Elliptic K3s, $T^4/\mathbb{Z}_2$ and Enriques involutions.

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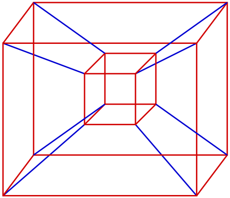

With R.Ebert, A.Hebecker, R.Valandro

Bonn - May 19th, 2009

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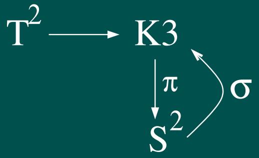

- In particular: a $K3$ described by a Weierstrass model.

Weierstrass model

$$y^2 = x^3 + f_8(a,b)xz^4 + g_{12}(a,b)z^6$$

- (y,x,z) are homogeneous coordinates of $\mathbb{P}_{1,2,3}$.

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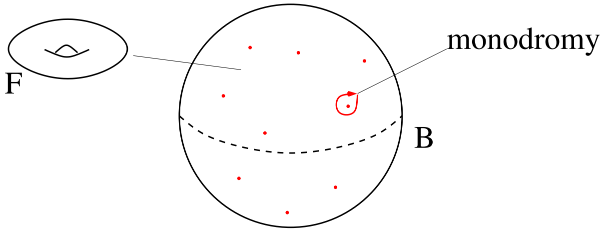

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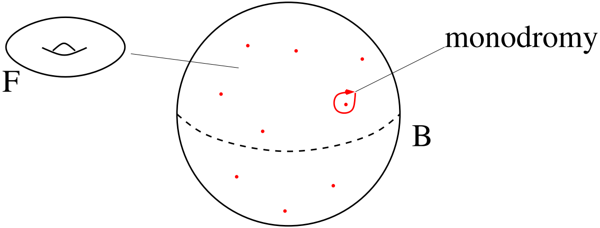

$$B, F \in H^{1,1}(K3,\mathbb{Z}) \to \Omega_{2,0} \cdot B = \Omega_{2,0} \cdot F = 0.$$

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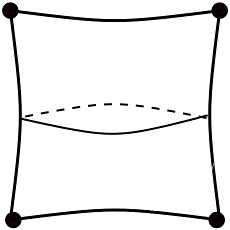

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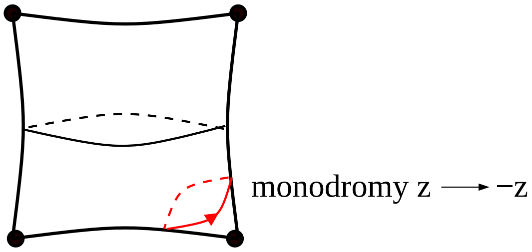

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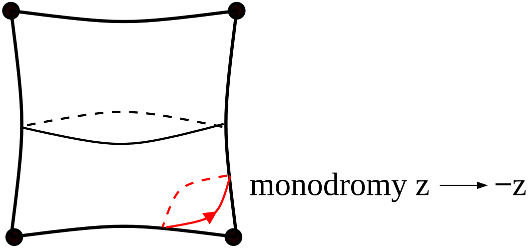

This is like $T^4/\mathbb{Z}_2$!

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The action of the Enriques Involution on $T^4/\mathbb{Z}_2$ is known:
(It is also known on the lattice of integral cycles of $K3$.)

$$(x_1, x_2, x_3, x_4) \mapsto (x_1, x_2 + \tfrac{1}{2}, -x_3, -x_4 + \tfrac{1}{2})$$

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Let us use this to learn something about possible Enriques involutions on elliptic $K3$ surfaces described by a Weierstrass model !

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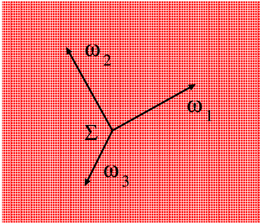
- **Enriques revisited**

K3 moduli space

Points in the moduli space of $K3$ are given by a 3-plane Σ (spanned by three orthogonal positive norm vectors ω_i) in

$$H^2(K3,\mathbb{Z}) = U^{\oplus 3} \oplus -E_8^{\oplus 2}, \qquad U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$



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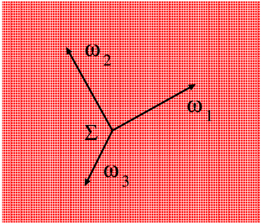

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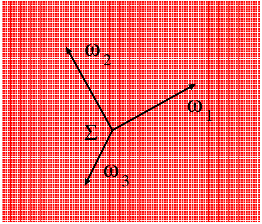

$$J = \sqrt{Vol}\,\omega_1, \qquad \Omega_{2,0} = \omega_2 + i\omega_3.$$

$J \mapsto J, \qquad \Omega_{2,0} \mapsto -\Omega_{2,0}$ under the Enriques involution.

K3 moduli space

$\gamma_i \in H^2(K3, \mathbb{Z})$ with $\gamma_i \cdot \gamma_i = -2$ (a "root") orthogonal to $\Sigma \rightarrow$ ADE-singularity.
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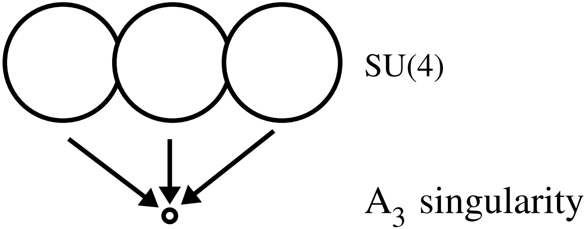

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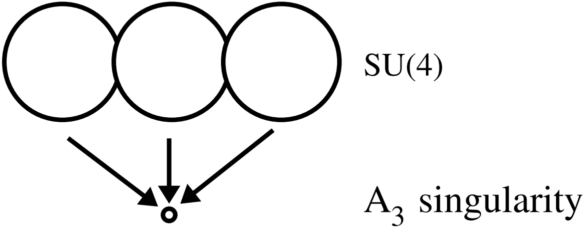

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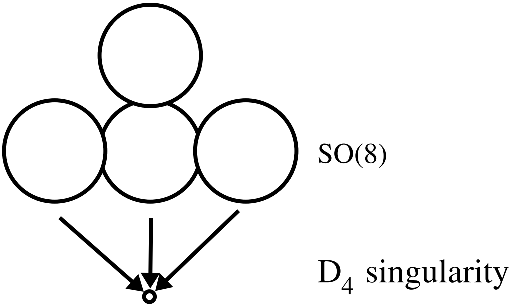


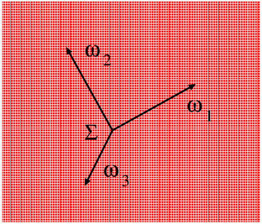
Gauge enhancement for F/M/IIA-theory compactifications !

Wilson Lines

Put Σ is entirely in the $U^{\oplus 3}$ lattice:

$$\omega_i = e_i + s_i e^i, \qquad e_i \cdot e^j = \delta_i^j$$

As the whole $-E_8 \times -E_8$ lattice is orthogonal to Σ, the singularity type/gauge enhancement is $E_8 \times E_8$.



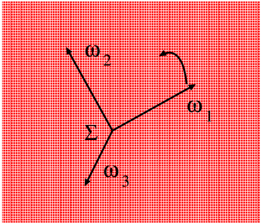
M-theory on this space is dual to het$_{E_8 \times E_8}$ on T^3.

Wilson Lines

Now rotate ω_1 into the E_8 lattices:

$$\omega_1 = e_1 + s(e^1 + W),$$

All roots of the E_8 lattice for which $W \cdot \gamma_k = n \in \mathbb{N}$ still lead to shrunk cycles: $\tilde{\gamma}_k = \gamma_k - ne_i$ now satisfies $\gamma_k \cdot \omega_i = 0$.
This is the same condition as for Wilson-line breaking.



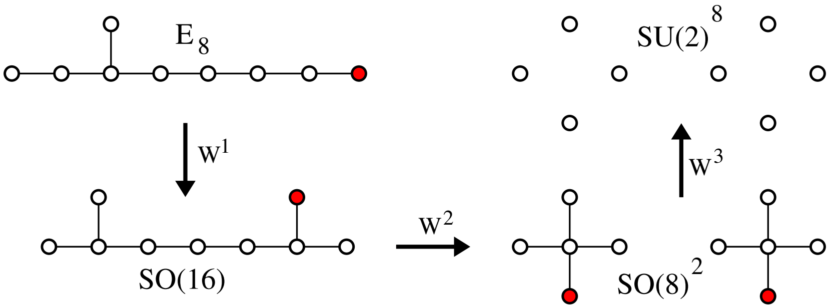
The resolution of singularities works like Wilson-line breaking !

Wilson Lines

We can reach $T^4/\mathbb{Z}_2$ by three Wilson-lines that break $E_8 \times E_8 \to SU(2)^{16}$

$$W^1 = (1, 0^7, \ldots) \quad W^2 = (0^4, \frac{1}{2}^4, \ldots)$$

$$W^3 = (0^2, -\frac{1}{2}, \frac{1}{2}, 0^2, -\frac{1}{2}, \frac{1}{2}, \ldots)$$



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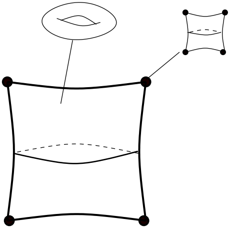
This gives an (explicit) embedding of

$$A_1^{\oplus 16} \oplus U(2)^3 \subset U^{\oplus 3} \oplus -E_8^{\oplus 2}.$$



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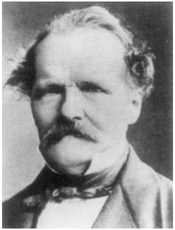
This is actually the original construction of $K3$ by Kummer as a "Kummer surface".



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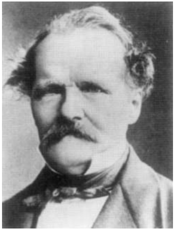

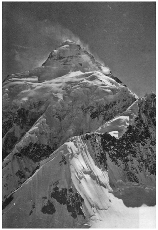

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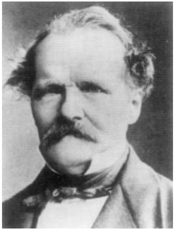

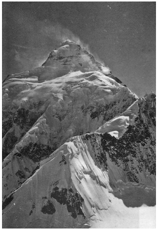

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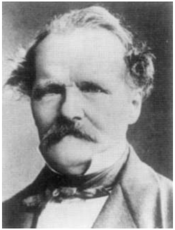

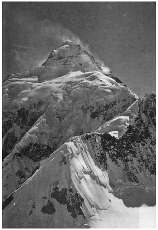

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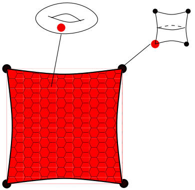
- Besides the $A_1^{\oplus 16} \oplus U(2)^3$ lattice we find naively, there are extra cycles that stem from the pillows.
- They complete $A_1^{\oplus 16} \oplus U(2)^3$ to $H^2(K3, \mathbb{Z}) = U^{\oplus 3} \oplus -E_8^{\oplus 2}$
- Over the reals, they can be expressed as

$$\sigma_{ij}^k = \frac{1}{2} \cdot (\pi_{ij} - \sum_i C_i).$$



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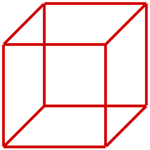

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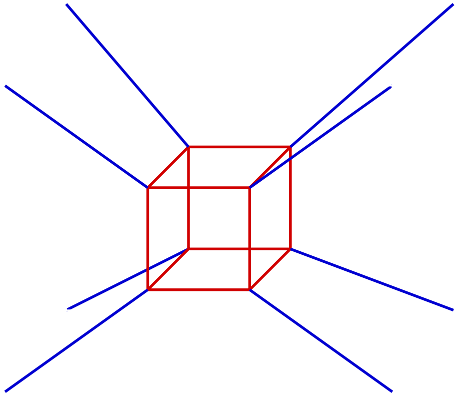

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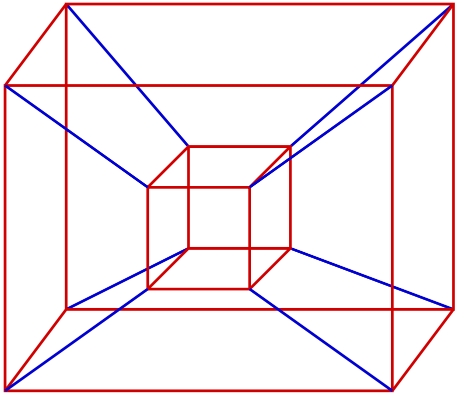

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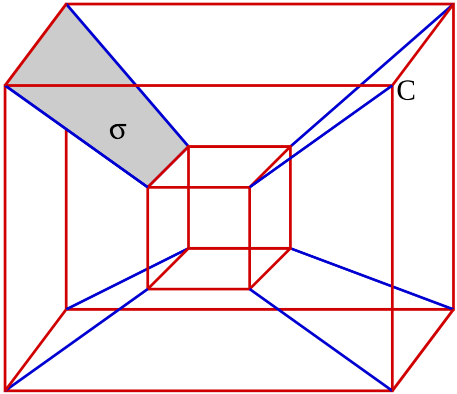

Enriques revisited

The Action of the Enriques involution on the $K3$ lattice is known,

$$e^1 \mapsto -e^1\,,\;\; e_1 \mapsto -e_1\,,\;\; e^2 \mapsto e^3\,,\;\; e_2 \mapsto e_3\,,$$
$$E_8 \leftrightarrow E_8\,,$$

we can compare it with the action on $T^4/\mathbb{Z}_2$:

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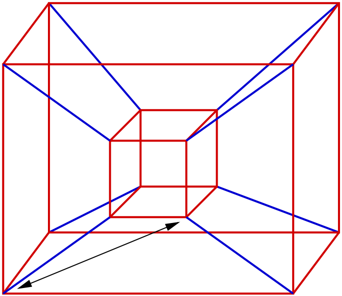

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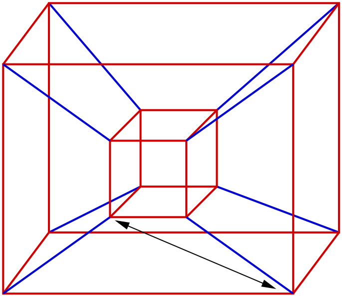

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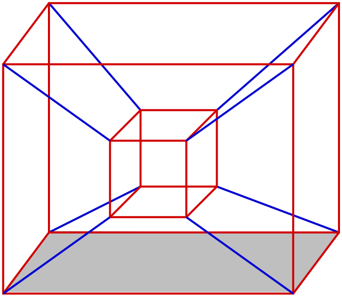

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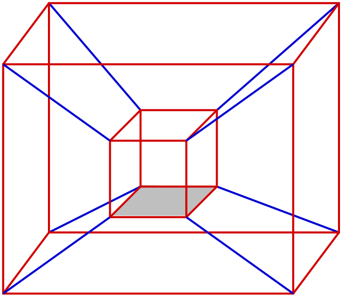

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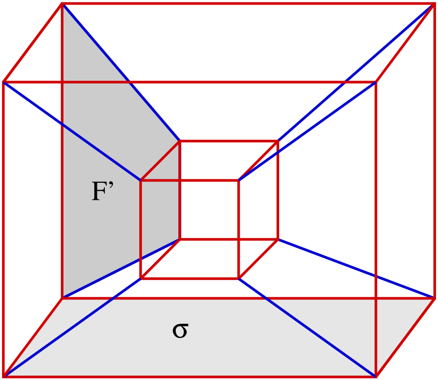

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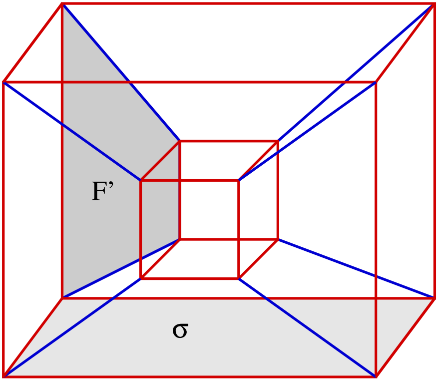
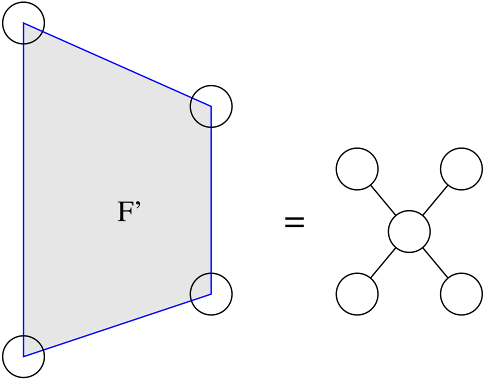


Going from $T^4/\mathbb{Z}_2$ to the Weierstrass model means blowing-up the singularities hitting the section and collapsing the singular fibres to produce the $D_4 \sim SO(8)$ singularities.

Enriques revisited



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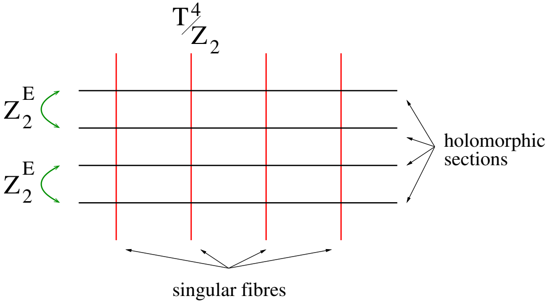
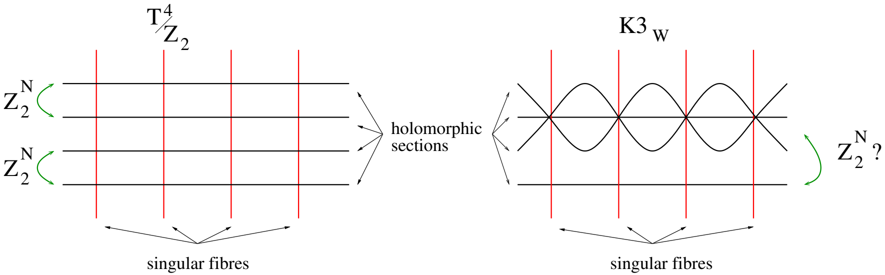
The structure of the holomorphic sections and the singular fibres for $T^4/\mathbb{Z}_2$ is symmetric under the Enriques involution:

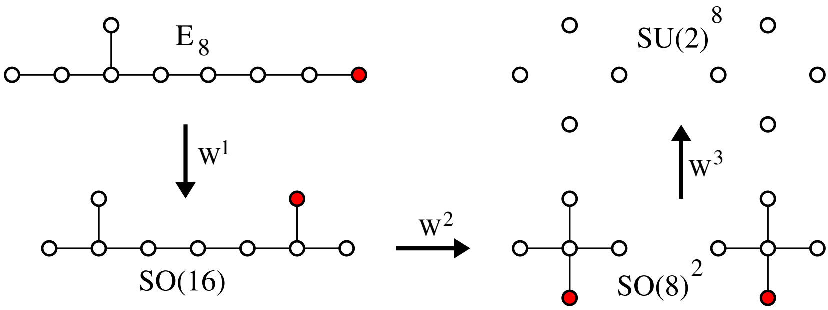


$K3$ as described by the Weierstrass model does not allow an Enriques involution that keeps the holomorphic section:



Enriques revisited

There is a further way to understand what goes wrong. Remember the Wilson lines:



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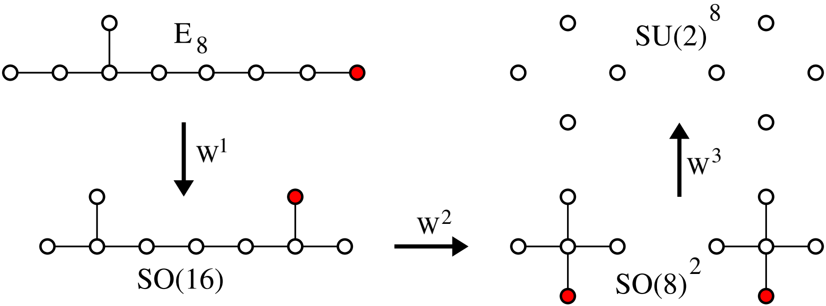

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We get to $SO(8)$ by switching off W^3.

Enriques revisited

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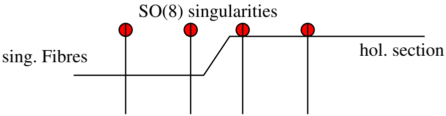

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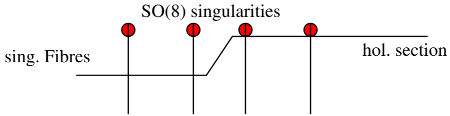

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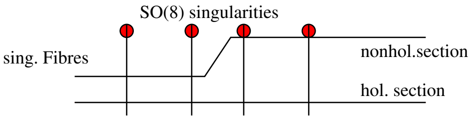

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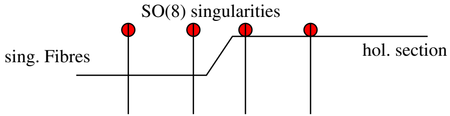

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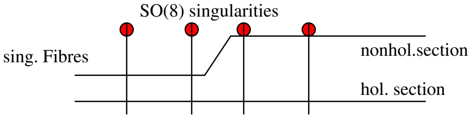

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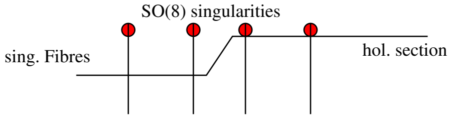

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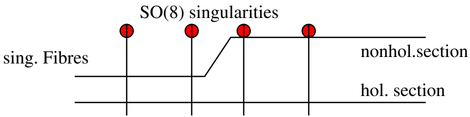

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- There is a clash
  hol. of the W. model section $\leftrightarrow$ hol. of Enriques involution

Conclusions

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- The Enriques involution acts as $y \mapsto -y, z \mapsto -z$ and exchanges the two sections.
- This equation describes a ten-dimensional family of elliptic $K3$ spaces, which agrees with the number of complex structure deformations of an Enriques surface.