Umbral Moonshine and String Duality

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Introduction

- K3 string theory: source of much of our understanding of non-perturbative string theory
  - How far can we get without even knowing a Calabi-Yau metric for a smooth K3 surface?
- Surprising role played by sporadic groups and special mock modular forms in some of the simplest string vacua.
  - Deep organizing principle that can help tame the chaos of the landscape?

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  - Deep organizing principle that can help tame the chaos of the landscape?

Introduction, continued

- Umbral Moonshine: know that modules exist, but no uniform construction for all of them (and, in fact, no explicit construction for many)
- What is relationship with K3 string theory (which, after all, initiated the subject)? (Especially tricky to incorporate 2-plane preserving group elements)

Goal of this talk

To explain (what I believe to be) the natural physical context for understanding Umbral moonshine.

BPS States

- Theories with extended supersymmetry have central charges: $[Q, Z] = 0$
- SUSY algebra trivially implies BPS bound. Roughly, $M \geq |Z|$.
- BPS states saturate this bound: $M = |Z|$
- Shortened representations of SUSY algebra (annihilated by some supercharges)

Supersymmetric Indices

- Partition function (which counts all states in the theory) is moduli-dependent, and so generically hard to compute
- If we restrict our attention to BPS states, we can define a moduli-independent index! Allows us to compute at an easy point in moduli space
- Examples:
  - Witten index: $\mathrm{Tr}\,(-1)^F$
  - Elliptic genus: $Z^X(q,y) = \mathrm{Tr}\,(-1)^{J_0+\overline{J_0}} q^{L_0} y^{J_0}$
- For K3, we have

$$Z^{K3}(q,y) = 2\chi_{0,1} = 8\left(\sum_{i=2}^{4} \frac{\theta_i(q,y)^2}{\theta_i(\tau,0)^2}\right)$$

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5d black hole microstate counting

- Type IIB string theory on $K3 \times S^1$. 16 supercharges
- Wrap Q_5 D5-branes and Q_1 D1-branes. The resulting bound state is described (when the volume of K3 is small) by $\mathrm{Sym}^{Q_1 Q_5}(K3)$.
- Elliptic genus $Z^{\mathrm{Sym}^{Q_1 Q_5}(K3)}(q, y)$ of this CFT counts $1/4$-BPS states with angular momentum J_0 and momentum L_0 on S^1
- If we get lucky, there aren't too many cancellations and in the classical limit we reproduce Bekenstein-Hawking black hole entropy [Strominger-Vafa '96]

- Take the 24 Niemeier lattices, i.e. the 24-dimensional negative-definite even unimodular lattices. One of them, the Leech lattice, Λ, has no roots, while the other 23 have roots. They are labeled by their ADE-type root system, X; we call the corresponding lattice L^X, and its corresponding Umbral group $G^X = O(L^X)/W^X$. We define W^Λ to be the trivial group and $G^\Lambda = Co_0$.

Symmetries of K3 NLSMs

- There is a simple classification of all SUSY-preserving discrete symmetry groups of type IIA string theory on K3 [Cheng-Harrison-Volpato-MZ '16]: they are the 4-plane preserving subgroups of the Umbral groups. If we restrict to non-singular points in moduli space (without enhanced gauge symmetry), then the Leech lattice suffices [Gaberdiel-Hohenegger-Volpato '11]. The quotient by the Weyl group enters essentially because it consists of gauge symmetries.
- The proof generalizes ideas in Kondo's proof [Kondo '98] of the Mukai theorem [Mukai '88], which classifies finite groups of K3 symplectomorphisms

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Symmetries of K3 NLSMs

- We now know the symmetry groups as abstract groups, but we still do not know their action on states.
- Useful tool: twisted-twining genus.

$$Z_{g,h}(q,y) = \mathrm{Tr}_{\mathcal{H}_g}\ h(-1)^{J_0+\overline{J_0}} q^{L_0} y^{J_0}$$

- Moduli-independent, so long as we stay at a point in moduli space where the symmetry is defined
- There are 81 different $O^+(\Gamma^{4,20})$ conjugacy classes, and therefore at most 81 different twining genera $Z_g = Z_{e,g}$ [Cheng-Harrison-Volpato-MZ '16]
- These can be determined by combining worldsheet constraints with constraints from 4d physics from a CHL orbifold. [Cheng-Harrison-Volpato-MZ '16, Paquette-Volpato-MZ '17]

Umbral Moonshine

- $M_{24} = G^{A_1^{24}}$ moonshine [Eguchi-Ooguri-Tachikawa '10]:

$$Z^{K3} = (\cdots) + (\cdots)(-2 + 90q + 462q^2 + 1540q^3 + \ldots)$$

- This generalizes (via a construction involving special mock modular forms) to all 23 Niemeier lattices with roots [Cheng-Duncan-Harvey '12, Cheng-Harrison '14]
- Coincidence? Can test: the same construction gives predictions for twining genera.
- There's also an independent construction for Λ [Duncan and Mack-Crane '14, 15]
- These are precisely the twining genera we find in string theory!

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Umbral Moonshine, continued

- More generally, for 4-plane preserving symmetries g we have
$$Z_g^{K3} = Z_g^{X,S} + Z_g^{X,U}$$
- $Z_g^{X,S}$ is twining genus of ALE gravitational instanton / du Val singularity corresponding to the root system X
- $Z_g^{X,U}$ is constructed from the vector-valued mock modular form H_g^X.
- Strongly suggests decomposition of K3 surface, but this is not possible: K3 NLSMs can develop only up to rank 20 singularities
- In particular, there is no K3 NLSM with M_{24} symmetry [Gaberdiel-Hohenegger-Volpato '11]

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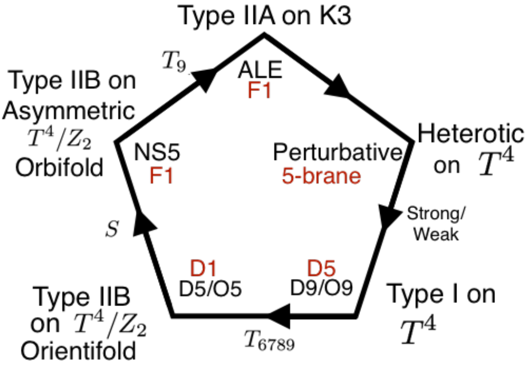
Physical Setting for Moonshine

- Can this be understood from physics? Note that 2-plane fixing conjugacy classes appear in that decomposition
- Go to 2d, on $K3 \times T^4$. [MZ '18]



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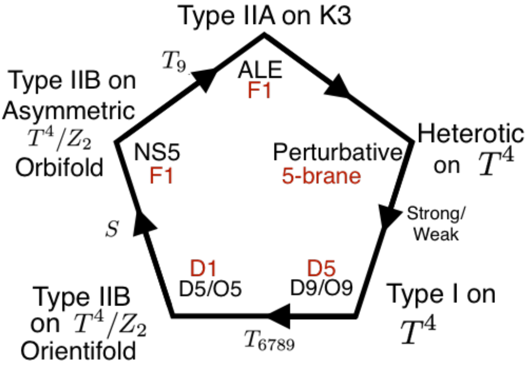
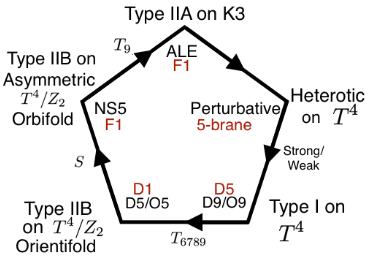
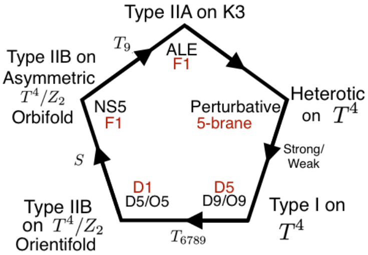


Type IIA on $K3 \times T^4$



- $$\hat{Z}_g = \text{Tr } g(-1)^{J_0+\bar{J_0}}(\bar{J}_0)^2 q^{L_0} y^{J_0}$$
  $$\hat{Z}_g^{K3\times T^4}(\tau, z) = Z_g^{K3}(\tau, z)\hat{Z}^{T^4}(\tau, z)$$

Heterotic on T^8



- Enhanced gauge symmetry

$$O(\Gamma^{24,8})\backslash O(24,8)/(O(24)\times O(8))$$

- Umbral group is either symmetry group at point of enhanced gauge symmetry, or if you deform slightly away in order to slightly Higgs the non-Abelian gauge symmetry [Kachru-Paquette-Volpato '16]

Heterotic on T^8, continued

- Compactified little string theory defines a 2d IR CFT (NLSM to moduli space) with Umbral symmetry near points with enhanced gauge symmetry. (Perturbative symmetries map 5-brane strings to 5-brane strings)
  - Tadpole blows up the universe? No! $g_s \to 0$
  - Analogous to studying worldvolume of an electron in 3d QED. Electron is not in S-matrix for any finite coupling, but perturbation theory still makes sense, and becomes valid in a larger and larger region of spacetime as the coupling vanishes. Our cutoff is set by Planck scale.
  - Can't couple to dynamical B-field – analogous to studying 2d or 6d theory with gravitational anomaly, so long as we decouple from gravity
  - In any case, if this duality frame displeases you, there will be others.

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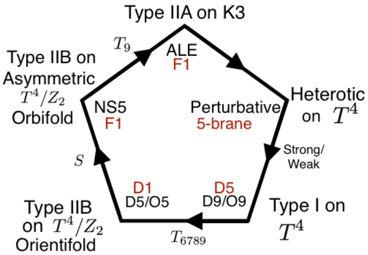
Type I on T^8

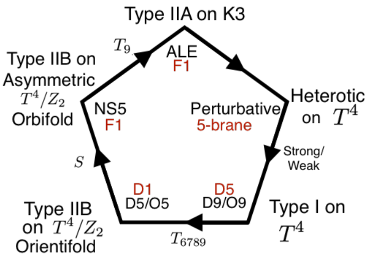


- Moduli space of D5-brane Wilson lines is moduli space of the heterotic little string theory. Can now attack this with gauge theory methods. (k D5-branes gives $Sp(k)$ gauge group with 8 supercharges)
- However, 6d gauge theory is only effective field theory

Type I on T^8, continued

- Generic point approximates a smooth K3 surface. What about points of enhanced spacetime gauge symmetry (or worldvolume global symmetry)?

Type IIB on T^8/Z_2 orientifold



- Can geometrize moduli space via T-duality. 2d gauge theory on D1-brane now gives accurate picture of moduli space near singularities, although it is not useful globally.

Type IIB on T^8/Z_2 orientifold, continued

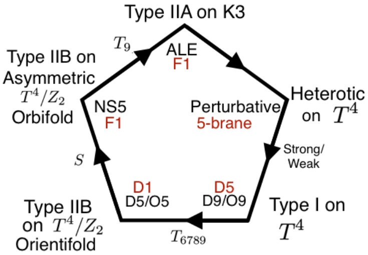
- Moduli space of $\mathcal{N} = (4,4)$ gauge theory [Diaconescu-Seiberg '97]

$$k = \begin{cases} N_f : U(1) \\ 2(N_f - 1) : SU(2) \end{cases} ,$$

$$ds^2 = \left(\frac{1}{g_{2d}^2} + \frac{k}{r^2} \right) d\vec{r}^2 , \quad H = -k \, d\Omega^3 ,$$

  where $d\Omega^3$ is the volume form on the unit 3-sphere centered at the origin.
- This gives metric and B-field describing NS5-branes
- Weird: spacetime non-Abelian gauge symmetry can have rank up to 24, whereas we only started with 16 D9-branes. But extra D-branes are generated non-perturbatively [Seiberg '96, and many others]

Type IIB on asymmetric T^8/Z_2 orbifold



- S-dualize, so that probe is a fundamental string. The geometry it probes *is* the target space background! So, imagine NS5-branes wrapping T^4, and sitting at a point on T^4/Z_2.
- Counting perturbative BPS states. Again, can take $g_s \to 0$ limit

Umbral NLSM

- $$\hat{Z}^{T^4 \times K3} = \hat{Z}^{T^4} Z^{K3} = \hat{Z}^{T^4} Z^{X,S} + \hat{Z}^{\text{rest}} \Rightarrow \hat{Z}^{\text{rest}} = \hat{Z}^{T^4} Z^{X,U}$$
- NS5-brane throats decompose the target space!
- Umbral group action on NS5-brane part of the target space is the obvious one from action on roots – that is, 24-dimensional permutation representation of D1-branes!
- Mock = non-compact
- Umbral NLSM: $\mathcal{N} = (4,4)$, $c = 12$ 2d CFT

Umbral NLSM. . . ?

- The picture these results seem to suggest is that we're dividing spacetime into two non-compact NLSMs that we glue together
- This should be exactly correct *at* the points of enhanced gauge symmetry
- But, this is where the throat CFT becomes singular.
- Fortunately, the problems that plague the throat don't plague the rest of spacetime – no diverging dilaton, sudden non-compactness.
- If the Umbral NLSM doesn't exist on its own, then the decomposition of the K3 NLSM has the physical meaning of discrete BPS states not mixing under Umbral group actions. Mock = ignoring continuum

Conclusions

- String dualities explain the Eguchi-Ooguri-Tachikawa and Cheng-Harrison decompositions of the K3 elliptic genus
- Are the Umbral points in moduli space special? Physics thus far seems to suggest any enhanced gauge symmetry gives moonshine (or at least natural decomposition of K3 elliptic genus). Either this is the case (interesting), or it should give a hint as to what makes the Umbral points special. (Obvious from representation theory – fewest trivial representations.) In case the former case is right, I'm dubbing it Antumbral moonshine. Possible role for 16 other weight one optimal mock Jacobi forms with rational coefficients [Cheng-Duncan '16]?

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Conclusions, continued

- Construct module of perturbative BPS states (Harvey-Moore ’95,’96; Gaberdiel-Hohenegger-Persson ’11, Paquette-Persson-Volpato ’16) to get Umbral moonshine modules and mock modular forms
- (Holographic?) explanation of Rademacher sum characterization of mock modular forms [Cheng-Duncan ’11; see also Dijkgraaf-Maldacena-Moore-Verlinde ’00]? Results of [Harvey-Murthy ’13, H-M-Nazaroglu ’14] give hope
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Ultimate hope

- Understand what Moonshine has to teach us about hidden structures in string theory and mathematics