This alternative approach leads to trivialization of braiding $(R_{(1)} \;\widetilde{\triangleright}\; \widehat{\phi})(R_{(2)} \;\widetilde{\triangleright}\; \widehat{\phi})$, because $R_{(1,2)} \;\widetilde{\triangleright}\; \widehat{\phi} = \in (R_{(1,2)})\widehat{\phi} = \widehat{\phi}$ (Fiore 2010)

If we consider the fields $\widehat{\phi}(\widehat{x})$ on noncommutative space-time the differential realization $\widehat{D}(\widehat{g})$ is defined on noncommutative Minkowski space; if after Weyl map we consider fields $\widehat{\phi}(x)$ we obtain

$$\widehat{g} \triangleright \widehat{\phi}(x) = D(\widehat{g})\widehat{\phi}(x)$$

where $D(\widehat{g})$ is the deformation of the classical differential realization $D^{(0)}(g)$ on classical Minkowski space when $g \;\longrightarrow\; \widehat{g}$

Important class of deformed quantum symmetries: twisted quantum symmetries generated by $F = F_{(1)} \otimes F_{(2)} \subset U(\mathcal{P}_{3,1}) \otimes U(\mathcal{P}_{3,1})$

Subclass: twisting of classical symmetries $\Uparrow$ twist factor

$$\widehat{\mathcal{H}}^{(0)} = (U(\mathcal{P}_{3,1}), m, \eta, \Delta^{(0)}, \varepsilon, S^{(0)}) \xrightarrow{F} \widehat{\mathcal{H}}^{F}(U(\mathcal{P}_{3,1}), m, \eta, \Delta^{F}, \varepsilon, S^{F})$$

Deformed twisted coproducts and antipodes ($g \in U(\mathcal{P}_{3,1})$)

$$\Delta^{F}(g) = F\, \Delta^{(0)}(g) F^{-1} \qquad \Delta^{(0)} = g \otimes \mathbb{1} + \mathbb{1} \otimes g$$

$$S^{F}(g) = v S^{(0)}(g) v^{-1} \qquad v = F_{(1)}\, S^{(0)}(g) F_{(2)}$$

Coassociativity of twisted coproduct ↔ two-cocycle condition:

$$F_{12}(\Delta^{(0)} \otimes id) F = F_{23}(id \otimes \Delta^{(0)}) F \qquad \begin{array}{l} F_{12} = F \otimes \mathbb{1} \\ F_{23} = \mathbb{1} \otimes F \end{array}$$

From twisted coproduct follows the formula for the R-matrix R^{F}

$$R^{F} = F_{21}\, F^{-1} \qquad R^{F} \subset U(\mathcal{P}_{3,1}) \otimes U(\mathcal{P}_{3,1})$$

Important property: for twisted theories the homomorphic Weyl map provides $\star$-product multiplication explicitly

$$f_1(\widehat{x}) \cdot f_2(\widehat{x}) \xrightarrow{W} f_1(x) \star f_2(x) = m[F^{-1} \triangleright (f_1(x) \otimes f_2(x))]$$
$$= (\overline{F}_{(1)} \triangleright f_1(x))(\overline{F}_{(2)} \triangleright f_2(x))$$

We use in $\Delta(F^{-1}) = \overline{F}_{(1)} \otimes \overline{F}_{(2)}$ the standard differential realizations of $g \in U(\mathcal{P}_{3,1})$ in classical space-time.
The $\star$-multiplication is noncommutative, however braided-commutative. One can show that (Aschieri, 2006)

$$f_1 \star f_2 = (R_{(2)} \triangleright f_2)(R_{(1)} \triangleright f_1) \equiv R_{21} \triangleright (f_2 \star f_1)$$

Braided commutativity $\Leftrightarrow$ vanishing braided commutator

$$[f_1, f_2]_\star^{BR} \equiv f_1 \star f_2 - R_{21} \triangleright (f_2 \star f_1) = 0$$

If $f_1 = x_\mu, f_2 = x_\nu$, in such braided form one can put as well the noncommutativity of deformed space-time coordinates.

Example: canonical deformation $([\hat{x}_\mu, \hat{x}_\nu] = i\theta^{(0)}_{\mu\nu} \equiv i\theta_{\mu\nu})$

$$F_\theta = \exp \tfrac{i}{2}\theta^{\mu\nu} P_\mu \otimes P_\nu \;\rightarrow\; \Delta^{(\theta)}(g) = F_\theta \Delta^{(0)}(g)(F_\theta)^{-1} = \Delta^{(\theta)}_{(1)} \otimes \Delta^{(\theta)}_{(2)}$$

Explicitly:

$$\Delta^{(\theta)}(P_\mu) = \Delta^{(0)}(P_\mu)$$

$$\Delta^{(\theta)}(M_{\mu\nu}) = \Delta^{(0)}(M_{\mu\nu}) - \theta^{\rho\sigma}[(\eta_{\rho\mu}P_\nu - \eta_{\rho\nu}P_\mu) \otimes P_\sigma + P_\rho \otimes (\eta_{\sigma\mu}P_\nu - \eta_{\sigma\nu}P_\mu)]$$

Generalized Heisenberg equations (after Weyl map)

$$P_\mu \blacktriangleright \widehat{\phi}(x) \equiv [P_\mu, \widehat{\phi}(x)] = i\partial_\mu \cdot \widehat{\phi}(x) \qquad \text{undeformed}$$

$$M_{\mu\nu} \blacktriangleright \widehat{\phi}(x) \equiv ad_{M_{\mu\nu}}\widehat{\phi}(x) = -D(M_{\mu\nu})\widehat{\phi}(x) \qquad \text{deformed}$$

$$ad_{M_{\mu\nu}}\widehat{\phi}(x) = [M_{\mu\nu}, \widehat{\phi}(x)] + \theta_{[\nu}{}^{\alpha} P_\alpha \widehat{\phi}(x) P_{\mu]} + \theta_{[\mu}{}^{\alpha} P_{\nu]} \widehat{\phi}(x) P_\alpha$$

$$D(M_{\mu\nu}) = -i(x_\mu\partial_\nu - x_\nu\partial_\mu) - \theta_\nu{}^\rho \partial_\rho \partial_\mu - \theta^\rho{}_\mu \partial_\rho \partial_\mu$$

Modified Heisenberg equation in Lorentz sector represents the equality of suitably adjusted deformed actions $\triangleright$ and $\blacktriangleright$.

4. COVARIANT BRAIDED FIELD COMMUTATORS AND BRAIDED LOCALITY OF NONCOMMUTATIVE QUANTUM FREE FIELDS

The quantum-covariant commutator of NC quantum fields is braided

$$[\widehat{\phi}(\widehat{x}), \widehat{\phi}(\widehat{y})]_{\bullet}^{BR} \equiv \widehat{\phi}(\widehat{x}) \bullet \widehat{\phi}(\widehat{y}) - (R_{(2)} \blacktriangleright \widehat{\phi}(\widehat{y}))(R_{(1)} \blacktriangleright \widehat{\phi}(\widehat{x}))$$

Remark: For covariant fields one can replace the actions $\blacktriangleright \longrightarrow \triangleright$

The quantum covariance of braided commutator is obtained in two steps:

i) We show that the product of quantum fields is covariant, i.e.

$$\widehat{g} \blacktriangleright (\widehat{\phi}(\widehat{x}) \bullet \widehat{\phi}(\widehat{y})) = m_{M \otimes H}[\Delta(\widehat{g}) \blacktriangleright (\widehat{\phi}(\widehat{x}) \bullet \widehat{\phi}(\widehat{y}))]$$
$$= (g_{(1)} \blacktriangleright \widehat{\phi}(\widehat{x})) \bullet (g_{(2)} \blacktriangleright \widehat{\phi}(\widehat{y}))$$

This relation is valid if the braid factor $\Psi_{\mathcal{M},\mathcal{H}}$ is given by R_{21}.

For $\widehat{f}\underline{\otimes}\widehat{h} \in \widehat{\phi}(\widehat{x})$ and $\widehat{f}'\underline{\otimes}\widehat{h}' \in \widehat{\phi}(\widehat{y})$ we have

(we denote $\Delta^{(4)}(\widehat{g}) = \widehat{g}_{(1)} \otimes \widehat{g}_{(2)} \otimes \widehat{g}_{(3)} \otimes \widehat{g}_{(4)}$)

$$
\begin{aligned}
&\widehat{g}\blacktriangleright[(\widehat{f}\underline{\otimes}\widehat{h})\bullet(\widehat{f}'\underline{\otimes}\widehat{h}')] = \\
&\qquad =[(\widehat{g}_{(1)}\blacktriangleright\widehat{f})\bullet(\widehat{g}_{(2)}R_{(2)}\blacktriangleright\widehat{f}')]\underline{\otimes}[(\widehat{g}_{(3)}R_{(1)}\blacktriangleright\widehat{h})\cdot(\widehat{g}_{(4)}\blacktriangleright\widehat{h}')] \overset{?}{=} \\
&\qquad \overset{?}{=} [(\widehat{g}_{(1)}\blacktriangleright\widehat{f})(R_{(2)}\widehat{g}_{(3)}\blacktriangleright\widehat{f}')]\otimes[(R_{(1)}\widehat{g}_{(2)}\blacktriangleright\widehat{h})(\widehat{g}_{(4)}\blacktriangleright\widehat{h}')] \\
&\qquad = [(\widehat{g}_{(1)}\blacktriangleright\widehat{f})\cdot(\widehat{g}_{(2)}\blacktriangleright\widehat{h})]\bullet[(\widehat{g}_{(3)}\blacktriangleright\widehat{f}')\cdot(\widehat{g}_{(4)}\blacktriangleright\widehat{h}')] \\
&\qquad = [\widehat{g}_{(1)}\blacktriangleright(\widehat{f}\underline{\otimes}\widehat{h})]\bullet[\widehat{g}_{(2)}\blacktriangleright(\widehat{f}'\underline{\otimes}\widehat{h}')]
\end{aligned}
$$

where $\overset{?}{=}$ follows from the definition of universal R-matrix

$$\sum_{I,J}(\widehat{g}^{I}_{(2)}R^{J}_{(2)} = R^{I}_{(2)}\widehat{g}^{J}_{(3)}) = 0 \quad \Leftrightarrow \quad \widehat{g}_{(2)}R_{(2)} = R_{(2)}\widehat{g}_{(3)} \quad \text{etc.}$$

We also use that $\widehat{\phi}(\widehat{y})$ is covariant what links $\rhd$ and $\blacktriangleright$

ii) We show that the braided commutator of quantum fields is covariant, i.e. we should have

$$\widehat{g}\blacktriangleright[\widehat{\phi}(\widehat{x}),\widehat{\phi}(\widehat{y})]_{\bullet}^{BR} \overset{?}{=} \widehat{g}\blacktriangleright(\widehat{\phi}(\widehat{x})\bullet\phi(\widehat{y})) - R_{21}\cdot\widehat{g}\blacktriangleright(\widehat{\phi}(\widehat{y})\bullet\widehat{\phi}(\widehat{x}))$$

First term follows from the covariance of product of fields, and second term is valid if we have

$$\widehat{g} \blacktriangleright (R_{(2)} \blacktriangleright \widehat{\phi}(\widehat{y})) \bullet (R_{(1)} \blacktriangleright \widehat{\phi}(\widehat{x})) =$$

$$= m_{M\otimes H}[\Delta(\widehat{g})R_{21} \blacktriangleright (\widehat{\phi}(\widehat{y}) \otimes \widehat{\phi}(\widehat{x}))] \overset{?}{=}$$

$$= m_{M\otimes H}[R_{21}\Delta_{21}(\widehat{g}) \blacktriangleright (\widehat{\phi}(\widehat{y}) \otimes \phi(\widehat{x}))]$$

where validity of ? implies the defining relation of the R-matrix

$$\Delta R_{21} = R_{21}\Delta_{21} \longleftrightarrow R\Delta_{21} = \Delta\, R$$

The braided field commutator is covariant for any quasitriangular deformation. Explicitly

$$\widehat{\phi}(\widehat{x}) \bullet \widehat{\phi}(\widehat{y}) = \tfrac{1}{(2\pi)^8} \int d^4p \int d^4q\, \delta(p^2 - m^2)\delta(p^2 - m^2)\cdot$$

$$\cdot m_{M\otimes H}[(\underbrace{e^{ip\widehat{x}} \cdot (R_{(2)} \triangleright e^{iq\widehat{y}}}_{\text{belongs to } \widehat{M}}) \otimes (\underbrace{R_{(1)} \blacktriangleright A(p))A(q)}_{\text{belongs to } \widehat{H}}]$$

If we perform the Weyl map (in twisted case)

$$m_{M\otimes H}(e^{ip\widehat{x}}\cdot(R_{(2)}\triangleright e^{iq\widehat{y}}))\otimes(R_{(1)}\blacktriangleright\widehat{A}(p))\widehat{A}(q)\overset{W}{\simeq}$$

$$\overset{W}{\simeq}(\underbrace{\overline{F}_{(1)}\triangleright e^{ipx})(\overline{F}_{(2)}R_{(2)}\triangleright}_{\text{c-number function}}e^{iqy})\cdot(R_{(1)}\blacktriangleright\underbrace{\widehat{A}(p))\widehat{A}}_{\text{element of algebra } H}(q))$$

we obtain for the braided field commutator after using the Weyl map (twist F arbitrary 2-cocycle)

$$[\widehat{\phi}(\hat{x}),\widehat{\phi}(\hat{y})]^{BR}_{\bullet}\overset{\mathrm{W}}{\simeq}\frac{1}{(2\pi)^8}\int d^4p\int d^4q\,\delta(p^2-m^2)\delta(q^2-m^2)\cdot$$

$$\cdot(\overline{F}_{(1)}\triangleright e^{ipx})(\overline{F}_{(2)}R_{(2)}\triangleright e^{iqy})(R_{(1)}\blacktriangleright\widehat{A}(p))\widehat{A}(q)$$

$$-(\overline{F}_{(1)}\triangleright e^{iqy})(\overline{F}_{(2)}R_{(2)}\triangleright e^{ipx})(R_{(2)}R_{(1)}\blacktriangleright\widehat{A}(q))(R_{(1)}\blacktriangleright\widehat{A}(p))$$

$$F^{-1}=\overline{F}_{(1)}\otimes\overline{F}_{(2)}\qquad R=R_{(1)}\cdot R_{(2)}=F_{21}\,F^{-1}$$

Explicite calculation: the canonical deformation described by twist $F_\theta = e^{\frac{i}{2}\theta^{\mu\nu}P_\mu \otimes P_\nu}$

In such a case one can factorize explicitly the modified commutation relations for deformed field oscillators.

We use

$$P_\mu \triangleright e^{ipx} = p_\mu e^{ipx} \qquad P_\mu \blacktriangleright \widehat{A}(p) = -p_\mu \widehat{A}(p)$$

and one gets

$$R^\theta_{21} \triangleright (e^{ipx} \otimes e^{iqy}) = e^{i\theta^{\mu\nu}p_\mu q_\nu}(e^{ipx} \otimes e^{iqy})$$

$$R^\theta_{21} \blacktriangleright (\widehat{A}(p) \otimes \widehat{A}(q)) = e^{i\theta^{\mu\nu}p_\mu q_\nu}(\widehat{A}(p) \otimes \widehat{A}(q))$$

We obtain:

$$[\widehat{\phi}(\hat{x})), \widehat{\phi}(\hat{y}))]^{BR}_{\bullet} \overset{\mathrm{W}}{\simeq} \frac{1}{(2\pi)^8} \int d^4p \int d^4q \, \delta(p^2 - m^2)\delta(q^2 - m^2) e^{ipx} \, e^{iqy} \cdot [\widehat{A}(p) \star_{\mathcal{H}} \widehat{A}(q) - R^\theta_{21} \blacktriangleright (\widehat{A}(q) \star_{\mathcal{H}} \widehat{A}(p))]$$

if we introduce the following modified multiplication $\star_{\mathcal{H}}$ in $\widehat{\mathcal{H}}$

$$\widehat{A}(p) \star_{\mathcal{H}} \widehat{A}(q) = m \circ F_\theta \blacktriangleright [\widehat{A}(p) \otimes \widehat{A}(q)] = e^{\frac{i}{2}\theta^{\mu\nu} p_\mu q_\nu} \widehat{A}(p) \widehat{A}(q)$$

If we postulate the braided covariant CCR for field oscillators

$$\delta(p^2 - m^2)\delta(q^2 - m^2)[\widehat{A}(p) \star_{\mathcal{H}} \widehat{A}(q) - R_{21} \blacktriangleright \widehat{A}(q) \star_{\mathcal{H}} \widehat{A}(p)] =$$

$$= \epsilon(p_0)\delta(p^2 - m^2)\delta^{(4)}(p+q)$$

one gets the braided local $\bullet$ - commutator for free noncommutative quantum fields

$$[\widehat{\phi}(\hat{x}), \widehat{\phi}(\hat{y})]^{BR}_{\bullet} \overset{\mathrm{W}}{\simeq} i\Delta(x - y; m^2) =$$

$$= -\frac{i}{(2\pi)^3} \int \frac{d^3\vec{p}}{\omega(\vec{p})} \sin[\omega(\vec{p})(x_0 - y_0)] e^{i\vec{p}(\vec{x}-\vec{y})}$$

The multiplication $\bullet$ of fields after Weyl map ($\widehat{\phi}(\widehat{x}) \xrightarrow{W} \widehat{\phi}(x)$) contains nonlocal $\star$-product multiplication in classical Minkowski space determined by F_θ^{-1} and the braid factor $\Psi_{\mathcal{M},\mathcal{H}}$.

5. BRAIDING AND κ-DEFORMED NC FREE QUANTUM FIELDS

Application of presented formalism implies

i) The knowledge of R-matrix. It depends on the choice of basis ↔ determines mass Casimir $C_\kappa(p)$ and $p_0 = \omega_\kappa(\vec{p})$

ii) The knowledge of $\star_\kappa$-product describing Weyl map for functions $\widehat{\mathcal{M}}_\kappa$ on κ-Minkowski space-time

i) R-matrix

$$R = exp(\frac{1}{\kappa} r_1 + \frac{1}{\kappa^2} r_2 + \frac{1}{\kappa^3} r_3 + \ldots)$$

r_1=classical
r-matrix

In standard basis with deformed Lorentz sector (J.L., Nowicki, Ruegg, Tolstoy 1991) one gets (Young, Zegers 2008)

$$r_1 = N_i \wedge P_i$$

$$r_2 = 0$$

$$r_3 = -\tfrac{1}{8}(P_0^2 N_i \wedge P_i + N_i \wedge P_0^2 P_i) - \tfrac{1}{12}(P_j M_{ji} \wedge P_0 P_k + N_k \wedge \vec{P}^2 P_k) + \text{four terms}$$

Bicrossproduct basis (Majid, Ruegg 1994) is better adjusted to the description of κ-deformation via braiding. One performs the basis transformation ($P_0', = P_0, M_i' = M_i$ unchanged)

$$P_i' = e^{-\frac{P_0}{2\kappa}} P_i \qquad N_i' = N_i e^{-\frac{P_0}{2\kappa}} - \frac{\in_{ijk}}{2\kappa} M_j P_k e^{-\frac{P_0}{2\kappa}}$$

One gets $r_1' = r_1$, but

$$r_2' = -(N_i P_0 \wedge P_i + N_i \wedge P_i P_0) - \frac{1}{2} \in_{ijk} \; M_j P_k \wedge P_i \neq 0$$

$$r_3' = r_3 + \ldots$$

It can be checked that the description of exchange relations for 2-particle states in a κ-deformed theory (Young, Zegers 2007) calculated in bicrossproduct basis can be derived from braided commutators of field oscillators with R-matrix given above.

ii) $\star_\kappa$-product and Weyl map

In NC κ-deformed fields one can introduce the NC plane waves with different orderings.

$$e^{ip\widehat{\vec{x}}},\ e^{i\vec{p}\widehat{\vec{x}}}\,e^{-ip_0\widehat{x}_0},\ e^{-\frac{i}{2}p_0\widehat{x}_0}\,e^{i\vec{p}\widehat{\vec{x}}}\,e^{-\frac{i}{2}p_0\widehat{x}_0} \qquad \text{etc.}$$

providing different Weyl maps.

If we use $e^{ip\widehat{x}}$, the $\star_\kappa$-product is obtained by applying the CBH formula

$$[\widehat{x}^\mu, \widehat{x}^\nu] = \frac{i}{\kappa}(\delta^\mu{}_0\,\widehat{x}^\nu - \delta^\nu{}_0\,\widehat{x}^\mu) \Rightarrow e^{ip\widehat{x}}\,e^{iq\widehat{x}} = e^{i\gamma(p,q)\widehat{x}}$$

i.e. we have

$$e^{ipx}\star_\kappa e^{iqx} = e^{i\gamma_\mu(p,q)x}$$

where (Kosinski, J.L., Maslanka 2000) $\gamma_0 = p_0 + q_0$ and

$$\gamma_i(p,q) = \frac{f_\kappa(p_0)e^{\frac{q_0}{\kappa}}p_i + f_\kappa(p_0)q_i}{f_\kappa(p_0+q_0)} \qquad f_\kappa(\alpha) = \frac{\kappa}{\alpha}(1 - e^{-\frac{\alpha}{\kappa}})$$

(J.L., M. Woronowicz, in preparation)

6. FINAL REMARKS

i) One can apply the formulation to more complicated twists, depending on P_k as well as $M_{\mu\nu}$; it may provide also quantum Minkowski spaces with Lie-algebraic commutation relations. (J.L., M.Woronowicz, 2006). Problem: factorization of braided algebra of oscillators not always possible.

ii) For general quasitriangular deformations, characterized by universal R-matrix one can introduce twist in the category of quasi-Hopf algebras, with nontrivial coassociator (Drinfeld 1990; Beggs, Majid (2004); Young, Zegers (2008))

iii) Interesting step to be made: to employ braided free commutators into the perturbative expansion of interacting NC quantum fields