Thompson and Penumbral Moonshine

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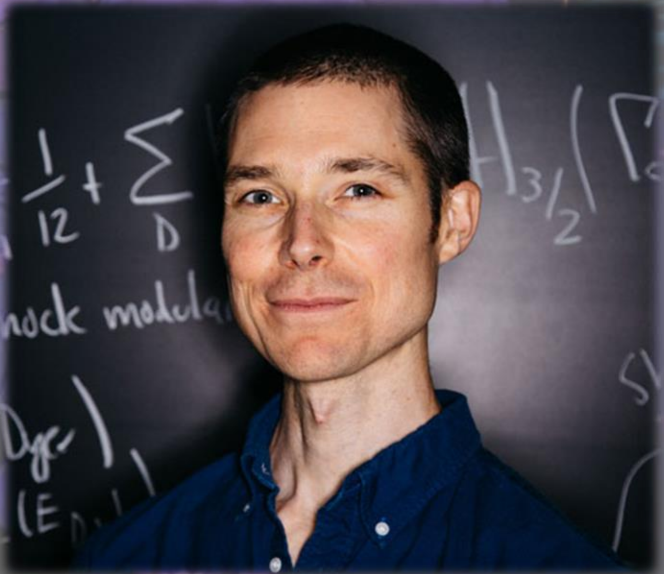
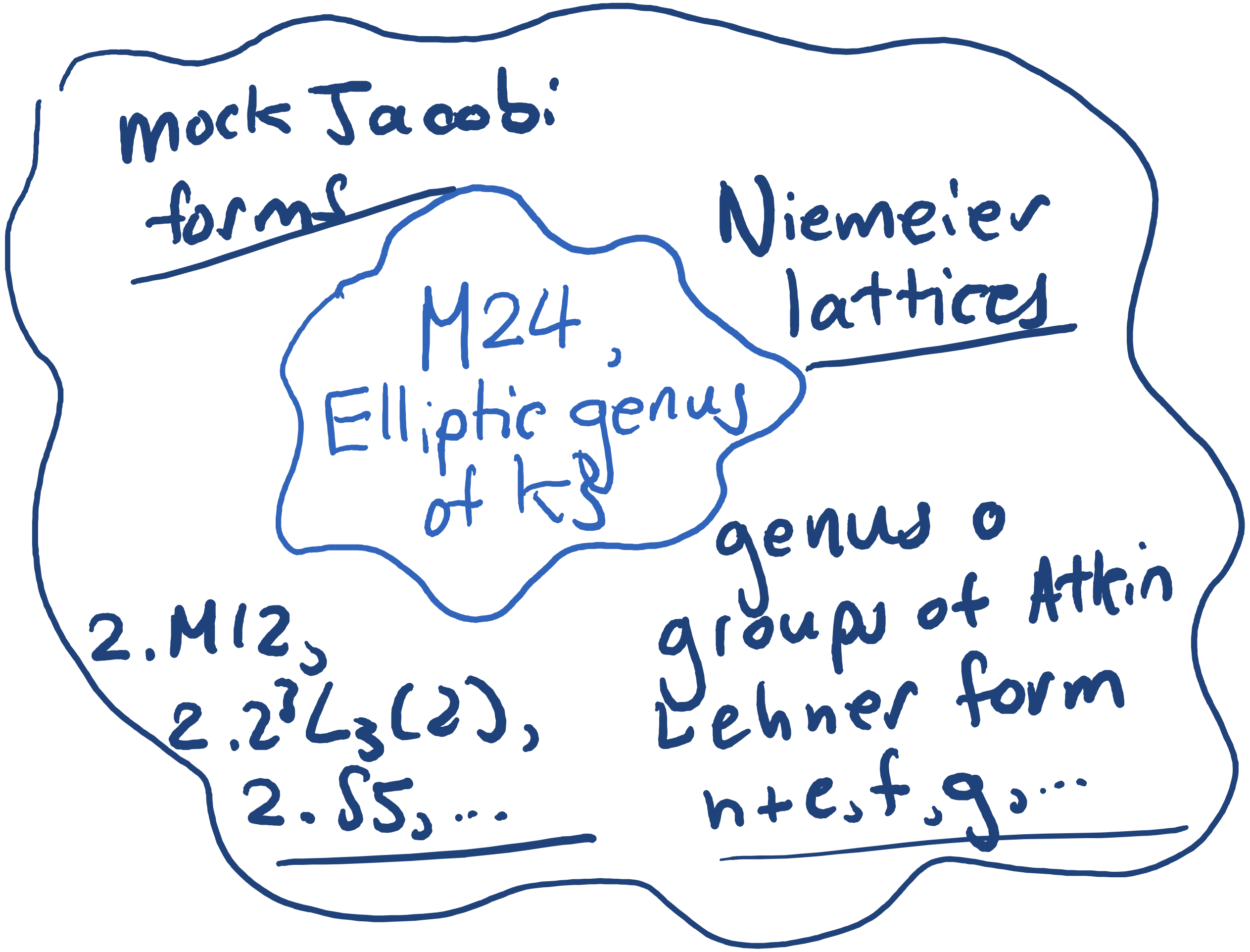


Umbral Moonshine



mock Jacobi forms

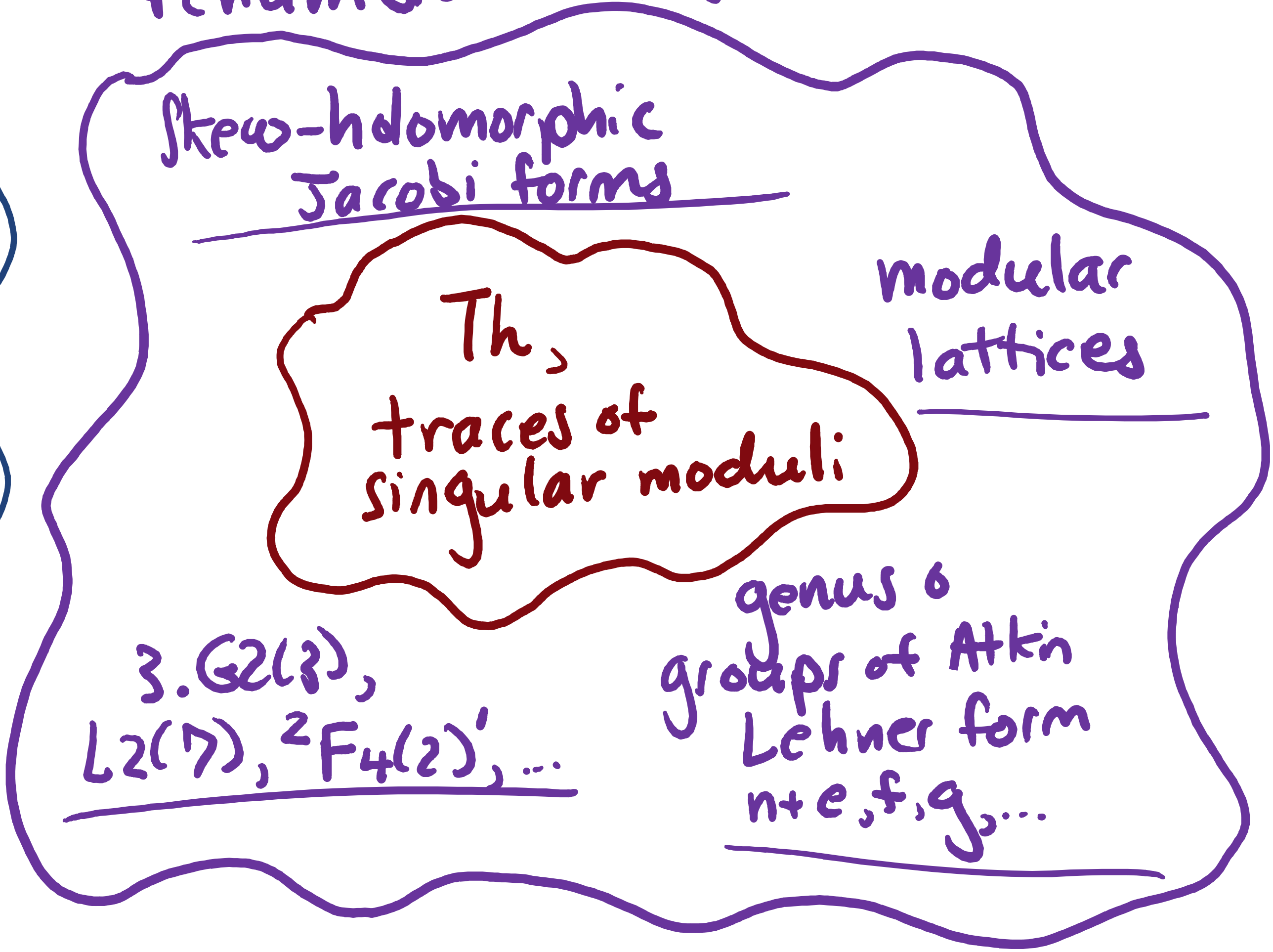
Niemeier lattices

M24,
Elliptic genus of K3

2.M12,
$2.2^3L_3(2)$,
2.S5, ...

genus 0 groups of Atkin Lehner form n+e,f,g,...

Penumbral Moonshine



Skew-holomorphic Jacobi forms

modular lattices

Th,
traces of singular moduli

3.G2(3),
L2(7), $^2F_4(2)'$, ...

genus 0 groups of Atkin Lehner form n+e,f,g,...

The automorphic objects

A weight 1, index *m* skew-holomorphic Jacobi form is equivalent to a vector valued modular form of weight ½ transforming under the Weil representation attached to the lattice $L_m = \sqrt{2m}\mathbb{Z}$, i.e. a function

$$\mathfrak{h} \to \mathbb{C}[\mathbb{Z}/2m\mathbb{Z}] \equiv \mathrm{span}_{\mathbb{C}}\{\mathfrak{e}_\mu \mid \mu = 0, \ldots, 2m-1\}$$

whose components

$$\mathfrak{f}(\tau) = \sum_{\mu \bmod 2m} f_\mu(\tau)\mathfrak{e}_\mu \quad \text{satisfy} \quad \mathfrak{f}\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^{1/2}\rho^{(m)}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\mathfrak{f}(\tau)$$

where $\rho^{(m)}(T)\mathfrak{e}_\mu = e^{\pi i\mu^2/2m}\mathfrak{e}_\mu$ and $\rho^{(m)}(S) = \frac{1}{\sqrt{2mi}} \sum_{\nu \bmod 2m} e^{-\pi i\mu\nu/m}\mathfrak{e}_\nu$

Examples

Theta functions $\theta_m(\tau) = \sum_{\mu \bmod 2m} \left(\sum_{\substack{n\in\mathbb{Z} \\ n\equiv\mu \bmod 2m}} q^{n^2/4m} \right) \mathfrak{e}_\mu$

Generating functions for traces of singular moduli, e.g. the scalar modular form

$$f_3(\tau) = q^{-3} - 248q + 26752q^4 - 85995q^5 + 1707264q^8 + O(q^9)$$

Zagier, Borcherds

can be repackaged into an $m=1$ skew-holomorphic Jacobi form.

The Thompson moonshine module has graded-dimension

$$\mathcal{F}_3(\tau) = 2\vec{f}_3(\tau) + 248\theta_1(\tau) = \begin{pmatrix} \overset{\overset{\mathbf{248}}{\nwarrow}}{248} + \overset{\overset{\mathbf{27000}+\overline{\mathbf{27000}}}{\uparrow}}{54000}q + \overset{\overset{\mathbf{1707264}\oplus\overline{\mathbf{1707264}}}{\nearrow}}{3414528}q^2 + 88660992q^3 + O(q^4) \\ 2q^{-3/4} - \underset{\underset{\mathbf{85995}\oplus\overline{\mathbf{85995}}}{\swarrow}}{171990}q^{5/4} - \underset{\underset{\mathbf{4096000}\oplus\overline{\mathbf{4096000}}}{\searrow}}{8192000}q^{9/4} + O(q^{17/4}) \end{pmatrix}$$

Moonshine for skew-holomorphic Jacobi forms through deconstruction

Start with a seed CFT/VOA $V^\natural$ with stress tensor $T(z)$

Find two additional dimension 2 operators $T^{(\pm)}(z)$ such that

$$T(z) = T^{(+)}(z) + T^{(-)}(z)$$

they sum to the original stress tensor

$$T^{(\pm)}(z)T^{(\pm)}(w) \sim \frac{c^\pm/2}{(z-w)^4} + \frac{2T^{(\pm)}(w)}{(z-w)^2} + \frac{\partial T^{(\pm)}(w)}{z-w}$$

they each have the OPE of a stress tensor

Organize the Hilbert space of the seed CFT into a direct sum of tensor products of modules of the two new Virasoro algebras. E.g. if $c^{(+)} < 1$ then

$$V^\natural = \bigoplus_h M(c^+, h) \otimes V_h^{(-)}$$

where the $M(c^+, h)$ are the (finite # of) highest weight modules of $\mathrm{Vir}(c^+)$

The $V_h^{(-)}$ often inherit symmetries from the seed CFT. Their counting functions often assemble into skew-holomorphic Jacobi forms.

Examples

Taking $V^\natural$ = Monster module and $c^+ = 1/2$ (Ising CFT) leads to a module with 2.B (Cent(2A) in M) symmetry whose counting function is an $m = 4$ skew-form

$$\eta(\tau)\mathrm{Tr}_{V_0^{(-)}} q^{L_0^{(-)}-c/24} = q^{-15/16} - q^{1/16} + \mathbf{96255}q^{17/16} + \mathbf{9550635}q^{33/16} + \cdots$$

$$\eta(\tau)\mathrm{Tr}_{V_{1/2}^{(-)}} q^{L_0^{(-)}-c/24} = \mathbf{4371}q^{9/16} + \mathbf{1139374}q^{25/16} + \mathbf{63532485}q^{41/16} + \cdots$$ (Hoehn)

$$\eta(\tau)\mathrm{Tr}_{V_{1/16}^{(-)}} q^{L_0^{(-)}-c/24} = \mathbf{96256}q + \mathbf{10506240}q^2 + \mathbf{410132480}q^3 + \cdots$$

Taking $V^\natural$ = Monster module and $c^+ = 4/5$ (3-state Potts CFT) leads to a module with 3.Fi24 (Cent(3A) in M) symmetry whose counting function is has $m = 30$ (Miyamoto)

Taking $V^\natural$ = Monster module and $c^+ = 2(n-1)/(n+2)$ the central charge of a parafermion model leads to a module with Cent(nX) symmetry whose counting function is a skew-form.

Taking $V^\natural$ = Conway module and $c^+ = 1/2$ also leads to an $m = 4$ skew-form.

Thompson moonshine and Borcherds lifting

The centralizer of 3C in the Monster is $\mathbb{Z}/3\mathbb{Z} \times \mathrm{Th}$

The 3C twisted sector $V^\natural_{3\mathrm{C}}$ of the Monster module has Thompson symmetry and graded dimension

$$Z(3\mathrm{C},1;\tau) = j(\tau)^{1/3} = q^{-1/3} + 248q^2 + 4124q^{5/3} + 34752q^{8/3} + \cdots$$

The weight ½ function of Zagier

$$f_{3,1}(\tau) = q^{-3} - 248q + 26752q^4 - 85995q^5 + \cdots = \sum c_1(n)q^n$$

which is closely related to the graded dimension of weight ½ Th moonshine, lifts to $Z(3\mathrm{C},1;\tau)$ in the sense of Borcherds:

$$Z(3\mathrm{C},1;\tau) = \mathrm{Lift}[f_{3,1}] = q^{-1/3}\prod_{n=1}^{\infty}(1-q^n)^{c_1(n^2)}$$

Can we promote this to a Th-equivariant structure?

$$\begin{tikzcd}
Z(3\mathrm{C}, 1; \tau) \arrow[r, "\text{twine}"] & Z(3\mathrm{C}, g; \tau) \\
f_{3,1}(\tau) \arrow[u, "\text{lift}"] \arrow[r, "\text{twine}"] & f_{3,g}(\tau) \arrow[u, "\text{lift}"']
\end{tikzcd}$$

Does the Th-module satisfy this equivariant notion of lifting?

$$\mathrm{Lift}[f_{3,g}(\tau)] = q^{-1/3} \exp\left(-\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} c_{g^k}(n^2)\frac{q^{nk}}{k}\right)$$

where $f_{3,g}(\tau) = \sum c_g(n)q^n$

No, but the next best thing happens.

Conjecture: there is *another* Th-module whose graded-dimension is Zagier's function $f_3(\tau) = q^{-3} - 248q + 26752q^4 + \cdots$ (up to minor adjustments) and which plays well with Borcherds lifting to $V^{\natural}_{3\mathrm{C}}$ in the sense of the previous slide.

Some properties:

1. Virtual representations needed at low order
2. McKay-Thompson series non-optimal (need to consider modular forms with poles at cusps other than infinity, have constructed almost all MT series using Rademacher series)

Physics outlook

Borcherds products/singular theta lifts appear in many physical settings (threshold corrections of N=2 d=4 string compactifications, elliptic genera of symmetric products, etc.) Can we borrow from existing ideas to make sense of Thompson moonshine in weight ½?

Strikingly similar to twisted denominator formulae in Borcherds/Hohn/Carnahan proof of generalized moonshine, and the more recent embedding of this structure into heterotic string theory by Paquette, Persson, Volpato. How much structure carries over (quantization functor, spacetime index, Lie algebra, CHL orbifolds, etc.)

Second-quantized BPS state count

$$T_g(\sigma) - T_g(\tau) = p^{-1} \exp\left(-\sum_{k=1}^{\infty} \sum_{m>0, n\in\mathbb{Z}} c_{g^k}(mn) \frac{p^{mk} q^{nk}}{k} \right) \qquad T_g(\tau) = \sum c_g(n) q^n$$

$$Z(3\mathrm{C}, g; \tau) = q^{-1/3} \exp\left(-\sum_{k=1}^{\infty} \sum_{n=1}^{\infty} c_{g^k}(n^2) \frac{q^{nk}}{k} \right) \qquad f_{3,g}(\tau) = \sum c_g(n) q^n$$

Math outlook

With Borcherds lifting machinery in place, can begin to take a more principled approach to realizing the MT series in weight ½ as generating functions for traces of singular moduli. Does $f_{3,g}(\tau)$ encode (suitably defined) traces of $Z(3\mathrm{C},g;\tau)^3$?

Extension to other cases of generalized moonshine. Many (all?) of the weight 0 functions appearing in generalized monstrous moonshine can be written as Borcherds lifts of suitable skew-holomorphic Jacobi forms. For example, for each prime *p* dividing the order of the Monster, one can construct an index *p* skew-form $f^{(p)}(\tau) = \sum c^{(p)}(n)q^n$ via Rademacher series whose Borcherds lift is the hauptmodul j_{p+} for $\Gamma_0(p)+$ (up to a constant)

$$j_{p+}(\tau) = q^{-h_p} \prod_{n=1}^{\infty} (1-q^n)^{c^{(p)}(n^2)}$$

Relation between the two versions of Th moonshine in weight ½?

Thanks!