Characterizing Borcherds-Kac-Moody algebras

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$\rightarrow$ Someone should find a suitable generalization of BKM algebras.

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$\mathfrak{g}$ is a *Borcherds-Kac-Moody algebra* iff

$$\mathfrak{g} = \mathfrak{g}(A) := \hat{\mathfrak{g}}(A)/\{h_{ij} : i \neq j\} \text{ for some } A.$$

(Convention of [Gannon 2006].)

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(b) $\mathfrak{g}(A)$ obeys (1)–(3).

using [Kac 1990, Borcherds 1988, Jurisich 1996]

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- If $x \in (\mathfrak{r} \cap \mathfrak{g}_{-\alpha}) \setminus \langle \mathrm{ad}_{f_i}^{1-a_{ij}}(f_j)\rangle$: reflect α "and x" into the *Weyl chamber*. Computations... contradiction to $2\rho\big(\nu^{-1}(\alpha)\big) = (\alpha, \alpha)$.

Questions?

Thank you for your attention!