BPS black holes, wall-crossing and mock modularity of higher depth

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S.A., B.Pioline arXiv:1808.08479

continuation of

S.A., S.Banerjee, J.Manschot, B.Pioline arXiv:1605.05945

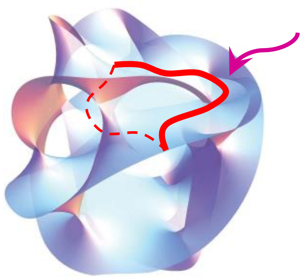
arXiv:1606.05495

arXiv:1702.05497

S.A., B.Pioline arXiv:1804.06928

The problem

- BPS black holes described by D4-D2-D0 bound states in Type IIA string theory compactified on a Calabi-Yau threefold



- electro-magnetic charge

$$\gamma = (0, p^a, q_a, q_0) \qquad\qquad a = 1, \dots, b_2(CY)$$

label 4- and 2-dim cycles
wrapped by D4 and D2-branes

- BPS index (black hole degeneracy) $\Omega(\gamma)$ — *generalized Donaldson-Thomas invariant*

Goal: understand modular properties of $\Omega(\gamma)$

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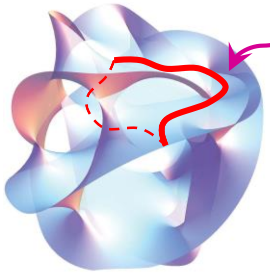

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Define a generating function: $$h^{\mathrm{DT}}_{\dots}(\tau) = \sum_{q_0 > 0} \Omega(\gamma)\, e^{2\pi \mathrm{i} q_0 \tau}$$

and study its properties under modular transformations: $$\tau \mapsto \frac{a\tau + b}{c\tau + d}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$$

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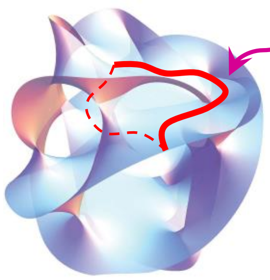

non-trivial cycle on CY

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Problems:

- Generating function depends on too many charges
- DT invariants depend on CY moduli (*wall-crossing*)

MSW invariants

Solution: consider *MSW invariants*

count states in SCFT constructed
in Maldacena,Strominger,Witten '97

$$\boxed{\Omega_\gamma^{\mathrm{MSW}} = \Omega(\gamma, z^a_\infty(\gamma))}$$

large volume
attractor point

$$z^a_\infty(\gamma) = \lim_{\lambda\to\infty} (-q^a + \mathrm{i}\lambda p^a)$$

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Properties:

- independent of CY moduli
- invariant under *spectral flow symmetry*

spectral flow

$$q_a \mapsto q_a - \kappa_{ab}\epsilon^b$$

$$q_0 \mapsto q_0 - \epsilon^a q_a + \tfrac{1}{2}\,\kappa_{ab}\epsilon^a\epsilon^b$$

$\kappa_{ab} = \kappa_{abc}p^c$ – quadratic form, given by intersection numbers of 4-cycles, of *indefinite* signature $(1, b_2 - 1)$

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$\hat q_0 \equiv q_0 - \frac12\, \kappa^{ab} q_a q_b$ – invariant charge bounded from above

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For irreducible cycle p^a:

[Gaiotto,Strominger,Yin '06]

$h_p(\tau)$ — modular form of weight $-\tfrac12\, b_2 - 1$

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What is modular behavior of $h_p(\tau)$ for generic divisor?

The logic of derivation

Type IIA / CY

D4-D2-D0 bl.h.

The logic of derivation

Type IIA / CY×S^1

D4-D2-D0 bl.h.

affect the metric on the QK moduli space $\mathcal{M}$

The logic of derivation

M-theory / $CY \times T^2$ ⟷ **Type IIA** / $CY \times S^1$ ⟷ **Type IIB** / $CY \times S^1$

T-duality

M5-brane / **divisor** ⟷ **D4-D2-D0 bl.h.** ⟷ **D3-D1-D(-1) inst.**

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$\mathcal{M}$ carries an isometric action of $SL(2,\mathbb{Z})$
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A function $\mathcal{G}$ on $\mathcal{M}$ constructed from DT-invariants
is *modular* of weight $\left(-\frac{3}{2},\frac{1}{2}\right)$

explained by Boris

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$$\mathcal{G} \sim \sum_{n=1}^{\infty}\left[\prod_{i=1}^{n}\sum_{p_i} h_{p_i}\right]\theta_{\boldsymbol{p}}$$

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Conclusion: for $n\geq 2$ there is a modular *anomaly* ⟶ **construct completion** $\widehat{h}_p(\tau,\bar{\tau})$

Theta series decomposition

Instanton expansion:

rational function of z_i
sum over labeled trees

$$\mathcal{G} = \sum_{n=1}^{\infty} \left[\prod_{i=1}^{n} \sum_{\gamma_i} \underbrace{\bar\Omega(\gamma_i) e^{-2\pi \mathrm{i} \hat q_{i,0} \tau}}_{h^{\rm DT}_{p_i,q_i}(\tau)} \underbrace{\int_{\ell_{\gamma_i}} \mathrm{d}z_i \, \mathcal{X}^{(\theta)}_{p_i,q_i}(z_i) \right] \mathcal{G}_n(\{\gamma_i, z_i\})}_{\text{do not depend on D0-brane charge}}$$

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Solution: express DT invariants in terms of *MSW invariants*

$$\bar\Omega(\gamma, z^a) = \sum_{\sum_{i=1}^n \gamma_i = \gamma} \underbrace{g_{{\rm tr},n}(\{\gamma_i\}, z^a)}_{\text{tree index}} \prod_{i=1}^{n} \bar\Omega_{p_i}(q_{i,0})$$

[Boris' talk]

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tree index

don't depend on D0-chrage

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theta series decomposition

$$\mathcal{G}=\frac{1}{\sqrt{\tau_2}}\sum_{n=1}^{\infty}\left[\prod_{i=1}^{n}\sum_{p_i}h_{p_i}\right]e^{-S^{\rm cl}_p}\,\vartheta_{\boldsymbol{p}}(\Phi_n^{\rm tot})$$

indefinite theta series with kernel $\Phi_n^{\rm tot}$ defined by iterated integrals of $\mathcal{G}_n$ and the tree index $g_{{\rm tr},n}$

Modularity of indefinite theta series

Our theta series fits in the class:

$$\vartheta_{\boldsymbol{p}}(\Phi,\lambda) = \tau_2^{-\lambda/2} \sum_{\boldsymbol{q}\in\boldsymbol{\Lambda}+\frac12\boldsymbol{p}} (-1)^{\boldsymbol{q}\cdot\boldsymbol{p}}\, \Phi(\sqrt{2\tau_2}(\boldsymbol{q}+\boldsymbol{b}))\, e^{-\pi \mathrm{i}\tau(\boldsymbol{q}+\boldsymbol{b})^2+2\pi\mathrm{i}\boldsymbol{c}\cdot(\boldsymbol{q}+\frac12\boldsymbol{b})}$$

d-dim. lattice

kernel

quadratic form of indefinite signature

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In our case:

- $\lambda = n-2$
- $\boldsymbol{\Lambda} = \oplus_{i=1}^n \Lambda_i$ — lattice of electric charges $q_{i,a}$; $d = nb_2$
- bilinear form $\boldsymbol{x}\cdot\boldsymbol{y} = \sum_{i=1}^n \kappa_{i,ab}\, x_i^a y_i^b = \sum_{i=1}^n \kappa_{abc} x_i^a y_i^b p_i^c$

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Simple criterion for modularity:

Vignéras '77

$$\boxed{\begin{array}{c}V_\lambda\cdot\Phi(\boldsymbol{x})=0\\ V_\lambda=\partial_{\boldsymbol{x}}^2+2\pi\,(\boldsymbol{x}\cdot\partial_{\boldsymbol{x}}-\lambda)\end{array}\;\Longrightarrow\;\begin{array}{l}\vartheta_{\boldsymbol{p}}(\Phi,\lambda)\text{ — modular}\\ \text{of weight }(\lambda+d/2,0)\end{array}}$$

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- bilinear form $\boldsymbol{x}\cdot\boldsymbol{y} = \sum_{i=1}^n \kappa_{i,ab}\, x_i^a y_i^b = \sum_{i=1}^n \kappa_{abc} x_i^a y_i^b p_i^c$

signature $(n, n(b_2-1))$

Simple criterion for modularity:

Vignéras '77

$$\boxed{\begin{array}{l} V_\lambda\cdot\Phi(\boldsymbol{x})=0 \\ V_\lambda = \partial_{\boldsymbol{x}}^2 + 2\pi\,(\boldsymbol{x}\cdot\partial_{\boldsymbol{x}}-\lambda) \end{array} \Longrightarrow \begin{array}{l} \vartheta_{\boldsymbol{p}}(\Phi,\lambda) \text{ — modular} \\ \text{of weight } (\lambda+d/2,0) \end{array}}$$

$\Downarrow$

It remains to check $V_{n-2}\cdot\Phi_n^{\mathrm{tot}} = 0$

Generalized error functions I

Solutions of Vignéras equation:

for convergence

- $n=1,\ \lambda=-1:\quad \Phi(\boldsymbol{x}) = e^{-\pi \frac{(\boldsymbol{x},\boldsymbol{v})^2}{\boldsymbol{v}^2}}$ $\qquad \boldsymbol{v}^2 > 0$
- $n=1,\ \lambda=0:\quad \Phi(\boldsymbol{x}) = \mathrm{Erf}\left(\sqrt{\pi}\,\frac{\boldsymbol{x}\cdot\boldsymbol{v}}{|\boldsymbol{v}|}\right) - \mathrm{Erf}\left(\sqrt{\pi}\,\frac{\boldsymbol{x}\cdot\boldsymbol{v}'}{|\boldsymbol{v}'|}\right)$ $\qquad \boldsymbol{v}\cdot\boldsymbol{v}' > 0$

Generalized error functions I

Solutions of Vignéras equation:

for convergence

- $n = 1,\ \lambda = -1: \quad \Phi(\boldsymbol{x}) = e^{-\pi \frac{(\boldsymbol{x},\boldsymbol{v})^2}{\boldsymbol{v}^2}}$ $\qquad \boldsymbol{v}^2 > 0$
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[Zwegers '02]

completion of *holomorphic mock* theta series with kernel $\Phi^{\mathrm{hol}}(\boldsymbol{x}) = \mathrm{sgn}(\boldsymbol{x},\boldsymbol{v}) - \mathrm{sgn}(\boldsymbol{x},\boldsymbol{v}')$

$$\mathrm{Erf}(u\sqrt{\pi}) = \mathrm{sgn}(u) - \mathrm{sgn}(u)\,\mathrm{Erfc}(|u|\sqrt{\pi})$$

smooth solution — holomorphic & discontinuous — exponentially decaying discontinuous solution

Generalized error functions I

Solutions of Vignéras equation:

for convergence

- $n=1,\ \lambda=-1:\quad \Phi(\boldsymbol{x}) = e^{-\pi\,\frac{(\boldsymbol{x},\boldsymbol{v})^2}{\boldsymbol{v}^2}}$ $\qquad \boldsymbol{v}^2>0$
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[Zwegers '02]

completion of *holomorphic mock* theta series with kernel $\Phi^{\rm hol}(\boldsymbol{x}) = \mathrm{sgn}(\boldsymbol{x},\boldsymbol{v}) - \mathrm{sgn}(\boldsymbol{x},\boldsymbol{v}')$

$$\int_{\mathbb{R}} \mathrm{d}u'\, e^{-\pi(u-u')^2}\mathrm{sgn}(u') = \mathrm{Erf}(u\sqrt{\pi})$$

smooth solution

$$-\mathrm{sgn}(u)\,\mathrm{Erfc}(|u|\sqrt{\pi}) = \frac{\mathrm{i}}{\pi}\int_{\mathbb{R}-\mathrm{i}u} \frac{\mathrm{d}z}{z}\, e^{-\pi z^2 - 2\pi\mathrm{i} z u}$$

exponentially decaying discontinuous solution

Generalized error functions I

Solutions of Vignéras equation:

for convergence

- $n=1,\ \lambda=-1: \quad \Phi(\boldsymbol{x}) = e^{-\pi\frac{(\boldsymbol{x},\boldsymbol{v})^2}{\boldsymbol{v}^2}}$ $\qquad \boldsymbol{v}^2>0$
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exponentially decaying
discontinuous solution

Generalization to arbitrary *n* :

[ABMP, Nazaroglu '16]

Generalized error functions

$$E_n(\mathcal{M};\mathbb{u}) = \int_{\mathbb{R}^n} \mathrm{d}\mathbb{u}'\, e^{-\pi(\mathbb{u}-\mathbb{u}')^{\mathrm{tr}}(\mathbb{u}-\mathbb{u}')} \prod \mathrm{sgn}(\mathcal{M}^{\mathrm{tr}}\mathbb{u}')$$

$$M_n(\mathcal{M};\mathbb{u}) = \left(\frac{\mathrm{i}}{\pi}\right)^n |\det\mathcal{M}|^{-1} \int_{\mathbb{R}^n-\mathrm{i}\mathbb{u}} \mathrm{d}^n z\, \frac{e^{-\pi\mathbb{z}^{\mathrm{tr}}\mathbb{z} - 2\pi\mathrm{i}\mathbb{z}^{\mathrm{tr}}\mathbb{u}}}{\prod(\mathcal{M}^{-1}\mathbb{z})}$$

$\mathbb{u},\mathbb{z}\in\mathbb{R}^n,\ \mathcal{M}$ — $n\times n$ matrix

Generalized error functions I

Solutions of Vignéras equation:

for convergence

- $n=1,\ \lambda=-1:\quad \Phi(\boldsymbol{x}) = e^{-\pi\frac{(\boldsymbol{x},\boldsymbol{v})^2}{\boldsymbol{v}^2}}$ $\qquad \boldsymbol{v}^2>0$
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[Zwegers '02]

completion of *holomorphic mock* theta series with kernel $\Phi^{\mathrm{hol}}(\boldsymbol{x}) = \mathrm{sgn}(\boldsymbol{x},\boldsymbol{v}) - \mathrm{sgn}(\boldsymbol{x},\boldsymbol{v}')$

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exponentially decaying discontinuous solution

Generalization to arbitrary *n* :

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Generalized error functions

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$$M_n(\mathcal{M};\mathbb{u}) = \left(\frac{\mathrm{i}}{\pi}\right)^n |\det\mathcal{M}|^{-1}\int_{\mathbb{R}^n-\mathrm{i}\mathbb{u}}\mathrm{d}^n z\,\frac{e^{-\pi\mathbb{z}^{\mathrm{tr}}\mathbb{z}-2\pi\mathrm{i}\mathbb{z}^{\mathrm{tr}}\mathbb{u}}}{\prod(\mathcal{M}^{-1}\mathbb{z})}$$

exponentially decaying, discontinuous

$\mathbb{u},\mathbb{z}\in\mathbb{R}^n,\ \mathcal{M}$ — $n\times n$ matrix

Generalized error functions II

Solutions of Vignéras equation for $\lambda = 0$ from generalized error functions:

$$\Phi_n^E(\mathcal{V};\boldsymbol{x}) = E_n(\mathcal{B}\cdot\mathcal{V};\mathcal{B}\cdot\boldsymbol{x})$$

$$\Phi_n^M(\mathcal{V};\boldsymbol{x}) = M_n(\mathcal{B}\cdot\mathcal{V};\mathcal{B}\cdot\boldsymbol{x})$$

$\mathcal{V} = (\boldsymbol{v}_1,\dots,\boldsymbol{v}_n)$ — $d\times n$ matrix

$\mathcal{B} = (\boldsymbol{e}_1,\dots,\boldsymbol{e}_n)^{\mathrm{tr}}$ — $n\times d$ matrix

$\boldsymbol{e}_i$ — orthonormal basis in the subspace spanned by $\boldsymbol{v}_i$

Φ_n^E provide modular *completions* for holomorphic indefinite theta series with quadratic form of signature $(n, d-n)$ and kernel $\Phi_n^{\mathrm{hol}}(\mathcal{V};\boldsymbol{x}) = \prod_{i=1}^n \mathrm{sgn}(\boldsymbol{v}_i,\boldsymbol{x})$

Φ_n^M — the part of the completion with fastest decay can be expressed as *n* iterated Eichler (period) integrals

Generalized error functions II

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⬇

Higher depth mock modularity

Generalized error functions II

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Φ_n^M — the part of the completion with fastest decay can be expressed as *n* iterated Eichler (period) integrals

⬇

Higher depth mock modularity

Lift to solution with $\lambda = m$:

$$\tilde\Phi_{n,m}^E(\mathcal{V},\tilde{\mathcal{V}};\boldsymbol{x}) = \left[\prod_{i=1}^{m}\mathcal{D}(\tilde{\boldsymbol{v}}_i)\right]\Phi_n^E(\mathcal{V};\boldsymbol{x})$$

$\mathcal{D}(\tilde{\boldsymbol{v}}) = \tilde{\boldsymbol{v}}\cdot\left(\boldsymbol{x} + \frac{1}{2\pi}\,\partial_{\boldsymbol{x}}\right)$ — covariant derivative raising λ by 1

Generalized error functions II

Solutions of Vignéras equation for $\lambda=0$ from generalized error functions:

$$\boxed{\begin{aligned}\Phi_n^E(\mathcal{V};\boldsymbol{x}) &= E_n(\mathcal{B}\cdot\mathcal{V};\mathcal{B}\cdot\boldsymbol{x})\\ \Phi_n^M(\mathcal{V};\boldsymbol{x}) &= M_n(\mathcal{B}\cdot\mathcal{V};\mathcal{B}\cdot\boldsymbol{x})\end{aligned}}$$

$\mathcal{V}=(\boldsymbol{v}_1,\dots,\boldsymbol{v}_n)$ — $d\times n$ matrix

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Lift to solution with $\lambda=m$:

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$$\underset{\boldsymbol{x}\to\infty}{\sim}\ \prod_{i=1}^m \mathcal{D}(\tilde{\boldsymbol{v}}_i)\prod_{j=1}^n \mathrm{sgn}(\boldsymbol{v}_j,\boldsymbol{x})$$

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Generalized error functions II

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Φ_n^M — the part of the completion with fastest decay can be expressed as *n* iterated Eichler (period) integrals

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Lift to solution with $\lambda=m$:

$$\boxed{\tilde\Phi^E_{n,m}(\mathcal{V},\tilde{\mathcal{V}};\boldsymbol{x}) = \left[\prod_{i=1}^m \mathcal{D}(\tilde{\boldsymbol{v}}_i)\right]\Phi_n^E(\mathcal{V};\boldsymbol{x})}$$

if $(\tilde{\boldsymbol{v}}_i,\tilde{\boldsymbol{v}}_j)=0$

$$\underset{\boldsymbol{x}\to\infty}{\sim}\prod_{i=1}^m(\tilde{\boldsymbol{v}}_i,\boldsymbol{x})\prod_{j=1}^n \mathrm{sgn}(\boldsymbol{v}_j,\boldsymbol{x})$$

$\mathcal{D}(\tilde{\boldsymbol{v}})=\tilde{\boldsymbol{v}}\cdot\left(\boldsymbol{x}+\frac{1}{2\pi}\,\partial_{\boldsymbol{x}}\right)$ — covariant derivative raising λ by 1

Kernels

In our case: $$\Phi_n^{\text{tot}} = \sum_{\substack{n_1 + \cdots n_m = n \\ n_k \geq 1}} \Phi_m^{\int} \prod_{k=1}^{m} \Phi_{n_k}^{g}$$ proportional to the tree index g_{tr,n_k}

given by iterated integrals of $\mathcal{G}_n$

Kernels

In our case: $\Phi_n^{\text{tot}} = \sum_{\substack{n_1+\cdots n_m=n \\ n_k\geq 1}} \Phi_m^{\int} \prod_{k=1}^{m} \Phi_{n_k}^{g}$ ← proportional to the tree index g_{tr,n_k}

given by iterated integrals of $\mathcal{G}_n$

locally $\Phi_{n_k}^{g} \sim \sum_{(\tilde{\boldsymbol{v}}_i,\tilde{\boldsymbol{v}}_j)=0} \prod_{i=1}^{n_k-1} (\tilde{\boldsymbol{v}}_i, \boldsymbol{x})$

Kernels

In our case: $\Phi_n^{\text{tot}} = \sum_{\substack{n_1+\cdots n_m=n \\ n_k\geq 1}} \Phi_m^{\int} \prod_{k=1}^{m} \Phi_{n_k}^{g}$ ← proportional to the tree index g_{tr,n_k}

given by iterated integrals of $\mathcal{G}_n$

locally $\Phi_{n_k}^{g} \sim \sum \prod_{i=1}^{n_k-1} (\tilde{\boldsymbol{v}}_i, \boldsymbol{x})$

$(\tilde{\boldsymbol{v}}_i, \tilde{\boldsymbol{v}}_j) = 0$

away from discontinuities

$$V_{n_k-1} \cdot \Phi_{n_k}^{g} = 0$$

Kernels

In our case: $\Phi_n^{\text{tot}} = \sum_{\substack{n_1+\cdots n_m=n \\ n_k\geq 1}} \Phi_m^{\int} \prod_{k=1}^{m} \Phi_{n_k}^{g}$ ← proportional to the tree index g_{tr,n_k}

given by iterated integrals of $\mathcal{G}_n$

$$\boxed{\Phi_n^{\int}(\boldsymbol{x}) = \frac{\Phi_1^{\int}(x)}{2^{n-1}n!} \sum_{\mathcal{T}\in\mathbb{T}_n^{\ell}} \tilde{\Phi}_{n-1}^{M}(\{\boldsymbol{u}_e\},\{\boldsymbol{v}_{s(e)t(e)}\};\boldsymbol{x})}$$

locally $\Phi_{n_k}^{g} \sim \sum \prod_{i=1}^{n_k-1} (\tilde{\boldsymbol{v}}_i, \boldsymbol{x})$

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$$V_{n_k-1}\cdot\Phi_{n_k}^{g} = 0$$

- $\Phi_1^{\int}(x) = (2\tau_2 t^2)^{-\frac{1}{2}}\, e^{-\pi\,\frac{(x,t)^2}{t^2}}$ — solution for $\lambda=-1$

Kernels

In our case: $\Phi_n^{\rm tot} = \sum_{\substack{n_1+\cdots n_m=n \\ n_k\geq 1}} \Phi_m^{\int} \prod_{k=1}^{m} \Phi_{n_k}^{g}$ ← proportional to the tree index $g_{{\rm tr},n_k}$

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$$V_{n_k-1} \cdot \Phi_{n_k}^{g} = 0$$

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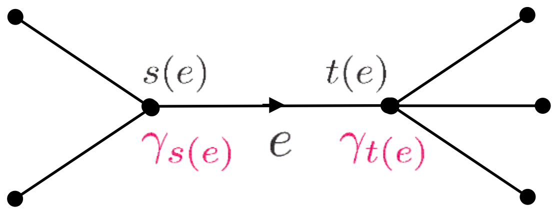
- $\mathcal{T} \in \mathbb{T}_n^{\ell}$ — labeled unrooted tree
  $V_{\mathcal{T}}$ — set of its vertices



Kernels

In our case: $\Phi_n^{\rm tot} = \sum_{\substack{n_1+\cdots n_m=n \\ n_k\ge 1}} \Phi_m^{\int} \prod_{k=1}^{m} \Phi_{n_k}^{g}$ ⟵ proportional to the tree index $g_{{\rm tr},n_k}$

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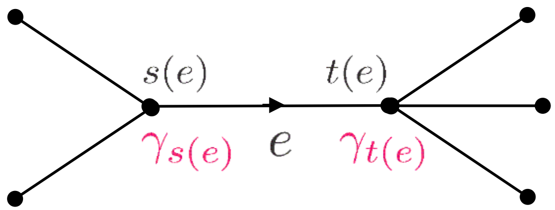
- $\boldsymbol{v}_{ij}, \boldsymbol{u}_{ij}$ — nb_2-dimensional vectors such that



$$\boldsymbol{v}_{ij}\cdot \boldsymbol{x} = \sqrt{2\tau_2}\langle \gamma_i, \gamma_j\rangle \qquad \boldsymbol{u}_{ij}\cdot \boldsymbol{x} = -\sqrt{2\tau_2}\,{\rm Im}(Z_{\gamma_i}\bar{Z}_{\gamma_j})$$

define stability condition of a bound state $\gamma_i+\gamma_j$

$$(\boldsymbol{v}_{ij}, \boldsymbol{x})(\boldsymbol{u}_{ij}, \boldsymbol{x}) < 0$$

Kernels

In our case: $\Phi^{\rm tot}_n = \sum_{\substack{n_1+\cdots n_m=n \\ n_k\geq 1}} \Phi^{\int}_m \prod_{k=1}^m \Phi^g_{n_k}$ proportional to the tree index $g_{{\rm tr},n_k}$

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locally $\Phi^g_{n_k} \sim \sum \prod_{i=1}^{n_k-1} (\tilde{\boldsymbol{v}}_i, \boldsymbol{x})$

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$$\Phi^{\int}_n(\boldsymbol{x}) = \frac{\Phi^{\int}_1(x)}{2^{n-1}n!} \sum_{\mathcal{T}\in \mathbb{T}^\ell_n} \tilde{\Phi}^M_{n-1}(\{\boldsymbol{u}_e\}, \{\boldsymbol{v}_{s(e)t(e)}\}; \boldsymbol{x})$$

away from discontinuities

$$V_{n_k-1}\cdot \Phi^g_{n_k} = 0$$

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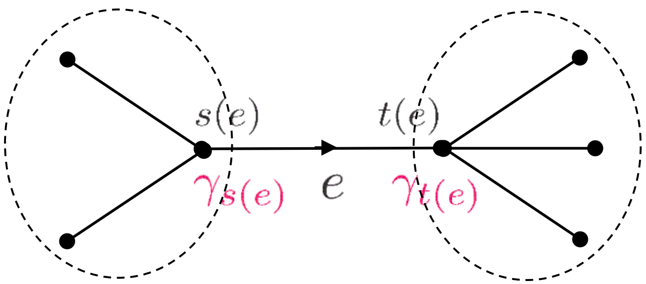

$\mathcal{T}^s_e$ $\mathcal{T}^t_e$

$$\boldsymbol{u}_e = \sum_{i\in V_{\mathcal{T}^s_e}} \sum_{j\in V_{\mathcal{T}^t_e}} \boldsymbol{u}_{ij}$$

Kernels

In our case: $\Phi_n^{\rm tot} = \sum_{\substack{n_1+\cdots n_m=n\\ n_k\geq 1}} \Phi_m^{\int} \prod_{k=1}^{m} \Phi_{n_k}^{g}$ ← proportional to the tree index $g_{{\rm tr},n_k}$

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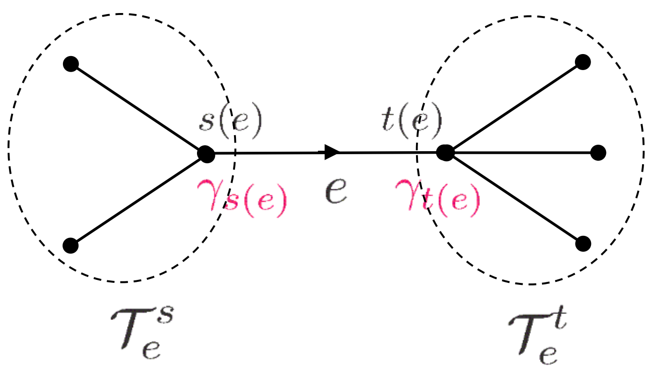


$$\boldsymbol{u}_e = \sum_{i\in V_{\mathcal{T}_e^s}} \sum_{j\in V_{\mathcal{T}_e^t}} \boldsymbol{u}_{ij}$$

- $\tilde{\Phi}_{n-1}^{M}$ — generalized error function of order $n-1$ lifted to solution for $\lambda=n-1$

Kernels

In our case: $\Phi^{\rm tot}_n = \sum_{\substack{n_1+\cdots n_m=n\\ n_k\ge 1}} \Phi^{\int}_m \prod_{k=1}^m \Phi^g_{n_k}$ ← proportional to the tree index $g_{{\rm tr},n_k}$

given by iterated integrals of $\mathcal{G}_n$

locally $\Phi^g_{n_k} \sim \sum \prod_{i=1}^{n_k-1} (\tilde{\boldsymbol{v}}_i, \boldsymbol{x})$

$(\tilde{\boldsymbol{v}}_i, \tilde{\boldsymbol{v}}_j) = 0$

$$\boxed{\Phi^{\int}_n(\boldsymbol{x}) = \frac{\Phi^{\int}_1(x)}{2^{n-1} n!} \sum_{\mathcal{T}\in\mathbb{T}^\ell_n} \tilde{\Phi}^M_{n-1}(\{\boldsymbol{u}_e\}, \{\boldsymbol{v}_{s(e)t(e)}\}; \boldsymbol{x})}$$

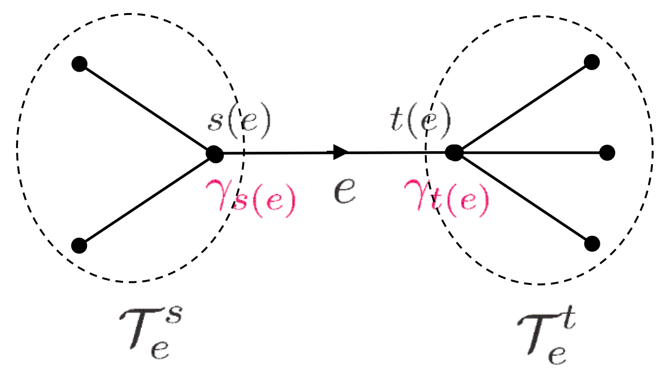
away from discontinuities
$V_{n_k-1}\cdot\Phi^g_{n_k} = 0$

- $\Phi^{\int}_1(x) = (2\tau_2 t^2)^{-\frac12}\, e^{-\pi\frac{(x,t)^2}{t^2}}$ — solution for $\lambda=-1$
- $\mathcal{T}\in\mathbb{T}^\ell_n$ — labeled unrooted tree
  $V_{\mathcal{T}}$ — set of its vertices
- $\boldsymbol{v}_{ij}, \boldsymbol{u}_{ij}$ — nb_2-dimensional vectors such that

$$\boldsymbol{v}_{ij}\cdot\boldsymbol{x} = \sqrt{2\tau_2}\langle\gamma_i,\gamma_j\rangle \qquad \boldsymbol{u}_{ij}\cdot\boldsymbol{x} = -\sqrt{2\tau_2}\,{\rm Im}(Z_{\gamma_i}\bar{Z}_{\gamma_j})$$

define stability condition of a bound state $\gamma_i+\gamma_j$

$$(\boldsymbol{v}_{ij},\boldsymbol{x})(\boldsymbol{u}_{ij},\boldsymbol{x})<0$$



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away from discontinuities
$V_{n-2}\cdot\Phi^{\int}_n = 0$

Modular anomaly

We do have $V_{n-2} \cdot \Phi_n^{\mathrm{tot}} = 0$ *away* from discontinuities.
But discontinuities can spoil Vignéras equation, and hence modularity of theta series, as it happens, for instance, for $\mathrm{sgn}(\boldsymbol{x}, \boldsymbol{v})$ in the case $n = 1$.

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Discontinuities of the kernels

discontinuity	origin	moduli dependence	cancel
walls of marginal stability	tree index & twistor integrals	yes	yes
fake walls	tree index	yes	yes
walls in the charge space $\langle \gamma_i, \gamma_j \rangle \sim (\boldsymbol{v}_{ij}, \boldsymbol{x}) = 0$	tree index	no	no

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$$V_{n-2} \cdot \Phi_n^{\rm tot} \neq 0$$

and both theta series and the generating function $h_p(\tau)$ have modular anomalies

Modular completion

It is possible to construct a *non-holomorphic completion* $\widehat{h}_p(\tau,\bar\tau)$
which transforms as a modular form of weight $-\frac{1}{2}\,b_2-1$

$$\boxed{\widehat{h}_p(\tau,\bar\tau)=h_p(\tau)-\sum_{n=2}^{\infty}\sum_{\sum_{i=1}^{n}\gamma_i=\gamma}R_n(\{\gamma_i\};\tau_2)\,e^{\pi \mathrm{i}\tau Q_n(\{\gamma_i\})}\prod_{i=1}^{n}h_{p_i}(\tau)}$$

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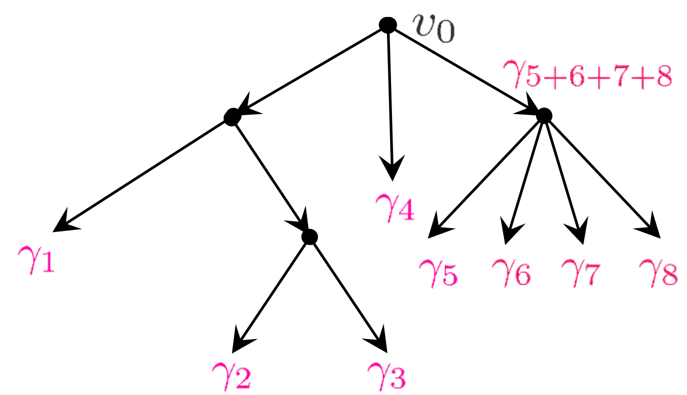

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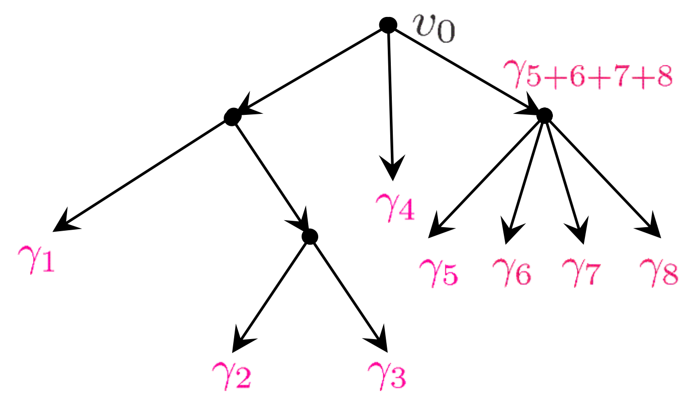

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at most polynomial in τ_2

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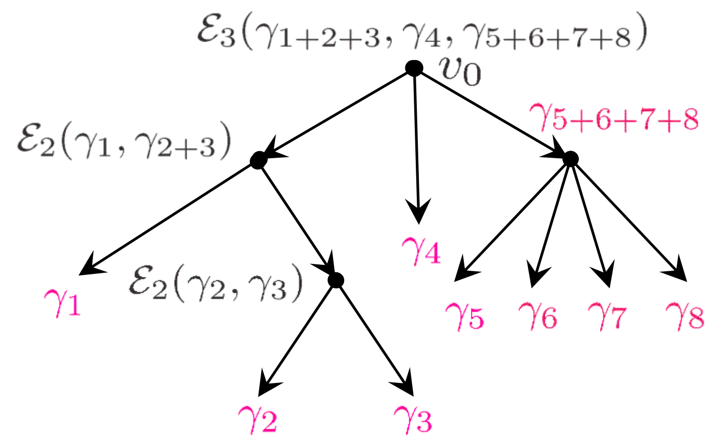

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number of children of v

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Completion and generalized error functions

Level 2: sum over *unrooted* trees

$\mathcal{E}_n(\{\gamma_i\};\tau_2)$ — smooth solutions of Vignéras equation for $\lambda = n-1$

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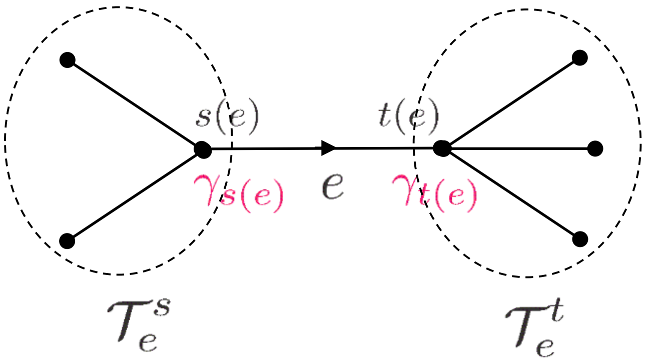

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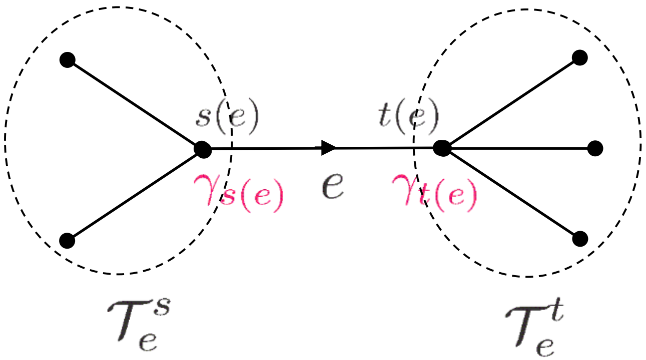

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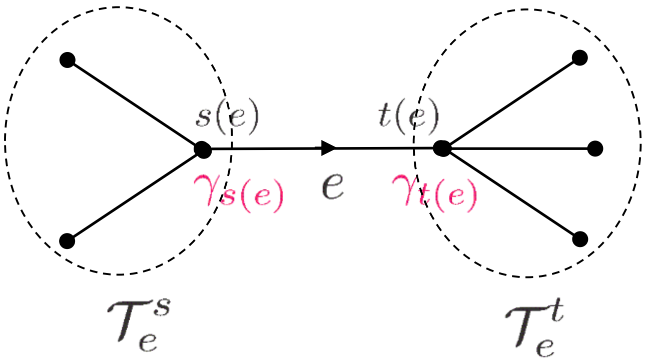

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Example: $n=2$ $\quad h_p(\tau)$ — usual mixed mock modular form

$$R_2 = -\frac{|\langle\gamma_1,\gamma_2\rangle|}{8\pi}\,\beta_{\frac{3}{2}}\left(\frac{2\tau_2\langle\gamma_1,\gamma_2\rangle^2}{(pp_1p_2)}\right)$$

$$\beta_{\frac{3}{2}}(x^2) = \frac{2}{|x|}e^{-\pi x^2} - 2\pi\,\mathrm{Erfc}(\sqrt{\pi}|x|)$$

$$(pp_1p_2) = \kappa_{abd}p^a p_1^b p_2^c$$

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This story has an extension which includes a *refinement* — parameter y conjugate to the angular momentum of black hole

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Modularity requires $\mathcal{E}_n^{(0)}(\{\gamma_i\}) = g_n^{(0)}(\{\gamma_i, \beta_i\})$

where $g_n^{(0)}(\{\gamma_i, c_i\})$ satisfies a recursive equation

A way to solve the equation: **refine it!** — replace a factor $\langle \gamma_i, \gamma_j \rangle$ by

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$$F_n^{(\mathrm{ref})}(\{c_i\}) = 2^{1-n} \sum_{\substack{n_1 + \cdots + n_m = n \\ n_k \geq 1}} \prod_{k=1}^{m} b_{n_k} \prod_{k=1}^{m-1} \mathrm{sgn}\left(\sum_{i=1}^{n_1 + \cdots n_k} c_i \right)$$

Why ?

Taylor coefficients of $\tanh(x)$ $\longrightarrow$ $b_n = \dfrac{2^n (2^n - 1)}{n!} B_n$

Bernoulli numbers

$$\boxed{\lim_{y \to 1} g_n^{(\mathrm{ref})}(\{\gamma_i, \beta_i\}, y) = \mathcal{E}_n^{(0)}(\{\gamma_i\})}$$

Refinement

This story has an extension which includes a *refinement* — parameter y conjugate to the angular momentum of black hole

$$\beta_i = \langle \gamma, \gamma_i \rangle$$
$$c_i = \mathrm{Im}(Z_{\gamma_i} \bar{Z}_\gamma)$$
$$\gamma = \gamma_1 + \cdots + \gamma_n$$

Modularity requires $\mathcal{E}_n^{(0)}(\{\gamma_i\}) = g_n^{(0)}(\{\gamma_i, \beta_i\})$

where $g_n^{(0)}(\{\gamma_i, c_i\})$ satisfies a recursive equation

A way to solve the equation: **refine it!** — replace a factor $\langle \gamma_i, \gamma_j \rangle$ by

$\dfrac{y^{\langle \gamma_i, \gamma_j \rangle} - y^{-\langle \gamma_i, \gamma_j \rangle}}{y - y^{-1}}$ — refined index of a 2-particle bound state

$$g_n^{(\mathrm{ref})}(\{\gamma_i, c_i\}, y) = \frac{\mathrm{Sym}\left\{ F_n^{(\mathrm{ref})}(\{c_i\})\, y^{\sum_{i<j} \langle \gamma_i \gamma_j \rangle} \right\}}{(y - y^{-1})^{n-1}}$$

$$F_n^{(\mathrm{ref})}(\{c_i\}) = 2^{1-n} \sum_{\substack{n_1 + \cdots + n_m = n \\ n_k \geq 1}} \prod_{k=1}^{m} b_{n_k} \prod_{k=1}^{m-1} \mathrm{sgn}\left(\sum_{i=1}^{n_1 + \cdots n_k} c_i \right)$$

Laurent polynomial, regular at $y \to 1$, if

$$\mathrm{Sym}\, F_n^{(\mathrm{ref})}(\{c_i\}) = 0$$

Why ?

Taylor coefficients of $\tanh(x)$ $\longrightarrow$ $b_n = \dfrac{2^n (2^n - 1)}{n!} B_n$

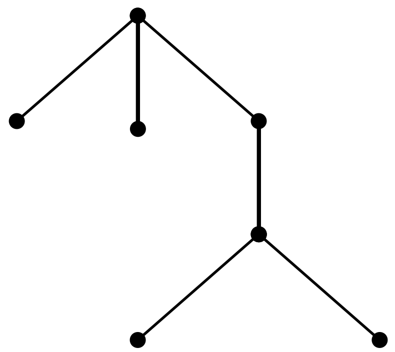
Bernoulli numbers

refined S-duality?

$$\boxed{\lim_{y \to 1} g_n^{(\mathrm{ref})}(\{\gamma_i, \beta_i\}, y) = \mathcal{E}_n^{(0)}(\{\gamma_i\})}$$

Fun with combinatorics

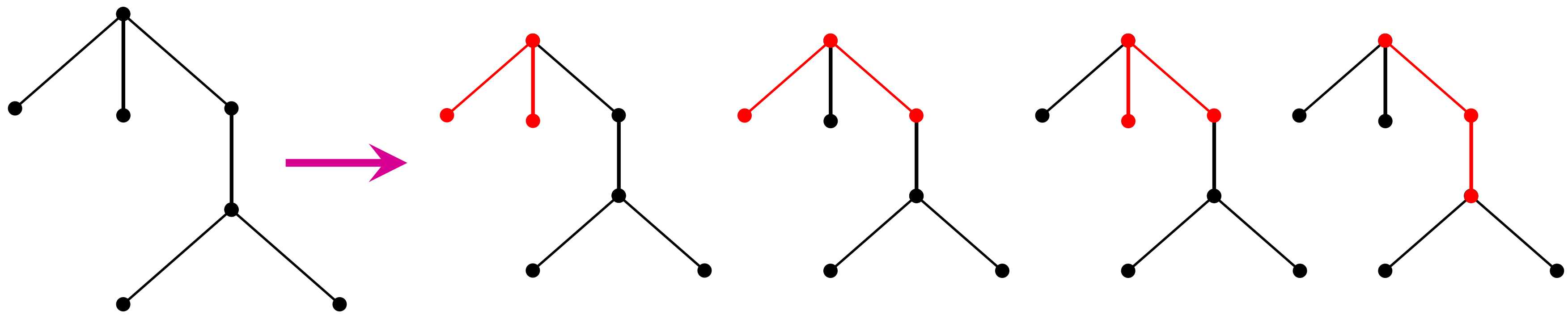
rooted tree T
with $n = 7$ vertices



Fun with combinatorics

rooted tree T
with $n = 7$ vertices

subtreees T' having the same root as T
and $m = 3$ vertices



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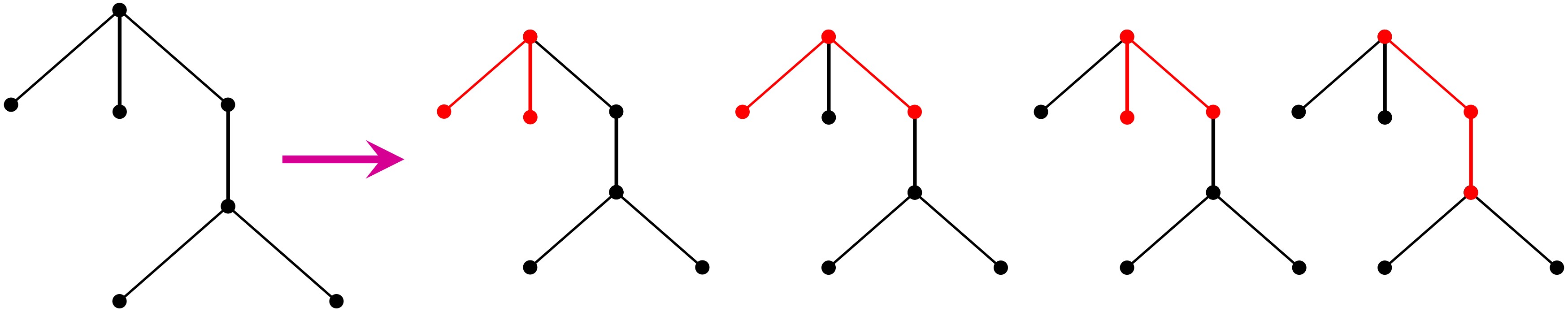

- for each vertex $v \in T'$ find $n_v(T)$ and $n_v(T')$
(number of vertices in the subtree rooted at v)

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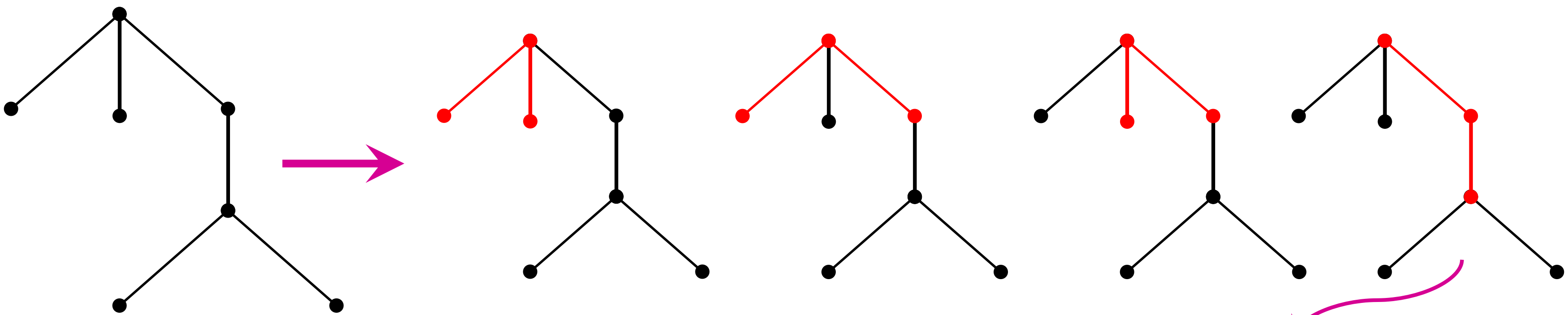

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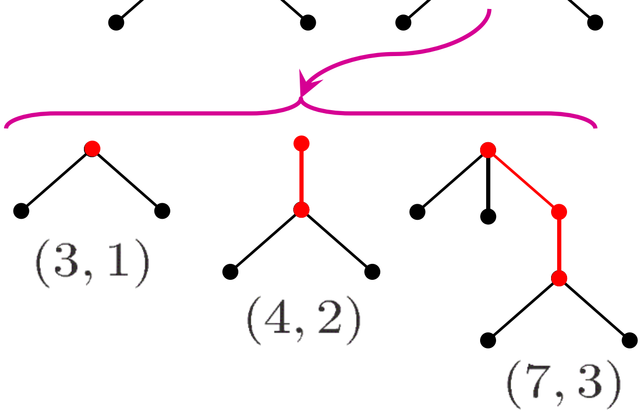

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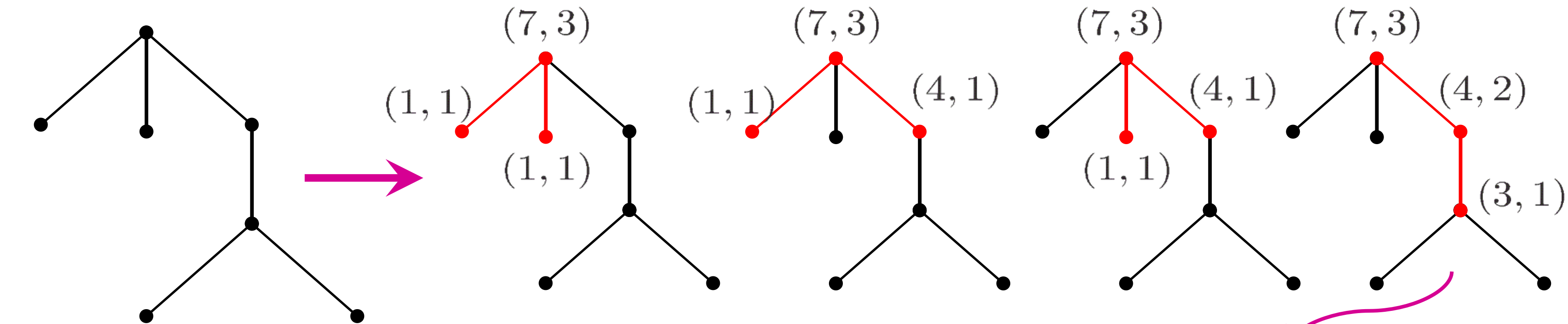

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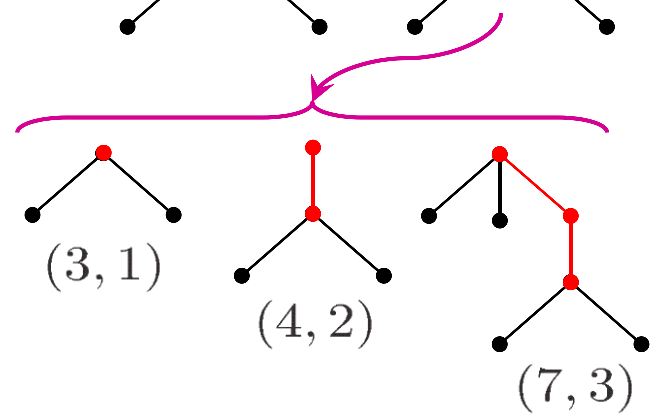

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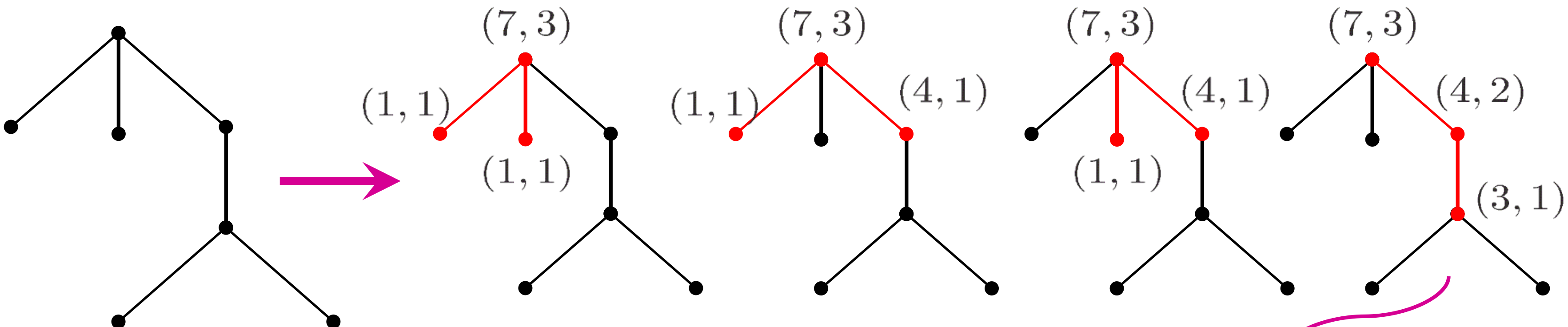
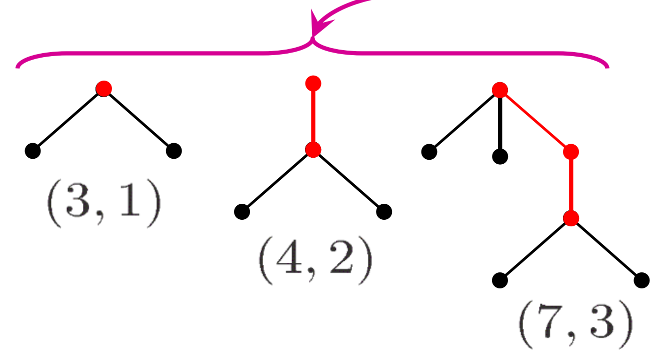


- for each vertex $v \in T'$ find $n_v(T)$ and $n_v(T')$ (number of vertices in the subtree rooted at v)
- take the ratio of the coefficients, multiply over all $v \in T'$ and sum over subtrees

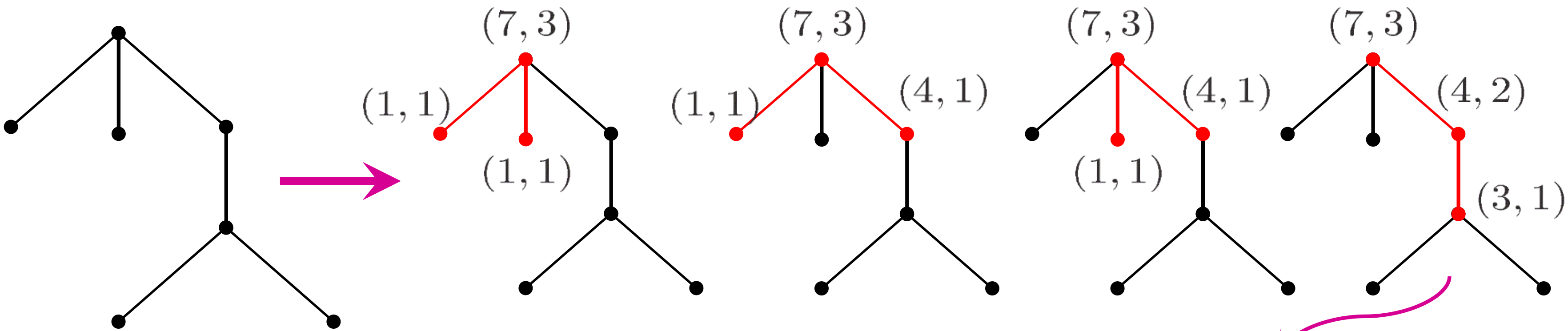
$$\frac{7}{3} + \frac{7 \cdot 4}{3} + \frac{7 \cdot 4}{3} + \frac{7 \cdot 4 \cdot 3}{3 \cdot 2} = 35 = \frac{7!}{3!\,4!}$$

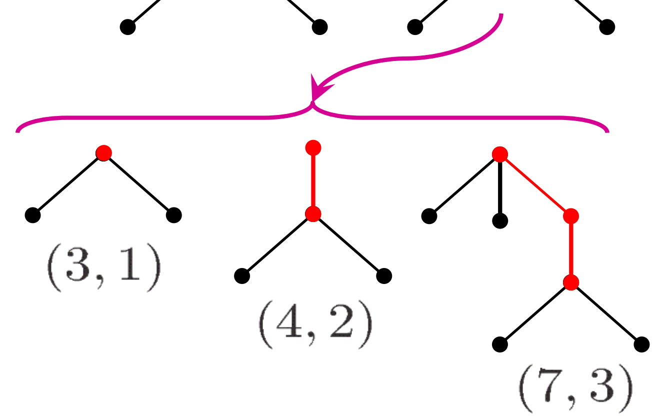


Fun with combinatorics

rooted tree T with $n = 7$ vertices

subtreees T' having the same root as T and $m = 3$ vertices





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Theorem: **for any rooted tree T with n vertices and $m < n$**

$$\sum_{T' \subset T} \prod_{v \in V_{T'}} \frac{n_v(T)}{n_v(T')} = \frac{n!}{m!(n-m)!}$$

Conclusions

The main result: the *explicit* form of the *modular completion* of the generating function of black hole degeneracies (DT invariants) at large volume attractor point for *arbitrary* divisor of CY.

$\longrightarrow$ $h_p(\tau)$ – higher depth (mixed) mock modular form

- Indications that S-duality is compatible with *refinement*.
- New unexpected results for combinatorics of trees.

Open problems and applications:

- Non-perturbative formulation of these results $\longrightarrow$ *integral equation on* $\widehat{h}_p(\tau)$*?*
- Understanding the completion from the point of view of world-volume theory on M5-brane wrapped on a reducible divisor.
- Geometric or physical meaning of the instanton generating function $\mathcal{G}$.
- Implications for black hole state counting $\longrightarrow$ restrictions on growth
- Vafa-Witten theory