Mathieu moonshine and K3 surfaces

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Mathieu Moonshine conjecture

In 2010, Eguchi, Ooguri and Tachikawa observed that when the elliptic genus of a K3 surface, the Jacobi form $2\phi_{0,1}(z;\tau)$ of weight 0 and index 1, is decomposed into a sum of the characters of the N = 4 superconformal algebra with central charge $c = 6$,

$$2\phi_{0,1}(z;\tau) = -2ch_{1,0,0}(z;\tau)+20ch_{1,1,0}(z;\tau)+2\sum_{n=1}^{\infty} A_n ch_{1,n,0}(z;\tau),$$

the first few coefficients A_n are sums of dimensions of irreducible representations of the largest Mathieu group M_{24}.

Mathieu moonshine conjecture

- Let
$$\Sigma(q) = q^{-\frac{1}{8}}(-2 + 2\sum_{n=1}^{\infty} A_n q^n). \quad (1)$$
It is a mock modular form of weight $\frac{1}{2}$.
- Their observation suggests the existence of a graded M_{24}-module $K = \sum_{n=0}^{\infty} K_n q^{n-1/8}$ with graded dimension $\Sigma(q)$. It is Mathieu analogue to the modular function $J(q)$ in the monstrous moonshine
- It is a special case of umbral moonshine. (2013, Cheng, Duncan and Harvey)

Mathieu McKay-Thompson series

- Subsequently the analogues to McKay-Thompson series in monstrous moonshine were proposed in several works Cheng and Duncan, Eguchi and Hikami, Gaberdiel, Hohenegger and Volpato.
- The McKay-Thompson series for g in M_{24} are of the form

$$\Sigma_g(q) = q^{-\frac{1}{8}} \sum_{n=0}^{\infty} q^n \,\mathrm{Trace}_{K_n}\, g = \frac{e(g)}{24}\Sigma_e(q) - \frac{f_g}{\eta(q)^3}$$

where $\Sigma_e(q) = \Sigma(q)$, $e(g)$ is the character of the 24-dimensional permutation representation of M_{24}, the series f_g is a certain explicit modular form of weight 2 for some subgroup $\Gamma^0(N_g)$ of $SL(2, \mathbb{Z})$ and η is the Dedekind eta function.

Gannon's result

Terry Gannon has proven that these McKay-Thompson series indeed determine a M_{24}-module:

Theorem (Gannon2012)

The McKay-Thompson series determine a virtual graded M_{24}-module $K = \sum_{n=0}^{\infty} K_n q^{n-1/8}$. For $n \geq 1$, the K_n are honest (and not only virtual) M_{24}-representations.

Symplectic automorphism groups of K3 surfaces

A complex automorphism g of a K3 surface X is called symplectic if it preserve the holomorphic symplectic 2-form,
The finite symplectic automorphism groups of K3 surfaces are all isomorphic to subgroups of the Mathieu group M_{23} of a particular type. M_{23} is isomorphic to a one-point stabilizer for the permutation action of M_{24} on 24 elements.

Theorem (Mukai1988)

A finite group H acting symplectically on a K3 surface is isomorphic to a subgroup of M_{23} with at least five orbits on the regular permutation representation of the Mathieu group $M_{24} \supset M_{23}$ on 24 elements.

Relation to K3 surfaces

Thomas Creutzig and Gerald Höhn showed that

- the complex elliptic genus of a K3 surface can be given the structure of a virtual M_{24}-module which is compatible with the H-module structure for all possible groups H of symplectic automorphisms of K3 surfaces under restriction.
- it is the graded character of a natural virtual module for the N $=4$ super conformal vertex algebra.

Relation to K3 surfaces

If g is a symplectic automorphism of a K3 surface, the functions $\Sigma_g(q)$ admit a geometric interpretation in terms of K3 surfaces.

Theorem (Creutzig-Höhn2012)

For a non-trivial finite symplectic automorphism g acting on a K3 surface X, the equivariant elliptic genus and the twining character determined by the McKay-Thompson series of Mathieu moonshine agree, i.e. one has

$$Ell_{X,g}(z;\tau) = \frac{e(g)}{12}\phi_{0,1} + f_g\phi_{-2,1}.$$

Goal

- We will construct a graded vector space

$$\mathcal{A}_X(q) = \sum_{n=1}^{\infty} \mathcal{A}_n(X) q^{n-\frac{1}{8}}.$$

The graded dimension of $\mathcal{A}_X(q)$ is $\Sigma(q) + 2q^{-\frac{1}{8}}$. For a finite symplectic automorphism g acting on a K3 surface X,

$$\Sigma_g(q) + 2q^{-\frac{1}{8}} = \sum_{n=1}^{\infty} q^{n-\frac{1}{8}} trace_{\mathcal{A}_n(X)} g = trace_{\mathcal{A}_X(q)} g.$$

- We will show $2A_n$ are even.(2012 Gannon)

original construction

Let Ω_X^{ch} be the chiral de Rham complex on X, by the following fact

- $H^0(X, \Omega_X^{ch})$ is the simple $N = 4$ vertex algebra with central charge $c = 6$;
- $H^2(X, \Omega_X^{ch}) \cong H^0(X, \Omega_X^{ch})$;
- The graded dimension of $H^0(X, \Omega_X^{ch}) - H^1(X, \Omega_X^{ch}) + H^2(X, \Omega_X^{ch})$ is the elliptic genus of X.
- $H^1(X, \Omega_X^{ch})$ is a representation of $H^0(X, \Omega_X^{ch})$

If the representaion is unitary, we immediantly get the decomposition formular of the elliptic genus. However we can not prove it. So we take a filtration $F^i H^1(X, \Omega_X^{ch})$ of $H^1(X, \Omega_X^{ch})$, such that its associated graded object is a unitary representation of $H^0(X, \Omega_X^{ch})$. The space of highest weight vectors is what we want.

A free system

Let V be an $2k$ dimensional complex vector space.
Let $\mathcal{W}(V)$ is be the vertex algebra generated by
even elements $\beta^{x'}(z)$, $\alpha^{x'}(z)$, $x' \in V$ $x \in V^*$
odd elements $b^{x'}(z)$, $c^{x}(z)$ $x' \in V$, $x \in V^*$
with their nontrivial OPEs are

$$\beta^{x'}(z)\alpha^{x}(w) \sim \langle x', x\rangle (z-w)^{-2}.$$

$$b^{x'}(z)c^{x}(w) \sim \langle x', x\rangle (z-w)^{-1}.$$

Here for $P = \beta, \alpha, b$ or c, we assume $a_1 P^{x_1} + a_2 P^{x_2} = P^{a_1 x_1 + a_2 x_2}$.
$\mathcal{W}(V)$ is a system of free bosons and free fermionics.

An Hermitian form on $\mathcal{W}(V)$

If there is an positive definite Hermitian form on V. Let $x_1', \cdots x_N'$ be an orthonormal basis of V and $x_1, \cdots x_N$ be its dual basis in V^*. $\mathcal{W}(V)$ is equipped with a positive definite Hermitian form $(-,-)$ with the following property:

$$(\beta_{(n)}^{x_i'}A, B) = (A, \alpha_{(-n)}^{x_i}B), \quad \text{for any } n \in \mathbb{Z}, n \neq 0, \forall A, B \in \mathcal{W}(V);$$
$$(b_{(n)}^{x_i'}A, B) = (A, c_{(-n-1)}^{x_i}B), \quad \text{for any } n \in \mathbb{Z}, \forall A, B \in \mathcal{W}(V).$$

A bilinear form

The state space of $\mathcal{W}(V)$ is the supercommutative ring freely generated by $c_{(n)}^{x_i}, b_{(n)}^{x_i'}, \alpha_{(n)}^{x_i}, \beta_{(n)}^{x_i'}$, $n < 0$. The is a nondegenerate symmetric bilinear form

$$\langle -, - \rangle : \mathcal{W}(V) \times \mathcal{W}(V) \to \mathbb{C},$$

given by $\langle \beta_n^{x'}, \alpha_m^x \rangle = \delta_m^n x'(x)$, $\langle b_n^{x'}, c_m^x \rangle = \delta_m^n x'(x)$

An involution

Let $\mathcal{I} : \mathcal{W}(V) \to \mathcal{W}(V)$ be the antilinear vertex algebra automorphism given by

$$\mathcal{I}(b^{x_i'}_{(n)}) = c^{x_i}_{(n)}, \qquad \mathcal{I}(c^{x_i}_{(n)}) = b^{x_i'}_{(n)},$$
$$\mathcal{I}(\beta^{x_i'}_{(n)}) = \alpha^{x_i}_{(n)}, \qquad \mathcal{I}(\alpha^{x_i}_{(n)}) = \beta^{x_i'}_{(n)}.$$

We have

$$\mathcal{I}^2(A) = A,$$
$$\langle A, B \rangle = (A, \mathcal{I}(B)).$$

N=4 superconformal vertex algebra

Let $\mathcal{V}(V)$ be $N = 4$ superconformal vertex algebra with central charge $c = 6k$ generated by the following elements:

$$L(z) = \sum_{i=1}^{2k} (:\beta^{x_i'}(z)\alpha^{x_i}(z): - :b^{x_i'}(z)\partial c^{x_i}(z):),$$

$$J(z) = -\sum_{i=1}^{2k} :b^{x_i'}(z)c^{x_i}(z):, \quad G(z) = \sum_{i=1}^{2k} :b^{x_i'}(z)\alpha^{x_i}(z):,$$

$$D(z) = \sum_{i=1}^{k} :b^{x_{2i-1}'}(z)b^{x_{2i}'}(z):, \quad E(z) = \sum_{i=1}^{k} :c^{x_{2i-1}}(z)c^{x_{2i}}(z):.$$

$$Q(z) = \sum_{i=1}^{2k} :\beta^{x_i'}(z)c^{x_i}(z):, \quad B(z) = Q_{(0)}D(z), \quad C(z) = G_{(0)}E(z):$$

N=4 superconformal vertex algebra

We have

$$Q^*_{(n)} = G_{(-n+1)}, \qquad J^*_{(n)} = J_{(-n)},$$
$$L^*_{(n)} = L_{(-n+2)} - (n-1)J_{(-n+1)}, \qquad D^*_{(n)} = -E_{(-n)}.$$
$$B^*_{(n)} = C_{(-n+1)},$$

So $\mathcal{W}(V)$ is a unitary representation of $\mathcal{V}(V)$.

$$\mathcal{I}(Q_{(n)}) = G_{(n)}, \qquad \mathcal{I}(J_{(n)}) = -J_{(n)},$$
$$\mathcal{I}(L_{(n)}) = L_{(n)} + nJ_{(n-1)}, \qquad \mathcal{I}(D_{(n)}) = E_{(n)}.$$
$$\mathcal{I}(B_{(n)}) = C_{(n)},$$

$\mathcal{I}$ maps irreducible representation of $\mathcal{V}(V)$ to irreducible representation of $\mathcal{V}(V)$.

Irreducible unitary representations

For an irreducible unitary representation $M_{k,h,l}$ of $N=4$ vertex algebra $\mathcal{V}(V)$ with central charge $c=6k$, there is a unique element (lowest weigbt vector) satisfying condition(*)

$$L_{(n)}v=0, n>1; \quad J_{(n)}v=0, n>0;$$

$$G_{(n)}v=0, \quad Q_{(n)}v=0, n>0;$$

$$B_{(n)}v=0; \quad C_{(n)}v=0, n\geq 0.$$

$$E_{(n)}v=0, n\geq 0; \quad D_{(n)}v=0, n>0;$$

$M_{k,h,l}$ is labeled by the conformal weight h and fermionic number of v

$$L_{(1)}v=hv, \quad J_{(0)}v=lv.$$

Character

There exist two types of unitary representations of $\mathcal{V}(V)$:
massless (BPS) : $h = 0$, $l = 0, 1, \cdots, k$
massive (non-BPS): $h > 0$, $l = 0, 1, \cdots, k - 1$.
The character of a representation M of the $N = 4$ vertex algebra is defined by

$$ch_V(z; \tau) = (-y)^{-k} trace_M (-y)^{J_{(0)}} q^{L_{(1)}}.$$

Let $ch_{k,h,l}(z; \tau)$ be the character of the representation $M_{k,h,l}$. We have $ch_{k,h+1,l}(z; \tau) = ch_{k,1,l}(z; \tau) q^h$, for $h \geq 0$.

Elliptic genus

For a complex manifold X, let $\mathcal{W}(TX)$ be the vector bundle of vertex algebra over X, with its fibre at $x \in X$ is $\mathcal{W}(T_x X)$.

$$\mathcal{W}(TX) \cong Sym^*(\bigoplus_{n=1}^{\infty}(T_{q^n} \oplus T^*_{q^n})) \bigotimes \wedge^*(\bigoplus_{n=1}^{\infty}(T_{-y^{-1}q^n} \oplus T^*_{-yq^{n-1}}))$$

The complex elliptic genus of X is

$$Ell_X(z;\tau) = y^{-\frac{\dim X}{2}} \sum (-1)^i \, \mathrm{Trace}_{H^i(X,\mathcal{W}(TX))} (-y)^{J_{(0)}} q^{L_{(1)}}.$$

For an automorphism g of X, the equivariant elliptic genus of X is

$$Ell_{X,g}(z;\tau) = y^{-\frac{\dim X}{2}} \sum (-1)^i \, \mathrm{Trace}_{H^i(X,\mathcal{W}(TX))} \, g(-y)^{J_{(0)}} q^{L_{(1)}}$$

Here $q = e^{2\pi i\tau}$ and $y = e^{2\pi iz}$

HyperKähler manifold

In this talk, we assume X is a HyperKähler manifold with dimension $2k$: a Kähler manifold with its holonomy group $Sp(k)$. Some properties:

- X has a holomorphic symplectic form w. If ω is the Kähler form, we can choose w such that $\frac{1}{k!k!}w^k \wedge \bar{w}^k = \frac{1}{(2k)!}\omega^{2k}$
- Its Ricci curvature vanishes.
- Its elliptic genus $Ell_X(z;\tau)$ is a Jacobi form.
- $H^0(X, \mathcal{W}(TX)) \cong \mathcal{W}(T_xX)^{Sp(k)}$, so Q, G, J, L, D, E, the generators of $N = 4$ vertex algebra are global sections of $\mathcal{W}(TX)$.

Example: K3 surface.

Duality operator 1

Let $\Omega^{0,l}(X, \mathcal{W}(TX))$ be the space of $(0, l)$ differential forms on X with values in $\mathcal{W}(TX)$.
The Käher form

$$\omega = \sqrt{-1} \sum \varphi_i \wedge \bar{\varphi}_i$$

in terms of a unitary coframe $\varphi_1, \cdots, \varphi_{2k}$.
For a set $I = \{i_1, \cdots, i_l\} \subset \{1, \cdots, 2k\}$, let $\bar{\varphi}_I = \bar{\varphi}_{i_i} \wedge \cdots \bar{\wedge} \varphi_{i_l}$ let $\bar{I} = \{1, \cdots, 2k\} - I$, let $\epsilon_I = \pm 1$ is the number such that

$$\epsilon_I \bar{\varphi}_I \wedge \bar{\varphi}_{\bar{I}} = \frac{1}{k!} \bar{w}^k.$$

Let

$$* : \Omega^{0,l}(X, \mathcal{W}(TX)) \to \Omega^{0,2k-l}(X, \mathcal{W}(TX))$$

such that for $\eta = \sum \eta_I \bar{\varphi}_I \in \Omega^{0,l}(X, \mathcal{W}(TX))$, $*\eta = \sum \epsilon_I J(\eta_I) \bar{\varphi}_{\bar{I}}$
We have

$$**\eta = (-1)^l \eta$$

Duality operator 2

- The bilinear form on $\mathcal{W}(T_x X)$ induce the bilinear form

$$\Omega^{0,l}(X, \mathcal{W}(TX)) \times \Omega^{0,2k-l}(X, \mathcal{W}(TX)) \to \mathbb{C}$$

$$\langle \eta, \psi \rangle = \int_X \sum_I \langle \eta, \psi_{\bar{I}} \rangle \frac{1}{k!} w^k \wedge \varphi_I \wedge \varphi_{\bar{I}}$$

- The Kähler metric induces an Hermitian metric on $\mathcal{W}(TX)$, so we have an Hermitian metric on $\Omega^{0,l}(X, \mathcal{W}(TX))$

$$(\eta, \eta') = \int_X \sum_i (\eta_I, \eta'_I) \frac{1}{k!k!} w^k \bar{w}^k$$

- We have

$$\langle \eta, *\psi \rangle = (\eta, \psi)$$

Duality operator 3

- We can use the complex $(\Omega^{0,*}(X, \mathcal{W}(TX)), \bar{\partial})$ to compute the cohomology $H^i(X, \mathcal{W}(TX))$. The dual of $\bar{\partial}$ is

$$\bar{\partial}^* = - * \bar{\partial} * .$$

  Let $\Delta = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$ be the Laplacian, the cohomology of $\mathcal{W}(TX)$ is isomorphic to $\mathrm{Ker}\,\Delta$.
- $H^i(X, \mathcal{W}(TX))$ is a unitary representation of the $N = 4$ vertex algebra.
- $*$ gives an antilinear isomorphism from $H^i(X, \mathcal{W}(TX))$to $H^{2k-i}(X, \mathcal{W}(TX))$.
- $*$ maps irreducible representation of the $N = 4$ vertex algebra to irreducible representation of the $N = 4$ vertex algebra.

An operator on $\Omega^{0,*}(X, \mathcal{W}(TX))$

- Let $F_1 : \Omega^{0,*}(X, \mathcal{W}(TX)) \to \Omega^{0,*+1}(X, \mathcal{W}(TX))$,

  $$F_1 = \sum \langle R(x_l')x_i', x^j\rangle :: (\Gamma^{x_l} c^{x_i} : b^{x_j'} :_{(0)} + \frac{1}{2} :: \Gamma^{x_l}\Gamma^{x_i} : \beta^{x_j'} :_{(0)}).$$

  Here $\Gamma^x(z) = \sum_{n\neq 0} \frac{-1}{n}\alpha^x_{(n)} z^{-n}$ and R is the curvature of TX.
- The dual of F_1 is $F_1^* = - * F_1 *$.
- F_1 commute with the action of the $N = 4$ vertex algebra.

The operators F_i

F_i comes from the chiral de Rham complex Ω_X^{ch}.
The chiral de Rham complex on X has a soft resolution $(\Omega_X^{ch,*}, \bar{Q})$ by "tensor" Ω_X^{ch} with $\Omega_X^{0,*}$, the sheaf of smooth $(0,*)$ forms.
There is a canonical linear isomorphism

$$\Phi : \Omega^{0,*}(X, \mathcal{W}(TX)) \to \Omega_X^{ch,*}(X).$$

$$\Phi^*(\bar{Q}) = \bar{\partial} + F_1 + F_2 + \cdots$$

$\bar{Q}^2 = 0$ implies $\bar{\partial} F_1 + F_1 \bar{\partial} = 0$ and $\bar{\partial} F_2 + F_2 \bar{\partial} = F_1 F_1$.

A new cohomology

- $\bar{\partial} F_1 + F_1 \bar{\partial} = 0$. We get

$$\mathcal{F}_1 : H^*(X, \mathcal{W}(TX)) \to H^{*+1}(X, \mathcal{W}(TX)).$$

- There is $F_2 : \Omega^{0,*}(X, \mathcal{W}(TX)) \to \Omega^{0,*+1}(X, \mathcal{W}(TX))$, such that $\bar{\partial} F_2 + F_2 \bar{\partial} = F_1 F_1$. So

$$\mathcal{F}_1^2 = 0.$$

- $(H^*(X, \mathcal{W}(TX)), \mathcal{F}_1)$ is a complex.
  Let $\mathcal{H}^*(X)$ be its cohomology.

$$\mathcal{H}^*(X) \cong \operatorname{Ker} \mathcal{F}_1 \cap \operatorname{Ker} \mathcal{F}_1^*$$

A new cohomology

- $\mathcal{H}^i(X)$ is a unitary representation of the $N=4$ vertex algebra with central charge $c=6k$.
- $*$ gives an isomorphism $\mathcal{H}^i(X)$ to $\mathcal{H}^{2k-i}(X)$. And $**=(-1)^i$.
- $$\mathcal{H}^0(X) \cong \mathcal{W}(T_xX)^{\mathfrak{g}} \cong H^0(X, \Omega_X^{ch}),$$

  here $\mathfrak{g}$ is the space of algebraic vector fields on T_xX which preserve $w|_x$.
  $\mathcal{H}^0(X)$ contain the $N=4$ vertex algebra. We expect it is exactly the $N=4$ vertex algebra. It is true of $K3$ surface.

Space of highest weight vectors

- Let $\mathcal{A}^i_{h,l}(X)$ be the space of the vectors satisfying condition (*) (highest weight vector) of the unitary representation $\mathcal{H}^i(X)$ of the $N = 4$ vertex algebra with conformal weight h and fermionic number l.
- $E^l_{(0)}*$ gives an antilinear isomorphism form $\mathcal{A}^i_{h,l}(X)$ to $\mathcal{A}^{2k-i}_{h,l}(X)$.
- $E^l_{(0)} * E^l_{(0)}* = (-1)^{l+i}$.
- If $l + k$ is odd. $E^l_{(0)} * E^l_{(0)}* = (-1)^{l+i} = -1$. $\dim \mathcal{A}^k_{h,l}(X)$ is even.

Decomposition of elliptic genus 1

$$\mathcal{H}^i(X) = (-1)^k(\bigoplus_{l=0}^{k} M_{k,0,l} \otimes \mathcal{A}^i_{0,l}(X)) \oplus (\bigoplus_{n=1}^{\infty} \bigoplus_{l=1}^{k} M_{k,h,l} \otimes \mathcal{A}^i_{h,l}(X)).$$

Let $A^i_{h,l} = \dim \mathcal{A}^i_{h,l}(X)$
The elliptic genus of X is

$$Ell_X(z;\tau) = \sum (-1)^i \,\mathrm{Trace}_{\mathcal{H}^i(X)}(-y)^{J_{(0)}} q^{L_{(1)}}$$

$$= \sum_{l=0}^{k} (\sum_{i=0}^{2k} ((-1)^i A^i_{0,l}) ch_{k,0,l}(z;\tau) + \sum_{h=1}^{\infty} \sum_{l=0}^{k-1} (\sum_{i=0}^{2k} ((-1)^i A^i_{h,l}) ch_{k,n,l}(z;\tau))$$

Decomposition of elliptic genus 2

Theorem

If X is a HyperKähler manifold, then its elliptic genus has decomposition

$$Ell_X(z;\tau) = \sum_{l=0}^{k} a_{0,l} ch_{k,0,l}(z;\tau) + \sum_{l=0}^{k-1} \sum_{h=0}^{\infty} a_{h+1,l} q^h ch_{k,1,l}(z;\tau))$$

Here $a_{h,l}$ is even when $k+l$ is odd.

Elliptic genus of K3 surface 1

We apply the previous result of elliptic genus to $K3$ surface. If X is a K3 surface.

- Unitary representation When $k = 1$:
  massless: $M_{1,0,0}$ and $M_{1,0,1}$
  Massive: $M_{1,h,1}$
- $\mathcal{H}^2(X) \cong \mathcal{H}^0(X)$ is isomorphic to $M_{1,0,0}$.
- $\dim H^{0,1}(X) = \dim H^{2,1}(X) = 0$ and $\dim H^{1,1}(X) = 20$.
  The weight zero part of $\mathcal{H}^1(X)$ is $H^{1,1}(X)$.
  Obviously $H^{1,1}(X) = \mathcal{A}^1_{0,1}(X)$ and $A^1_{0,1}(X) = 0$.
- $\mathcal{H}^1(X, \Omega^{ch}_X) = M_{1,0,1} \otimes H^{1,1}(X) \oplus (\oplus_{h=1}^{\infty} M_{1,h,0} \otimes \mathcal{A}^1_{n,0}(X))$.

Elliptic genus of K3 surface 2

- We get the decomposition of the elliptic genus of K3 surface,

$$Ell_X(z;\tau) = -2ch_{1,0,0}(z;\tau) + 20ch_{1,0,1}(z;\tau) + 2\sum_{h=1}^{\infty} A_h ch_{1,n,0}(z;\tau).$$

Here $A_h = \frac{1}{2}A^1_{h,0}(X)$.

- Since $A^1_{h,0}$ is even, A_h is integer .
- We have constructed a graded vector space

$$\mathcal{A}_X(q) = \sum_{h=1}^{\infty} \mathcal{A}^1_{h,0}(X) q^{h-\frac{1}{8}},$$

which has the exact dimension for Mathieu moonshine.

Equivariant Elliptic genus of K3 surface

If g is a finite symplectic automorphism of X, elements of $\mathcal{H}^0(X)$ and $\mathcal{H}^2(X)$ are g invariant. Action of g commute with the action of $N = 4$ vertex algebra.

$$Ell_{X,g}(z;\tau) = -2ch_{1,0,0}(z;\tau) + ch_{1,0,1}(z;\tau)trace_{H^{1,1}(X)}g + \sum_{h=1}^{\infty} trace_{\mathcal{A}^1_{h,0}(X)}g\, ch_{1,h,0}(z;\tau).$$

Compare with Thomas Creutzig and Gerald Höhn's result

$$Ell_{X,g}(z;\tau) = \frac{e(g)}{12}\phi_{0,1} + f_g\phi_{-2,1},$$

we get :

$$\Sigma_g(q) + 2q^{\frac{1}{8}} = \sum_{n=1}^{\infty} q^{n-\frac{1}{8}} trace_{\mathcal{A}^1_{n,2}(X)}g = trace_{\mathcal{A}_X(q)}g.$$

Questions

By Gannon's result, $A_X(q)$ is an M_{24} module. How to construct a concrete M_{24} module structure on $A_X(q)$?
By our construction, we can construct a vector bundle on the moduli space of HyperKäher structure of K3 surfaces.
Can we glue the finite symplectic automorphism in different K3 surface?

Thank you!