Mixed mock modular forms and BPS black hole entropy

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Introduction

- Discuss a physical realization of interesting number theoretic objects.
- Relationship between `String Theory`, `Number Theory` and `Supergravity`.

"My dream is that I will live to see the day when our young physicists, struggling to bring the predictions of superstring theory into correspondence with the facts of nature, will be led to enlarge their analytic machinery to include not only theta-functions but mock theta-functions... But before this can happen, the purely mathematical exploration of the mock-modular forms and their mock-symmetries must be carried a great deal further."

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- This "purely mathematical exploration" of mock modular forms has now been "carried a great deal further".
- In physics: BPS black holes, supersymmetric localization and string theory make use of this "analytic machinery".

1/4-BPS black hole degeneracies

- Physical system: 1/4-BPS dyons in IIB string theory compactified on K3 $\times T^2$ with D1-D5-P-KK. Characterized by charge vector $\Gamma^i{}_\alpha = (Q^i,\, P^i)$.

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- Degeneracy of states in terms of the (T-duality invariant) combinations

$$n = Q^2/2\,, \quad m = P^2/2\,, \quad \ell = Q \cdot P$$

given by the Fourier coeff. of the inverse of a Siegel modular form:

[Dijkgraaf,Verlinde,Verlinde'97;...]

$$d_{1/4}(n, m, \ell) = (-1)^{\ell+1} \int_{\mathcal{C}} \frac{e^{-2i\pi(n\tau + m\sigma + \ell z)}}{\Phi_{10}(\tau, \sigma, z)}\, d\tau\, d\sigma\, dz$$

- Φ_{10} is the Igusa cusp form (weight 10). Has second order zeroes at $z = 0$ and $Sp(2, \mathbb{Z})$ images. Thus contour dependence $\mathcal{C}$ in $d_{1/4}$.

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- Φ_{10} is the Igusa cusp form (weight 10). Has second order zeroes at $z = 0$ and $Sp(2, \mathbb{Z})$ images. Thus contour dependence $\mathcal{C}$ in $d_{1/4}$.
- Physically, due to existence of bound states in the spectrum which are stable only in some chambers of the moduli space and can appear/decay.

Wall-crossing, meromorphicity and mock modularity

- [Dabholkar,Murthy,Zagier'12] showed that this wall-crossing is manifest in the Fourier-Jacobi expansion of the Igusa cusp form:

$$\Phi_{10}^{-1}(\tau, z, \sigma) = \sum_{m \geq -1} \psi_m(\tau, z) p^m \quad \text{with} \quad \psi_m(\tau, z) = \psi_m^{\mathcal{F}}(\tau, z) + \psi_m^{\mathcal{P}}(\tau, z)$$

- $\psi_m^{\mathcal{P}}$ has same poles and residues as ψ_m, generating function of bound states. $\psi_m^{\mathcal{F}}$ counts "immortal" 1/4-BPS states (stable throughout moduli space). Holomorphic in z but *mock* Jacobi since part of the spectrum is removed.

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- Further, can impose an "optimal" growth condition

$$\psi_m^{\mathcal{F}}(\tau,z) = \Delta(\tau)^{-1}\big(\varphi_{2,m}(\tau,z) + \varphi_{2,m}^{\text{opt}}(\tau,z)\big) \quad \text{where} \quad \Delta(\tau) = \eta(\tau)^{24}$$

  $\varphi_{2,m}$ is a weakly holomorphic Jacobi form of weight 2 and index m.
  $\varphi_{2,m}^{\text{opt}}$ can be chosen to be a strongly holomorphic *mock* Jacobi form involving gen. function of Hurwitz-Kronecker class numbers (for m a prime power).

Mixed mock expansion

- The function $\Delta^{-1}\varphi_{2,m}$ admits a standard theta-decomposition, and its vector-valued modular components a standard Rademacher expansion for their Fourier coefficients.
- The function $\Delta^{-1}\varphi^{\mathrm{opt}}_{2,m}$ admits a theta-decomposition, but its vector-valued components are mixed mock modular forms

$$\Delta^{-1}\varphi^{\mathrm{opt}}_{2,m} = \sum_{\ell\in\mathbb{Z}/2m\mathbb{Z}} h^{\mathrm{MM}}_{\ell}(\tau)\,\vartheta_{m,\ell}(\tau,z)$$

The shadow of h^{MM}_{ℓ} is $\Delta^{-1}\vartheta_{m,\ell}(\tau,z=0)$, weakly holomorphic modular form.

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  The shadow of h^{MM}_{ℓ} is $\Delta^{-1}\vartheta_{m,\ell}(\tau,z=0)$, weakly holomorphic modular form.
- Extending the analysis of [Bringmann,Manschot'10], obtain the Fourier coefficients of h^{MM}_{ℓ} in terms of Hurwitz-Kronekcer numbers (for m prime). Non-trivial contribution from the shadow. [Ferrari,VR'17]
- Summing the two expansions, we obtain the expression for the Fourier coefficients of $\psi^{\mathcal{F}}_{m}$, i.e. the degeneracy of immortal dyons, in terms of the polar coefficients of $\Delta^{-1}\varphi_{2,m}$ and the Hurwitz-Kronecker numbers.

A macroscopic derivation

- We can aim to reproduce this result macroscopically. 1/4-BPS dyons have description in terms of dyonic blakc holes at strong string coupling.
- These black holes have near-horizon $\mathrm{AdS}_2 \times S^2$. Their quantum entropy $\mathcal{S} = k_B \log d_{\text{macro}}$ is given by a path-integral in supergravity [Sen'08]

$$e^{\mathcal{S}(p,q)} = d_{\text{macro}}(q,p) = \left\langle \exp\left[-iq_I \oint A^I_\tau \, d\tau\right] \right\rangle^{\text{finite}}_{(\mathrm{E})\mathrm{AdS}_2}$$

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- Thanks to supersymmetry, can use localization techniques to evaluate this functional integral exactly in certain cases.

[Dabholkar,Gomes,Murthy'10-'14;de Wit,Murthy,VR'18;...]

- The path-integral localizes to BPS field configurations

$$d_{\text{macro}}(q,p) = \int_{-\infty}^{\infty} \prod_{I=0}^{n_v} [d\phi^I] \exp\left[-\pi q_I \phi^I + 4\pi \,\mathrm{Im} F\left(\frac{\phi^I + ip^I}{2}\right)\right] Z_{\text{1-loop}}(\phi^I)$$

A macroscopic derivation (cont.)

- For the $\mathcal{N}=4$ theory under consideration, $n_v+1=24$ and the prepotential is [Harvey,Moore'96;Cardoso,de Wit,Mohaupt'99]

$$F(X^I) = -\frac{X^1 X^a C_{ab} X^b}{X^0} + \frac{1}{2\pi i} \log \eta^{24}\Big(\frac{X^1}{X^0}\Big) , \quad a,b = 2\ldots 23$$

- Performing the integrals over the ϕ's, we obtain the degeneracies as seen from supergravity in the macroscopic picture. [Murthy,VR'15]

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- Despite these discrepancies, the structure in supergravity is interesting: all the theory knows about is $\Delta(\tau)$ via the $F(X)$. Natural question: is there a way to rebuild the Fourier coefficients of $\psi_m^{\mathcal{F}}$ from the ones of $\Delta(\tau)$ only?

- This seems to be the case. The formula we are looking for is motivated by physical considerations of a black hole/brane system.

[Chowdhury,Kidambi,Murthy,VR,Wrase - WIP]

Thank you for your attention