Quark superfluidity in the two-fluid formalism


- Motivation: hydrodynamics of CFL
- Superfluids as two-component fluids
- Link microscopic physics with hydro
  - $T = 0$: one fluid
  - $T > 0$: two fluids
• **Motivation: hydrodynamics in compact stars**

• What are compact stars made of?
  Are they ...
  ... neutron stars?
  ... hybrid stars?
  ... quark stars?

• For various properties, need hydrodynamics:
  – pulsar glitches e.g., B. Link, MNRAS 422, 1640 (2012)
  – magnetohydrodynamics e.g., P. D. Lasky, B. Zink, K. D. Kokkotas, arXiv:1203.3590
• **CFL quark matter in the core of a compact star?**

Color-flavor locked (CFL) quark matter ...
M. Alford, K. Rajagopal, F. Wilczek, NPB 537, 443 (1999)

... is the ground state of asymptotically dense, 3-flavor quark matter ...
breaks *color*, *chiral*, and *baryon number* symmetries spontaneously

\[
SU(3)_c \times SU(3)_L \times SU(3)_R \times U(1)_B \rightarrow SU(3)_{c+L+R} \times \mathbb{Z}_2
\]

→ octet of pseudo-Goldstones $K^0$, $K^\pm$, $\pi^0$, ...
→ (exactly massless) Goldstone $\phi$ (”phonon”)

... has a kaon condensate for finite $m_s$ (”CFL-$K^0$”)

→ $U(1)_S$ broken spontaneously
→ pseudo-Goldstone ($U(1)_S$ expl. broken by weak interactions)
- Towards the hydrodynamics of CFL ...

**Astrophysicist:** How many fluid components does CFL have?

**Particle physicist:** ???

**Astrophysicist:** Is CFL a superfluid?

**Particle physicist:** Yes, CFL breaks $U(1)_B$.

**Astrophysicist:** ???

**Particle physicist:** CFL-$K^0$ also breaks $U(1)_S$, but that’s only an approximate symmetry.

**Astrophysicist:** ???
• **Two-fluid picture of a superfluid (Helium-4)** (page 1/2)

L. Tisza, Nature 141, 913 (1938); L. Landau, Phys. Rev. 60, 356 (1941)

• “superfluid component”: condensate, carries no entropy

• “normal component”: excitations (Goldstone mode), carries entropy

• Hydrodynamic eqs.
  \[ \frac{\partial^2 \rho}{\partial t^2} = \Delta P \]
  \[ \frac{\partial^2 S}{\partial t^2} = \frac{S^2 \rho_s}{\rho_n} \Delta T \]

⇒ two wave eqs.

⇒ two sound velocities:

\[ u_1 = \sqrt{\frac{\partial P}{\partial \rho}} \]  
\[ u_2 = \sqrt{\frac{s^2 T \rho_s}{\rho c_V \rho_n}} \]
• Two-fluid picture of a superfluid (Helium-4) (page 2/2)

• 1st sound: total density oscillates

• 2nd sound: relative densities of superfluid and normal components oscillate

• exp. sound velocities of $^4$He

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E. Taylor et al., PRA 80, 053601 (2009)
according to K.R. Atkins et al. (1953);
V.P. Peshkov (1960)

→ How does the two-fluid picture arise from a microscopic theory?
• Bose condensation and superfluid velocity (page 1/2)

• start with simplest case: \( \varphi^4 \) model

\( \rightarrow \) from chiral Lagrangian for CFL mesons

Bedaque, Schäfer, NPA 697, 802 (2002); Alford, Braby, Schmitt, JPG 35, 025002 (2008)

\[
\mathcal{L} = (\partial \varphi)^2 - m^2|\varphi|^2 - \lambda|\varphi|^4
\]

\[
m^2 = m_{K^0}^2 = \frac{m_s^2 - m_d^2}{2\mu}
\]

\[
\lambda = \frac{4\mu_{K^0}^2 - m_{K^0}^2}{6f_\pi^2}
\]

• \( \varphi \rightarrow \phi + \varphi \), condensate \( \phi = \frac{\rho}{\sqrt{2}} e^{i\psi} \)

• first step: no fluctuations (\( T = 0 \))

• minimize \( V(\rho) = -\mathcal{L} \)

\[
\rho^2 = \frac{(\partial \psi)^2 - m^2}{\lambda} \quad \text{(assumption: } \rho, \partial \psi \text{ const.)}
\]
**Bose condensation and superfluid velocity (page 2/2)**

- “translation” at zero temperature (single fluid!) \((m = 0)\)

<table>
<thead>
<tr>
<th></th>
<th>Field-theoretically</th>
<th>Hydrodynamically</th>
</tr>
</thead>
<tbody>
<tr>
<td>(j^\mu)</td>
<td>(\frac{(\partial \psi)^2}{\lambda} \partial^\mu \psi)</td>
<td>(n v^\mu)</td>
</tr>
<tr>
<td>(T^{\mu\nu})</td>
<td>(\frac{(\partial \psi)^2}{\lambda} \partial^\mu \psi \partial^\nu \psi - g^{\mu\nu} \mathcal{L})</td>
<td>((\epsilon + P) v^\mu v^\nu - g^{\mu\nu} P)</td>
</tr>
</tbody>
</table>

- With \(\epsilon + P = \mu n\):

\[
P = \frac{(\partial \psi)^4}{4\lambda}, \quad \epsilon = \frac{3(\partial \psi)^4}{4\lambda}
\]

\[
\mu = |\partial \psi|, \quad n = \frac{|\partial \psi|^3}{\lambda}
\]

- **superfluid velocity**

\[
v^\mu = \frac{\partial^\mu \psi}{\mu}
\]

\[
\Rightarrow \text{superfluid is curl-free,} \quad \nabla \times \mathbf{v}_s = 0
\]
• **From one fluid** \((T = 0)\) **to two fluids** \((T > 0)\)

• qualitative change:
  - one fluid: \(\exists\) frame in which pressure is isotropic
  - two fluids: pressure anisotropic \(\forall\) frames

• formulation in terms of superfluid and normal fluid:

\[
j^\mu = n_s v^\mu + n_n u^\mu
\]

\[
T^{\mu\nu} = (\epsilon_s + P_s) v^\mu v^\nu - g^{\mu\nu} P_s + (\epsilon_n + P_n) u^\mu u^\nu - g^{\mu\nu} P_n
\]


• formulation in terms of entropy current and conserved current:

I.M. Khalatnikov and V.V. Lebedev, Phys. Lett. 91A, 70 (1982)

B. Carter and I. M. Khalatnikov, PRD 45, 4536 (1992)
• Microscopic calculation at nonzero $T$ (page 1/2)

• calculation for all $T \leq T_c$ needs self-consistent formalism;

• here: one-loop (small $T$) effective action

\[
\frac{T}{V} \Gamma_{\text{eff}} = \frac{(\partial \psi)^4}{4 \lambda} - \frac{1}{2V} \sum_k \text{Tr} \ln \frac{S^{-1}(k)}{T^2}
\]

• inverse tree-level propagator (at the $T = 0$ stationary point)

\[
S^{-1}(k) = \begin{pmatrix}
-k^2 + 2(\partial \psi)^2 & 2ik \cdot \partial \psi \\
-2ik \cdot \partial \psi & -k^2
\end{pmatrix}
\]

• anisotropic phonon dispersion (\(\rightarrow\) first sound)

\[
\epsilon(\theta, k) = \frac{f(\theta)}{\sqrt{3}} k + \ldots , \quad f(\theta) = \frac{\sqrt{1 - v_s^2} \sqrt{1 - \frac{v_s^2}{3} (1 + 2 \cos^2 \theta)} + \frac{2 |v_s|}{\sqrt{3}} \cos \theta}{1 - \frac{v_s^2}{3}}
\]
• **Microscopic calculation at nonzero $T$ (page 2/2)**

• compute current and stress-energy tensor

\[
j^\mu = \frac{\sigma^2}{\lambda} \partial^\mu \psi - \frac{1}{2V} \sum_k \text{Tr} \left[ S \frac{\partial S^{-1}}{\partial \partial^\mu \psi} \right]
\]

\[
T^{\mu\nu} = \frac{(\partial \psi)^2}{\lambda} \partial^\mu \psi \partial^\nu \psi - g^{\mu\nu} \frac{(\partial \psi)^4}{4\lambda} - \frac{T}{V} \sum_k \text{Tr} \left[ S \frac{\partial S^{-1}}{\partial g^{\mu\nu}} - u^\mu u^\nu \right]
\]

[where $u^\mu = (1, 0, 0, 0)$]

• can be evaluated analytically for small $T$, $|v_s|$, e.g.,

\[
T^{00} = \frac{\mu^4}{4\lambda} (3 - 2v_s^2) + \frac{\pi^2}{10\sqrt{3}} (3 + 4v_s^2) T^4 - \frac{4\pi^2}{21\sqrt{3}} (3 + 19v_s^2) \frac{T^6}{\mu^2} + \ldots
\]
• RELATIVISTIC TWO-FLUID FORMALISM (PAGE 1/2)

• Write stress-energy tensor as

\[ T^{\mu\nu} = -g^{\mu\nu}\Psi + j^{\mu}\partial^{\nu}\psi + s^{\mu}\Theta^{\nu} \]

• "generalized pressure" \( \Psi \):
  - \( \Psi \) is transverse pressure in superfluid and normal rest frames
  - \( \Psi \) depends on momenta \( \partial^{\mu}\psi, \Theta^{\mu} \)
  \[ \Psi = \Psi[(\partial\psi)^2, \Theta^2, \partial\psi \cdot \Theta] \]

• "generalized energy density" \( \Lambda \equiv -\Psi + j \cdot \partial\psi + s \cdot \Theta \)
  - \( \Lambda \) is Legendre transform of \( \Psi \)
  - \( \Lambda \) depends on currents \( j^{\mu}, s^{\mu} \)
  \[ \Lambda = \Lambda[j^2, s^2, j \cdot s] \]
• **Relativistic two-fluid formalism (page 2/2)**

\[ j^\mu = \frac{\partial \Psi}{\partial (\partial_{\mu} \psi)} = \mathcal{B} \partial^\mu \psi + \mathcal{A} \Theta^\mu \]

\[ s^\mu = \frac{\partial \Psi}{\partial \Theta_{\mu}} = \mathcal{A} \partial^\mu \psi + \mathcal{C} \Theta^\mu \]

\[ \mathcal{B} = 2 \frac{\partial \Psi}{\partial (\partial \psi)^2}, \quad \mathcal{C} = 2 \frac{\partial \Psi}{\partial \Theta^2} \]

\[ \mathcal{A} = \frac{\partial \Psi}{\partial (\partial \psi \cdot \Theta)} \]

“entrainment coefficient”

• conservation equations \( \partial_\mu T^{\mu\nu} = 0, \quad \partial_\mu j^\mu = 0 \) become

\[ \partial_\mu j^\mu = 0, \quad \partial_\mu s^\mu = 0, \quad s_\mu \left( \partial^\mu \Theta^\nu - \partial^\nu \Theta^\mu \right) = 0 \]

“vorticity”

• in “mixed” form, we recover superfluid/normal formulation

\[ T^{\mu\nu} = -g^{\mu\nu} \Psi + \frac{\mathcal{B} \mathcal{C} - \mathcal{A}^2}{\mathcal{C}} \partial^\mu \psi \partial^\nu \psi + \frac{1}{\mathcal{C}} s^\mu s^\nu \]
• **Connect microscopic calculation with hydro**

• microscopic calculation done in “normal rest frame” \( s^\mu = (s^0, 0, 0, 0) \)

• one can then show that

\[
\frac{T}{V} \Gamma_{\text{eff}} = \Psi
\]

• 8 independent degrees of freedom from 16 \( (\partial^\mu \psi, \Theta^\mu, j^\mu, s^\mu) \)

\[
(\mu, \mu v^i_s, T) = (\partial^0 \psi, \partial^i \psi, \Theta^0) + \text{ constraint } s^i = 0
\]

• obtain current \( j^\mu \) and entropy \( s^0 \) microscopically

• determine \( A, B, C \), (and \( \Theta^i \)), for instance

\[
A = \frac{s^0}{\partial^0 \psi} \left[ j^0 + \frac{\vec{j} \cdot \nabla \psi}{(\nabla \psi)^2} \partial^0 \psi \right] \left[ j^0 + \frac{\vec{j} \cdot \nabla \psi}{(\nabla \psi)^2} \partial^0 \psi + s^0 \Theta^0 \right]^{-1}
\]

etc.
• **Compute properties of the superfluid (page 1/2)**

• **superfluid** and **normal** charge densities
  (measured in normal rest frame)

\[
n_s = \mu \frac{BC - A^2}{C} = \frac{\mu^3}{\lambda} (1 - v_s^2) - \frac{4\pi^2}{5\sqrt{3}} \left( 1 + \frac{47}{6} v_s^2 \right) \frac{T^4}{\mu} + \ldots
\]

\[
n_n = s \frac{A}{C} = \frac{4\pi^2}{5\sqrt{3}} \left( 1 + \frac{41}{6} v_s^2 \right) \frac{T^4}{\mu} + \ldots
\]

![Graph showing charge density fraction vs. T/μ for superfluid and normal states.

(effect exaggerated by choosing \(\lambda\) very large)

• one fluid gets converted into the other by heating
• Compute properties of the superfluid (page 2/2)

• sound velocities (measured in normal rest frame)

\[ u_1 = \frac{1}{\sqrt{3}} \left( 1 + \frac{2 \cos \theta}{\sqrt{3}} |v_s| - \frac{1 + \cos^2 \theta}{3} v_s^2 \right) + \ldots \]

\[ u_2 = \frac{1}{3} \left( 1 + \frac{\cos \theta}{3} |v_s| - \frac{17 - \cos^2 \theta}{18} v_s^2 \right) + \frac{20}{21} \left[ 1 - \frac{4 \cos \theta}{3} |v_s| + \left( \frac{2702}{243} - \frac{\cos^2 \theta}{2} \right) v_s^2 \right] \left( \frac{\pi T}{\mu} \right)^2 + \ldots \]

\[ |v_s| = 0 \]

\[ |v_s| = \frac{0.5}{\sqrt{3}} \]

\[ |v_s| = \frac{0.95}{\sqrt{3}} \]
• **Summary**

• The hydrodynamics of CFL is nontrivial ...
  ... and poses fundamental questions regarding relativistic superfluid hydrodynamics and its microscopic, field-theoretical description.

• For the case of a $\varphi^4$ model we have connected the microscopic theory (at finite $T$) with the two-fluid formalisms of Son and Khalatnikov/Lebedev
• **Outlook**

• go **beyond small-$T$ expansion**
  
  – solve stationarity eqs with superflow numerically
  
  – compute superfluid density etc for all $T < T_c$

• how does the picture change with approximate (not exact) $U(1)_S$ symmetry? is superfluidity lost completely?

• start from fermionic microscopic theory to account for $U(1)_B$

• put all this together for **hydrodynamics of CFL-$K^0$**

• include dissipation
  
  – various viscosity coefficients already computed
    
    
  
  – three bulk viscosity coefficients due to two fluids
    
    M. Mannarelli and C. Manuel, PRD 81, 043002 (2010)