Equilibration in φ^4 theory in 3+1 dimensions

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Outline

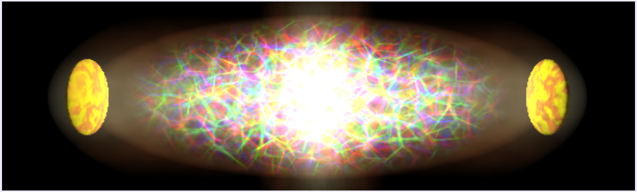
Motivation

Why is equilibration interesting?

Early Universe

- (P)reheating during inflation ($\longrightarrow$ Baryogenesis)

Heavy-Ion Collisions

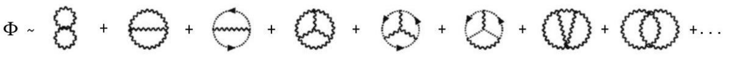


Is a thermalized QGP achieved during the collisions?

- Hydrodynamics point to short thermalization time ($\tau \sim 1$ fm/c).
- Traditional QCD estimates give a larger thermalization time.

2PI Effective action as a tool to study equilibration

- Exact representation of path integral in terms of a functional depending solely on the connected 1- and 2-point functions ϕ and G.
- Evolution equations derived from variational principle on the functional (Φ(Functional)-derivable approximations)



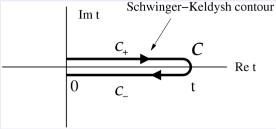
- Energy conservation
- Respect global symmetries [Baym, Kadanoff'61]
- Renormalization? Possible and systematic [van Hees, Knoll'02; Blaizot, Iancu, Reinosa'04;Cooper,Mihaila, Dawson '04; Berges, Borsányi, Reinosa'05]
- Gauge invariance? Not completely [AA, Smit '02; Carrington,Kunstatter,Zaraket '03]

- Recent out-of-equilibrium studies:
  - Equilibration in scalar fields (1+1 dim) [Berges,Cox '00; Aarts,Berges'01; Berges'02, Cooper,Dawson,Mihaila'03. . .]
  - Equilibration in scalar fields (2+1 dim) [Cassing,Juchem,Greiner'02]
  - Equilibration of fermions and scalars (3+1 dim) [Berges,Borsányi,Serreau'03]
  - Preheating [Berges,Serreau'03; AA,Tranberg,Smit'04]

2PI Effective Action in scalar theory

Scalar $\lambda\varphi^4$ theory

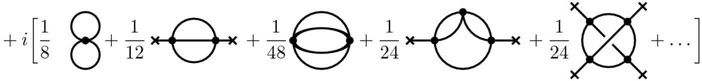
$$S[\varphi] = \int_{\mathcal{C}} d^4x \left[\frac{1}{2}\partial_\mu\varphi(x)\partial^\mu\varphi(x) - \frac{1}{2}m^2\varphi(x)^2 - \frac{\lambda}{4!}\varphi(x)^4 \right]$$



- Symmetric phase: $v = \langle\varphi\rangle_{T=0} = 0$
- Broken phase: $v \neq 0$, $v_{\text{tree}} = \sqrt{6|m^2|/\lambda}$

2PI Effective Action

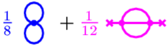
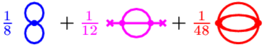
$$\Gamma[\phi, G] = S[\phi] - \frac{i}{2}\mathrm{Tr}\ln G + \frac{i}{2}\mathrm{Tr}\left[(G_0^{-1} - G^{-1})\cdot G\right]$$



with $G_0^{-1}(x, y) = \left(-\partial^2 - m^2 - \frac{1}{2}\lambda\phi^2\right)\delta_{\mathcal{C}}(x, y)$.

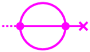
Truncations

Truncations of the 2PI Effective Action

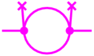
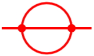
Truncation	*Order*	$i\Phi[\phi, G]$
Hartree approximation	$\mathcal{O}(\lambda)$	
Two-loop approximation	2 loops	
"Basketball" approximation	$\mathcal{O}(\lambda^2)$	

Evolution equations obtained from variational principle

$$\frac{\delta\Gamma[\phi,G]}{\delta\phi} = 0 \Longrightarrow \frac{\delta S[\phi]}{\delta\phi(x)} + \frac{1}{2}\lambda G(x,x)\phi(x) = -\frac{\delta\Phi[\phi,G]}{\delta\phi(x)} = \frac{i}{6}$$



$$\frac{\delta\Gamma[\phi,G]}{\delta G} = 0 \Longrightarrow \delta_C(x,y) = \int_C d^4z\, G_0^{-1}(x,z)G(z,y) + i\int_C d^4z\, \Sigma(x,z)G(z,y)$$

$$\Sigma(x,y) = -2\frac{\delta\Phi[\phi,G]}{\delta G(y,x)} = \frac{i}{2}$$  $+\frac{i}{2}$  $+\frac{i}{6}$ 

2-point functions

2-point functions on the contour

- Evolution equations are defined on the Schwinger-Keldysh contour $\mathcal{C}$

$$G(x,y) = \Theta_{\mathcal{C}}(x_0-y_0)G^{>}(x,y)+\Theta_{\mathcal{C}}(y_0-x_0)G^{<}(x,y) \text{ with } \begin{cases} G^{>}(x,y) \equiv \langle\varphi(x)\varphi(y)\rangle \\ G^{<}(x,y) \equiv \langle\varphi(y)\varphi(x)\rangle \end{cases}$$

- Real scalar theory $\left[G^{>}(x,y)\right]^{\star} = G^{<}(x,y) \rightarrow$ only 2 independent real functions.

$$G^{>}(x,y) = F(x,y) - \frac{i}{2}\rho(x,y),$$
$$G^{<}(x,y) = F(x,y) + \frac{i}{2}\rho(x,y).$$

- The functions F/ρ contain statistical/spectral information

$$F(x,y) = \frac{1}{2}\langle\{\varphi(x),\varphi(y)\}\rangle \quad , \qquad \rho(x,y) = i\,\langle[\varphi(x),\varphi(y)]\rangle$$

Evolution Equations

2-point functions

$$\left[\partial_x^2 + M^2(x)\right] F(x,y) = \int_0^{x_0} dz_0 \int d^3z\, \Sigma^\rho(x,z) F(z,y) - \int_0^{y_0} dz_0 \int d^3z\, \Sigma^F(x,z)\rho(y,z),$$

$$\left[\partial_x^2 + M^2(x)\right] \rho(x,y) = \int_{y_0}^{x_0} dz_0 \int d^3z\, \Sigma^\rho(x,z)\rho(z,y),$$

with $M^2(x) = m^2 + \frac{\lambda}{2}\phi(x)^2 + \frac{\lambda}{2}F(x,x)$

$$\Sigma^F(x,y) = \frac{\lambda^2}{2}\phi(x)\phi(y)\left[F^2(x,y) - \frac{\rho^2(x,y)}{4}\right] + \frac{\lambda^2}{6}F(x,y)\left[F^2(x,y) - \frac{3\rho^2(x,y)}{4}\right]$$

$$\Sigma^\rho(x,y) = \lambda^2\phi(x)\phi(y)\left[F(x,y)\rho(x,y)\right] + \frac{\lambda^2}{6}\rho(x,y)\left[3F^2(x,y) - \frac{\rho^2(x,y)}{4}\right]$$

1-point function

$$\left[\partial_x^2 + M^2(x)\right] \phi(x) = \frac{\lambda}{3}\phi(x)^3 + \int_0^{x_0} dz_0 \int d^3z\, \widetilde{\Sigma}^\rho(x,z)\phi(z),$$

with $\widetilde{\Sigma}^\rho(x,z) = -\frac{\lambda^2}{6}\rho(x,z)\left[3F(x,z)^2 - \frac{\rho(x,z)^2}{4}\right]$

Initial Conditions

Spatially homogeneous situation

$$\left\{F(x,y)=F(t,t',\mathbf{x}-\mathbf{y}),\ \rho(x,y)=\rho(t,t',\mathbf{x}-\mathbf{y})\right\} \Longrightarrow \left\{F_{\mathbf{k}}(t,t'),\ \rho_{\mathbf{k}}(t,t')\right\}$$

Mean Field

$\phi=0$ Symmetric Phase $\quad \phi=v_{\text{tree}}$ Broken Phase

Spectral Function

$$\rho_{\mathbf{k}}(t,t)=0, \qquad \partial_t\rho_{\mathbf{k}}(t,t')\big|_{t=t'}=1$$

Symmetric Function

$$F_{\mathbf{k}}(t,t')\big|_{t=t'=0}=\langle\{\varphi_{\mathbf{k}}(t),\varphi_{-\mathbf{k}}(t')\}\rangle|_{t=t'=0}=\frac{1}{\omega_{\mathbf{k}}}\left[n_{\mathbf{k}}+\frac{1}{2}\right]$$

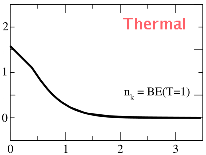
$$\partial_t F_{\mathbf{k}}(t,t')\big|_{t=t'=0}=\langle\{\pi_{\mathbf{k}}(t),\varphi_{-\mathbf{k}}(t')\}\rangle|_{t=t'=0}=0$$

$$\partial_t\partial_{t'} F_{\mathbf{k}}(t,t')\big|_{t=t'=0}=\langle\{\pi_{\mathbf{k}}(t)\pi_{-\mathbf{k}}(t')\}\rangle|_{t=t'=0}=\omega_{\mathbf{k}}\left[n_{\mathbf{k}}+\frac{1}{2}\right]$$

Thermal

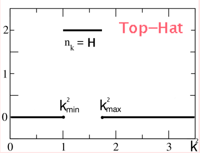
$$n_{\mathbf{k}}=\frac{1}{e^{(\omega_{\mathbf{k}}/T_{\text{in}})}-1}$$

with $\omega_{\mathbf{k}}=\sqrt{m_{in}^2+\mathbf{k}^2}$



"Top-Hat"

$$n_{\mathbf{k}}=H\,\Theta(\mathbf{k}_{max}^2-\mathbf{k}^2)\Theta(\mathbf{k}^2-\mathbf{k}_{min}^2)$$



Observables

Quasiparticle distribution function $n_{\mathbf{k}}(t) + \frac{1}{2} = c_{\mathbf{k}} \sqrt{\partial_t \partial_{t'} F_{\mathbf{k}}(t,t')\big|_{t=t'} F_{\mathbf{k}}(t,t)}$

Dispersion relation $\omega_{\mathbf{k}}(t) = \sqrt{\partial_t \partial_{t'} F_{\mathbf{k}}(t,t')\big|_{t=t'} / F_{\mathbf{k}}(t,t)}$

Total Particle number density $n_{\text{tot}}(t) = \int_{\mathbf{k}} n_{\mathbf{k}}(t)$

Close to equilibrium

- Effective quasiparticle mass m_{eff} $\quad \omega_{\mathbf{k}}^2(t) = c^2(t)\left(m_{\text{eff}}(t)^2 + \mathbf{k}^2\right)$
- Effective Temperature T_{eff} and chemical potential μ_{eff} $\quad n_{\mathbf{p}}(t) = \dfrac{1}{e^{\left[\omega_{\mathbf{p}}(t) - \mu_{\text{eff}}(t)\right]/T_{\text{eff}}(t)} - 1}$

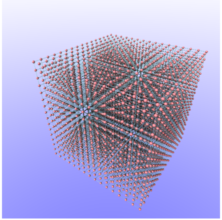
Energy and Memory Kernels

- We monitor the memory kernels, i.e. $\Sigma^F(t,t')$, $\Sigma^\rho(t,t')$ and $\widetilde{\Sigma}^\rho(t,t')$
- Only a finite memory is kept, i.e. $\Sigma(t,t') \rightarrow$ for $|t - t'| > t_{\text{cut}}$
- We check that the energy $E(t) = \int d^3x\, T^{00}(\mathbf{x},t)$ is conserved

Numerical Implementation

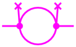
- The system is discretized on a $N^3 = 16^3$ spatial lattice of spacing a.
- Time is discretized with spacing a_t

$$S_{\mathrm{lat}}[\varphi] = a^3\, a_t \sum_{\mathbf{x},t} \left[\frac{1}{2} (\partial_t \varphi(\mathbf{x},t))^2 - \frac{1}{2} \sum_i (\partial_i \varphi(\mathbf{x},t))^2 - \frac{1}{2} m_0^2 \varphi(\mathbf{x},t)^2 - \frac{1}{4!} \lambda_0 \varphi(\mathbf{x},t)^4 \right]$$



Renormalization

- General method quite involved (solution of Bethe-Salpeter equations)
- In our discretized case we use an *approximate* 2-loop renormalization

$m_0^2 = m^2 - \frac{i}{2}$  $+ \frac{i}{2}$  $\Big|_{T=0,G_0}$

$$= m^2 - \frac{\lambda}{2a^2} \frac{1}{N^3} \sum_{\mathbf{k}} \frac{1}{2\sqrt{a^2m^2 + \lambda a^2 v^2/2 + \mathbf{k}^2}} - \frac{\lambda^2 v^2}{2} \frac{1}{N^3} \sum_{\mathbf{k}} \frac{1}{4\sqrt{(a^2m^2 + \lambda a^2 v^2/2 + \mathbf{k}^2)}^3}$$

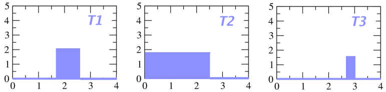
$\frac{1}{\lambda_0} = \frac{1}{\lambda} - \frac{1}{2}$ 

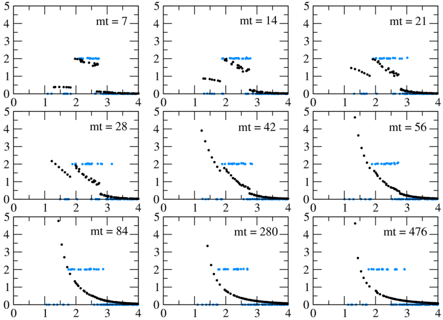
$$= \frac{1}{\lambda} - \frac{1}{N^3} \sum_{\mathbf{k}} \frac{1}{4\sqrt{(a^2m^2 + \lambda a^2 v^2/2 + \mathbf{k}^2)}^3}$$

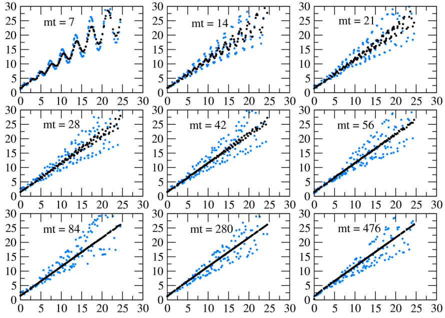
Symmetric Phase: Equilibration

Simulation Parameters: $\phi = 0$, $am = 0.7$, $\lambda = 6$, $a_t = 0.1a$, $mt_{\text{cut}} = 28$

- T1, T2 and T3: same energy
- T1 and T2: similar total particle number density





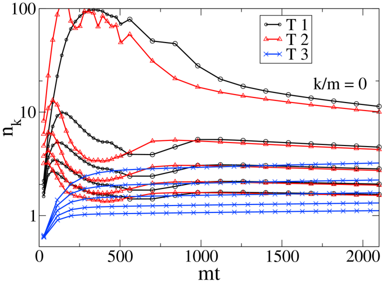


Distribution function
$n_{\mathbf{k}}$ vs. $\omega_{\mathbf{k}}$, Hartree and Basketball for T1

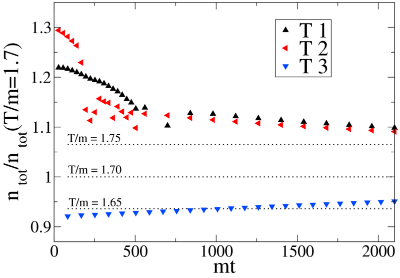
Dispersion relation
$\omega_{\mathbf{k}}^2$ vs. $\mathbf{k}^2$, Hartree and Basketball for T1

Equilibration seems to occur at mt $\sim$ 100!

Kinetic *vs.* Chemical Equilibration

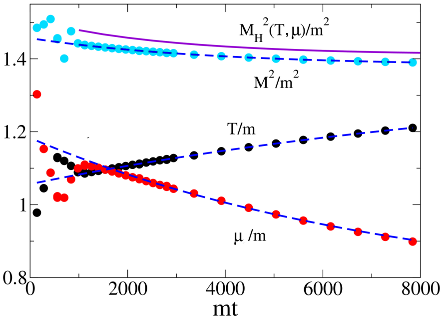


Evolution of individual modes



Evolution of total particle number n_{tot}

- Kinetic equilibration occurs relatively fast ($mt \sim 100$), dominated by $2 \leftrightarrow 2$ processes
- Chemical equilibration is much slower (caused by $1 \leftrightarrow 3$, $2 \leftrightarrow 4, \ldots$ processes).
- Kinetically preequilibrated state remembers the initial particle number.



Evolution of effective mass, temperature and chemical potential

- Very slow evolution towards final equilibrium ($m\tau \sim 10^{4-5}$)
- Exponential fits suggest asympotic values $T/m = 1.36$ and $\mu/m = 0.7$(!)
- Chemical equilibration seems to be much smaller than in 2+1 dimensions [Juchem,Cassing,Greiner '03]
- Effective mass: Comparison with Hartree estimate $M_H(T_{\text{eff}}, \mu_{\text{eff}})$ indicates that the contribution to the mass from the basketball not very large.

Symmetric Phase: Damping

- Close to thermal equilibrium (Initial conditions: Thermal)
- Mean field slightly displaced from $\phi = 0$

$$\ddot{\phi}(t) + M^2(T, t)\phi(t) = -\frac{\lambda}{6}\phi(t)^3 - \int_0^t dt'\, \tilde{\Sigma}_{\mathbf{0}}^{\rho}(t, t')\, \phi(t')$$

⇓ Linearization

$$\ddot{\phi}(t) + M^2(T)\phi(t) = -\int_0^t dt'\, \tilde{\Sigma}_{\mathbf{0}}^{\rho}(t - t')\, \phi(t')$$

⇓ Solvable

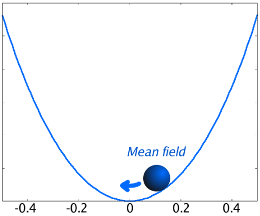
$$\phi(t) = \frac{2\phi_i}{\pi}\int_0^\infty d\omega \frac{\omega\, \mathrm{Im}\tilde{\Sigma}_{\mathbf{0}}^R(\omega)\, \cos(\omega t)}{\left[\omega^2 - M^2 - \mathrm{Re}\tilde{\Sigma}_{\mathbf{0}}^R(\omega)\right]^2 + \mathrm{Im}\tilde{\Sigma}_{\mathbf{0}}^R(\omega)^2}$$

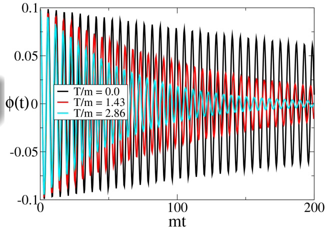
⇓ Narrow width

$$\phi(t) \approx \phi_i Z e^{-\gamma t} \cos\left(M_{\mathrm{eff}}\, t - \alpha\right),$$

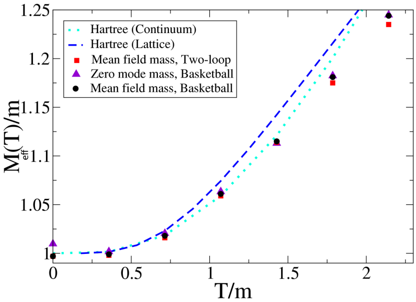
$$\gamma = Z\frac{\mathrm{Im}\tilde{\Sigma}_{\mathbf{0}}^R(M_{\mathrm{eff}})}{M_{\mathrm{eff}}}, \qquad M_{\mathrm{eff}}^2 = M^2 + \mathrm{Re}\tilde{\Sigma}_{\mathbf{0}}^R(\omega)$$

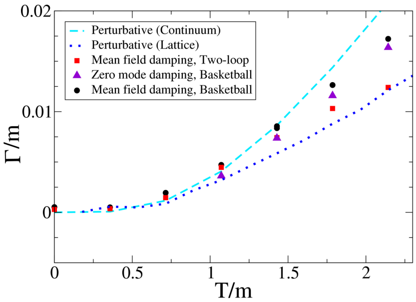
- Spectral Function $\rho_{\mathbf{k}}(t, t') = \frac{1}{\omega_{\mathbf{k}}} e^{-\gamma_{\mathbf{k}}|t-t'|} \sin\left[\omega_{\mathbf{k}}(t - t')\right]$



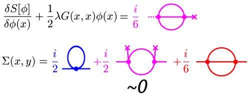


Damping: 2-loop *vs.* Basketball



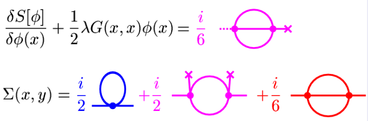


- Effective masses almost identical and close to Hartree
- Basketball damping slightly larger than 2-loop damping
- Basketball damping (20-40)% larger than Perturbative
- Spectral function zero-mode mass and damping closely follow mean field values



Broken Phase: Equilibration

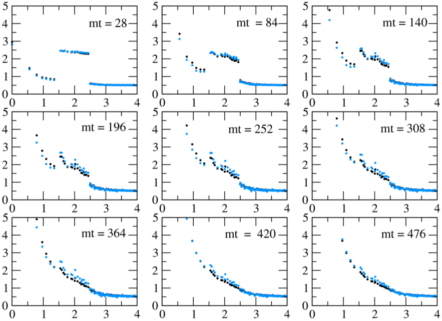
- $\phi \neq 0$ allows to compare 2-loop and basketball for the equilibration of 2-point functions



- The 2-loop perturbative approximation contains no on-shell scattering,
- But the 2-loop Φ-derivable approximation contains on-shell scattering (through resummation of higher orders)
- We take $\phi = v_{\text{tree}} \approx v$ so that the time evolution of $\phi(t)$ does not affect the dynamics of the 2-point functions

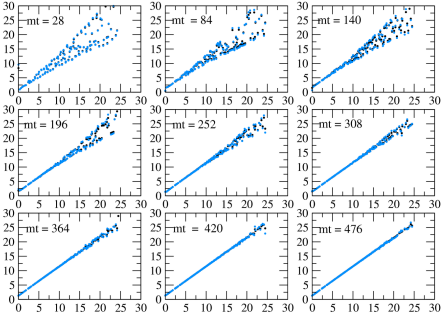
Broken Phase: Equilibration

Simulation Parameters: $\phi = v_{\text{tree}}$, $am = 0.7$, $\lambda = 1$, $a_t = 0.1a$, $mt_{\text{cut}} = 84$,



Distribution function

$n_{\mathbf{k}}$ vs. $\omega_{\mathbf{k}}$, 2-loop and Basketball for T1



Dispersion relation

$\omega_{\mathbf{k}}^2$ vs. $\mathbf{k}^2$, 2-loop and Basketball for T1

- Early equilibration in 2-loop almost as fast as in Basketball
- Further chemical and final equilibration very slow

Broken Phase: Damping

- Close to thermal equilibrium (Initial conditions: Thermal)
- Mean field $\phi = v_{\rm tree}$ slightly displaced from true v
- Linearization around v: $\phi(t) = v + \sigma(t)$

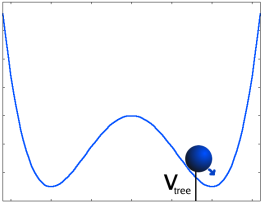
$$\ddot{\sigma}(t) + M^2(T,t)\sigma(t) = -\int_0^t dt'\, \tilde{\Sigma}^{\rho}_{\mathbf{0}}(t,t')\sigma(t')$$

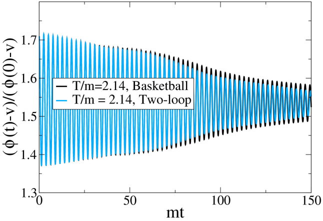
- Vacuum expectation value v

$$M^2(T,t)v - \frac{\lambda}{3}v^3 + \int_0^t dt'\, \tilde{\Sigma}^{\rho}_{\mathbf{0}}(t,t')v = 0$$

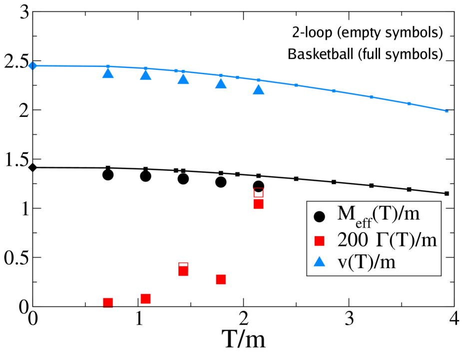
- Close enough to equilibrium

$$\sigma(t) \approx \sigma_{\rm in} Z e^{-\gamma t}\cos\left(M_{\rm eff}\, t - \alpha\right)$$





Damping: 2-loop *vs.* Basketball



- Effective masses and *v* practically identical and close to Hartree
- Similar damping in both approximations (rough estimates)

Conclusions

Equilibration stages

Early Kinetic Equilibration (“Stabilization” of occupation numbers and dispersion relation)

Late Chemical and final equilibration

Prethermalization? (J. Berges and S. Borsányi’s talks)

Hartree/2-loop/Basketball Φ-derivable approximations

- Hartree *vs.* 2-loop/Basketball: Not large changes in masses and v
- Enhanced mean field damping (w.r.t perturbation theory)
- Possible to study 2-point function equilibration in 2-loop (broken phase)
- Equilibration almost as fast in 2-loop as in Basketball (broken phase)

- Larger couplings: Secular-like Instabilities?, Renormalization?