Indefinite Theta Series and Applications

Caner Nazaroglu

Universität zu Köln

Based on
C.N. arXiv:1609.01224 and K. Bringmann, C.N. arXiv:1803.09270

Theta Functions for Positive Definite Lattices

Let Λ be a positive-definite, even lattice. Theta functions are defined as:

$$\Theta_\mu^Q(\tau, \mathfrak{z}) := \sum_{n \in \Lambda + \mu} q^{\frac{1}{2}n^2} e^{2\pi i \mathfrak{z} \cdot n}, \qquad \mu \in \Lambda^*/\Lambda$$

Theta Functions for Positive Definite Lattices

Let Λ be a positive-definite, even lattice. Theta functions are defined as:

$$\Theta^Q_\mu(\tau,\mathfrak{z}) := \sum_{n\in\Lambda+\mu} q^{\frac{1}{2}n^2} e^{2\pi i\mathfrak{z}\cdot n}, \qquad \mu\in\Lambda^*/\Lambda$$

- Elliptic transformations:

$$\Theta^Q_\mu(\tau,\mathfrak{z}+m) = \Theta^Q_\mu(\tau,\mathfrak{z}) \quad \text{for } m\in\Lambda,$$

$$\Theta^Q_\mu(\tau,\mathfrak{z}+m\tau) = q^{-\frac{1}{2}m^2} e^{-2\pi i\mathfrak{z}\cdot m}\,\Theta^Q_\mu(\tau,\mathfrak{z}) \quad \text{for } m\in\Lambda.$$

- Modular Transformations:

$$\Theta^Q_\mu(\tau+1,\mathfrak{z}) = e^{\pi i\mu^2}\,\Theta^Q_\mu(\tau,\mathfrak{z}),$$

$$\Theta^Q_\mu\left(-\frac{1}{\tau},\frac{\mathfrak{z}}{\tau}\right) = \frac{(-i\tau)^{N/2}}{\sqrt{|\Lambda^*/\Lambda|}} e^{\pi i\mathfrak{z}^2/\tau} \sum_{\nu\in\Lambda^*/\Lambda} e^{-2\pi i\mu\cdot\nu}\Theta^Q_\nu(\tau,\mathfrak{z})\,.$$

Theta Functions and Indefinite Signature Lattices?

Theta functions we defined do not converge for indefinite signature lattices, signature (N_+, N_-), since $q^{n^2/2}$ factors grow unbounded along negative-definite directions.

- However, we can form real analytic modular forms for indefinite signature lattices.

Theta Functions and Indefinite Signature Lattices?

Theta functions we defined do not converge for indefinite signature lattices, signature (N_+, N_-), since $q^{n^2/2}$ factors grow unbounded along negative-definite directions.

- However, we can form real analytic modular forms for indefinite signature lattices.
- Preserving holomorphy?
  Example: $\displaystyle\sum_{n\in(\Lambda+\mu)\cap\mathcal{C}} q^{n^2/2}e^{2\pi i n\cdot \mathfrak{z}}$ where $\mathcal{C}$ is a non-negative cone.

Theta Functions and Indefinite Signature Lattices?

Theta functions we defined do not converge for indefinite signature lattices, signature (N_+, N_-), since $q^{n^2/2}$ factors grow unbounded along negative-definite directions.

- However, we can form real analytic modular forms for indefinite signature lattices.
- Preserving holomorphy?
  Example: $\sum_{n \in (\Lambda+\mu)\cap \mathcal{C}} q^{n^2/2} e^{2\pi i n\cdot \mathfrak{z}}$ where $\mathcal{C}$ is a non-negative cone.

An even more specific example:

$$\sum_{n\in\Lambda+\mu} \left[\frac{1}{2^{N_-}} \prod_{j=1}^{N_-} \left(\operatorname{sgn}\left(c_j \cdot n\right) - \operatorname{sgn}\left(c_j' \cdot n\right)\right)\right] q^{\frac{1}{2}n^2}.$$

Lorentzian Lattices and Mock Modularity

For signature $(n-1,1)$ (Lorentzian) lattices,

$$\sum_{n\in\Lambda+\mu} \frac{1}{2}\left[\operatorname{sgn}(c\cdot n) - \operatorname{sgn}(c'\cdot n)\right] q^{n^2/2},$$

the modular completion for signature $(n-1,1)$ (Lorentzian) lattices is given by [Zwegers 2002]:

$$\operatorname{sgn}(c\cdot n) - \operatorname{sgn}(c'\cdot n) \to \boxed{\operatorname{erf}\left(c\cdot n\sqrt{2\pi\tau_2}\right) - \operatorname{erf}\left(c'\cdot n\sqrt{2\pi\tau_2}\right),}$$

where $c^2 = c'^2 = -1$ and $c\cdot c' < 0$.

Lorentzian Lattices and Mock Modularity

For signature $(n-1,1)$ (Lorentzian) lattices,

$$\sum_{n\in\Lambda+\mu} \frac{1}{2}\left[\mathrm{sgn}\,(c\cdot n) - \mathrm{sgn}\,(c'\cdot n)\right] q^{n^2/2},$$

the modular completion for signature $(n-1,1)$ (Lorentzian) lattices is given by [Zwegers 2002]:

$$\mathrm{sgn}\,(c\cdot n) - \mathrm{sgn}\,(c'\cdot n) \to \boxed{\mathrm{erf}\,(c\cdot n\sqrt{2\pi\tau_2}) - \mathrm{erf}\,(c'\cdot n\sqrt{2\pi\tau_2}),}$$

where $c^2 = c'^2 = -1$ and $c\cdot c' < 0$.

- $\mathrm{erf}(x\sqrt{\pi}) = \int_{\mathbb{R}} \mathrm{d}u\, e^{-\pi(x-u)^2}\mathrm{sgn}(u)$

Lorentzian Lattices and Mock Modularity

For signature $(n-1,1)$ (Lorentzian) lattices,

$$\sum_{n\in\Lambda+\mu}\frac{1}{2}\left[\operatorname{sgn}(c\cdot n)-\operatorname{sgn}(c'\cdot n)\right]q^{n^2/2},$$

the modular completion for signature $(n-1,1)$ (Lorentzian) lattices is given by [Zwegers 2002]:

$$\operatorname{sgn}(c\cdot n)-\operatorname{sgn}(c'\cdot n)\to\boxed{\operatorname{erf}\left(c\cdot n\sqrt{2\pi\tau_2}\right)-\operatorname{erf}\left(c'\cdot n\sqrt{2\pi\tau_2}\right),}$$

where $c^2=c'^2=-1$ and $c\cdot c'<0$.

- $\operatorname{erf}(x\sqrt{\pi})=\int_{\mathbb{R}}\mathrm{d}u\,e^{-\pi(x-u)^2}\operatorname{sgn}(u)$ Convolution: $\operatorname{sgn}(x)\star e^{-\pi x^2}$

Lorentzian Lattices and Mock Modularity

For signature $(n-1,1)$ (Lorentzian) lattices,

$$\sum_{n\in\Lambda+\mu} \frac{1}{2}\left[\operatorname{sgn}(c\cdot n) - \operatorname{sgn}(c'\cdot n)\right] q^{n^2/2},$$

the modular completion for signature $(n-1,1)$ (Lorentzian) lattices is given by [Zwegers 2002]:

$$\operatorname{sgn}(c\cdot n) - \operatorname{sgn}(c'\cdot n) \to \boxed{\operatorname{erf}\left(c\cdot n\sqrt{2\pi\tau_2}\right) - \operatorname{erf}\left(c'\cdot n\sqrt{2\pi\tau_2}\right),}$$

where $c^2 = c'^2 = -1$ and $c\cdot c' < 0$.

- $\operatorname{erf}(x\sqrt{\pi}) = \int_{\mathbb{R}} \mathrm{d}u\, e^{-\pi(x-u)^2}\operatorname{sgn}(u)$ — Convolution: $\operatorname{sgn}(x)\star e^{-\pi x^2}$

- In physics [Akhoury, Comtet]: $H = \frac{p^2}{2} + \frac{1}{2}\left(W'(x)\right)^2 + \frac{1}{2}W''(x)\sigma_3$. With $\phi_\pm = W'(\pm\infty)$, the supersymmetric partition function is:

$$\frac{1}{2}\left(\frac{\phi_+}{|\phi_+|} - \frac{\phi_-}{|\phi_-|}\right) - \frac{1}{2}\left[\operatorname{erfc}\left(\phi_+\sqrt{\beta/2}\right) - (+\leftrightarrow -)\right].$$

Indefinite Theta Series for All Signatures

[Alexandrov, Banerjee, Manschot, Pioline 2016], [N 2016], [Westerholt-Raum 2016], [Kudla 2016], [Funke, Kudla 2017], [Zagier, Zwegers]

Indefinite Theta Series for All Signatures

[Alexandrov, Banerjee, Manschot, Pioline 2016], [N 2016], [Westerholt-Raum 2016], [Kudla 2016], [Funke, Kudla 2017], [Zagier, Zwegers]

$$\sum_{n\in\Lambda+\mu}\left[\frac{1}{2^{N_-}}\prod_{j=1}^{N_-}\left(\operatorname{sgn}\left(c_j\cdot n\right)-\operatorname{sgn}\left(c_j'\cdot n\right)\right)\right]q^{\frac{1}{2}n^2}$$

Modular completion is achieved by replacing

$$\prod_{j=1}^{r}\operatorname{sgn}\left(-f^{(j)}\cdot n\right)\to E_r\left((f^{(1)},\ldots,f^{(r)}),\sqrt{2\tau_2}x\right)$$

where the ‘generalized error function’ is defined by (in an orthonormal basis):

$$E_r(\mathcal{M};u)\ :=\ \int_{\mathbb{R}^r}\mathrm{d}^r u'\,e^{-\pi(u-u')^T(u-u')}\operatorname{sgn}\left(\mathcal{M}^T u'\right).$$

Indefinite Theta Series for All Signatures

[Alexandrov, Banerjee, Manschot, Pioline 2016], [N 2016], [Westerholt-Raum 2016], [Kudla 2016], [Funke, Kudla 2017], [Zagier, Zwegers]

$$\sum_{n\in\Lambda+\mu}\left[\frac{1}{2^{N_-}}\prod_{j=1}^{N_-}\left(\mathrm{sgn}\left(c_j\cdot n\right)-\mathrm{sgn}\left(c_j'\cdot n\right)\right)\right]q^{\frac{1}{2}n^2}$$

Modular completion is achieved by replacing

$$\prod_{j=1}^{r}\mathrm{sgn}\left(-f^{(j)}\cdot n\right)\to E_r\left((f^{(1)},\ldots,f^{(r)}),\sqrt{2\tau_2}x\right)$$

where the 'generalized error function' is defined by (in an orthonormal basis):

$$E_r(\mathcal{M};u)\ :=\int_{\mathbb{R}^r}\mathrm{d}^r u'\,e^{-\pi(u-u')^T(u-u')}\mathrm{sgn}\left(\mathcal{M}^T u'\right).$$

[Imbimbo, Mukhi]: $H=\frac{p_i^2}{2}+\frac{1}{2}\left(\partial_i W(x)\right)^2-\frac{i}{2}\chi_1^i\chi_2^j\partial_i\partial_j W(x)$. Defining $y^j=\partial_j W(x)$ and B the range of this mapping, the susy partition function is:

$$N\int_B \mathrm{d}^r y\,\left(\frac{\beta}{2\pi}\right)^{r/2}e^{-\beta(y^i)^2/2}.$$

Indefinite Theta Series for All Signatures

[Alexandrov, Banerjee, Manschot, Pioline 2016], [N 2016], [Westerholt-Raum 2016], [Kudla 2016], [Funke, Kudla 2017], [Zagier, Zwegers]

$$\sum_{n\in\Lambda+\mu}\left[\frac{1}{2^{N_-}}\prod_{j=1}^{N_-}\left(\operatorname{sgn}\left(c_j\cdot n\right)-\operatorname{sgn}\left(c_j'\cdot n\right)\right)\right]q^{\frac{1}{2}n^2}$$

Modular completion is achieved by replacing

$$\prod_{j=1}^{r}\operatorname{sgn}\left(-f^{(j)}\cdot n\right)\to E_r\left((f^{(1)},\ldots,f^{(r)}),\sqrt{2\tau_2}x\right)$$

where the 'generalized error function' is defined by (in an orthonormal basis):

$$E_r(\mathcal{M};u)\;:=\;\int_{\mathbb{R}^r}\mathrm{d}^r u'\,e^{-\pi(u-u')^T(u-u')}\operatorname{sgn}\left(\mathcal{M}^T u'\right).$$

[Imbimbo, Mukhi]: $H=\frac{p_i^2}{2}+\frac{1}{2}\left(\partial_i W(x)\right)^2-\frac{i}{2}\chi_1^i\chi_2^j\partial_i\partial_j W(x)$. Defining $y^j=\partial_j W(x)$ and B the range of this mapping, the susy partition function is:

$$N\int_B \mathrm{d}^r y\,\left(\frac{\beta}{2\pi}\right)^{r/2}e^{-\beta(y^i)^2/2}.$$

ex: Elliptic genera of squashed toric sigma models [Gupta, Murthy, N 2018].

Shadows or 'What are these functions?'

The $\bar{\tau}$ derivative of $E_r\left(F;\sqrt{2\tau_2}x\right)$ is:

$$-\frac{i}{\sqrt{2\tau_2}}\sum_{j=1}^{r} f^{(j)}\cdot x\, e^{2\pi\tau_2\left(f^{(j)}\cdot x\right)^2}\; E_{r-1}\left(F_{[r]/\{j\}\perp\{j\}};\sqrt{2\tau_2}x\right)$$

Shadows or 'What are these functions?'

The $\bar{\tau}$ derivative of $E_r\left(F;\sqrt{2\tau_2}x\right)$ is:

$$-\frac{i}{\sqrt{2\tau_2}}\sum_{j=1}^{r} f^{(j)}\cdot x\, e^{2\pi\tau_2\left(f^{(j)}\cdot x\right)^2}\; E_{r-1}\left(F_{[r]/\{j\}\perp\{j\}};\sqrt{2\tau_2}x\right)$$

$\partial_{\bar{\tau}}$ [Indefinite theta series of signature (N_+, N_-)]

$= \tau_2^{-1/2}\sum$ [Weight $\frac{3}{2}$ unary theta function]* $\times$

[Indefinite theta series of signature $(N_+, N_- - 1)$]

Shadows or 'What are these functions?'

The $\bar{\tau}$ derivative of $E_r\left(F;\sqrt{2\tau_2}x\right)$ is:

$$-\frac{i}{\sqrt{2\tau_2}}\sum_{j=1}^{r} f^{(j)}\cdot x\, e^{2\pi\tau_2\left(f^{(j)}\cdot x\right)^2}\; E_{r-1}\left(F_{[r]/\{j\}\perp\{j\}};\sqrt{2\tau_2}x\right)$$

$\partial_{\bar{\tau}}$ [Indefinite theta series of signature (N_+, N_-)]

$= \tau_2^{-1/2}\sum$ [Weight $\frac{3}{2}$ unary theta function]* $\times$

[Indefinite theta series of signature $(N_+, N_- - 1)$]

- Mixed mock modular forms are defined as real analytic modular forms satisfying $\frac{\partial}{\partial\bar{\tau}}\widehat{h}(\tau,\bar{\tau}) \in \bigoplus_j \left(\tau_2^{r_j} M_{k+r_j}\otimes\overline{M_{2+r_j}}\right)$.

Shadows or 'What are these functions?'

The $\bar{\tau}$ derivative of $E_r\left(F; \sqrt{2\tau_2}x\right)$ is:

$$-\frac{i}{\sqrt{2\tau_2}} \sum_{j=1}^{r} f^{(j)} \cdot x\, e^{2\pi\tau_2 \left(f^{(j)}\cdot x\right)^2} E_{r-1}\left(F_{[r]/\{j\}\perp\{j\}}; \sqrt{2\tau_2}x\right)$$

$\partial_{\bar{\tau}}$ [Indefinite theta series of signature (N_+, N_-)]
$= \tau_2^{-1/2} \sum$ [Weight $\frac{3}{2}$ unary theta function]* $\times$

[Indefinite theta series of signature $(N_+, N_- - 1)$]

- Mixed mock modular forms are defined as real analytic modular forms satisfying $\frac{\partial}{\partial\bar{\tau}}\widehat{h}(\tau,\bar{\tau}) \in \bigoplus_j \left(\tau_2^{r_j} M_{k+r_j} \otimes \overline{M_{2+r_j}}\right)$.
- Indefinite theta functions lead us to a more general space where $\bar{\tau}$ derivative could involve mock modular forms (of higher depth).

An Example: Vafa-Witten Invariants

$\mathrm{U}(N)$ Vafa-Witten Invariants on $\mathbb{P}^2$:

- $\mathrm{U}(2)$ case: $\frac{h_{2,\alpha}(\tau)}{\eta(\tau)^6}$, $\alpha = 0, 1$ where $h_{2,\alpha}$ is three times the Hurwitz class number generator function. [Vafa, Witten 1995]

An Example: Vafa-Witten Invariants

$\mathrm{U}(N)$ Vafa-Witten Invariants on $\mathbb{P}^2$:

- $\mathrm{U}(2)$ case: $\frac{h_{2,\alpha}(\tau)}{\eta(\tau)^6}$, $\alpha = 0, 1$ where $h_{2,\alpha}$ is three times the Hurwitz class number generator function. [Vafa, Witten 1995]
- $\mathrm{U}(3)$ case: $\frac{h_{3,\mu}(\tau)}{\eta(\tau)^9}$, $\mu = 0, 1, -1$ where [...]

$$h_{3,0}(\tau) = \frac{1}{9} - q + 3q^2 + 17q^3 + 41q^4 + 78q^5 + 120q^6 + 193q^7 + \ldots,$$

$$h_{3,1}(\tau) = h_{3,-1}(\tau) = 3q^{\frac{5}{3}} + 15q^{\frac{8}{3}} + 36q^{\frac{11}{3}} + 69q^{\frac{14}{3}} + 114q^{\frac{17}{3}} + \ldots.$$

These functions can be written in terms of signature $(2,2)$ indefinite theta series [Manschot 2014, 2017].

An Example: Vafa-Witten Invariants

$\mathrm{U}(N)$ Vafa-Witten Invariants on $\mathbb{P}^2$:

- $\mathrm{U}(2)$ case: $\frac{h_{2,\alpha}(\tau)}{\eta(\tau)^6}$, $\alpha = 0, 1$ where $h_{2,\alpha}$ is three times the Hurwitz class number generator function. [Vafa, Witten 1995]
- $\mathrm{U}(3)$ case: $\frac{h_{3,\mu}(\tau)}{\eta(\tau)^9}$, $\mu = 0, 1, -1$ where [...]

$$h_{3,0}(\tau) = \frac{1}{9} - q + 3q^2 + 17q^3 + 41q^4 + 78q^5 + 120q^6 + 193q^7 + \ldots,$$

$$h_{3,1}(\tau) = h_{3,-1}(\tau) = 3q^{\frac{5}{3}} + 15q^{\frac{8}{3}} + 36q^{\frac{11}{3}} + 69q^{\frac{14}{3}} + 114q^{\frac{17}{3}} + \ldots.$$

These functions can be written in terms of signature $(2, 2)$ indefinite theta series [Manschot 2014, 2017].

- Modular completion can be written as:

$$\widehat{h}_{3,\mu}(\tau,\bar{\tau}) = h_{3,\mu}(\tau) - \frac{3\sqrt{3}i}{2\sqrt{2}\pi} \sum_{\alpha \bmod 2} \int_{-\bar{\tau}}^{i\infty} \mathrm{d}w \, \frac{\widehat{h}_{2,\alpha}(\tau,-w)\ \vartheta_{\frac{2\mu+3\alpha}{6}}(3w)}{(-i(w+\tau))^{\frac{3}{2}}}.$$

Circle Method

Goal: A Rademacher series for $f_{3,\mu}(\tau) := \frac{h_{3,\mu}(\tau)}{\eta(\tau)^9}$. [Bringmann, N 2018]

extending results of [Bringmann, Manschot 2010] on mixed mock modular forms.

Circle Method

Goal: A Rademacher series for $f_{3,\mu}(\tau) := \frac{h_{3,\mu}(\tau)}{\eta(\tau)^9}$. [Bringmann, N 2018]

extending results of [Bringmann, Manschot 2010] on mixed mock modular forms.

Writing $f_{3,\mu}(\tau) = \sum_{n=0}^{\infty} \alpha_{3,\mu}(n)\, q^{n-\Delta_\mu}$ we have

$$\alpha_{3,\mu}(n) = \int_i^{i+1} f_{3,\mu}(\tau) e^{-2\pi i(n-\Delta_\mu)\tau} d\tau.$$

Circle Method

Goal: A Rademacher series for $f_{3,\mu}(\tau) := \frac{h_{3,\mu}(\tau)}{\eta(\tau)^9}$. [Bringmann, N 2018]

extending results of [Bringmann, Manschot 2010] on mixed mock modular forms.

Writing $f_{3,\mu}(\tau) = \sum_{n=0}^{\infty} \alpha_{3,\mu}(n)\, q^{n-\Delta_\mu}$ we have

$$\alpha_{3,\mu}(n) = \int_i^{i+1} f_{3,\mu}(\tau) e^{-2\pi i (n-\Delta_\mu)\tau} d\tau.$$

If $f_{3,\mu}(\tau)$ were a weakly holomorphic modular form, this would be an easy task. We would use an integration path that gets closer and closer to the real line near the rationals and use $f_{3,\mu}$'s behavior at $i\infty$:

$$f_{3,\mu}(\tau) = \frac{1}{9} q^{-\frac{3}{8}} \delta_{\mu,0} + \text{non-polar terms.}$$

Circle Method

Goal: A Rademacher series for $f_{3,\mu}(\tau) := \frac{h_{3,\mu}(\tau)}{\eta(\tau)^9}$. [Bringmann, N 2018]

extending results of [Bringmann, Manschot 2010] on mixed mock modular forms.

Writing $f_{3,\mu}(\tau) = \sum_{n=0}^{\infty} \alpha_{3,\mu}(n)\, q^{n-\Delta_\mu}$ we have

$$\alpha_{3,\mu}(n) = \int_i^{i+1} f_{3,\mu}(\tau) e^{-2\pi i (n-\Delta_\mu)\tau} d\tau.$$

If $f_{3,\mu}(\tau)$ were a weakly holomorphic modular form, this would be an easy task. We would use an integration path that gets closer and closer to the real line near the rationals and use $f_{3,\mu}$'s behavior at $i\infty$:

$$f_{3,\mu}(\tau) = \frac{1}{9} q^{-\frac{3}{8}} \delta_{\mu,0} + \text{non-polar terms}.$$

That is not enough to determine the behavior near rationals, e.g.

$$f_{3,\mu}\left(-\frac{1}{\tau}\right) = \tau^{-3/2} \sum \psi_{\nu,\mu} \Bigg[f_{3,\nu}(\tau) + c_1 \frac{h_{2,\alpha}(\tau)}{\eta(\tau)^9} \int_0^{i\infty} \frac{\vartheta_{\frac{2\nu+3\alpha}{6}}(3w)}{(-i(w+\tau))^{\frac{3}{2}}} dw$$

$$+ c_2 \frac{1}{\eta(\tau)^9} \int_0^{i\infty} \int_{w_1}^{i\infty} \frac{\vartheta_{\frac{\alpha}{2}}(w_2) \vartheta_{\frac{2\nu+3\alpha}{6}}(3w_1)}{(-i(w_2+\tau))^{\frac{3}{2}} (-i(w_1+\tau))^{\frac{3}{2}}} dw_2\, dw_1 \Bigg].$$

Application to Vafa-Witten Invariants

Each of $f_{3,\nu}(\tau)$, $\frac{h_{2,\alpha}(\tau)}{\eta(\tau)^9}$ and $\frac{1}{\eta(\tau)^9}$ has $q^{-3/8}$ pieces that contribute to the Fourier coefficients. How about the integrals?

Application to Vafa-Witten Invariants

Each of $f_{3,\nu}(\tau)$, $\frac{h_{2,\alpha}(\tau)}{\eta(\tau)^9}$ and $\frac{1}{\eta(\tau)^9}$ has $q^{-3/8}$ pieces that contribute to the Fourier coefficients. How about the integrals?

For this purpose, we find Mordell-type representations, e.g. the two dimensional integral can be expressed as

$$\int_{\mathbb{R}^2} d^2w\, g(w)\, q^{\frac{1}{3}\left(w_1^2+w_2^2+w_1w_2\right)}.$$

Application to Vafa-Witten Invariants

Each of $f_{3,\nu}(\tau)$, $\frac{h_{2,\alpha}(\tau)}{\eta(\tau)^9}$ and $\frac{1}{\eta(\tau)^9}$ has $q^{-3/8}$ pieces that contribute to the Fourier coefficients. How about the integrals?

For this purpose, we find Mordell-type representations, e.g. the two dimensional integral can be expressed as

$$\int_{\mathbb{R}^2} d^2w\, g(w)\, q^{\frac{1}{3}\left(w_1^2+w_2^2+w_1w_2\right)}.$$

Comparing with $q^{-3/8}$ term, the exponentially growing part is given by restricting the integral to the $w_1^2+w_2^2+w_1w_2 \leq \frac{9}{8}$ region.

Application to Vafa-Witten Invariants

Each of $f_{3,\nu}(\tau)$, $\frac{h_{2,\alpha}(\tau)}{\eta(\tau)^9}$ and $\frac{1}{\eta(\tau)^9}$ has $q^{-3/8}$ pieces that contribute to the Fourier coefficients. How about the integrals?

For this purpose, we find Mordell-type representations, e.g. the two dimensional integral can be expressed as

$$\int_{\mathbb{R}^2} d^2w\, g(w)\, q^{\frac{1}{3}\left(w_1^2+w_2^2+w_1w_2\right)}.$$

Comparing with $q^{-3/8}$ term, the exponentially growing part is given by restricting the integral to the $w_1^2+w_2^2+w_1w_2 \leq \frac{9}{8}$ region.

Now, the circle method can be used with these modified polar terms.

- Fourier coefficients of depth-two $f_{3,\mu}(\tau)$ are expressed as a series, taking as input only the polar terms of the involved holomorphic functions and some information on the shadow.
- The mock terms in the modular transformation contribute power-law suppressed terms to the leading asymptotics of Fourier coefficients.

Thank you!