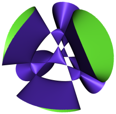
A supersymmetric index for a class of 2d σ-models with large $\mathcal{N} = 4$ superconformal symmetry

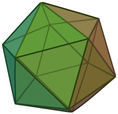
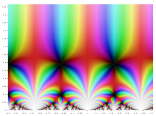
Based on 1804.09987 and WIP with Anne Taormina, Xin Tang

Sam Fearn

Durham University

September 12th, 2018



Introduction & Motivation

Mathieu Moonshine appears when considering a quantity known as the *elliptic genus* of a $K3$ theory, which is an $\mathcal{N} = (4, 4)$ theory with $c = 6$.

Introduction & Motivation

Mathieu Moonshine appears when considering a quantity known as the *elliptic genus* of a $K3$ theory, which is an $\mathcal{N} = (4,4)$ theory with $c = 6$.

Definition

The Elliptic Genus of an $\mathcal{N} = (4,4)$ *conformal field theory is defined as*

$$\varepsilon_{\mathcal{C}}(\tau, z) := \mathrm{Tr}_{\mathcal{H}^R}\left((-1)^F q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{\bar{c}}{24}} y^{2J_0^3}\right) \tag{1}$$

Introduction & Motivation

Mathieu Moonshine appears when considering a quantity known as the *elliptic genus* of a $K3$ theory, which is an $\mathcal{N} = (4,4)$ theory with $c = 6$.

Definition

The Elliptic Genus of an $\mathcal{N} = (4,4)$ *conformal field theory is defined as*

$$\varepsilon_{\mathcal{C}}(\tau, z) := \mathrm{Tr}_{\mathcal{H}^R}\left((-1)^F q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L_0} - \frac{\bar{c}}{24}} y^{2J_0^3}\right) \quad (1)$$

The elliptic genus is a moduli space invariant given in terms of the partition function as

$$\varepsilon_{\mathcal{M}}(\tau, z) := Z_{\tilde{R}}(\tau, z; \bar{\tau}, \bar{z} = 0). \quad (2)$$

Introduction & Motivation

Mathieu Moonshine appears when considering a quantity known as the *elliptic genus* of a $K3$ theory, which is an $\mathcal{N} = (4,4)$ theory with $c = 6$.

Definition

The Elliptic Genus of an $\mathcal{N} = (4,4)$ conformal field theory is defined as

$$\varepsilon_{\mathcal{C}}(\tau, z) := \mathrm{Tr}_{\mathcal{H}^R}\left((-1)^F q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{\bar{c}}{24}} y^{2J_0^3}\right) \quad (1)$$

The elliptic genus is a moduli space invariant given in terms of the partition function as

$$\varepsilon_{\mathcal{M}}(\tau, z) := Z_{\tilde{R}}(\tau, z; \bar{\tau}, \bar{z} = 0). \quad (2)$$

In terms of $\mathcal{N} = 4$ characters we can expand the elliptic genus of $K3$ as

$$\varepsilon_{K3}(\tau, z) = 24\, \mathrm{ch}^{\tilde{R}}_{l=0}(\tau, z) + 2 h_2(\tau) q^{\frac{1}{8}} \hat{\mathrm{ch}}^{\tilde{R}}_{l=1/2}(\tau, z) \quad (3)$$

Unitary Ramond HW Representations of A_γ

σ-models on group manifolds can possess a SCA larger than the usual $\mathcal{N} = 4$, called A_γ[1]. This is an $\mathcal{N} = 4$ SCA with an $SU(2) \oplus SU(2) \oplus U(1)$ Kac-Moody subalgebra and four free fermionic fields.

[1] Philippe Spindel et al. "Complex structures on parallelised group manifolds and supersymmetric σ-models". In: *Physics Letters B* 206.1 (1988), pp. 71–74.

Unitary Ramond HW Representations of A_γ

σ-models on group manifolds can possess a SCA larger than the usual $\mathcal{N}=4$, called A_γ[1]. This is an $\mathcal{N}=4$ SCA with an $SU(2)\oplus SU(2)\oplus U(1)$ Kac-Moody subalgebra and four free fermionic fields.

Characters for A_γ are defined by

$$\mathrm{Ch}^{A_\gamma,R} = \mathrm{Tr}_{\mathcal{H}^R}(q^{L_0-c/24} z_+^{2T_0^{+3}} z_-^{2T_0^{-3}} \chi^{iU_0}). \tag{4}$$

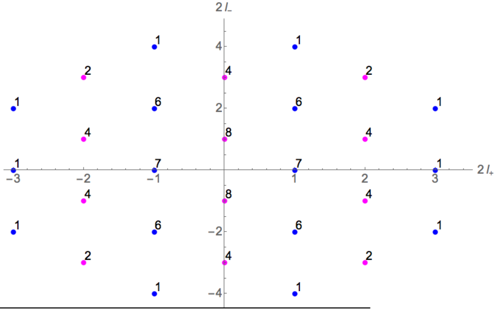
[1] Spindel et al., "Complex structures on parallelised group manifolds and supersymmetric σ-models".

Unitary Ramond HW Representations of A_γ

σ-models on group manifolds can possess a SCA larger than the usual $\mathcal{N} = 4$, called A_γ[1]. This is an $\mathcal{N} = 4$ SCA with an $SU(2) \oplus SU(2) \oplus U(1)$ Kac-Moody subalgebra and four free fermionic fields.

Characters for A_γ are defined by

$$\mathrm{Ch}^{A_\gamma,R} = \mathrm{Tr}_{\mathcal{H}^R}(q^{L_0 - c/24} z_+^{2T_0^{+3}} z_-^{2T_0^{-3}} \chi^{iU_0}). \quad (4)$$



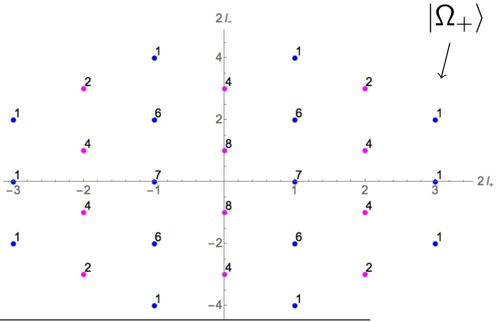
[1] Spindel et al., "Complex structures on parallelised group manifolds and supersymmetric σ-models".

Unitary Ramond HW Representations of A_γ

σ-models on group manifolds can possess a SCA larger than the usual $\mathcal{N} = 4$, called A_γ[1]. This is an $\mathcal{N} = 4$ SCA with an $SU(2) \oplus SU(2) \oplus U(1)$ Kac-Moody subalgebra and four free fermionic fields.

Characters for A_γ are defined by

$$\mathrm{Ch}^{A_\gamma, R} = \mathrm{Tr}_{\mathcal{H}^R}(q^{L_0 - c/24} z_+^{2T_0^{+3}} z_-^{2T_0^{-3}} \chi^{iU_0}). \quad (4)$$



Considering the norm $|Q_0^{-k} G_0^{-k} |\Omega_+\rangle|$ leads to a unitarity bound

$$(h - \frac{c}{24})k \geq u^2 + (l_+^+ + l_+^-)^2 \quad (5)$$

[1] Spindel et al., "Complex structures on parallelised group manifolds and supersymmetric σ-models".

The Question

Does there exist a story similar to that of Mathieu moonshine for this larger algebra?

The Index I_1

Massive characters of A_γ have a double zero at $z_+ = z_-$, while massless characters only have a single zero. This is due to the contribution of the zero modes of the free fermions.

$$\mathrm{Ch}^{A_\gamma} = \mathrm{Ch}^{A_{QU}} \times \mathrm{Ch}^{\tilde{A}_\gamma} \tag{6}$$

[2] Sergei Gukov et al. "An index for 2D field theories with large $\mathcal{N} = 4$ superconformal symmetry". In: *arXiv preprint* (2004). eprint: hep-th/0404023.

The Index I_1

Massive characters of A_γ have a double zero at $z_+ = z_-$, while massless characters only have a single zero. This is due to the contribution of the zero modes of the free fermions.

$$\mathrm{Ch}^{A_\gamma} = \mathrm{Ch}^{A_{QU}} \times \mathrm{Ch}^{\tilde{A}_\gamma} \tag{6}$$

Using this, one can construct an index for A_γ theories which is deformation invariant.[2]

[2] Gukov et al., "An index for 2D field theories with large $\mathcal{N} = 4$ superconformal symmetry".

The Index I_1

Massive characters of A_γ have a double zero at $z_+ = z_-$, while massless characters only have a single zero. This is due to the contribution of the zero modes of the free fermions.

$$\mathrm{Ch}^{A_\gamma} = \mathrm{Ch}^{A_{QU}} \times \mathrm{Ch}^{\tilde{A}_\gamma} \tag{6}$$

Using this, one can construct an index for A_γ theories which is deformation invariant.[2]

Definition

The index I_1 of a theory $\mathcal{D}$ with partition function $Z^{\mathcal{D}}$, is given by

$$\begin{aligned} I_1(\mathcal{D})(q, z_+, z_-, \bar{q}, \bar{z}) &:= -\bar{z}_+ \frac{\partial}{\partial \bar{z}_-} Z^{\mathcal{D}}_{\mathscr{H}^{\tilde{R}}}(q, z_+, z_-, \bar{q}, \bar{z}_+, \bar{z}_-)\Big|_{\bar{z}_+ = \bar{z}_- = \bar{z}}, \\ &= \mathrm{Tr}_{\mathscr{H}^R}\left(-F_R (-1)^F q^{L_0 - c/24} \bar{q}^{\bar{L}_0 - \bar{c}/24} z_+^{2T_0^{+3}} z_-^{2T_0^{-3}} \bar{z}^{2(\bar{T}_0^{+3} + \bar{T}_0^{-3})}\right), \end{aligned} \tag{7}$$

[2] Gukov et al., "An index for 2D field theories with large $\mathcal{N} = 4$ superconformal symmetry".

Charges of Contributing States

The contribution to the index of a massless A_γ representation is given by[3]

$$I_1\left(\mathrm{Ch}_0^{A_\gamma,\tilde{R}}(k^+,k^-,l^+,l^-,u)\right)=(-1)^{2l^- -1}q^{u^2/k}\Theta^-_{\mu,k}(\tau,\omega), \tag{8}$$

where $\mu = 2(l^+ + l^-) - 1$,

$$\Theta^-_{\mu,k} = q^{\mu^2/4k}\sum_{n\in\mathbb{Z}} q^{kn^2+n\mu}(z^{2kn+\mu} - z^{-2kn-\mu}). \tag{9}$$

[3] Gukov et al., "An index for 2D field theories with large $\mathcal{N}=4$ superconformal symmetry".

Charges of Contributing States

The contribution to the index of a massless A_γ representation is given by[3]

$$I_1\left(\mathrm{Ch}_0^{A_\gamma,\tilde{R}}(k^+,k^-,l^+,l^-,u)\right)=(-1)^{2l^--1}q^{u^2/k}\Theta^-_{\mu,k}(\tau,\omega), \tag{8}$$

where $\mu=2(l^++l^-)-1$,

$$\Theta^-_{\mu,k}=q^{\mu^2/4k}\sum_{n\in\mathbb{Z}}q^{kn^2+n\mu}(z^{2kn+\mu}-z^{-2kn-\mu}). \tag{9}$$

From here we can read that contributing states must satisfy

$$L_0-\frac{c}{24}=\frac{u^2}{k}+\frac{1}{k}\left((T_0^{+3}+T_0^{-3})^2\right). \tag{10}$$

[3] Gukov et al., "An index for 2D field theories with large $\mathcal{N}=4$ superconformal symmetry".

The Zero Mode Subalgebra of A_γ and Supertableaux

In the Ramond sector, the zero mode subalgebra of A_γ is the direct sum of a $\mathfrak{u}(1)$ algebra and the simple Lie superalgebra $A(1|1)$,

$$A_{\gamma_0} = A(1|1) \oplus \mathfrak{u}(1). \tag{11}$$

The Zero Mode Subalgebra of A_γ and Supertableaux

In the Ramond sector, the zero mode subalgebra of A_γ is the direct sum of a $\mathfrak{u}(1)$ algebra and the simple Lie superalgebra $A(1|1)$,

$$A_{\gamma_0} = A(1|1) \oplus \mathfrak{u}(1). \tag{11}$$

We can therefore study the branching of representations of A_γ into representations of (the complexification) of $\mathfrak{su}(2|2)$, which can be classified by supertableaux[4].

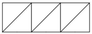
[4] A Baha Balantekin and Itzhak Bars. "Dimension and character formulas for Lie supergroups". In: *Journal of Mathematical Physics* 22.6 (1981), pp. 1149–1162; Sam Fearn. "Young Supertableaux and the large $\mathcal{N} = 4$ superconformal algebra". In: *arXiv preprint* (2018). eprint: 1804.09987.

The Zero Mode Subalgebra of A_γ and Supertableaux

In the Ramond sector, the zero mode subalgebra of A_γ is the direct sum of a $\mathfrak{u}(1)$ algebra and the simple Lie superalgebra $A(1|1)$,

$$A_{\gamma_0} = A(1|1) \oplus \mathfrak{u}(1). \tag{11}$$

We can therefore study the branching of representations of A_γ into representations of (the complexification) of $\mathfrak{su}(2|2)$, which can be classified by supertableaux[4].



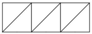
[4] Balantekin and Bars, "Dimension and character formulas for Lie supergroups"; Fearn, "Young Supertableaux and the large $\mathcal{N} = 4$ superconformal algebra".

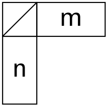
The Zero Mode Subalgebra of A_γ and Supertableaux

In the Ramond sector, the zero mode subalgebra of A_γ is the direct sum of a $\mathfrak{u}(1)$ algebra and the simple Lie superalgebra $A(1|1)$,

$$A_{\gamma_0} = A(1|1) \oplus \mathfrak{u}(1). \tag{11}$$

We can therefore study the branching of representations of A_γ into representations of (the complexification) of $\mathfrak{su}(2|2)$, which can be classified by supertableaux[4].



$l_1\Big($  $\Big) = (-1)^n(z^{-m-n-1} - z^{m+n+1})$ (12)

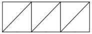
[4] Balantekin and Bars, "Dimension and character formulas for Lie supergroups"; Fearn, "Young Supertableaux and the large $\mathcal{N} = 4$ superconformal algebra".

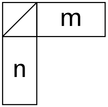
The Zero Mode Subalgebra of A_γ and Supertableaux

In the Ramond sector, the zero mode subalgebra of A_γ is the direct sum of a $\mathfrak{u}(1)$ algebra and the simple Lie superalgebra $A(1|1)$,

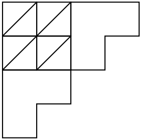
$$A_{\gamma_0} = A(1|1) \oplus \mathfrak{u}(1). \tag{11}$$

We can therefore study the branching of representations of A_γ into representations of (the complexification) of $\mathfrak{su}(2|2)$, which can be classified by supertableaux[4].





$$l_1\big(\ \cdot\ \big) = (-1)^n(z^{-m-n-1} - z^{m+n+1}) \tag{12}$$



$$l_1\big(\ \cdot\ \big) = 0 \tag{13}$$

[4] Balantekin and Bars, "Dimension and character formulas for Lie supergroups"; Fearn, "Young Supertableaux and the large $\mathcal{N} = 4$ superconformal algebra".

The Sum Rules

The coset construction of A_γ has been used[5] to construct character sum rules for $\tilde{A}_\gamma$ ($k^- = 2$).

[5] Hirosi Ooguri, Jens Lyng Petersen, and Anne Taormina. "Modular invariant partition functions for the doubly extended $\mathcal{N} = 4$ superconformal algebras". In: *Nuclear Physics B* 368.3 (1992), pp. 611–624; Jens Lyng Petersen and Anne Taormina. "Coset construction and character sum rules for the doubly extended $\mathcal{N} = 4$ superconformal algebras". In: *Nuclear Physics B* 398.2 (1993), pp. 459–495.

The Sum Rules

The coset construction of A_γ has been used[5] to construct character sum rules for $\tilde{A}_\gamma$ ($k^- = 2$).

$$\mathcal{H}^{WS} \otimes \mathcal{H}_{\Lambda}^{\widehat{su(3)}_{\tilde{k}^+}} = \bigoplus_i \left(\mathcal{H}^{\tilde{A}_\gamma}{}_{0,l_i^+,l_i^-} \otimes \mathcal{H}_{m_i}^{\mathcal{A}_{3k}} \right) \bigoplus_j \left(\oplus_n \mathcal{H}^{\tilde{A}_\gamma}{}_{h_n,l_j^+} \otimes \mathcal{H}_{m_j}^{\mathcal{A}_{3k}} \right)$$

[5] Ooguri, Petersen, and Taormina, "Modular invariant partition functions for the doubly extended $\mathcal{N} = 4$ superconformal algebras"; Petersen and Taormina, "Coset construction and character sum rules for the doubly extended $\mathcal{N} = 4$ superconformal algebras".

The Sum Rules

The coset construction of A_γ has been used[5] to construct character sum rules for $\tilde{A}_\gamma$ ($k^- = 2$).

$$\mathcal{H}^{WS} \otimes \mathcal{H}_\Lambda^{\widehat{su(3)}_{\tilde{k}^+}} = \bigoplus_i \left(\mathcal{H}^{\tilde{A}_\gamma}{}_{0,l_i^+,l_i^-} \otimes \mathcal{H}_{m_i}^{\mathcal{A}_{3k}} \right) \bigoplus_j \left(\oplus_n \mathcal{H}^{\tilde{A}_\gamma}{}_{h_n,l_j^+} \otimes \mathcal{H}_{m_j}^{\mathcal{A}_{3k}} \right)$$

In the $\tilde{R}$ sector, the sum rules for A_γ representations are of the form

$$\frac{\theta_1(q, z_+ z_-)\theta_1(q, z_+^{-1} z_-)}{\eta^2(q)} \cdot \frac{\theta_1(q, z_- z_y)\theta_1(q, z_- z_y^{-1})}{\eta^2(q)} \chi_\Lambda^{\mathfrak{su}(3)}(q, z_+, z_y)$$

$$= \sum_{L=0}^{k-2} \eta(q) M_\Lambda^L(q, z_y) \ \mathrm{Ch}_0^{A_\gamma, \tilde{R}}(L; q, z_\pm)$$

$$+ \sum_{2\tilde{l}^+=0}^{\tilde{k}^+-1} \sum_{n \in \mathbb{Z}_k} \hat{\mathrm{Ch}}_m^{A_\gamma, \tilde{R}}(l^\pm; q, z_\pm)\, \eta(q) \chi^{3k}_{-2a_1+2a_2+6\tilde{l}^+ +6n}(q, z_y) F^\Lambda_{2\tilde{l}^+, n}(q),$$

[5] Ooguri, Petersen, and Taormina, "Modular invariant partition functions for the doubly extended $\mathcal{N} = 4$ superconformal algebras"; Petersen and Taormina, "Coset construction and character sum rules for the doubly extended $\mathcal{N} = 4$ superconformal algebras".

A_γ Theories From Diagonal $\mathfrak{su}(3)$ Invariants

The sum rules give a way to construct modularly invariant partition functions for A_γ theories using $\widehat{\mathfrak{su}(3)}_{\tilde{k}^+}$ invariants.

A_γ Theories From Diagonal $\mathfrak{su}(3)$ Invariants

The sum rules give a way to construct modularly invariant partition functions for A_γ theories using $\widehat{\mathfrak{su}(3)}_{\tilde{k}^+}$ invariants.
In order to calculate the index we restrict to the $\tilde{R}\tilde{R}$ sector of the theory where the (restricted) partition function is given by

$$Z_{\tilde{R},\tilde{R}}(q, z_+, z_-, z_y) = \sum_{\Lambda \in P_+^{\tilde{k}^+}} \left| \frac{\theta_1(q, z_+ z_-)\theta_1(q, z_+^{-1} z_-)}{\eta^2(q)} \cdot \frac{\theta_1(q, z_- z_y)\theta_1(q, z_- z_y^{-1})}{\eta^2(q)} \chi_\Lambda^{\mathfrak{su}(3)}(q, z_+, z_y) \right|^2, \tag{14}$$

A_γ Theories From Diagonal $\mathfrak{su}(3)$ Invariants

The sum rules give a way to construct modularly invariant partition functions for A_γ theories using $\widehat{\mathfrak{su}(3)}_{\tilde{k}^+}$ invariants.

Labelling the $\tilde{R}\tilde{R}$ sector of the partition function of the diagonal theory as $Z^{D_{\tilde{k}^+}}_{\tilde{R},\tilde{R}}$ and the theory itself as $\mathcal{D}_{\tilde{k}^+}$, we can now calculate the index I_1 of this theory as,

$$\begin{aligned} I_1(\mathcal{D}_{\tilde{k}^+})(q, z_+, z_-, z_y; \bar{q}, \bar{z}, \bar{z}_y) :=&\ -\bar{z}_+ \frac{\partial}{\partial \bar{z}_-} Z^{D_{\tilde{k}^+}}_{\tilde{R},\tilde{R}} \Big|_{\bar{z}_+ = \bar{z}_-}\ , \\ = |\eta(q)|^2 \sum_{\Lambda \in P_+^{\tilde{k}^+}} &\left(\sum_{L=0}^{k-2} M^L_\Lambda(q, z_y)\, \mathrm{Ch}_0^{A_\gamma, \tilde{R}}(L; q, z_\pm) \right. \\ &\left. + \sum_{2\tilde{l}^+=0}^{\tilde{k}^+-1} \sum_{n \in \mathbb{Z}_k} \hat{M}^\Lambda_{2l_+, n}(q, z_+, z_-, z_y) F^\Lambda_{2\tilde{l}^+, n}(q) \right) \\ &\cdot I_1 \left(\sum_{L=0}^{k-2} M^L_\Lambda(\bar{q}, \bar{z}_y)\, \mathrm{Ch}_0^{A_\gamma, \tilde{R}}(L; \bar{q}, \bar{z}_\pm) \right) . \end{aligned} \tag{15}$$

The Functions $F_i(q)$ for $\tilde{k}^+ = 2$

In the simplest case we consider, $\tilde{k}^+ = 2$, there are 6 independent functions $F_i(q)$.

The Functions $F_i(q)$ for $\tilde{k}^+ = 2$

In the simplest case we consider, $\tilde{k}^+ = 2$, there are 6 independent functions $F_i(q)$. Up to the offset, the coefficients of these functions agree (to at least the first 29 terms) with some known functions. In particular, the q-expansions we have obtained agree with,

$$F_2(q) \sim q^{2/5} \frac{f(-q^5)^2}{f(-q^2,-q^3)}, \qquad F_4(q) \sim q^{1/5} \frac{f(-q^5)^2}{f(-q,-q^4)}, \qquad (16)$$
$$F_3(q) \sim q^{-2/5}\Psi_1(q), \qquad F_5(q) \sim \Psi_0(q),$$

where

$$f(a,b) = \sum_{n\in\mathbb{Z}} a^{n(n+1)/2} b^{n(n-1)/2}, \qquad f(-a) = f(-a,-a^2), \qquad (17)$$

is the Ramanujan general theta function, and $\Psi_0(q)$ and $\Psi_1(q)$ are 5^{th} order mock theta functions.

The Functions $F_i(q)$ for $\tilde{k}^+ = 2$

In the simplest case we consider, $\tilde{k}^+ = 2$, there are 6 independent functions $F_i(q)$. Up to the offset, the coefficients of these functions agree (to at least the first 29 terms) with some known functions. In particular, the q-expansions we have obtained agree with,

$$F_2(q) \sim q^{2/5}\frac{f(-q^5)^2}{f(-q^2,-q^3)}, \qquad F_4(q) \sim q^{1/5}\frac{f(-q^5)^2}{f(-q,-q^4)}, \qquad (16)$$
$$F_3(q) \sim q^{-2/5}\Psi_1(q), \qquad F_5(q) \sim \Psi_0(q),$$

where

$$f(a,b) = \sum_{n\in\mathbb{Z}} a^{n(n+1)/2} b^{n(n-1)/2}, \qquad f(-a) = f(-a,-a^2), \qquad (17)$$

is the Ramanujan general theta function, and $\Psi_0(q)$ and $\Psi_1(q)$ are 5^{th} order mock theta functions. Work to prove that these are indeed the functions $F_i(q)$ is ongoing.

Conclusions and Summary

- In the Mathieu Moonshine story, we consider the moduli space invariant elliptic genus of K3. This index gives the Witten Index of the right-movers, counting only massless reps.

Conclusions and Summary

- In the Mathieu Moonshine story, we consider the moduli space invariant elliptic genus of K3. This index gives the Witten Index of the right-movers, counting only massless reps.
- There exists a larger $\mathcal{N} = 4$ algebra A_γ, which is found by considering WZW models on symmetric spaces (Wolf spaces). The index I_1 is an invariant of such theories, generalising the elliptic genus.

Conclusions and Summary

- In the Mathieu Moonshine story, we consider the moduli space invariant elliptic genus of K3. This index gives the Witten Index of the right-movers, counting only massless reps.
- There exists a larger $\mathcal{N} = 4$ algebra A_γ, which is found by considering WZW models on symmetric spaces (Wolf spaces). The index I_1 is an invariant of such theories, generalising the elliptic genus.
- The index I_1 receives contributions only from massless states, but unlike the elliptic genus, states throughout the representation.

Conclusions and Summary

- In the Mathieu Moonshine story, we consider the moduli space invariant elliptic genus of K3. This index gives the Witten Index of the right-movers, counting only massless reps.
- There exists a larger $\mathcal{N} = 4$ algebra A_γ, which is found by considering WZW models on symmetric spaces (Wolf spaces). The index I_1 is an invariant of such theories, generalising the elliptic genus.
- The index I_1 receives contributions only from massless states, but unlike the elliptic genus, states throughout the representation.
- We can classify the representations of the 0-mode algebra by supertableau and indentify the index contributing tableau.

Conclusions and Summary

- In the Mathieu Moonshine story, we consider the moduli space invariant elliptic genus of K3. This index gives the Witten Index of the right-movers, counting only massless reps.
- There exists a larger $\mathcal{N} = 4$ algebra A_γ, which is found by considering WZW models on symmetric spaces (Wolf spaces). The index I_1 is an invariant of such theories, generalising the elliptic genus.
- The index I_1 receives contributions only from massless states, but unlike the elliptic genus, states throughout the representation.
- We can classify the representations of the 0-mode algebra by supertableau and indentify the index contributing tableau.
- Using the character sum rules and the diagonal $\widehat{\mathfrak{su}(3)}$ invariant we can construct partition functions for this class of A_γ theories and calculate their indices.

Conclusions and Summary

- In the Mathieu Moonshine story, we consider the moduli space invariant elliptic genus of K3. This index gives the Witten Index of the right-movers, counting only massless reps.
- There exists a larger $\mathcal{N} = 4$ algebra A_γ, which is found by considering WZW models on symmetric spaces (Wolf spaces). The index I_1 is an invariant of such theories, generalising the elliptic genus.
- The index I_1 receives contributions only from massless states, but unlike the elliptic genus, states throughout the representation.
- We can classify the representations of the 0-mode algebra by supertableau and indentify the index contributing tableau.
- Using the character sum rules and the diagonal $\widehat{\mathfrak{su}(3)}$ invariant we can construct partition functions for this class of A_γ theories and calculate their indices.
- The functions $F_i(q)$ describing the massive content of the partition function, agree in some cases with 5^{th}-order mock theta functions.

Thanks for listening!

Spectral Flow Orbits

There exists an automorphism of A_γ known as spectral flow which moreover induces an isomorphism of representations of A_γ.

Spectral Flow Orbits

There exists an automorphism of A_γ known as spectral flow which moreover induces an isomorphism of representations of A_γ. Under two-fold symmetric spectral flow with $n \in \mathbb{Z}$, the Virasoro and $SU(2)$ zero modes are transformed as

$$\begin{aligned} L_0^{2n,2n} &= L_0 - 2n(T_0^{+3} + T_0^{-3}) + n^2, \\ T_0^{2n,2n;+3} &= T_0^{+3} - nk^+, \\ T_0^{2n,2n;-3} &= T_0^{-3} - nk^-. \end{aligned} \tag{18}$$

Spectral Flow Orbits

There exists an automorphism of A_γ known as spectral flow which moreover induces an isomorphism of representations of A_γ. Under two-fold symmetric spectral flow with $n \in \mathbb{Z}$, the Virasoro and $SU(2)$ zero modes are transformed as

$$\begin{aligned} L_0^{2n,2n} &= L_0 - 2n(T_0^{+3} + T_0^{-3}) + n^2, \\ T_0^{2n,2n;+3} &= T_0^{+3} - nk^+, \\ T_0^{2n,2n;-3} &= T_0^{-3} - nk^-. \end{aligned} \tag{18}$$

Under this spectral flow, the massless condition

$$L_0 - \frac{c}{24} = \frac{u^2}{k} + \frac{1}{k}\left((T_0^{+3} + T_0^{-3})^2\right), \tag{19}$$

is preserved.

Spectral Flow Orbits

There exists an automorphism of A_γ known as spectral flow which moreover induces an isomorphism of representations of A_γ. Under two-fold symmetric spectral flow with $n \in \mathbb{Z}$, the Virasoro and $SU(2)$ zero modes are transformed as

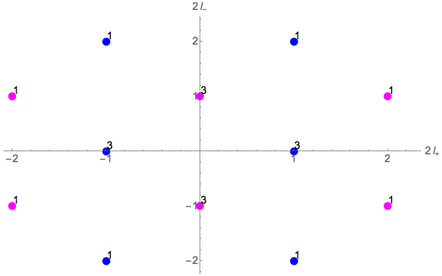
$$\begin{aligned} L_0^{2n,2n} &= L_0 - 2n(T_0^{+3} + T_0^{-3}) + n^2, \\ T_0^{2n,2n;+3} &= T_0^{+3} - nk^+, \\ T_0^{2n,2n;-3} &= T_0^{-3} - nk^-. \end{aligned} \tag{18}$$

Under this spectral flow, the massless condition

$$L_0 - \frac{c}{24} = \frac{u^2}{k} + \frac{1}{k}\left((T_0^{+3} + T_0^{-3})^2\right), \tag{19}$$

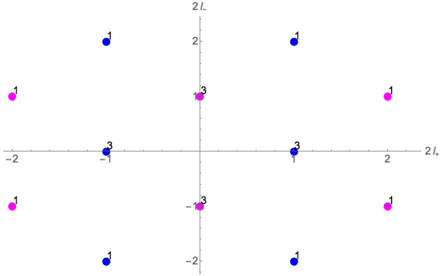
is preserved. The index I_1 is therefore seen to count two-fold symmetric spectral flow orbits of the extremely charged states.

A_γ in Supertableaux

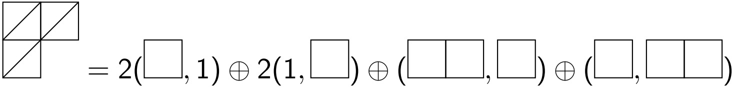


Ground level for $k^+ = 3$, $k^- = 2$, $l^+ = l^- = 1$.

A_γ in Supertableaux

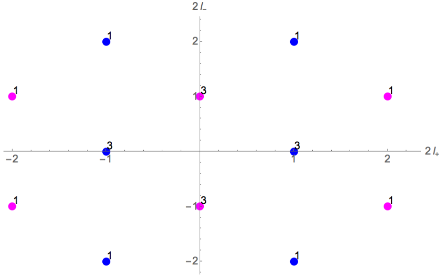


Ground level for $k^+ = 3$, $k^- = 2$, $l^+ = l^- = 1$.

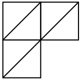


(20)

A_γ in Supertableaux



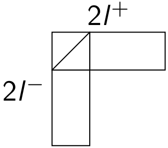
Ground level for $k^+ = 3$, $k^- = 2$, $l^+ = l^- = 1$.



$$= 2(\square, 1) \oplus 2(1, \square) \oplus (\square\square, \square) \oplus (\square, \square\square) \qquad (20)$$

In general the ground level is always described by a single tableau

$$\mathrm{Ch}_0^{A_\gamma, R} = \left(\vphantom{\Big|} \right.$$



$$\left. \vphantom{\Big|} \right) q^{h-c/24} + \dots \qquad (21)$$

The Functions $F_i(q)$

The functions $F^{\Lambda}_{2\tilde{l}^+,n}$ describe how the massive representation of A_γ embed in the representation space.

The Functions $F_i(q)$

The functions $F^{\Lambda}_{2\tilde{l}^+,n}$ describe how the massive representation of A_γ embed in the representation space. A priori there are

$$N_D(\tilde{k}^+) = \frac{1}{2}\tilde{k}^+(\tilde{k}^+ + 1)(\tilde{k}^+ + 2)(\tilde{k}^+ + 3) \tag{22}$$

such functions in the character sum rules, though symmetries in the character sum rules can be used to show that there are only

$$N_I(\tilde{k}^+) = \frac{N_D(\tilde{k}^+)}{12} + \frac{1}{2}\left\lceil \frac{\tilde{k}^+}{2} \right\rceil \left\lceil \frac{\tilde{k}^+ + 2}{2} \right\rceil, \tag{23}$$

independent such functions.

The Functions $F_i(q)$

The functions $F^{\Lambda}_{2\tilde{l}^+,n}$ describe how the massive representation of A_γ embed in the representation space. A priori there are

$$N_D(\tilde{k}^+) = \frac{1}{2}\tilde{k}^+(\tilde{k}^+ + 1)(\tilde{k}^+ + 2)(\tilde{k}^+ + 3) \tag{22}$$

such functions in the character sum rules, though symmetries in the character sum rules can be used to show that there are only

$$N_I(\tilde{k}^+) = \frac{N_D(\tilde{k}^+)}{12} + \frac{1}{2}\left\lceil \frac{\tilde{k}^+}{2} \right\rceil \left\lceil \frac{\tilde{k}^+ + 2}{2} \right\rceil, \tag{23}$$

independent such functions.

Generically we label the independent such functions as $F_i(q)$. We have calculated the first $\sim$15 terms of these q-series for $\tilde{k}^+ \in \{2, 3, 4, 5\}$ using Mathematica.