APPLICATIONS OF MOONSHINE:

STRING COMPACTIFICATIONS

AND BLACK HOLES

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INTRODUCTION

- Consider the Elliptic genus of $K3$.

$$F(K3;\tau,z)=$$
$$\mathrm{Tr}_{RR}\left((-1)^{F^{K3}+\bar{F}^{K3}}e^{2\pi i z F^{K3}}e^{2\pi i\tau(L_0-c/24)}\bar{e}^{-2\pi i\bar{\tau}(\bar{L}_0-\hat{c}/24)}\right)$$
$$=\sum_{m\geq 0,l}c(4m-l^2)e^{2\pi i m\tau}e^{2\pi i l z}$$

The trace is taken over the Ramond sector.

The elliptic genus is holomorphic in τ, z.

Only the ground states of the left movers are counted.

Evaluating the index we obtain

$$\begin{aligned} F(K3;\tau,z) &= 8\left[\frac{\theta_2(\tau,z)^2}{\theta_2(\tau,0)^2}+\frac{\theta_3(\tau,z)^2}{\theta_3(\tau,0)^2}+\frac{\theta_4(\tau,z)^2}{\theta_4(\tau,0)^2}\right] \\ &\equiv 8A(\tau,z) \end{aligned}$$

The Hodge diamond of $K3$ is given by

$$\begin{aligned} h_{(0,0)} = h_{(2,2)} = h_{(0,2)} = h_{(2,0)} &= 1, \\ h_{(1,1)} &= 20 \end{aligned}$$

Focus on 2 aspects in which the elliptic genus of $K3$ plays an important role.

• Counting $1/4$ BPS, dyons in $\mathcal{N} = 4$ string compactifications of type II string theory on $K3 \times T^2$.

Dual to heterotic on T^6.

• One loop corrections to gravitational/gauge couplings of $\mathcal{N} = 2$ string compactifications of heterotic string theory on $K3 \times T^2$.

Dual to type II on Calabi-Yau

Consider generalizations:

- Consider type II B on $K^3 \times T^2$.

Orbifold this with g':
Acts as a Z_2 involution on $K3$ together with a $1/2$ shift on one of the circles of S^1.

The action preserves $\mathcal{N} = 4$ supersymmetry.
The Hodge Diamond of the quotiented $K3$

$$h_{(0,0)} = h_{(2,2)} = h_{(0,2)} = h_{(2,0)} = 1, \\ h_{(1,1)} = 12$$

• On the heterotic side, this compactification is dual to the simplest of the CHL compactifications.

Exchange the 2 $E_8 \times E_8$ of the heterotic together with a $1/2$ shift on one of the circles of T^6.

Notice that the rank is reduced by 8.

To be explicit, let us provide an orbifold realization of this compactification.

Realize the $K3$: $T^4/\mathbb{Z}_2$

$$g : (y^1, y^2, y^3, y^4, y^5, y^6) \rightarrow (y^1, y^2, -y^3, -y^4, -y^5, -y^6)$$

The involution g' acts as

$$g' : (y^1, y^2, y^3, y^4, y^4, y^6) \rightarrow (y^1 + \pi, y^2, y^3 + \pi, y^4, y^5, y^6)$$

Orbifolding by g produces the $K3 \times T^2$ manifold.

Further orbifolding by g' produces $(K3 \times T^2)/\mathbb{Z}_2$

Lets us now evaluate the following twisted elliptic genus of $K3$.

$$\begin{aligned} & F^{(r,s)}(\tau, z) \\ = \; & \frac{1}{N} \mathrm{Tr}^{K3}_{RR;g'^r} \left((-1)^{F^{K3}+\bar{F}^{K3}} g'^s e^{2\pi i z F^{K3}} e^{2\pi i \tau (L_0 - c/24)} \bar{q}^{-2\pi i \bar{\tau}(\bar{L}_0 - c/24)} \right) \\ = \; & \sum_{m \geq 0 \in \mathbb{Z}/2, l \in \mathbb{Z}} c^{(r,s)}(4m - l^2) e^{2\pi i m \tau} e^{2\pi i l z} \end{aligned}$$

$$0 \leq r, s, \leq 1.$$

Using the orbifold realization of $K3$ and the g' action, we obtain

$$F^{(0,0)} = \frac{8}{2} A(\tau, z),$$

$$F^{(0,1)} = \frac{8}{6} A(\tau, z) - \frac{2}{3} B(\tau, z) E_2(\tau),$$

$$F^{(1,0)} = \frac{8}{6} A(\tau, z) + \frac{1}{3} B(\tau, z) E_2(\frac{\tau}{2}),$$

$$F^{(1,1)} = \frac{8}{6} A(\tau, z) + \frac{1}{3} B(\tau, z) E_2(\frac{\tau+1}{2})$$

$$B(\tau, z) = \frac{\theta_1^2(\tau, z)}{\eta^6(\tau)},$$

$$E_N(\tau) = \frac{12i}{\pi(N-1)} \partial_\tau \ln \frac{\eta(\tau)}{\eta(N\tau)}$$

Is a modular form of weight 2 of the group $\Gamma_0(N)$

These twisted elliptic genera for the $\mathbb{Z}_N$ quotients of $K3$ by g' with $N = 2, 3, 5, 7$ have been written down in David, Jatkar, Sen (2006) .

g' is a $\mathbb{Z}_N$ automorphism of $K3$.

The Hodge diamond of $K3/\mathbb{Z}_N$ becomes

$$h_{(0,0)} = h_{(2,2)} = h_{(0,2)} = h_{(2,0)} = 1,$$
$$h_{(1,1)} = 2\left(\frac{24}{N+1} - 2\right) = 2k$$

N	$h_{(1,1)}$	k
1	20	10
2	12	6
3	8	4
5	4	2
7	2	1

Let us call these CHL orbifolds. In fact there are 3 additional CHL orbifolds $N = 4, 6, 8$. with $k = 3, 2, 1$ respectively.

The heterotic duals of these orbifolds have been studied earlier.
Chaudhuri, Hockney, Lykken (1995), Chaudhuri, Lowe (1995)

The twisted elliptic genera on these quotients of $K3$ is the crucial input

for constructing the generating function of

$1/4$ BPS dyons in compactifications of type II on $K3 \times T^2/\mathbb{Z}_N$

($\mathcal{N} = 4$ supersymmetry).

With the discovery of Mathieu Moonshine symmetry in $K3$ it was found that to each conjugacy class g' in M_{24}, one can construct the twining character

$$F^{(0,1)}(\tau, z) = \frac{1}{N}\mathrm{Tr}_{RR}^{K3}\left((-1)^{F^{K3}+\bar{F}^{K3}} g' e^{2\pi i z F^{K3}} e^{2\pi i\tau(L_0-c/24)}\bar{q}^{-2\pi i\bar{\tau}(\bar{L}_0-c/24)}\right),$$

There are 26 twining characters in all.
Cheng (2005), Eguchi, Hikami (2005), Gaberdiel, Hohenegger, Volpato (2006)

The twining character character of the $N = 2, 3, 5, 7$ CHL constructed earlier coincides with the class pA, with $p = 2, 3, 5, 7$.

Conjucay Class	Order	Cycle shape	Cycle
1A	1	1^{24}	()
2A	2	$1^{8} \cdot 2^{8}$	(1, 8)(2, 12)(4, 15)(5, 7)(9, 22)(1
3A	3	$1^{6} \cdot 3^{6}$	(3, 18, 20)(4, 22, 24)(5, 19, 17)(6, 1
5A	4	$1^{4} \cdot 5^{4}$	(2, 21, 13, 16, 23)(3, 5, 15, 22, 14)(4, 12
7A	7	$1^{3} \cdot 7^{3}$	(1, 17, 5, 21, 24, 10, 6)(2, 12, 13, 9, 4, 2
11A	11	$1^{2} \cdot 11^{2}$	(1, 3, 10, 4, 14, 15, 5, 24, 13, 17, 18)(2, 21,
23A	23	$1^{1} \cdot 23^{1}$	(1, 7, 6, 24, 14, 4, 16, 12, 20, 9, 11, 5, 15, 1
23B	23	$1^{1} \cdot 23^{1}$	(1, 4, 11, 18, 8, 6, 12, 15, 17, 21, 14, 9, 19,
4B	4	$1^{4} \cdot 2^{2} \cdot 4^{4}$	(1, 17, 21, 9)(2, 13, 24, 15)(3, 23)(4, 14
6A	6	$1^{2} \cdot 2^{2} \cdot 3^{2} \cdot 6^{2}$	(1, 8)(2, 24, 11, 12, 23, 18)(3, 20, 10)(4, 15
8A	8	$1^{2} \cdot 2^{1} \cdot 4^{1} \cdot 8^{2}$	(1, 13, 17, 24, 21, 15, 9, 2)(3, 16, 23, 6)(4,
14A	14	$1^{1} \cdot 2^{1} \cdot 7^{1} \cdot 14^{1}$	(1, 12, 17, 13, 5, 9, 21, 4, 24, 23, 10, 20, 6, 2
14B	14	$1^{1} \cdot 2^{1} \cdot 7^{1} \cdot 14^{1}$	(1, 13, 21, 23, 6, 12, 5, 4, 10, 2, 17, 9, 24, 20
15A	15	$1^{1} \cdot 3^{1} \cdot 5^{1} \cdot 15^{1}$	(2, 13, 23, 21, 16)(3, 7, 9, 5, 4, 18, 15, 12, 19
15B	15	$1^{1} \cdot 3^{1} \cdot 5^{1} \cdot 15^{1}$	(2, 23, 16, 13, 21)(3, 12, 24, 15, 17, 18, 14, 4

Table: Conjugacy classes of $M_{23} \subset M_{24}$ (Type 1)

Conjucay Class	Order	Cycle shape	
2B	4	2^{12}	(1, 8)(2, 10)(3, 20)(4, 22)(5, 17)(6, 11
3B	9	3^{8}	(1, 10, 3)(2, 24, 18)(4, 13, 22)(5, 19,
12B	144	12^{2}	(1, 12, 24, 23, 10, 8, 18, 6, 3, 21, 2,
6B	36	6^{4}	(1, 24, 10, 18, 3, 2)(4, 11, 13, 20, 22
4C	16	4^{6}	(1, 23, 18, 21)(2, 12, 10, 6)(3, 7, 24,
10A	20	$2^{2} \cdot 10^{2}$	(1, 8)(2, 18, 21, 19, 13, 10, 16, 24, 2
21A	63	$3^{1} \cdot 21^{1}$	(1, 3, 9, 15, 5, 12, 2, 13, 20, 23, 17,
21B	63	$3^{1} \cdot 21^{1}$	(1, 12, 17, 22, 16, 5, 23, 21, 11, 15,
4A	8	$2^{4} \cdot 4^{4}$	(1, 4, 8, 15)(2, 9, 12, 22)(3, 6)(5, 24,
12A	24	$2^{1} \cdot 4^{1} \cdot 6^{1} \cdot 12^{1}$	(1, 15, 8, 4)(2, 19, 24, 9, 11, 7, 12, 1

Table: Conjugacy classes of $M_{24} \not\in M_{23}$ (Type 2)

Using:

modular transformations and

correspondence with cycle structure in M_{24} (needed for conjugacy class with composite orders)

We can construct the twisted elliptic genera given the twining character for all classes $g' \in M_{23} \subset M_{24}$.

Chattopadhyaya, David (2017),
earlier Gaberdiel, Persson, Ronellenfitsch, Volpato (2012)

The classes are:

Conjucay Class	Order
1A	1
2A	2
3A	3
5A	5
7A	7
11A	11
23A/B	23
4B	4
6A	6
8A	8
14A/B	14
15A/B	15

Table: The classes 2A, 3A, 5A, 7A, 4B, 6A, 8A are the CHL orbifolds.

The twisted elliptic genera is of the form:

$$\begin{aligned} F^{(0,0)}(\tau, z) &= \alpha_{g'}^{(0,0)} A(\tau, z), \\ F^{(r,s)}(\tau, z) &= \alpha_{g'}^{(r,s)} A(\tau, z) + \beta_{g'}^{(r,s)}(\tau) B(\tau, z), \\ & \qquad r, s \in \{0, 1, \cdots N-1\} \text{ with } (\mathrm{r}, \mathrm{s}) \neq (0,0) \end{aligned}$$

$\alpha_{g'}^{(r,s)}$ are numerical constants

$\beta_{g'}^{(r,s)}$ is a weight 2 modular form under $\Gamma_0(N)$ which have been explicitly constructed.

With the twisted genera we can explore its role in

• Dyons in type IIB on $(K3 \times T^2)/\mathbb{Z}_N$: $\mathcal{N} = 4$ compactifications.

• One loop corrections: gravitational/gauge
on heterotic $E_8 \times, E_8$ theory on $(K3 \times T^2)/\mathbb{Z}_N$: $\mathcal{N} = 2$
compactifications.

The rest of the talk will focus on the second aspect.

HETEROTIC $E_8 \times E_8$ on

$(K3 \times T^2)/\mathbb{Z}_N$

Let us describe the heterotic compactification briefly.

Consider first the $E_8 \times E_8$ theory on $K3 \times T^2$.

This is well studied compactificatiion in the context of $\mathcal{N} = 2$ string duality.

This theory is dual to type II A on a Calabi-Yau.

To preserve supersymmetry one needs to embed the spin connection of $K3$ in a $SU(2)$ of the gauge group.

The simplest way to see this is from the supersymmetry variations.

This requires

$$\mathrm{Tr}(F \wedge F) - \mathrm{Tr}(R \wedge R) = 0$$

We must pick the gauge connection from $SU(2)$ from say one of the E_8's and set it equal to the spin connection of $K3$ to ensure this equation is true.

Lets describe this more from the conformal field theory point of view.

We take the right moving sector to be supersymmetric.

Consider the fermionic description of the E_8 in terms of left moving fermions
$\lambda^I, I = 1, \cdots 16$.

We break this lattice to $D2$ with $I = 1, 2, 3, 4$ and the rest $D6$.

The world sheet has a term of

$$\mathcal{G} = \sum_{I,J=1}^{4} \lambda^I B_a^{IJ} \lambda^J \bar{\partial} X^a$$

B_a refers to the $SU(2)$ spin connection.
The 4 fermions get coupled to the bosons of $K3$.
The remaining fermions are free.

Thus the internal CFT splits into

$$\mathcal{H}^{internal} = \mathcal{H}^{(6,6)}_{D2K3} \otimes \mathcal{H}^{(6,0)}_{D6} \otimes \mathcal{H}^{(8,0)}_{E_8} \otimes \mathcal{H}^{(2,3)}_{T^2} .$$

The left moving fermions together with bosons and their right moving bosons and their super partners form the $(6,6)$ SCFT of $K3$.

With this decomposition, we can specify the action of g'.

The g' acts as a $\mathbb{Z}_N$ automorphism on the $(6,6)$ CFT $\mathcal{H}_{D2K3}$ together with a $1/N$ shift on one of the circles in $\mathcal{H}^{(2,3)}_{T^2}$.

- These compactifications preserve $\mathcal{N} = 2$ supersymmetry.
- The un-broken gauge group is $U(1) \times E_7 \times E_8$
- The orbifold by g' reduces the number of hypers in the theory.

Note that for the $K3 \times T^2$ compactification due to the up-lift to 6 dimensions, there is a constraint on number of

$$N_h - N_v = 240$$

These compactifications cannot be lifted to $6d$, (note the shift by $1/N$ on one of the radii of T^2.

• Recall the heterotic $E_8 \times E_8$ string theory on $K3 \times T^2$ with the standard embedding is dual to type II on Calabi-Yau (K3 fibered) with

$$\chi = -480$$

Kachru, Vafa (1995), Klemm, Lerche, Mayr (1995)

• Compactifications on orbifolds $(K3 \times T^2)/\mathbb{Z}_N$ give rise to generalizations of this heterotic-type II string duality

- The dual theory is type II string compactified on Calabi-Yau.

The properties of this Calabi-Yau can be predicted by evaluating one loop corrections/ certain supersymmetric indices on the heterotic side.

The one loop gravitational corrections contain information of Gopakumar-Vafa invariants.

- With this aim, we examine the one loop correction

$$S = \int F_g(y, \bar{y}) F^{2g-2} R^2$$

F, R are the self dual part of the gravi-photon and the Riemann curvature.

The coupling $F_g(y, \bar{y})$ depends on the vector multiplet moduli.

We will restrict our attention to the Kähler and complex structure of the torus.

NEW SUPERSYMMETRY INDEX

AND

THE TWISTED ELLIPTIC GENUS

• The coupling F_g is obtained by performing a one loop integral over the fundamental domain.

The input to obtain the integrand is the new supersymmetric index

$$\mathcal{Z}_{\rm new} = \frac{1}{\eta^2(\tau)} \mathrm{Tr}_R[(-1)^{\bar{F}} \bar{F} q^{L_0 - \frac{c}{24}} \bar{q}^{\tilde{L}_0 - \frac{\tilde{c}}{24}}].$$

F refers to the right moving fermions number.
Sum of the right moving fermion number of the T^2 CFT and right moving fermion number of the $K3$ CFT

$$\bar{F} = \bar{F}^{T^2} + \bar{F}^{K3}.$$

The trace is taken over the Ramond sector of the right moving supersymmetric internal conformal field theory.
Note that $(c, \tilde{c}) = (22, 9)$.

The new supersymmetric index is a different quantity from the Elliptic genus/twisted elliptic genus of $K3$

• We will show that the twisted elliptic genus determines the new supersymmetric index.

The internal conformal field theory

$$\mathcal{H}^{internal} = \mathcal{H}_{D2K3}^{(6,6)} \otimes \mathcal{H}_{D6}^{(6,0)} \otimes \mathcal{H}_{E_8}^{(8,0)} \otimes \mathcal{H}_{T^2}^{(2,3)} .$$

The CFT $\mathcal{H}_{D2K3}^{(6,6)}$ is orbifolded by g' with $1/N$ shifts in the $\mathcal{H}_{T^2}^{(2,3)}$.

Keeping track of the zero modes on the torus, evaluating the trace

$$\mathcal{Z}_{\rm new} = \frac{1}{\eta^2(\tau)}\frac{\Gamma^{(r,s)}_{2,2}(q,\bar{q})}{\eta^2(\tau)} \times \left[\frac{\theta_2^6(\tau)}{\eta^6(\tau)}\Phi_R^{(r,s)} + \frac{\theta_3^6(\tau)}{\eta^6(\tau)}\Phi_{NS^+}^{(r,s)} - \frac{\theta_4^6(\tau)}{\eta^6(\tau)}\Phi_{NS^-}^{(r,s)}\right] \times \frac{E_4(q)}{\eta^8(\tau)}.$$

The sum over the sectors (r,s) is implied and r,s run from 0 to $N-1$.

Let us understand the origin of each term

$\frac{\Gamma_{2,2}^{(r,s)}}{\eta^2}$ arises from the lattice sum on T^2 together with the left moving bosonic oscillators.

The lattice sum

$$
\begin{aligned}
\Gamma_{2,2}^{(r,s)}(q,\bar{q}) &= \sum_{\substack{m_1,m_2,n_2\in\mathbb{Z},\\ n_1=\mathbb{Z}+\frac{r}{N}}} q^{\frac{p_L^2}{2}}\bar{q}^{\frac{p_R^2}{2}} e^{2\pi i m_1 s/N}, \\
\frac{1}{2}p_R^2 &= \frac{1}{2T_2U_2}|-m_1U+m_2+n_1T+n_2TU|^2, \\
\frac{1}{2}p_L^2 &= \frac{1}{2}p_R^2+m_1n_1+m_2n_2\,.
\end{aligned}
$$

- T, U are the Kähler and complex structure of the T^2.

The lattice sum is the only part of the index that contains anti-holomorphic dependence.

- The insertion of $(g')^s$ and the twisted sectors of $(g')^s$:

Result in the phase $e^{2\pi i m_1 s/N}$

The winding modes to shift from integers by $\frac{r}{N}$.

• The term $\frac{E_4(q)}{\eta^8(\tau)}$ arises from the partition function of the second E_8 gauge group which is untouched in the standard embedding.

E_4 is the Eisenstein series of weight 4.

• The terms in the square bracket arises from evaluating the index on the $D6$ lattice together with the combined $D2K3$.

The partition function on the $D6$ lattice in the various sectors are given by

$$\mathcal{Z}_R(D6;q) = \frac{\theta_2^6}{\eta^6}, \quad \mathcal{Z}_{NS^+}(D6;q) = \frac{\theta_3^6}{\eta^6}, \quad \mathcal{Z}_{NS^-}(D6;q) = \frac{\theta_4^6}{\eta^6}.$$

• The indices on the combined $D2K3$, $(6,6)$ conformal field theory are given by

$$\begin{aligned}
\Phi_R^{(r,s)} &= \frac{1}{N}\mathrm{Tr}_{R\,R,g^r}[g^s(-1)^{F_R}q^{L_0-c/24}\bar{q}^{\bar{L}_0-\bar{c}/24}], \\
&= F^{(r,s)}(\tau,\frac{1}{2}) \\
\Phi_{NS^+}^{(r,s)} &= \frac{1}{N}\mathrm{Tr}_{NS\,R,g^r}[g^s(-1)^{F_R}q^{L_0-c/24}\bar{q}^{\bar{L}_0-\bar{c}/24}], \\
&= q^{1/4}F^{(r,s)}(\tau,\frac{\tau+1}{2}), \\
\Phi_{NS^-}^{(r,s)} &= \frac{1}{N}\mathrm{Tr}_{NS\,R,g^r}[g^s(-1)^{F_R+F_L}q^{L_0-c/24}\bar{q}^{\bar{L}_0-\bar{c}/24}], \\
&= q^{1/4}F^{(r,s)}(\tau,\frac{\tau}{2})
\end{aligned}$$

Use the expressions for the twisted elliptic genus and using identities relating theta functions and Eisenstein series we obtain

$$\mathcal{Z}_{\rm new}(q,\bar{q}) = -2\frac{1}{\eta^{24}}\Gamma_{2,2}^{(r,s)}E_4\left[\frac{1}{4}\alpha_{g'}^{(r,s)}E_6 - \beta_{g'}^{(r,s)}f_{g'}^{(r,s)}E_4\right].$$

- Only the lattice sum is dependent on both $(\tau,\bar{\tau})$.
- The Eisenstein series E_6, E_4 as well as $f^{(r,s)}$ are holomorphic in τ.
- The sum over r,s from $0,\cdots N-1$ is understood.

• This result for the new supersymmetric index, though obtained using some what abstract arguments was verified using explicit orbifold limits of $K3$.

Datta, David, Lust (2015) Chattopadhyaya, David (2016)

• When there is no-orbifold: the new supersymmetric index reduces to well known

$$\frac{E_4 E_6}{\eta^{24}}$$

Harvey, Moore (1995)

We read out $N_h - N_v$ from the new supersymmetric index.

$$\begin{aligned} f^{(r,s)}(\tau) &= \frac{1}{2\eta^{24}(\tau)} E_4 \left[\frac{1}{4} \alpha_{g'}^{(r,s)} E_6 - \beta_{g'}^{(r,s)}(\tau) E_4 \right], \\ &= \sum_{l \in \frac{\mathbb{Z}}{N}} c_{-1}^{(r,s)}(l) q^l. \end{aligned}$$

$$N_h - N_v = -\sum_{s=0}^{N-1} c_{-1}^{(0,s)}(0).$$

Evaluating this for each of the orbifolds in M_{23} we obtain

Orbifold	$N_h - N_v$	χ
1A	240	-480
2A	-16	32
3A	-138	276
5A	-260	520
7A	-321	642
11A	-380	760
23A	- 442	884
4B	-200	400
6A	-262	524
8A	-322	644
14A	-382	764
15A	-382	764

Table: $N_h - N_v, \ \chi$

GRAVITATIONAL CORRECTIONS

- Given the new supersymmetric index, we can evaluate the gravitational corrections.

Following the analysis of
Atoniadis, Gava, Narain, Taylor (1995)

The one loop string amplitude with 2 graviton and $2g-2$ one gravi-photon insertions for the orbifold theory we arrive at the following.

Consider the generating function

$$F(\lambda, T, U) = \sum_{g=1}^{\infty} \lambda^{2g} F_g(T, U).$$

Then the F_g 's can be obtained by performing the integral

$$\begin{aligned} F(\lambda, T, U) &= \frac{1}{\pi^2} \int \frac{d^2\tau}{\tau_2} \frac{1}{\eta^{24}(\tau)} \Gamma_{2,2}^{(r,s)} E_4 \left[\frac{1}{4} \alpha_{g'}^{(r,s)} E_6 - \beta_{g'}^{(r,s)}(\tau) E_4 \right] \\ &\times \left[\left(\frac{2\pi i \lambda \eta^3}{\theta_1(\tilde{\lambda}, \tau)} \right)^2 e^{-\frac{\pi \tilde{\lambda}^2}{\tau_2}} \right]^{(r,s)} . \end{aligned}$$

where

$$\tilde{\lambda} = \frac{p_R^{(r,s)} \lambda}{\sqrt{2 T_2 U_2}}.$$

• The integral can be performed by the unfolding method.

• The integral is a generalization of that performed by Harvey, Moore (1998) The integrand involves modular form under $\Gamma_0(N)$.

• The result for the holomorphic part, topological part of the integral is the following.

Start from the twisted elliptic genus

$$
\begin{aligned}
f^{(r,s)}(\tau) &= \frac{1}{2\eta^{24}(\tau)} E_4 \left[\frac{1}{4} \alpha_{g'}^{(r,s)} E_6 - \beta_{g'}^{(r,s)}(\tau) E_4 \right], \\
G_{2k} &= 2\zeta(2k) E_{2k}.
\end{aligned}
$$

Define:

$$f^{(r,s)}(\tau)\mathcal{P}_{2g}(G_2, G_4, G_6, \cdots, G_{2g}) = \sum_{l\in\frac{\mathbb{Z}}{N}} c_{g-1}^{(r,s)}(l,0)q^l,$$

$\mathcal{P}_{2g}$ is related to the Schur polynomial $\mathcal{S}$ of order g by

$$\mathcal{P}_{2g}(x_1, x_2, \cdots x_g) = -\mathcal{S}(x_1, \frac{1}{2}x_2, \cdots \frac{1}{g}x_g).$$

$$\mathcal{P}_0 = -1, \quad \mathcal{P}_2(\hat{G}_2) = -\hat{G}_2, \quad \mathcal{P}_4(\hat{G}_2, G_4) = -\frac{1}{2}(\hat{G}_2^2 + G_4),$$
$$\mathcal{P}_3(\hat{G}_2, G_4, G_6) = -\frac{1}{6}(\hat{G}_2^3 + \hat{G}_2\hat{G}_4) - \frac{1}{3}G_6.$$

where the G's are normalized Eisenstein series

$$G_{2k} = 2\zeta(2k)E_{2k}, \qquad \hat{E}_2(\tau) = E_2 - \frac{3}{\pi\tau_2}.$$

Then the topological amplitude $\bar{F}_g$ is given by

$$\begin{aligned}
&\bar{F}_g^{\mathrm{hol}}(y) \\
= \;& \frac{(-1)^{g-1}}{\pi^2} \sum_{s=0}^{N-1} \left(\sum_{m>0} e^{-2\pi i n_2 s/N} c_{g-1}^{(r,s)}(m^2/2, 0) \mathrm{Li}_{3-2g}(e^{2\pi i m \cdot y}) \right. \\
& \left. + \frac{1}{2} c_{g-1}^{(0,s)}(0,0) \zeta(3-2g) \right) .
\end{aligned}$$

The sum over lattice points $m > 0$ refers to the following lattice points (n_1, n_2), $n_1 \in \frac{\mathbb{Z}}{N}$, $n_2 \in \mathbb{Z}$ with the restrictions

$$\begin{aligned}
& n_1, n_2 \geq 0, \quad \text{but } (n_1, n_2) \neq (0,0), \\
(r/N, -n_2), \qquad & \text{with } \; n_2 > 0 \; \text{ and } \; r n_2 \leq N.
\end{aligned}$$

$y = (T, U)$ is the Kähler and complex structure of the torus T^2, $m^2 = 2 n_1 n_2$ and $m \cdot y = n_1 T + n_2 U$.
The functions Li_{3-2g} are polylogarithm functions of order $3 - 2g$.

• We see that indeed it is the
coefficients of the twisted elliptic genus of $K3$ which
forms the basic input data for the topological amplitude $\bar{F}_g(y)$.

• It is in this sense M_{24} symmetry of $K3$ is carried over to the topological amplitude.

• This is a generalization of the observation
Cheng, Dong, Duncan, Harvey, Kachru, Wrase (2013) in which the elliptic genus of $K3$ which results in
$E_4 E_6/\eta^{24}$,
for the new supersymmetric index
is the crucial input data for the topological amplitude for the unorbifolded model.

GOPAKUMAR-VAFA INVARIANTS

The genus g topological amplitude on a Calabi-Yau admits the following expansion For $g > 1$, this is given by

$$F_g^{\rm GV} = \frac{(-1)^g |B_{2g} B_{2g-2}| \chi(X)}{4g(2g-2)(2g-2)!}$$
$$+ \sum_m \left[\frac{|B_{2g}| n_m^0}{2g(2g-2)!} + \frac{2(-1)^g n_m^2}{(2g-2)!} \pm \ldots - \frac{g-2}{12} n_m^{g-1} + n_m^g \right] \mathrm{Li}_{3-2g}(e^{2\pi i m\cdot y})$$

We have included the constant term which is the contribution to the topological amplitude due to holomorphic maps from genus g surface to a single point.

Review by Marino (2002)

For $g = 0$ we have we obtain

$$F_0^{\mathrm{GV}} = \zeta(3)\frac{\chi(X)}{2} + \sum_{m>0} n_m^0 \mathrm{Li}_3(e^{2\pi i m\cdot y}),$$

where have included the contribution due to the Euler characteristic of the Calabi-Yau target.
Finally for $g = 1$ we have

$$F_1^{\mathrm{GV}} = \sum_{m>0} \left(\frac{1}{12} n_m^0 + n_m^1\right) \mathrm{Li}_1(e^{2\pi i m\cdot y}).$$

• It is not obvious that the topological amplitude evaluated for the orbifolds g': $K3 \times T^2/\mathbb{Z}_N$
can be written in the Gopakumar-Vafa form with integer invariants n_m^g.

• Comparing the constant terms in the Gopakumar-Vafa form of the topological amplitude and that evaluated using the one loop calculation
we fix the normalization relating the amplitudes

$$F_g^{\rm GV} = \frac{(-1)^{g+1}}{2(2\pi)^{2g-2}} \bar{F}_g^{\rm hol}.$$

• Once the normalization is fixed:
We read out the invariants n_m^g in terms of $c_{g-1}^{(r,s)}(m^2/2, 0)$

which in turn is determined from the expansion of the twisted elliptic genus of $K3$.

- For example

$$n^0_{(n_1,n_2)} = 2\sum_{s=0}^{N-1} e^{-\frac{2\pi i n_2 s}{N}} c^{(r,s)}_{-1}(m^2/2,0), \qquad r = n_2 N \bmod N$$

$$= -2\sum_{s=0}^{N-1} e^{-\frac{2\pi i n_2 s}{N}} c^{(r,s)}(n_1 n_2).$$

• We have evaluated the GV invariants for all the orbifolds g' corresponding to the M_{23} conjugacy classes M_{23} for $g = 0, 1, 2, 3$ and shown that they are integers.

In fact the way this works is the formulae are such that once the genus zero invariant n_m^0 are integers,

Then the we can show that the higher genus ($g = 1, 2, 3)$ GV invariants are integers.
A list of the GV invariants is provided
Chattopadhyaya, David (2017)

(n_1, n_2)	$(1, -1)$	(1,0)	(1,1)	(1,2)	(
$n^0_{(n_1,n_2)}$	-2	480	282888	17058560	477

(n_1, n_2)	(1,4)	(1,5)	(1,6)	(1,7)	(
$n^0_{(n_1,n_2)}$	8606976768	115311621680	1242058447872	11292809553810	895500

Table: $N = 1$, $K3$ itself.

(n_1, n_2)	$(0, n_2)$	(1,1)	(1,2)	(1,3)	(1,4)
$n^0_{(n_1,n_2)}$	-32	151552	8387328	47890048	4294949632
(n_1, n_2)	$(1/2, 0)$	(1/2,2)	(1/2,4)	(1/2,6)	(1/2,8)
$n^0_{(n_1,n_2)}$	512	151552	8671232	240009216	4312027136
(n_1, n_2)	$(1/2, -1)$	(1/2,1)	(3/2,1)	(5/2,1)	(7/2,1)
$n^0_{(n_1,n_2)}$	16	8128	1212576	47890048	1055720304

Table: $N = 2$, 2A orbifold.

• We can locate conifold sigularities from the toplogical amplitude.

These occur at $m \cdot y \to 0$.

These singularities occur only in the twisted sector at lattice points

$$m = (\frac{r}{N}, -n_2), \qquad r, n_2 > 0, \ \frac{m^2}{2} = -\frac{rn_2}{N}, \ rn_2 \leq N.$$

There are no singularities in the untwisted sector.

• The strength of the singularity is

$$\bar{F}_g^{\mathrm{hol}}|_{m\cdot y\to 0} = \frac{1}{2\pi^2}(2\pi)^g \chi(\mathcal{M}_g) \times n^0_{(\frac{r}{N},-n_2)} \frac{1}{\{1-e^{2\pi i m\cdot y}\}^{2g-2}}$$

It is determined by the genus zero GV invariant.

• For the un-orbifolded $K3 \times T^2$ the conifold singularity lies at only the lattice point $m = (1,-1)$ and $n^0_{(1,-1)} = -2$.

GAUGE COUPLING CORRECTIONS

• We have also studied the one loop corrections to gauge coupling constants, including the an additional Wilson line V along the T^2.

It can be shown that for the standard embedding, the difference of the one loop corrections between the the two gauge groups: E_7 and E_8
is determined by Siegel modular forms $\Phi_k(U, T, V)$.

Consider 1-loop corrections to the gauge couplings

$$\frac{1}{g^2(E_7)} = \Delta_{G'}(T,U,V), \qquad \frac{1}{g^2(E_8)} = \Delta_{G'}(T,U,V)$$

which depend on the Kähler and complex structure moduli T, U of the torus T^2.

We turn on the Wilson line

$$V = A_1 + iA_2$$

with values in say a $U(1)$ of the unbroken E_8.

• We show that the difference in one loop threshold corrections

$$\Delta_G(T,U,V) - \Delta_{G'}(T,U,V) = -48 \log \left[(\det \mathrm{Im}\Omega)^k \, |\Phi_k(T,U,V)|^2 \right] ,$$

where Φ^k is a weight k modular form transforming under subgroups of $Sp(2,\mathbb{Z})$ with k

Datta, David, Lust (2015) Chattopadhyaya, David (2016)

• Again, the gauge coupling corrections also depend only of the coefficients of the twisted elliptic genus.

- This generalises the result due to
Stieberger (1998)
who observed this phenomenon for the unorbifolded theory. In this case the Seigel modular form is the Igusa cusp form $\Phi_{10}(U, T, V)$.

CONCLUSIONS

• We have studied the $E_8 \times E_8$ heterotic string theory compactified on orbifolds g': $K3 \times T^2/\mathbb{Z}_N$ inspired by the Moonshine symmetry of $K3$.

• We have shown that the data which determines one loop corrections are the coeffiicents elliptic genus of $K3$ twisted by g'.

- Our study provides some topological information of the type II theory on the putative Calabi-Yau.

- It will be interesting to explore the type II theory directly in detail.