Squashed Toric Manifolds and Mock Modular Forms

Rajesh Gupta

King's College London

ESI Moonshine Workshop, 2018

Ref: Gupta and Murthy [1705.00649]

Gupta, Murthy and Nazaroglu [1808.00012]

QFT,Geometry and Number theory

There is an intimate relation between 2 dim. gauged linear sigma model (GLSM), compact CY- manifolds and modular/Jacobi forms.

Low energy description of GLSM described by SCFT with target space a compact CY manifold.

The elliptic genera of SCFT give rise to certain geometric invariants of CY manifold. These elliptic genera has interesting number theoretic properties.

Elliptic Genus

The elliptic genus is defined by

$$\chi_{\rm ell}(M;\tau,z,\{\beta\}) = {\rm Tr}_{\mathcal{H}_{\rm RR}}\left[(-1)^F q^{L_0}\,\bar{q}^{\bar{L}_0}\,\zeta_z^{J_0^R}\prod_{\ell=1}^{d}\exp\left(2\pi i\beta_\ell\int j_\ell\right)\right]$$

$\mathcal{H}_{\rm RR}$: Ramond-Ramond sector

J_0^R : Left moving R-charge

j_ℓ : a global symmetry current

The elliptic genus receive contributions from $\bar{L}_0 = 0$ and thus, are holomorphic function of τ.

Non Compact CY Toric Mfd.

It is easy to construct GLSM which flows in IR to a CY toric manifold.

The constriction of the GLSM involves $U(1)^{n-d}$ gauge theory coupled to n- number of matter fields Φ_i with conditions that

$$\sum_{i=1}^{n} Q_i^a = 0 \qquad\qquad \forall \quad a = 1, ...(n-d)$$

IR is a SCFT with the target space d -complex dim. toric manifold

Elliptic genus of a non compact CY manifold e.g.CY toric manifold is subtle to define.

Deformations of Non Compact CY Toric Mfd.

The toric manifold obtained by considering a certain "deformations" of original CY toric manifold has nice modular properties.

These deformations consist of gauging the d dim. flavor symmetries and introducing equal number of compensator fields

$$P_\ell \to P_\ell + i\,\Lambda_\ell \qquad \text{for} \qquad A^\ell_\mu \to A^\ell_\mu + \partial_\mu \Lambda_\ell$$

Geometrically:

$$\mu_a = \sum_{i=1}^{n} Q^a_i\,|\phi_i|^2 - r_a\,, \quad a = 1, \ldots, n-d$$

$$\mu'_\ell = \sum_{i=1}^{n} F^\ell_i\,|\phi_i|^2 + k_\ell\,\mathrm{Re}P_\ell \qquad \ell = 1, ..., d$$

The squashed toric manifold is

$$\mu^{-1}(0)/(U(1)^{n-d} \times U(1)^d) \subset \mathbb{C}^{n+d}\,, \qquad \mu := (\mu_a, \mu'_\ell)$$

Elliptic Genus of Squashed Toric Mfd.

The elliptic genus of the squashed toric sigma models are given in terms of d-dim. torus integral

$$\chi_{\rm ell}(\widetilde{M}_{\rm tor};\tau,z) = \int_{E_\tau^d} \prod_{\ell=1}^{d} \frac{d^2 u'_\ell}{\tau_2} H_\ell(\tau,z,u'_\ell)\, \chi_{\rm ell}(M_{\rm tor};\tau,z,u')$$

integral over holonomy of flavor gauge fields

elliptic genus of the original toric model in presence of holonomy of flavor gauge fields

here

$$H_\ell(\tau,z,u) = k_\ell \sum_{m,w\in\mathbb{Z}} e^{2\pi i b_\ell w z - \frac{\pi k_\ell}{\tau_2}\left(w\tau+m+u+\frac{b_\ell z}{k_\ell}\right)\left(w\bar\tau+m+\bar u+\frac{b_\ell z}{k_\ell}\right)}$$

$\chi_{\rm ell}(\widetilde{M}_{\rm tor};\tau,z)$ transforms like a holomorphic Jacobi form of wt. 0

and index

$$m = \frac{d}{2} + \sum_{\ell=1}^{d} \frac{b_\ell^2}{k_\ell}$$

central charge
$c = 6m$

A General Integral

A typical integral we encounter in the expression for the elliptic genus of a squashed toric sigma model is of the following type

$$\int_{\mathbb{C}^N} \frac{d^{2N}u'}{\tau_2^N} \prod_{j=1}^{N} \left[\widetilde{k}_j \, \frac{\theta_1(\tau, -z + \mu^{(j)T} u')}{\theta_1(\tau, \mu^{(j)T} u')} \, e^{-\frac{\pi \widetilde{k}_j}{\tau_2} \left(u'_j + \frac{z}{\widetilde{k}_j}\right)\left(\overline{u}'_j + \frac{z}{\widetilde{k}_j}\right)} \right],$$

integral over holonomy of flavor gauge fields

Not holomorphic in τ.

We find that the above integral is the completion of indefinite theta function associated with lattice of type (N, N).

d=1 Example

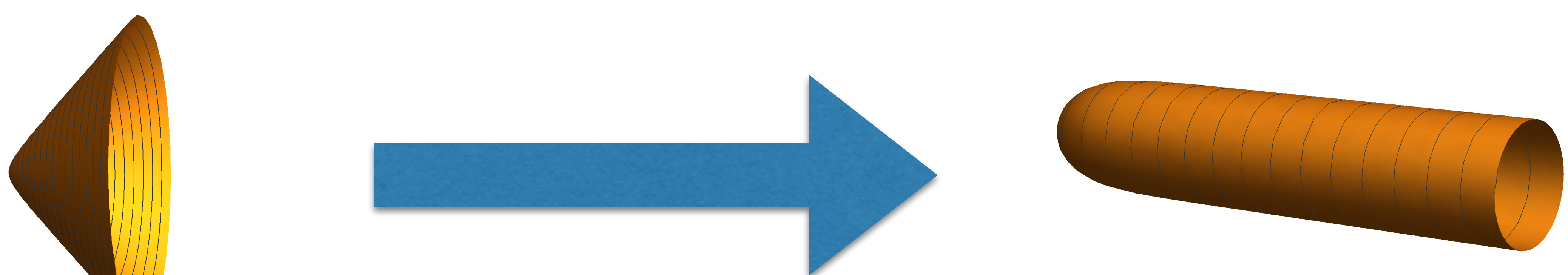
Consider following 2 dim gauge theory

	$U(1)_g$	$U(1)_F$
Φ_1	1	F1
Φ_2	-1	F2

$$b = F1 + F2$$

In addition, there is a compensator field which transform in homogeneously under flavour gauge transformation

$$P_\ell \to P_\ell + i\,\Lambda_\ell \qquad \text{for} \qquad A_\mu \to A_\mu + \partial_\mu \Lambda$$



The Elliptic Genus of Cigar

The elliptic genus is

$$\chi(\widetilde{\mathbb{C}/\mathbb{Z}_2};\tau,z) = \frac{\kappa}{\tau_2}\int_{\mathbb{C}} d^2u\, \frac{\theta_1(\tau,-z+bu)}{\theta_1(\tau,bu)} e^{-\frac{\pi\kappa}{\tau_2}(u+\frac{bz}{\kappa})(\bar{u}+\frac{bz}{\kappa})}$$

$$\tau_2 = \mathrm{Im}\tau$$

transforms like a holomorphic Jacobi form of wt. 0 and index

$$m = \frac{1}{2} + \frac{1}{k}$$

Integrating, we get

$$\chi\left(\widetilde{\mathbb{C}/\mathbb{Z}_2};\tau,z\right) = \frac{-i\theta_1(\tau,z)}{\eta(\tau)^3}\sum_{n,w\in\mathbb{Z}} \frac{q^{nw}\,\zeta_z^{w+\frac{n}{k}}}{1-\zeta_z^{-1}\,q^{-n}}$$
$$\times\frac{1}{2}\left[\mathrm{erf}\left(\sqrt{k\pi\tau_2}\left(\frac{n}{k}-w\right)\right) - \mathrm{erf}\left(\sqrt{k\pi\tau_2}\left(\frac{n}{k}-w-1\right)\right)\right].$$

has wt. -1

A Holomorphic Anomaly

It satisfies holomorphic anomaly equation

$$-\frac{2i}{\sqrt{k}}\tau_2^{1/2}\,\partial_{\bar\tau}\chi\left(\widetilde{\mathbb{C}/\mathbb{Z}_2};\tau,z\right)=\frac{-i\theta_1(\tau,z)}{k\eta(\tau)^3}\sum_{\alpha,\beta\in\mathbb{Z}/k\mathbb{Z}}e^{\frac{2\pi i\alpha\beta}{k}}q^{\frac{\alpha^2}{k}}\zeta_z^{\frac{2\alpha}{k}}\sum_{\ell(\mathrm{mod}2k)}\overline{\theta^{(1)}_{k,\ell}(\tau)}\,\theta_{k,\ell}(\tau,\frac{z+\alpha\tau+\beta}{k})$$

where

$$\theta_{k,\ell}(\tau,z)=\sum_{\substack{\lambda\in\mathbb{Z}\\ \lambda\equiv\ell(\mathrm{mod}2k)}}q^{\frac{\lambda^2}{4k}}\zeta_z^{\lambda} \longrightarrow \text{theta function of wt. } \frac{1}{2}$$

$$\theta^{(1)}_{k,\ell}(\tau)=\sum_{\substack{\lambda\in\mathbb{Z}\\ \lambda\equiv\ell(\mathrm{mod}2k)}}\lambda\,q^{\frac{\lambda^2}{4k}} \longrightarrow \text{theta series of wt. } \frac{3}{2}$$

This is an example of mixed mock Jacobi form.

Indefinite Theta Function

The holomorphic part (mock part) of the elliptic genus is

$$\chi^{\text{hol}}\left(\widetilde{\mathbb{C}/\mathbb{Z}_2};\tau,z\right) = \frac{i\,\theta(\tau,z)}{2\eta(\tau)^3}\sum_{\gamma=0}^{k} a_{\gamma,k}\sum_{m,w\in\mathbb{Z}}\left[\mathrm{sgn}(m+\epsilon) - \mathrm{sgn}(-kw+\gamma-\frac{\mathrm{Im}z}{\mathrm{Im}\tau})\right]$$
$$\times q^{k(w^2+mw)-\gamma(m+w)}\zeta_z^{2w+m-\frac{\gamma}{k}}$$

This is an indefinite theta function associated with a lattice of type (1,1).

$$a_{\gamma,k} = \begin{cases} 1/2 & \text{if } \gamma = 0, k, \\ 1 & \text{if } 0<\gamma<k, \\ 0 & \text{otherwise,} \end{cases}$$

$$\mathrm{sgn}(x) = \begin{cases} 1 & \text{if } x>0, \\ 0 & \text{if } x=0, \\ -1 & \text{if } x<0. \end{cases}$$

e.g. $k=2, \quad \gamma=1$

$$Q = 2\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$$

$$c_1 = \frac{1}{2}\begin{pmatrix} 1 \\ -2 \end{pmatrix}, \quad c_1' = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \qquad ;\ c_1^2 = -1, \quad {c_1'}^2 = 0$$

The Elliptic Genus of Squashed A1

The squashed toric A1 $(\mathbb{C}^2/\mathbb{Z}_2)$ model is defined by following fields

	$U(1)_g$	$U(1)_{F_1}$	$U(1)_{F_2}$
Φ_1	1	1	0
Φ_2	-2	1	1
Φ_3	1	0	1

together with compensator fields P_1 and P_2

The elliptic genus is given by

$$\chi_{\text{ell}}(\widetilde{A}_1, k; \tau, z) = 2k^2 \int_{\mathbb{C}} \frac{d^2u_1'}{\tau_2} \int_{\mathbb{C}} \frac{d^2u_2'}{\tau_2} \frac{\theta_1(\tau, -z - \frac{1}{2}u_1' + \frac{1}{2}u_2')}{\theta_1(\tau, -\frac{1}{2}u_1' + \frac{1}{2}u_2')} \frac{\theta_1(\tau, -z + \frac{3}{2}u_1' + \frac{1}{2}u_2')}{\theta_1(\tau, \frac{3}{2}u_1' + \frac{1}{2}u_2')} \times e^{-\frac{\pi k}{\tau_2}(u_1' + \frac{z}{k})(\bar{u}_1' + \frac{z}{k})} e^{-\frac{\pi k}{\tau_2}(u_2' + \frac{z}{k})(\bar{u}_2' + \frac{z}{k})}.$$

transforms like a holomorphic Jacobi form of wt. 0 and index

$$m = 1 + \frac{2}{k}$$

Indefinite Theta Function (2,2)

Integrating

$$\chi(\widetilde{A}_1;\tau,z) = 2\left(\frac{-i\theta_1(\tau,z)}{\eta(\tau)^3}\right)^2 \sum_{n,w\in\mathbb{Z}^2} \frac{q^{n_1w_1+n_2w_2}\,\zeta_z^{w_1+w_2+\frac{2n_2}{k}}}{\left(1-\zeta_z^{-1}q^{-n_1}\right)\left(1-\zeta_z^{-1}q^{-n_2}\right)}$$

$$\times\frac{1}{4}\sum_{c_1,c_2\in\{0,1\}}(-1)^{c_1+c_2}E_2\left(M^{-T};\frac{1}{2k}\sqrt{\frac{\tau_2}{2}}M\begin{pmatrix} n_1-n_2-2k(w_1+c_1)\\ 5n_2-n_1-2k(w_2+c_2)\end{pmatrix}\right).$$

E_2 → 2 dim. error function

The holomorphic part

$$\chi^{\mathrm{hol}}(\widetilde{A}_1;\tau,z) = h(0,0;\tau,z)+h(2,1;\tau,z)+h(3,1;\tau,z)+h(4,1;\tau,z)$$

where

$$h(p_1,p_2;\tau,z) = 2\left(\frac{-i\,\theta_1(\tau,z)}{\eta(\tau)^3}\right)^2 \sum_{w,r\in\mathbb{Z}^2}\frac{1}{4}\left[\mathrm{sign}(-4r_1+p_1-p_2)-\mathrm{sign}(5w_1+w_2+p_1+\frac{\mathrm{Im}(z)}{\tau_2})\right]$$

$$\times\left[\mathrm{sign}(-4r_2-p_1+5p_2)-\mathrm{sign}(w_1+w_2+p_2+\frac{\mathrm{Im}(z)}{\tau_2})\right]\zeta_z^{2w_1+2w_2+r_1+r_2+p_2}$$

$$\times q^{5w_1^2+2w_1w_2+w_2^2+r_1(5w_1+w_2)+r_2(w_1+w_2)}\,q^{p_1(w_1+r_1)+p_2(w_2+r_2)},$$

This is an indefinite theta function associated with a lattice of type (2,2).

Summary

- Elliptic genus of 2 dim GLSM flowing to a compact SCFT provides interesting examples of relation between low energy spectrum and Jacobi forms.

- Elliptic genus of 2 dim GLSM flowing to a non-compact SCFT provides examples of mock modular (Jacobi) form.

- These mock Jacobi form is characterised by the presence of "d" dim. Error function.

- These examples realises the modular completion of an indefinite theta function associated with a lattice of type (d,d).