A module for a case of umbral moonshine

based on arXiv:1709.01952 with Miranda Cheng and Sarah Harrison

Vassilis Anagiannis, UvA

ESI moonshine workshop, Vienna

11 September 2018

Outline

1. Umbral moonshine modules
2. Moonshine and K3
3. Final module

Umbral moonshine modules

- 23 cases of umbral moonshine

Umbral moonshine modules

- 23 cases of umbral moonshine $\leftarrow$ 23 Niemeier lattices N^X

Umbral moonshine modules

- 23 cases of umbral moonshine $\leftarrow$ 23 Niemeier lattices N^X
  - X: rank 24 unions of ADE root systems with the same Coxeter number

Umbral moonshine modules

- 23 cases of umbral moonshine $\leftarrow$ 23 Niemeier lattices N^X
  - X: rank 24 unions of ADE root systems with the same Coxeter number
- The umbral groups are $G^X := \text{Aut}\left(N^X\right)/\text{Weyl}(X)$

Umbral moonshine modules

- 23 cases of umbral moonshine $\leftarrow$ 23 Niemeier lattices N^X
  - X: rank 24 unions of ADE root systems with the same Coxeter number
- The umbral groups are $G^X := \mathrm{Aut}\left(N^X\right)/\mathrm{Weyl}(X)$
- MT series $H^X_{g,r}$ are weight $1/2$ vector-valued mock modular forms

Umbral moonshine modules

- 23 cases of umbral moonshine $\leftarrow$ 23 Niemeier lattices N^X
  - X: rank 24 unions of ADE root systems with the same Coxeter number
- The umbral groups are $G^X := \mathrm{Aut}\left(N^X\right)/\mathrm{Weyl}(X)$
- MT series $H^X_{g,r}$ are weight $1/2$ vector-valued mock modular forms
- Modules have been constructed for E_8^3, A_6^4, A_{12}^2, D_6^4, D_8^3, D_{12}^2, D_{24}

Umbral moonshine modules

- 23 cases of umbral moonshine $\leftarrow$ 23 Niemeier lattices N^X
  - X: rank 24 unions of ADE root systems with the same Coxeter number
- The umbral groups are $G^X := \mathrm{Aut}\left(N^X\right)/\mathrm{Weyl}(X)$
- MT series $H^X_{g,r}$ are weight $1/2$ vector-valued mock modular forms
- Modules have been constructed for E_8^3, A_6^4, A_{12}^2, D_6^4, D_8^3, D_{12}^2, D_{24}
- Small umbral groups, order ≤ 24

Umbral moonshine modules

- 23 cases of umbral moonshine $\leftarrow$ 23 Niemeier lattices N^X
  - X: rank 24 unions of ADE root systems with the same Coxeter number
- The umbral groups are $G^X := \mathrm{Aut}\left(N^X\right)/\mathrm{Weyl}(X)$
- MT series $H^X_{g,r}$ are weight 1/2 vector-valued mock modular forms
- Modules have been constructed for E_8^3, A_6^4, A_{12}^2, D_6^4, D_8^3, D_{12}^2, D_{24}
- Small umbral groups, order ≤ 24
- We constructed a module for D_4^6

Umbral moonshine modules

- 23 cases of umbral moonshine $\leftarrow$ 23 Niemeier lattices N^X
  - X: rank 24 unions of ADE root systems with the same Coxeter number
- The umbral groups are $G^X := \mathrm{Aut}\left(N^X\right)/\mathrm{Weyl}(X)$
- MT series $H^X_{g,r}$ are weight 1/2 vector-valued mock modular forms
- Modules have been constructed for E_8^3, A_6^4, A_{12}^2, D_6^4, D_8^3, D_{12}^2, D_{24}
- Small umbral groups, order ≤ 24
- We constructed a module for D_4^6, where $G^{D_4^6} \cong 3.S_6$

Umbral moonshine modules

- 23 cases of umbral moonshine $\leftarrow$ 23 Niemeier lattices N^X
  - X: rank 24 unions of ADE root systems with the same Coxeter number
- The umbral groups are $G^X := \mathrm{Aut}\left(N^X\right)/\mathrm{Weyl}(X)$
- MT series $H^X_{g,r}$ are weight $1/2$ vector-valued mock modular forms
- Modules have been constructed for E_8^3, A_6^4, A_{12}^2, D_6^4, D_8^3, D_{12}^2, D_{24}
- Small umbral groups, order ≤ 24
- We constructed a module for D_4^6, where $G^{D_4^6} \cong 3.S_6$, $|3.S_6| \sim 10^3$

Umbral moonshine and K3

- *Umbral Moonshine and K3 Surfaces* [Cheng, Harrison]

Umbral moonshine and K3

- *Umbral Moonshine and K3 Surfaces* [Cheng, Harrison]:

$$\mathbf{EG}_g(\tau, z; K3) = \mathbf{EG}_g(\tau, z; X) + \frac{\theta_1(\tau, z)^2}{2\eta(\tau)^6} \left(\frac{1}{2\pi i} \frac{\partial}{\partial \omega} \Psi_g^X(\tau, \omega) \right) \Bigg|_{\omega=0}$$

Umbral moonshine and K3

- *Umbral Moonshine and K3 Surfaces* [Cheng, Harrison]:

$$\mathbf{EG}_g(\tau, z; K3) = \mathbf{EG}_g(\tau, z; D_4^6) + \frac{\theta_1(\tau, z)^2}{2\eta(\tau)^6} \left(\frac{1}{2\pi i} \frac{\partial}{\partial \omega} \Psi_g^{D_4^6}(\tau, \omega) \right) \Big|_{\omega=0}$$

Umbral moonshine and K3

- *Umbral Moonshine and K3 Surfaces* [Cheng, Harrison]:

$$\mathbf{EG}_g(\tau, z; K3) = \mathbf{EG}_g(\tau, z; D_4^6) + \frac{\theta_1(\tau, z)^2}{2\eta(\tau)^6} \left(\frac{1}{2\pi i} \frac{\partial}{\partial \omega} \Psi_g^{D_4^6}(\tau, \omega) \right) \Big|_{\omega=0}$$

- $\Psi_g^{D_4^6} \;\rightarrow\;$ (twined) mock Jacobi form associated with $X = D_4^6$ umbral

Umbral moonshine and K3

- *Umbral Moonshine and K3 Surfaces* [Cheng, Harrison]:

$$\mathbf{EG}_g(\tau, z; K3) = \mathbf{EG}_g(\tau, z; D_4^6) + \frac{\theta_1(\tau, z)^2}{2\eta(\tau)^6} \left(\frac{1}{2\pi i} \frac{\partial}{\partial \omega} \Psi_g^{D_4^6}(\tau, \omega) \right) \Big|_{\omega=0}$$

- $\Psi_g^{D_4^6}$ $\rightarrow$ (twined) mock Jacobi form associated with $X = D_4^6$ umbral
- $\mathbf{EG}_g(\tau, z; D_4^6)$ $\rightarrow$ (twined) "singularity" elliptic genus

Umbral moonshine and K3

- *Umbral Moonshine and K3 Surfaces* [Cheng, Harrison]:

$$\mathbf{EG}_g(\tau, z; K3) = \mathbf{EG}_g(\tau, z; D_4^6) + \frac{\theta_1(\tau, z)^2}{2\eta(\tau)^6} \left(\frac{1}{2\pi i} \frac{\partial}{\partial \omega} \Psi_g^{D_4^6}(\tau, \omega) \right) \Big|_{\omega=0}$$

- $\Psi_g^{D_4^6} \;\rightarrow\;$ (twined) mock Jacobi form associated with $X = D_4^6$ umbral
- $\mathbf{EG}_g(\tau, z; D_4^6) \;\rightarrow\;$ (twined) "singularity" elliptic genus
  - Related to ADE singularities of K3

Umbral moonshine and K3

- $\mathbf{EG}_g(\tau, z; D_4^6)$

Umbral moonshine and K3

- $\mathbf{EG}_g(\tau, z; D_4^6) := \mathrm{tr}_* \left(\widehat{\Omega}_g^{D_4^6} \cdot \Xi \right) = \mathrm{str}_{V_{\mathrm{sing}}}^{(q,y)} g \ , \quad g \in 3.S_6$

Umbral moonshine and K3

- $\mathbf{EG}_g(\tau, z; D_4^6) := \mathrm{tr}_* \left(\widehat{\Omega}_g^{D_4^6} \cdot \Xi \right) = \mathrm{str}_{V_{\mathrm{sing}}}^{(q,y)} g \ , \quad g \in 3.S_6$
- $\Xi \rightarrow$ contains characters of CFT describing a D_4-type K3 singularity [Ooguri, Vafa]

Umbral moonshine and K3

- $\mathbf{EG}_g(\tau, z; D_4^6) := \mathrm{tr}_* \left(\widehat{\Omega}_g^{D_4^6} \cdot \Xi \right) = \mathrm{str}_{V_{\mathrm{sing}}}^{(q,y)} g \ , \quad g \in 3.S_6$

- $\Xi \rightarrow$ contains characters of CFT describing a D_4-type K3 singularity [Ooguri, Vafa]:

$$\left(\mathcal{N} = 2 \text{ minimal } \otimes \ \mathcal{N} = 2 \left(\frac{\mathrm{SL}(2,\mathbb{R})}{\mathrm{U}(1)} \right) \text{ coset} \right) / (\mathbb{Z}/6\mathbb{Z})$$

Umbral moonshine and K3

- $\mathbf{EG}_g(\tau, z; D_4^6) := \mathrm{tr}_* \left(\widehat{\Omega}_g^{D_4^6} \cdot \Xi \right) = \mathrm{str}_{V_{\mathrm{sing}}}^{(q,y)} g \ , \quad g \in 3.S_6$

- $\Xi \rightarrow$ contains characters of CFT describing a D_4-type K3 singularity [Ooguri, Vafa]:

$$\left(\mathcal{N} = 2 \text{ minimal } \otimes \ \mathcal{N} = 2 \ \left(\frac{\mathrm{SL}(2,\mathbb{R})}{\mathrm{U}(1)} \right) \text{ coset} \right) / (\mathbb{Z}/6\mathbb{Z})$$

- $\widehat{\Omega}_g^{D_4^6} \rightarrow$ Cappelli–Itzykson–Zuber type omega matrices for $D_4^6 := D_4^{\oplus 6}$

Umbral moonshine and K3

- $\mathbf{EG}_g(\tau, z; D_4^6) := \mathrm{tr}_* \left(\widehat{\Omega}_g^{D_4^6} \cdot \Xi \right) = \mathrm{str}_{V_{\mathrm{sing}}}^{(q,y)} g \ , \quad g \in 3.S_6$

- $\Xi \to$ contains characters of CFT describing a D_4-type K3 singularity [Ooguri, Vafa]:

$$\left(\mathcal{N} = 2 \text{ minimal } \otimes \ \mathcal{N} = 2 \ \left(\frac{\mathrm{SL}(2, \mathbb{R})}{\mathrm{U}(1)} \right) \text{ coset} \right) / (\mathbb{Z}/6\mathbb{Z})$$

- $\widehat{\Omega}_g^{D_4^6} \to$ Cappelli–Itzykson–Zuber type omega matrices for $D_4^6 := D_4^{\oplus 6}$

- $V_{\mathrm{sing}} = \left(\bar{R} \otimes V_5^1 \right) \oplus \left(R \otimes V_1^5 \right) \oplus \left(\check{R} \otimes V_3^3 \right) \ , \quad \mathrm{sdim}^{(q,y)} V_s^r = \Xi_s^r$

Conway moonshine and K3

$$\mathbf{EG}_g(\tau, z; K3) = \mathbf{EG}_g(\tau, z; D_4^6) + \frac{\theta_1(\tau, z)^2}{2\eta(\tau)^6} \left(\frac{1}{2\pi i} \frac{\partial}{\partial \omega} \Psi_g^{D_4^6}(\tau, \omega) \right) \Big|_{\omega=0}$$

- Main idea: "invert" this relation to get a module

Conway moonshine and K3

$$\mathbf{EG}_g(\tau, z; K3) = \mathbf{EG}_g(\tau, z; D_4^6) + \frac{\theta_1(\tau, z)^2}{2\eta(\tau)^6} \left(\frac{1}{2\pi i} \frac{\partial}{\partial \omega} \Psi_g^{D_4^6}(\tau, \omega) \right) \Big|_{\omega=0}$$

- Main idea: "invert" this relation to get a module
- $3.S_6$ is not a symmetry of a single K3 (doesn't fix a 4-plane in $\mathbf{24}_{\mathrm{Co}_0}$)

Conway moonshine and K3

$$\mathbf{EG}_g(\tau, z; K3) = \mathbf{EG}_g(\tau, z; D_4^6) + \frac{\theta_1(\tau, z)^2}{2\eta(\tau)^6} \left(\frac{1}{2\pi i} \frac{\partial}{\partial \omega} \Psi_g^{D_4^6}(\tau, \omega) \right) \Big|_{\omega=0}$$

- Main idea: "invert" this relation to get a module
- $3.S_6$ is not a symmetry of a single K3 (doesn't fix a 4-plane in $\mathbf{24}_{\mathrm{Co}_0}$)
- $\mathbf{EG}_g(\tau, z; K3)$ coincide with functions ϕ_g from the Conway moonshine module $V_{\mathrm{tw}}^{s\natural}$ [Duncan, Mack-Crane]

Conway moonshine and K3

$$\mathbf{EG}_g(\tau, z; K3) = \mathbf{EG}_g(\tau, z; D_4^6) + \frac{\theta_1(\tau, z)^2}{2\eta(\tau)^6} \left(\frac{1}{2\pi i} \frac{\partial}{\partial \omega} \Psi_g^{D_4^6}(\tau, \omega) \right) \Big|_{\omega=0}$$

- Main idea: "invert" this relation to get a module
- $3.S_6$ is not a symmetry of a single K3 (doesn't fix a 4-plane in $\mathbf{24}_{\mathrm{Co}_0}$)
- $\mathbf{EG}_g(\tau, z; K3)$ coincide with functions ϕ_g from the Conway moonshine module $V^{s\natural}_{\mathrm{tw}}$ [Duncan, Mack-Crane]
- The $U(1)$-grading of ϕ_g is *not* preserved by $3.S_6 \subset \mathrm{Co}_0$

Final module

- <u>Solution:</u> construct a VOA that reproduces ϕ_g but also preserves the $U(1)$-grading

Final module

- Solution: construct a VOA that reproduces ϕ_g but also preserves the $U(1)$-grading:

$$\mathcal{T}: \ \left(12 \text{ complex fermions} + 2\ \beta\gamma \text{ systems} + 2\ bc \text{ systems}\right)/\mathbb{Z}_2$$

Final module

- Solution: construct a VOA that reproduces ϕ_g but also preserves the $U(1)$-grading:

$$\mathcal{T}: \ (12 \text{ complex fermions} + 2 \ \beta\gamma \text{ systems} + 2 \ bc \text{ systems}) \, / \mathbb{Z}_2$$

- Fermions carry the $3.S_6$ action

Final module

- Solution: construct a VOA that reproduces ϕ_g but also preserves the $U(1)$-grading:

$$\mathcal{T}: \ \left(12 \text{ complex fermions} + 2\ \beta\gamma \text{ systems} + 2\ bc \text{ systems}\right)/\mathbb{Z}_2$$

- Fermions carry the $3.S_6$ action

- bc–$\beta\gamma$ systems provide the $U(1)$-grading $\rightarrow$ trivial $3.S_6$ action

Final module

- Solution: construct a VOA that reproduces ϕ_g but also preserves the $U(1)$-grading:

$$\mathcal{T}: \ (12 \text{ complex fermions} + 2\ \beta\gamma \text{ systems} + 2\ bc \text{ systems})\ /\mathbb{Z}_2$$

- Fermions carry the $3.S_6$ action
- bc–$\beta\gamma$ systems provide the $U(1)$-grading $\rightarrow$ trivial $3.S_6$ action
- Key point: all $g \in 3.S_6$ fix a 4-plane in $\mathbf{24}_{\mathrm{Co}_0}$, albeit not the same one

Final module

- Solution: construct a VOA that reproduces ϕ_g but also preserves the $U(1)$-grading:

$$\mathcal{T}: \ (12 \text{ complex fermions} + 2\ \beta\gamma \text{ systems} + 2\ bc \text{ systems})\,/\mathbb{Z}_2$$

- Fermions carry the $3.S_6$ action
- bc–$\beta\gamma$ systems provide the $U(1)$-grading $\rightarrow$ trivial $3.S_6$ action
- Key point: all $g \in 3.S_6$ fix a 4-plane in $\mathbf{24}_{\mathrm{Co}_0}$, albeit not the same one
- Modules K^r such that $H_{g,r} = \mathrm{str}_{K^r}^{(q,y)} g\ , \quad g \in 3.S_6$

Final module

- Solution: construct a VOA that reproduces ϕ_g but also preserves the $U(1)$-grading:

$$\mathcal{T}: \ (12 \text{ complex fermions} + 2\ \beta\gamma \text{ systems} + 2\ bc \text{ systems})\,/\mathbb{Z}_2$$

- Fermions carry the $3.S_6$ action
- bc–$\beta\gamma$ systems provide the $U(1)$-grading $\rightarrow$ trivial $3.S_6$ action
- Key point: all $g \in 3.S_6$ fix a 4-plane in $\mathbf{24}_{\mathrm{Co}_0}$, albeit not the same one
- Modules K^r such that $H_{g,r} = \mathrm{str}^{(q,y)}_{K^r} g\,, \quad g \in 3.S_6$

$$K^r = \mathcal{H}^r_{\mathrm{aux}} \otimes \begin{cases} W|P & r = 1,5 \\ W|(\mathbf{1} - P) & r = 3 \end{cases}, \quad W = \mathcal{T} \ominus V_{\mathrm{sing}}$$

The end

Thank you!