Black hole degeneracies and

Siegel modular forms from Mathieu moonshine

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and ongoing work with Justin R. David

Introduction and Results

- We construct the Siegel modular forms associated with the theta lift of twisted elliptic genera of $K3$ orbifolded with g' corresponding to the conjugacy classes of the Mathieu group M_{23}
- These forms satisfy the required properties for them to be generating functions of $1/4$ BPS dyons of type II string theories
- Inverse of these Siegel modular forms admit a Fourier expansion with integer coefficients and the correct sign as predicted from black hole physics (as conjectured by Sen)

Introduction and Results

- The correct sign is observed for dyons for all the 7 CHL compactifications and also some non-geometric orbifolds of $K3$
- Using saddle point analysis to estimate the statistical entropy we encounter and fix a constant associated with the two kinds of Siegel modular forms related to each other by $Sp(2, \mathbb{Z})$ transformation.

Twisted Elliptic genus

There exists $\mathbb{Z}_N$ quotients of $K3$ corresponding to the CHL orbifolds listed below:

N	$h_{(1,1)}$	k
1	20	10
2	12	6
3	8	4
4	6	3
5	4	2
6	4	2
7	2	1
8	2	1

Let us refer to these $\mathbb{Z}_N$ action by g'.

• Let g' be action of this quotient,
the twisted elliptic genus of $K3$ is defined as $F^{(r,s)}(\tau, z)$

$$= \frac{1}{N}\mathrm{Tr}^{K3}_{RR;g'^r}\left((-1)^{F^{K3}+\bar{F}^{K3}} g'^s e^{2\pi i z F^{K3}} e^{2\pi i\tau(L_0-c/24)}\bar{q}^{-2\pi i\bar{\tau}(\bar{L}_0-c/24)}\right)$$

$$= \sum_{b=0}^{1}\sum_{m\geq 0\in\mathbb{Z}/N, l\in 2\mathbb{Z}+b} c_b^{(r,s)}(4m-l^2)e^{2\pi i m\tau}e^{2\pi i l z}$$

$$0\leq r,s,\leq (N-1)$$

The trace is taken over the Ramond sector and it is holomorphic in τ, z.

The twisted elliptic genera for the $\mathbb{Z}_N$ quotients of $K3$ by g' with $N = 2, 3, 5, 7$ have been written down in
David, Jatkar, Sen (2006)

Twisted elliptic genera and counting Black Hole degeneracy

Consider type II B/A theory on $K3 \times T^2/\mathbb{Z}_N$ where the $\mathbb{Z}_N$ action is g' on $K3$ and a shift of $1/N$ on one of the circles of T^2.

These compactifications preserve $\mathcal{N} = 4$ supersymmetry in $d = 4$.

This gives a class of new $\mathcal{N} = 4$ string vacua.

Each of these vacua admit $1/4$ BPS states.
These are dyons with both electric and magnetic charges.

For large charges they can be identified with supersymmetric black hole solutions.

The generating function for the degeneracy (index) of dyons in these $\mathcal{N} = 4$ theories is given by

$$-B_6 = -(-1)^{Q\cdot P} \int_{\mathcal{C}} \mathrm{d}\rho \mathrm{d}\sigma \mathrm{d}v \; e^{-\pi i (N\rho Q^2 + \sigma/N P^2 + 2vQ\cdot P)} \frac{1}{\tilde{\Phi}(\rho, \sigma, v)},$$

where $\mathcal{C}$ is a contour in the complex 3-plane. Q, P refer to the electric and magnetic charge of the dyons.

Dijkgraaf, Verlinde, Verlinde (1996), Jatkar Sen (2005), David, Jatkar, Sen (2006), David, Sen (2006), Dabholkar Nampuri (2006)

The contour $\mathcal{C}$ is defined over a 3 dimensional subspace of the 3 complex dimensional space $(\rho = \rho_1 + i\rho_2, \sigma = \sigma_1 + i\sigma_2, v = v_1 + iv_2)$.

$$\rho_2 = M_1, \qquad \sigma_2 = M_2, \qquad v_2 = -M_3,$$
$$0 \leq \rho_1 \leq 1, \qquad 0 \leq \sigma_1 \leq N, \qquad 0 \leq v_1 \leq 1.$$
$$M_1, M_2 >> 0, \quad M_3 << 0, \qquad |M_3| << M_1, M_2$$

$\tilde{\Phi}(\rho, \sigma, v)$ is the Siegel modular form associated with the twisted elliptic genus is given by

$$\tilde{\Phi}(\rho, \sigma, v) = e^{2\pi i(\tilde{\alpha}\rho + \tilde{\beta}\sigma + v)}$$

$$\prod_{b=0,1} \prod_{r=0}^{N-1} \prod_{\substack{k' \in \mathcal{Z} + \frac{r}{N}, l \in \mathcal{Z}, \\ j \in 2\mathcal{Z} + b \\ k', l \geq 0, j < 0 k' = l = 0}} (1 - e^{2\pi i(k'\sigma + l\rho + jv)})^{\sum_{s=0}^{N-1} e^{2\pi i s l/N} c_b^{r,s}(4k'l - j^2)}.$$

where

$$\tilde{\beta} = \frac{1}{N}, \qquad \tilde{\alpha} = 1$$

Here N is the order of the orbifold action.
This Siegel modular form transforms as a weight k form under appropriate sub-groups of $Sp(2, \mathbb{Z})$.

The modular property is defined as follows. Let

$$\Omega = \begin{pmatrix} \rho & v \\ v & \sigma \end{pmatrix}$$

Then

$$\tilde{\Phi}_k((C\Omega + D)^{-1}(A\Omega + B)) = [\det(C\Omega + D)]^k \tilde{\Phi}_k(\Omega)$$

where

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^T \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}_{4\times 4} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

A, B, C, D are 2×2 matrices with integer elements.

The weight k is related to the low lying coefficients of the twisted elliptic genus and is given by

$$k = \frac{1}{2} \sum_{0}^{N-1} c^{(0,s)}(0)$$

$-B_6$

In the context of N= 4 supersymmetric string theories in four dimensions the 6th helicity trace index B_6 which corresponds to 12 broken supersymmetries (1/4 BPS dyons) can be given by

$$B_6 = \frac{1}{6!}\mathrm{Tr}((-1)^{2h}(2h)^6)$$

where h is the third component of the angular momentum of a state in the rest frame, and the trace is taken over all states carrying a given set of charges.

From the above definition one can show that $-B_6$ is positive for single centered, spherically symmetric black holes.

The index $-B_6$ was first introduced by Kiritsis 97

A test for this degeneracy formula

The coefficients $-B_6(Q, P)$ certainly must be integers.
It was conjectured that:
from the fact that for single centered black holes,
due to spherical symmetry and the regularity of the horizon,
the only angular momentum it caries is from the fermionic zero modes.

$-B_6(Q, P)$ for single centered black holes must be positive.
Sen (2010)

We have $-B_6 \sim e^{S_{BH}}$, where S_{BH} is the extremal black hole entropy.

The sufficient condition which ensures the positivity property is that for charges which satisfy

$$Q \cdot P \geq 0, \qquad (Q \cdot P)^2 < Q^2 P^2, \qquad Q^2, P^2 > 0.$$

the coefficient $-B_6(Q, P)$ evaluated from the Fourier expansion of the Siegel modular form should be positive.

This gives a non-trivial condition on the Fourier expansion of the inverse of Siegel modular forms which are generating functions for the index $-B_6(Q, P)$

For the case of $1A$, (compactification of type II on $K3 \times T^2$) for a specific class of charges, this conjecture has been proved by Bringmann, Murthy (2013)

For the orbifolds corresponding to classes pA, $p = 2, 3, 5, 7,$ it has been verified by explicit computation of the Fourier coefficients of $-B_6(Q, P)$ for low lying charges.
Sen (2010)

We constructed the twisted elliptic genus for orbifolds corresponding to the conjugacy classes in M_{23} and $2B,\ 3B$

Using this we can explicitly evaluate the Fourier coefficients which evaluate $-B_6$ of the dyons for low lying charges.

Results for 11A, 4B, 2B are listed.

$(Q^2,\ P^2)\quad \backslash Q\cdot P$	-2	0	1	2	3
(1/2, 2)	-512	176	8	0	0
(1/2, 4)	-1536	896	80	0	0
(1/2, 6)	-4544	3616	480	0	0
(1/2, 8)	11752	12848	2176	24	0
(1,4)	-4592	5024	832	16	0
(1,6)	-13408	22464	36786	224	0
(1,8)	-33568	88320	26176	1760	0
(3/2, 6)	-37330	112316	36786	2998	38
(3/2, 8)	-80896	491920	196960	23616	592

Table: Some results for the index $-B_6$ for the $4B$ orbifold of $K3$ for different values of Q^2, P^2 and $Q \cdot P$

$(Q^2,\ P^2)\quad \backslash Q\cdot P$	-2	0	1	2	3
(2/11, 2)	-50	10	0	0	0
(2/11, 4)	-100	30	0	0	0
(2/11, 6)	-200	82	1	0	0
(4/11, 6)	-400	276	18	0	0
(6/11, 6)	-800	806	83	0	0
(6/11, 8)	-1438	2064	314	2	0
(6/11, 10)	-2584	4962	937	16	0
(6/11, 12)	-4328	11132	2558	72	0
(6/11, 22)	-34000	366378	139955	12760	114

Table: Some results for the index $-B_6$ for the $11A$ orbifold of $K3$ for different values of Q^2, P^2 and $Q \cdot P$

$(Q^2,\ P^2)\quad \backslash Q\cdot P$	-2	0	1	2	3
(1/2, 2)	320	288	24	0	0
(1/2, 4)	0	512	256	0	0
(1/2, 6)	-752	1120	888	48	0
(1/2, 8)	384	3328	2048	384	0
(1,4)	32	4416	2240	32	0
(1,6)	-2304	22464	13248	224	0
(1,8)	5920	42944	27328	5920	64
(3/2, 6)	-2008	102380	66172	9032	28
(3/2, 8)	59392	372736	243712	59392	2048

Table: Some results for the index B_6 for the $2B$ orbifold of $K3$ for different values of Q^2, P^2 and $Q\cdot P$

Remarks

It is interesting to note that the non-geometric orbifolds $11A, 23A, 23B, 2B, 3B$ also satisfy the positivity constraints.

The test for positivity of $-B_6$ was also carried out for some torus orbifolds and some of the values turned out to be negative.

Saddle point analysis

One can do the integral for $-B_6$ using saddle point analysis for large values of $Q^2,\ P^2,\ Q\cdot P$

$$
\begin{aligned}
d(Q,P) \quad = \quad & \frac{(i)^{-k}}{NC_1}(-1)^{Q\cdot P+1} \\
& \int_{\mathcal{C}'} d\rho d\sigma dv\ (2v-\rho-\sigma)^{-k-3}\ e^{-\pi i(\tilde\rho Q^2+\tilde\sigma P^2+2\tilde v Q\cdot P)} \\
& \frac{1}{\hat\Phi(\rho,\sigma,v)}
\end{aligned}
$$

$$
\tilde\rho = \frac{1}{N}\frac{1}{2v-\rho-\sigma}, \quad \tilde\sigma = N\frac{v^2-\rho\sigma}{2v-\rho-\sigma}, \quad \tilde v = \frac{v-\rho}{2v-\rho-\sigma},
$$

$$\tilde{\Phi}_k(\tilde{\rho}, \tilde{\sigma}, \tilde{v}) = -i^k C_1 (2v - \rho - \sigma)^k \hat{\Phi}_k(\rho, \sigma, v).$$

$$C_1 = 1 \quad \text{for } K3$$

In the limit $v \to 0$

$$\hat{\Phi}_k(\rho, \sigma, v) \sim -4\pi v^2 h^{k+2}(\rho) h^{k+2}(\sigma)$$

where h^{k+2} is the eta product related to $\hat{\Phi}_k$.

Using saddle point analysis we can find the degeneracy and entropy for large charges

$$S^1(Q,P) = \pi\sqrt{Q^2P^2-(Q\cdot P)^2}$$
$$+\ln(h^{(k+2)}(\tau))+\ln(h^{(k+2)}(-\bar{\tau}))-(k+2)\ln(2\tau_2)$$
$$-\ln(NC_1)$$

with

$$\tau_1 = \frac{Q\cdot P}{P^2}, \qquad \tau_2 = \frac{1}{P^2}\sqrt{Q^2P^2-(Q\cdot P)^2}$$

for an orbifold corresponding to $[M_{23}]$

$$C_1 = N^{\frac{k+2}{2}}$$

Comparison of statistical entropy and asymptotic entropy

$(Q^2, P^2\, Q.P)$	d^{stat}	S^{stat}	S^1	δ	δ_{C_1}
(1, 2, 0)	2164	7.67971	7.28409	5.15	50.28
(1, 2, 1)	360	5.8861	5.34077	9.26	68.14
(1, 4, 1)	4352	8.37839	8.39542	-0.2	41.16
(2, 4, 0)	198144	12.1967	11.727	3.85	32.27
(1, 6, 1)	36024	10.4919	11.1568	-6.33	26.7
(3, 6, 0)	15219528	16.5381	16.1699	2.22	23.18
(3, 6, 3)	149226	11.9132	11.624	2.43	31.52
(3, 6, 4)	2164	7.67971	7.28409	5.15	50.28

Table: 2A orbifold; δ is the relative percentage difference between S^{stat} and S^1, while δ_{C_1} is the difference in percentage if $-\ln NC_1$ is not included

The modular functions which determine the sub-leading corrections are given by

Conjugacy Class	$h^{(k+2)}(\rho)$
pA	$\eta^{k+2}(\rho)\eta^{k+2}(p\rho)$
4B	$\eta^4(4\rho)\eta^2(2\rho)\eta^4(\rho)$
6A	$\eta^2(\rho)\eta^2(2\rho)\eta^2(3\rho)\eta^2(6\rho)$
8A	$\eta^2(\rho)\eta(2\rho)\eta(4\rho)\eta^2(8\rho)$
14A	$\eta(\rho)\eta(2\rho)\eta(7\rho)\eta(14\rho)$
15A	$\eta(\rho)\eta(3\rho)\eta(5\rho)\eta(15\rho)$

Table: $p \in \{1, 2, 3, 5, 7, 11, 23\}$

Summary

The leading term in the asymptotic formula for the entropy is the Hawking Bekenstein entropy of the corresponding black hole.

The moduli dependent sub-leading term gives the contribution of entropy from the Gauss Bonnet term in the effective action and matches with the Wald formula.

The constant term $-\ln NC_1$ predicts a constant in the Gauss Bonnet term in the effective action.
Would be nice to fix it from the bulk side!

- The Fourier coefficients of the Siegel modular forms associated with the twisted elliptic genus of $K3$ have the correct sign as predicted from black hole physics when the $K3$ is orbifolded by $g' \in [M_{24}]$, else there are violations.

THANK YOU