Homotopy algebras and string field theory

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Witten's cubic string field theory

Action

$$S[a] = \frac{1}{2}\omega(Qa, a) + \frac{1}{3}\omega(a * a, a)$$

- $A = \oplus_n A_n$ denotes the state space of the underlying CFT
- conventions on grading: classical string field $a \in A$ carries ghost number zero
- ω is the odd symplectic structure induced by the bpz inner product
- $Q : A \to A$ and $* : A^{\otimes 2} \to A$ satisfy the axioms of a differential graded associative algebra (DGA), i.e.

$$Q^2 = 0 \tag{1}$$

$$Q \circ (\cdot * \cdot) + (Q \cdot) * \cdot + \cdot * (Q \cdot) = 0 \tag{2}$$

$$\cdot * (\cdot * \cdot) + (\cdot * \cdot) * \cdot = 0 \tag{3}$$

A_∞-algebra

more general: [Zwiebach, Gaberdiel 97]

Any consistent classical open string field theory defines an A_∞-algebra on the space of string fields A.

- action:

$$S[a] = \sum_{n=1}^{\infty} \tfrac{1}{n+1}\omega\big(m_n(a^{\otimes n}), a\big)$$

  where

$$m_n : A^{\otimes n} \to A$$

- A_∞-relations:

$$\sum_{i+j+k=n} m_{i+1+k} \circ (1^{\otimes i} \otimes m_j \otimes 1^{\otimes k}) = 0$$

BV structure on moduli spaces

Geometric data [Zwiebach 93]

Geometrically, string vertices represent subspaces of moduli spaces.

- $\mathcal{P}$ denotes the appropriate moduli spaces, e.g.
  - (classical) open SFT: discs
  - classical closed SFT: spheres
  - closed SFT: closed Riemann surfaces of arbitrary genus
  - open-closed SFT: bounded Riemann surfaces with arbitrary number of boundary components and arbitrary genus
- geometric string vertices $\mathcal{V} \in C^\bullet(\mathcal{P})$
- BV structure on $C^\bullet(\mathcal{P})$ induced by sewing punctures
- consistency condition: single cover of moduli spaces via Feynman rules $\Rightarrow$ string vertices satisfy the BV master equation

$$\partial\mathcal{V} + \hbar\Delta\mathcal{V} + \tfrac{1}{2}(\mathcal{V},\mathcal{V}) = 0$$

From geometry to algebra

The conformal field theory of bosonic string theory of matter X^μ and ghosts b, c maps geometric vertices to algebraic vertices [Zwiebach 93, Zwiebach 98].

- operator formalism: construct forms on moduli spaces, e.g. in closed SFT

$$\begin{aligned} C^{(k)}(\mathcal{P}^{g,n}) &\to \mathrm{Hom}(A^{\wedge n}, \mathbb{C}) \\ \mathcal{C} &\mapsto \int_{\mathcal{C}} \Omega^{g,n,(k)} \end{aligned} \tag{4}$$

- BV structure on multilinear maps on A via odd symplectic structure ω [Schwarz 93]

$$\Delta f := \tfrac{1}{2}\mathrm{div} X_f \ , \quad (f,g) := X_f(g)$$

  where

$$i_{X_f}\omega = -df$$

- the map (4) defines a morphism of BV algebras
- $\Rightarrow$ algebraic vertices $\int_{\mathcal{V}} \Omega^{g,n,(0)}$ satisfy the BV master equation on $\mathrm{Hom}(SA, \mathbb{C})$

- solutions to BV master equation on space of multilinear maps are in one-to-one correspondence with algebras over the Feynman transform/cobar transform of some modular/cyclic operad [Barannikov 07]
- the corresponding operads encode the symmetry properties of moduli spaces
- algebras over the Feynman transform/cobar transform of a modular operad/cyclic operad are homotopy algebras

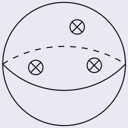
algebraic vertices satisfy BV master equation $\Leftrightarrow$ homotopy algebra

- examples:
  - classical open SFT: A_∞-algebra [Zwiebach, Gaberdiel 97]
  - classical closed SFT: L_∞-algebra [Zwiebach 93]
  - 'classical' open-closed SFT: OCHA [Kajiura, Stasheff 06]
  - quantum closed SFT: loop homotopy algebra [Zwiebach 93, Markl 01]
  - quantum open-closed SFT: QOCHA [M., Sachs 11]
- open-closed algebras lead to relations between open and closed strings

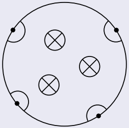
Open-closed homotopy algebra

OCHA: Geometric data of 'classical' open-closed SFT

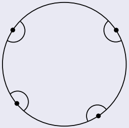
spheres with closed string punctures



discs with open and closed punctures



discs with only open string punctures



- operations on surfaces:
  - sewing an open string puncture on one surface with an open string puncture on another surface ↔ odd Poisson bracket (anti bracket) of open strings
  - sewing a closed string puncture on a sphere with another closed string puncture on a sphere or a disc
  - exclude the operation of a sewing two closed string puncture, each living on a disc, since this operation generates an annulus
- the closed string is treated as an external field

OCHA: algebraic formulation

- the closed vertices corresponding to spheres define a L_∞-algebra $l = \sum_n l_n$ on the state space of closed strings A_c, i.e. $\hat{l} \in \mathrm{Coder}(SA_c)$ of degree one and $\hat{l}^2 = 0$

$$\mathrm{Coder}(SA_c) \cong \mathrm{Hom}(SA_c, A_c)$$

- the open string vertices corresponding to discs with only open string punctures define an A_∞-algebra $m = \sum_n m_n$ on the state space of open strings A_o, i.e. $\hat{m} \in \mathrm{Coder}(TA_o)$ of degree one and $\hat{m}^2 = 0$

$$\mathrm{Coder}(TA_o) \cong \mathrm{Hom}(TA_o, A_o)$$

- $\hat{m}$ endows the (cyclic) Hochschild complex $\mathrm{Coder}(TA_o)$ with the structure of a differential graded Lie algebra

$$d_h = [\hat{m}, \cdot]_G$$
$$[D_1, D_2]_G = D_1 \circ D_2 - (-1)^{D_1 D_2} D_2 \circ D_1$$

$$(A_c, \{l_n\}_{n>0}) \xrightarrow{L_\infty-\text{morphism}} (\text{Coder}(TA_o), d_h, [\cdot,\cdot]_G)$$

- the open-closed vertices associated to discs with open and closed punctures, define a L_∞-morphism

$$f : SA_c \to \text{Coder}(TA_o)$$

from $(A_c, \{l_n\}_{n>0})$ to $(\text{Coder}(TA_o), d_h, [\cdot,\cdot]_G)$, i.e.

$$f \circ \hat{l} = d_h \circ f + \tfrac{1}{2}[f, f]_G \circ \Delta \, , \tag{5}$$

where $\Delta : SA_c \to SA_c^{\otimes 2}$ denotes the comultiplication

$$\Delta(c_1 \wedge \ldots \wedge c_n) = \sum_{i=0}^{n} \sum_{\sigma \in Sh(i,j)} c_{\sigma_1} \wedge \ldots \wedge c_{\sigma_i} \otimes c_{\sigma_{i+1}} \wedge \ldots \wedge c_{\sigma_n}$$

Infinitesimal deformations

- consider the map f_1 of the L_∞-morphism f corresponding to discs with just one closed string puncture
- f_1 defines a chain map, i.e.

$$f_1 \circ Q_c = d_h \circ f_1$$

- cohomology of Q_c is the space of physical closed string states
- cohomology of d_h classifies infinitesimal deformations of $\hat{m}$
- $\Rightarrow$ physical closed states map to infinitesimal deformations of open SFT
- f_1 indeed induces an isomorphism on cohomologies [Sachs, Moeller 11]

Finite deformations

- Maurer Cartan elements of L_∞-algebra $(A, \{l_n\}_{n>0})$: $c \in A$ of degree zero, satisfying

$$\sum_n \tfrac{1}{n!} l_n(c^{\wedge n}) = l_1(c) + \frac{1}{2} l_2(c, c) + \cdots = 0$$

- Maurer Cartan elements of $(A_c, \{l_n\}_{n>0})$: solutions of the equations of motion, i.e. closed string backgrounds
- Maurer Cartan elements of $(\mathrm{Coder}(TA_o), d_h, [\cdot, \cdot]_G)$: finite deformations of $\hat{m}$
- L_∞-morphisms preserve Maurer Cartan elements
- $\Rightarrow$ the open-closed vertices map closed string backgrounds into finite deformations of open string field theory

- gauge transformations:
  - $c_0, c_1 \in \mathcal{MC}(A, \{l_n\}_{n>0})$
  - $c_0 \sim c_1$, if $\exists c(t) \in \mathcal{MC}(A, \{l_n\}_{n>0})$ and $\exists \lambda(t) \in A$

$$\frac{d}{dt}c(t) = \sum_n \frac{1}{n!} l_{n+1}(\lambda(t) \wedge c(t)^{\wedge n}) \ , \quad c(0) = c_0 \ , \quad c(1) = c_1$$

- moduli space of an L_∞-algebra: space of Maurer Cartan elements modulo gauge transformations
- L_∞-quasi-isomorphism: f is called a quasi-isomorphism, if f_1 induces an isomorphism on cohomology

Theorem ([Kontsevich 97])

L_∞-quasi-isomorphisms induce isomorphism on moduli spaces.

- recall: f_1 induces isomorphism on cohomologies, i.e. f is a L_∞-quasi-isomorphism

open-closed correspondence

Closed string backgrounds modulo gauge transformations are in one-to-one correspondence with inequivalent open SFTs.

example: deformation quantization [Kontsevich 97]

- 'closed' side:
  - poly-vectorfields:
    $$A_c = T_p(M) = \bigoplus_n \Gamma(M, \Lambda^n TM)$$
  - Schouten-Nijenhuis bracket: $l_2 = [\cdot,\cdot]_{SN}$ and $l_n = 0$ for $n \neq 2$
  - Maurer Cartan elements of $(T_p(M), [\cdot,\cdot]_{SN})$ define Poisson structures on M
- 'open' side:
  - smooth functions:
    $$A_o = C^\infty(M)$$
  - pointwise multiplication: $m_2(f,g) = f \cdot g$ and $m_n = 0$ for $n \neq 2$
  - consider the subset $D_p(M) \subset \mathrm{Hom}(TA_o, A_o)$ of poly differential operators
  - Maurer Cartan elements of $(D_p(M), d_h, [\cdot,\cdot]_G)$ define star products
- formality: construction of L_∞-quasi-isomorphism from $(T_p(M), l_2)$ to $(D_p(M), d_h, [\cdot,\cdot]_G)$
  $\Rightarrow$ star products on $C^\infty(M)$ in one-to-one correspondence
  with Possion structure on M (modulo gauge transformations)

Quantum open-closed SFT

geometric data

Moduli spaces $\mathcal{P}$ of Riemann surfaces with arbitrary number of boundary components, arbitrary genus, closed string punctures in the bulk and open string punctures on the boundaries.

- BV structure on singular chain complex $C^\bullet(\mathcal{P})$ induces by
  - twist sewing of closed string punctures
  - sewing of open string punctures
- five distinct sewing operations
  - sewing closed string punctures on distinct surfaces $\leftrightarrow (\cdot,\cdot)_c$
  - sewing closed string punctures on one surface $\leftrightarrow \Delta_c$
  - sewing open string punctures on distinct surfaces $\leftrightarrow (\cdot,\cdot)_o$
  - sewing open string punctures on one surface but distinct boundary components $\leftrightarrow \Delta_o$
  - sewing open string punctures on one boundary component $\leftrightarrow \Delta_o$

Algebraic counterpart - QOCHA

- In 'classical' open-closed SFT, the open-closed vertices define a L_∞-morphism from the L_∞-algebra of closed strings to the Lie algebra controlling deformations of open SFT.
- generalization to quantum level: we have to consider homotopy Lie bialgebras (IBL_∞) rather than homotopy Lie algebras (L_∞)
- what are the corresponding IBL_∞-algebras on the closed string and open string side?

Closed string side

- closed string vertices for arbitrary genus g:

$$l^g_n : A_c^{\wedge n} \to A_c$$

- attaching handles $\leftrightarrow$ apply inverse symplectic structure ω_c^{-1}
- algebraic structure of closed SFT (loop homotopy algebra) [Markl 01]:

$$\mathfrak{L}_c := \sum_g \hbar^g \hat{l}^g + \hbar \widehat{\omega_c^{-1}} , \quad l^g = \sum_n l^g_n ,$$

with

$$\mathfrak{L}_c^2 = 0 \quad \Leftrightarrow \quad \sum_{\substack{i+j=n \\ g_1+g_2=g}} \sum_{\sigma \in Sh(i,j)} l^{g_1}_{1+j} \circ (l^{g_2}_i \wedge 1^{\wedge j}) \circ \sigma + l^{g-1} \circ (\omega_c^{-1} \wedge 1^{\wedge n})$$

- loop homotopy algebra is a special case of an IBL_∞-algebra

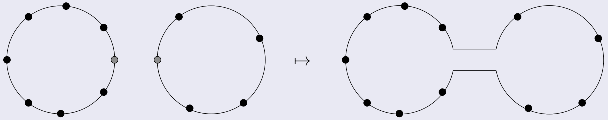
Open string side

- note:

$$\mathrm{Coder}(TA_o) \cong \mathrm{Hom}(TA_o, A_o) \overset{\omega_o}{\cong} \mathrm{Hom}(TA_o, \mathbb{C})$$

- here it is convenient to work with $\mathcal{A}_o := \mathrm{Hom}(TA_o, \mathbb{C})$
- the graded commutator on $\mathrm{Coder}(TA_o)$ induces a Lie bracket $[\cdot,\cdot]_G : \mathcal{A}_o^{\wedge 2} \to \mathcal{A}_o$
- elements in $\mathcal{A}_o$ represent boundary components $\to$ $[\cdot,\cdot]_G$ is the algebraic counterpart of sewing two open strings on distinct boundary components

geometric interpretation of $[\cdot,\cdot]_G$



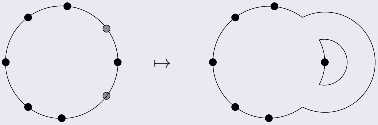
- define operation that takes account of sewing two open string punctures on one boundary component

$$\delta : \mathcal{A}_o \to \mathcal{A}_o^{\wedge 2}$$

$$\begin{aligned}&(\delta f)(a_1, \ldots, a_n)(b_1, \ldots, b_m) \\ &\quad := \pm f(e_i, \tau(a_1, \ldots, a_n), e^i, \tau(b_1, \ldots, b_m)) \,,\end{aligned}$$

where τ is the map that cyclically permutes the inputs and $\omega_o^{-1} = e_i \otimes e^i$

geometric interpretation of δ



- $[\cdot,\cdot]_G$ and δ define a involutive Lie bialgebra [Chen 10]
- equivalently:

$$\mathfrak{L}_o = \hat{d}_h + \widehat{[\cdot,\cdot]}_G + \hbar\hat{\delta} \,,$$

$$\mathfrak{L}_o^2 = 0$$

Definition of QOCHA

The open-closed vertices define a IBL_∞-morphism from the homotopy loop algebra of closed string to the involutive Lie bialgebra on the cyclic Hochschild complex of open strings.

$$(A_c, \mathfrak{L}_c) \xrightarrow{IBL_\infty-\text{morphism}} (\mathcal{A}_o, \mathfrak{L}_o) \ .$$

- open-closed vertices: $f^{b,g} : SA_c \to \mathcal{A}_o^{\wedge b}$

$$\mathfrak{f} = \sum_{b=1}^{\infty} \sum_{g=0}^{\infty} \hbar^{b+g-1} f^{b,g} \ .$$

- QOCHA: $\mathfrak{L}_c^2 = 0$, $\mathfrak{L}_o^2 = 0$

$$e^{\mathfrak{f}} \circ \mathfrak{L}_c = \mathfrak{L}_o \circ e^{\mathfrak{f}}$$

quantum open-closed correspondence?

- IBL_∞-morphisms preserve Maurer Cartan elements
- Maurer Cartan elements of $(\mathcal{A}_o, \mathfrak{L}_o)$ represent deformations of the classical open SFT $\hat{m}$ into a consistent quantum SFT of only open strings
- $\mathfrak{f}$ is an IBL_∞-quasi-isomorphism
- IBL_∞-quasi-isomorphisms induce isomorphisms on moduli spaces
- $\Rightarrow$ quantum open SFTs are in one-to-one correspondence with Maurer Cartan elements of closed string homotopy loop algebra

Maurer Cartan equation of homotopy loop algebra

- $\mathfrak{L}_c(e^{\mathfrak{c}}) = 0$, where $\mathfrak{c} = c + \mathcal{O}(\hbar^1)$ and c a classical closed string background
- Maurer Cartan equation implies that the background shifted BRST differential $Q_c[c]$ has to have trivial cohomology
- $\Rightarrow$ no physical closed string states
- further investigation: $\mathfrak{L}_c$ does not admit any Maurer Cartan elements $\leftrightarrow$open String field theory inconsistent due to closed string poles arising in open string loops

Outlook

- A_∞-algebra and boundary SFT (background independent)
- algebraic structure of super SFT
- topological strings and quantum open-closed correspondence

Thank you for your attention!

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