

Introduction to superfluidity – Field-theoretical approach and applications

Summer Semester 2013

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I. INTRODUCTION AND OUTLINE

Superfluidity is a very general phenomenon occurring on vastly different energy scales. Understanding its underlying mechanism therefore means understanding the basics of very different phenomena in various fields of physics. This is, in a nutshell, the motivation for this course. We shall discuss the general properties of any superfluid but also aim at getting some insight in the specific consequences of superfluidity in a number of different systems.

What is the general phenomenon that connects all physical systems discussed in this course? The short answer is spontaneous symmetry breaking. This term refers to a ground state being not invariant under a certain symmetry of the underlying Lagrangian. In the case of a superfluid, this symmetry is a global symmetry, i.e., independent of space-time. In the case of a superconductor, this symmetry is a gauge symmetry, whose transformations do depend on space-time. Despite of this difference, there are many similarities, at least on the theoretical level, between superfluidity and superconductivity. Therefore, both phenomena will be discussed in this course.

It may seem challenging to pack the physics of very different fields into one lecture. After all, we shall be talking about systems whose characteristic (here: critical) temperatures span 17 orders of magnitude (from 200 nK for atomic gases to $\sim 10^{10}$ K for nuclear and quark matter). Moreover, we shall be interested in both fermionic and bosonic systems as well as relativistic and nonrelativistic systems. It is therefore not the intention of the course to lead systematically, building one argument onto another, through all these different systems. We shall rather aim at developing the underlying physics systematically and then apply them, *mutatis mutandi*, to the physics of superfluid helium, cold atomic gases, neutron stars etc. For definiteness the systematic calculations will be preferably done in a relativistic field theory. We shall explain the following important concepts

- spontaneous symmetry breaking
- Bose-Einstein condensation (BEC)
- Goldstone boson
- Cooper pairing and Bardeen-Cooper-Schrieffer (BCS) Theory
- Meissner mass

This will be sufficient to understand the basic properties of “usual” superfluids/superconductors. More specific topics, either going beyond the elementary properties superfluids or including more exotic systems, are

- BEC-BCS crossover (ultra-cold atomic gases)
- hydrodynamics of a superfluid
- anisotropic superfluids (^3He)
- vortices in a superfluid (e.g., applications in neutron stars)
- “mismatched” Cooper pairing (relevant for cold atoms & quark matter)

There are other unconventional phenomena and systems in this context for which we will have no time in this course. One of them is high- T_c superconductivity.

Literature: These lecture notes will grow during the course and be the primary source of literature. I will use several sources some of which are research papers, to be quoted later where relevant. I will make use of elements of thermal field theory, see my own lecture notes [1] and references therein. General textbooks about superfluidity are for instance refs. [2, 3]. More specific literature about superfluidity/superconductivity in various systems can be found in refs. [4, 5] (ultra-cold fermionic atoms), [6] (color-superconducting quark matter), [7] (superfluidity/superconductivity of nuclear matter in compact stars), [8] (superfluid helium-3).

[End of 1st lecture, March 4th, 2013.]

II. SUPERFLUID ^4He

In this section we shall discuss superfluid ^4He whose atoms are bosons and whose superfluidity can thus be understood in terms of Bose-Einstein condensation. Most of what follows in this section can be found in the textbooks by Khalatnikov [2] and by Landau/Lifshitz [9]. In 1908, ^4He was first liquefied by H. Kamerlingh Onnes who cooled it below the liquid/gas transition temperature of 4.2 K. Later, in 1927, M. Wolfke and W.H. Keesom realized that there is another phase transition at lower temperatures, around 2.17 K. This phase transition had manifested itself

in a discontinuity of the specific heat, whose curve as a function of temperature resembles the Greek letter λ , and thus the transition point was called λ point. The remarkable superfluid properties of liquid helium below the λ point were experimentally established by Kapitza in 1938 (and independently by Allen and Misener in the same year). Kapitza had set up an experiment with two cylinders that were connected by a thin tube which was $0.5\,\mu\text{m}$ thick. Only below the λ point helium was flowing easily through the tube, suggesting a strikingly low viscosity. Kapitza received the Nobel prize for this discovery in 1978 (interestingly, together with Penzias and Wilson who received it for the completely unrelated discovery of the cosmic microwave background radiation). It is remarkable that Kapitza already talked about superfluidity as an analogue of superconductivity in his paper of 1938. Although superconductivity had been observed much earlier, in 1911, a microscopic understanding was only achieved in 1957 by Bardeen, Cooper, and Schrieffer. Only with the help of their “BCS theory” it is possible to appreciate the deep connection between electronic superconductivity through Bose condensation of electron Cooper pairs and superfluidity in ^4He through Bose condensation of helium atoms. It is also interesting that, in spite of the already conjectured analogy of superfluidity and superconductivity, it was quite controversial for a long time whether Bose condensation is indeed the underlying mechanism for superfluidity in ^4He . For instance Landau never believed in Bose condensation in ^4He , although his phenomenological two-fluid model – to be discussed in Sec. IIC – is completely consistent with the idea of Bose-condensed atoms in the liquid. If you are interested in more details about the interesting history of the discovery of superfluidity and early theoretical developments put forward by London, Tisza, and Landau, you may consult Refs. [10, 11].

In this course we shall work out the underlying theory for both superfluidity and superconductivity and thus we shall understand that Kapitza was right with his conjecture that both phenomena are connected. In both cases, the most fundamental property is that charge (in a general sense: electric charge in an electronic superconductor, mass in liquid helium) can be transported without dissipation. A theory of this phenomenon has to explain (i) what carries the charge and (ii) why the collective excitations of the system that potentially dissipate energy are absent, or at least suppressed.

What is special about Helium, i.e., why can it become superfluid at low temperatures? One of the necessary conditions is that it remains fluid for arbitrarily small temperatures. Only because of this property can quantum effects become important in a liquid. Most other elements solidify at a certain, sufficiently small, temperature. The reason why helium does not solidify has to do with (i) the lightness of its atoms and (ii) the weakness of its inter-atomic attractive interactions. To understand why these properties are crucial, recall why a solid, i.e., a crystal, usually forms at small temperatures: from quantum mechanics we know that all atoms in the crystal have a zero-point motion which seeks to drive them out of their fixed lattice position. This effect is countered by the mass of the atoms and by the depth of the inter-atomic potential. More precisely, thinking of the zero-point energy E_0 as the energy of a quantum harmonic oscillator, we have $E_0 = \frac{3}{2}\hbar\omega_0$ with the frequency $\omega_0 \propto [\epsilon_0/(mr_0^2)]^{1/2}$, where ϵ_0 is the depth of the inter-atomic potential, r_0 the location of the minimum of this potential, i.e., the inter-atomic distance, and m the atomic mass. The energy turns out to be $E_0 \simeq 7\text{ meV} \sim 70\text{ K}$ which is too large to keep the helium atoms on a fixed lattice position. Hence helium remains fluid for arbitrarily small temperatures (if the pressure is smaller than a certain critical value). The phase diagram of ^4He is shown in Fig. 1.

A. Landau’s critical velocity

Under which conditions does a fluid flow without friction? If we have a Bose-condensed system we already fulfil a necessary condition, because the condensate may constitute the current and cannot be excited by arbitrarily small energies, i.e., it cannot lose energy while flowing. For the Bose-condensed fluid to lose energy we thus have to look at the collective excitations, or quasiparticles, which can potentially result in dissipation. So let us consider a superfluid moving through a capillary with velocity \mathbf{v} . In the rest frame of the fluid (where the capillary moves with velocity $-\mathbf{v}$), let the energy of such an excitation and its corresponding momentum be $E_0 = \epsilon_p$ and $\mathbf{P}_0 = \mathbf{p}$. Now, in the rest frame of the capillary, the energy of the fluid is given by the kinetic energy E_{kin} plus the energy of the elementary excitations, transformed into the new frame¹,

$$E = E_{\text{kin}} + \epsilon_p + \mathbf{p} \cdot \mathbf{v}. \quad (1)$$

¹ When we speak about ^4He we use a nonrelativistic treatment, i.e., the transformation of the energy follows a Galilei transformation, not a general Lorentz transformation. In this course it is unavoidable to change between relativistic and nonrelativistic descriptions; I will still try to keep the notation as consistent as possible (for instance I will not reinstate factors of c , \hbar , and k_B in this section).

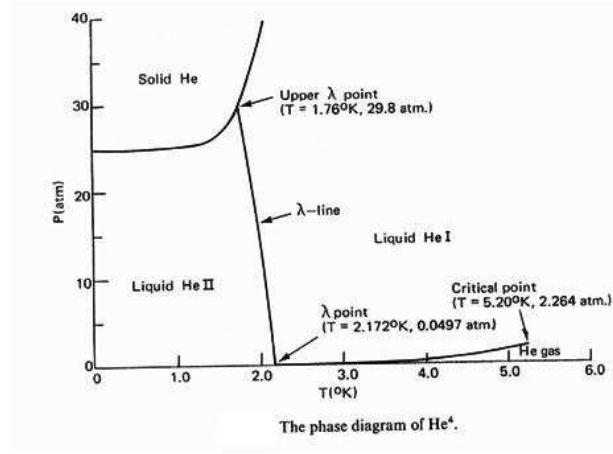


FIG. 1: Phase diagram of ^4He in the pressure/temperature plane. Below a certain pressure, helium remains fluid for arbitrarily small temperatures, allowing for a superfluid phase below a critical temperature, sometimes called λ -temperature because the shape of the curve for the specific heat vs. temperature resembles the letter λ due to the discontinuity at the phase transition. Superfluid and normal fluid phases are denoted by He II and He I, respectively. This terminology has historical origin and was given to the two phases after the discovery of the phase transition, but before the discovery of superfluidity of He II.

Since dissipation means that the fluid *loses* energy, we must have

$$\epsilon_p + \mathbf{p} \cdot \mathbf{v} < 0. \quad (2)$$

The left-hand side can only be negative if its minimal value is negative, $\epsilon_p - pv < 0$. Consequently, the system cannot dissipate energy and thus behaves as a superfluid for velocities smaller than the critical velocity

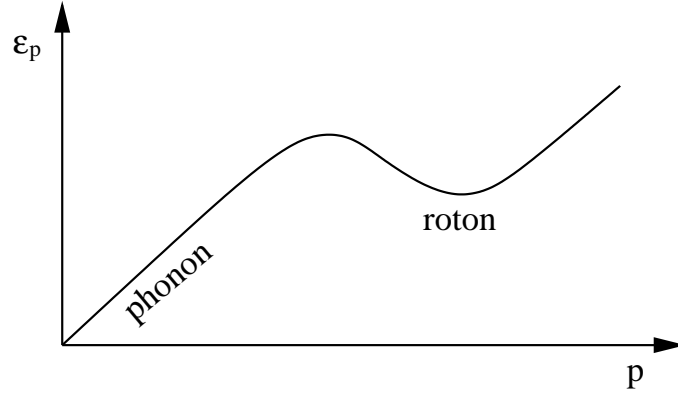
$$v_c = \min \frac{\epsilon_p}{p}. \quad (3)$$

This simple argument has been pointed out by Landau and is of fundamental importance for the understanding of a superfluid. Note in particular that the existence of superfluidity requires the critical velocity to be larger than zero. We can write the minimum of ϵ_p/p as the solution of

$$0 = \frac{\partial}{\partial p} \frac{\epsilon_p}{p} \Rightarrow \frac{\partial \epsilon_p}{\partial p} = \frac{\epsilon_p}{p}. \quad (4)$$

For a given point on the curve ϵ_p we are thus asking whether the slope of the curve is identical to the slope of a straight line from the origin through the given point. Or, in other words, to check the superfluidity of a system (provided it has a condensate), take a horizontal line through the origin in the ϵ_p - p plane and rotate it upwards. If you can do so by a finite amount before touching the dispersion curve tangentially, the system supports superfluidity. The point where you touch the curve gives you the critical velocity according to Landau above which superfluidity is destroyed. In particular, any gapless dispersion with slope zero in the origin must lead to dissipation for any nonzero velocity.

We shall see later, in our field-theoretical calculation, that Bose-Einstein condensation (more generally, spontaneous symmetry breaking of a global symmetry of the Lagrangian) is always accompanied by a gapless mode. One might think that such a gapless mode can easily be excited (for instance, due to the gaplessness, such a mode becomes populated for arbitrarily small temperatures). Landau's argument, however, suggests that the Bose-condensed system can be a superfluid despite the Goldstone mode. Typically, and as we shall see, the dispersion of a Goldstone mode is linear for small momenta, and therefore the critical velocity is larger than zero and identical to the slope of this dispersion. This linearity leads to sparser available states than for instance a quadratic dispersion, where the density of states $d(\epsilon) \sim p^2 \left(\frac{\partial \epsilon}{\partial p} \right)^{-1}_{p=p(\epsilon)}$, $p(\epsilon)$ being the solution of $\epsilon_p = \epsilon$, would be larger for small energies: $d(\epsilon) \propto \epsilon^2$ for $\epsilon_p \propto p$, while $d(\epsilon) \propto \epsilon^{1/2}$ for $\epsilon_p \propto p^2$.

FIG. 2: Goldstone mode for superfluid ^4He .

B. Thermodynamics of ^4He

While the existence of a Goldstone mode and the linearity at small p are very general features, the details of the complete dispersion of this mode depend on the details of the interaction in a given system. In ^4He , it turns out that the mode has the form shown in Fig. 2. It can effectively be described by two different excitations, one accounting for the linear small-momentum part – this is called the “phonon” – and one for the local minimum of the excitation at a finite value of p – this is called the “roton”. We can write these two dispersions as

$$\epsilon_p = cp \quad (\text{“phonon”}), \quad (5a)$$

$$\epsilon_p = \Delta + \frac{(p - p_0)^2}{2m} \quad (\text{“roton”}). \quad (5b)$$

Let us first compute some of the thermodynamic properties of ^4He . We start from the general expression for the pressure,

$$P = -T \int \frac{d^3\mathbf{p}}{(2\pi)^3} \ln(1 - e^{-\epsilon_p/T}) = \frac{1}{3} \int \frac{d^3\mathbf{p}}{(2\pi)^3} p \frac{\partial \epsilon_p}{\partial p} f(\epsilon_p), \quad (6)$$

where the second step is seen with the help of partial integration and where

$$f(\epsilon_p) = \frac{1}{e^{\epsilon_p/T} - 1} \quad (7)$$

is the Bose distribution function.

The phonon contribution to the pressure is

$$P_{\text{ph}} = \frac{c}{6\pi^2} \int_0^\infty dp \frac{p^3}{e^{cp/T} - 1} = \frac{T^4}{6\pi^2 c^3} \underbrace{\int_0^\infty dy \frac{y^3}{e^y - 1}}_{\pi^4/15} = \frac{\pi^2 T^4}{90 c^3}. \quad (8)$$

If the dispersion were linear for all p , this result would be valid for any T . However, the dispersion is linear only for small p . Since the corrections to the linear behavior become important at larger temperatures, this result cannot be trusted for all T (obviously, the critical temperature is another, absolute, limit above which this result is inapplicable). For $c = 1$ the result is the same as the one for a relativistic, massless, bosonic degree of freedom. Since a photon has two degrees of freedom, multiplying the result by 2 gives the Stefan-Boltzmann pressure of blackbody radiation.

We can now compute the entropy and the specific heat per unit volume from the usual thermodynamic definitions,

$$s_{\text{ph}} = \frac{\partial P_{\text{ph}}}{\partial T} = \frac{2\pi^2 T^3}{45 c^3}, \quad (9)$$

and

$$c_{V,\text{ph}} = T \frac{\partial s_{\text{ph}}}{\partial T} = \frac{2\pi^2 T^3}{15 c^3} = 3 s_{\text{ph}}, \quad (10)$$

where the subscript V indicates that the specific heat is computed at fixed volume (as opposed to fixed pressure).

The roton contribution is a bit more complicated to compute,

$$P_{\text{rot}} = \frac{1}{6\pi^2 m} \int_0^\infty dp p^3 (p - p_0) \frac{1}{e^{\frac{\Delta}{T} + \frac{(p-p_0)^2}{2mT}} - 1}. \quad (11)$$

In general, this integral has to be solved numerically. Here we proceed by making the assumption $T \ll \Delta$, such that

$$\frac{1}{e^{\frac{\Delta}{T} + \frac{(p-p_0)^2}{2mT}} - 1} \simeq e^{-\Delta/T} e^{-\frac{(p-p_0)^2}{2mT}}, \quad (12)$$

and thus

$$P_{\text{rot}} \simeq \frac{e^{-\Delta/T}}{6\pi^2 m} \int_0^\infty dp p^3 (p - p_0) e^{-\frac{(p-p_0)^2}{2mT}}. \quad (13)$$

This expression already gives the leading exponential suppression. This is due to the energy gap Δ of the roton dispersions. In other words, for the roton to contribute significantly to the pressure, the temperature has to be at least of the order of this gap. To obtain the subleading temperature dependence, we introduce the new integration variable $q = p - p_0$ and drop terms in the integrand of higher order in q since they would lead to higher powers in the temperature,

$$P_{\text{rot}} \simeq \frac{e^{-\Delta/T} p_0^2}{2\pi^2 m} \int_{-p_0}^\infty dq q^2 e^{-\frac{q^2}{2mT}} \simeq \frac{e^{-\Delta/T} p_0^2}{2\pi^2 m} (2mT)^{3/2} \underbrace{\int_{-\infty}^\infty dy y^2 e^{-y^2}}_{\pi^{1/2}/4} = \sqrt{\frac{m}{2\pi^3}} p_0^2 T^{3/2} e^{-\Delta/T}. \quad (14)$$

(In the first step, we have dropped the p_0^3 contribution from the factor p^3 in the integrand, anticipating the approximation in the second step where the lower boundary is approximated by $-\infty$; due to this approximation, antisymmetric parts of the integrand do not contribute, i.e., $(p_0 + q)^3 \rightarrow 3p_0^2 q$.) Again we may compute entropy and specific heat,

$$s_{\text{rot}} \simeq \sqrt{\frac{m}{2\pi^3}} \frac{p_0^2 \Delta}{T^{1/2}} e^{-\Delta/T}, \quad (15)$$

and

$$c_{V,\text{rot}} \simeq \sqrt{\frac{m}{2\pi^3}} \frac{p_0^2 \Delta^2}{T^{3/2}} e^{-\Delta/T}, \quad (16)$$

where we have neglected the terms of higher order in Δ/T .

[End of 2nd lecture, Mar 11th, 2013.]

C. Two-fluid model

The hydrodynamics of a superfluid is often described within a so-called two-fluid model, suggested by Tisza and Landau shortly after the discovery of superfluidity. A priori, this was a purely phenomenological description. We shall discuss later how it emerges as a kind of effective theory from a microscopic theory or model. In the two-fluid picture, the system is formally divided into two components, the *superfluid* and *normal* components. These two components interpenetrate each other. Importantly, they do so without dissipation, i.e., one component can flow “through” the other component without losing energy. The superfluid component consists of the condensate (although, as remarked above, this is not the way Landau originally viewed it) while the normal component contains the elementary excitations, i.e., the phonon and roton excitations in the case of superfluid ^4He . This picture suggests that at zero temperature, the normal component vanishes. Then, upon heating up the system, the normal component will become more and more important until the superfluid component vanishes at and above the critical temperature. Phenomenologically, this model for instance explains nicely the two sound modes in a superfluid that we shall discuss in Sec. IID. Originally, the model served to explain the “viscosity paradox” which had appeared from two apparently contradicting behaviors of superfluid helium: damping times of the oscillations of a torsion pendulum in liquid helium suggested a viscosity, in apparent contrast to the observation flow of liquid helium through a thin tube, which suggested zero viscosity. In the two-fluid picture it is only the superfluid component that can flow through the thin tube while

the pendulum “sees” both fluid components, i.e., the excitations of the normal fluid were responsible for the damping of the pendulum.

The model describes the flow of the system as being composed of two fluids with different velocities. For instance the momentum density \mathbf{g} (which corresponds to certain components of the stress-energy tensor, $g_i = T_{0i}$) can be thought of receiving contributions from two components,

$$\mathbf{g} = \rho_s \mathbf{v}_s + \rho_n \mathbf{v}_n, \quad (17)$$

where \mathbf{v}_s and \mathbf{v}_n are the velocities of the superfluid and normal components, respectively, and ρ_s and ρ_n are the superfluid and normal mass densities, such that the total mass density is $\rho = \rho_n + \rho_s$.

To compute the normal density, we consider the rest frame of the superfluid component, in which the normal component is moving with velocity $\mathbf{w} \equiv \mathbf{v}_n - \mathbf{v}_s$. In this frame, the momentum density is

$$\mathbf{g}_0 = \rho_n \mathbf{w}. \quad (18)$$

On the other hand, we can write the momentum density as

$$\mathbf{g}_0 = \int \frac{d^3 \mathbf{p}}{(2\pi)^3} \mathbf{p} f(\epsilon_p - \mathbf{p} \cdot \mathbf{w}), \quad (19)$$

where we have taken into account that the energy of the elementary excitations, as seen in the frame of the superfluid component, is shifted by $\mathbf{p} \cdot \mathbf{w}$. Equating the right-hand sides of Eqs. (18) and (19) yields

$$\begin{aligned} \rho_n \mathbf{w} &= \int \frac{d^3 \mathbf{p}}{(2\pi)^3} \mathbf{p} f(\epsilon_p - \mathbf{p} \cdot \mathbf{w}), \\ \Rightarrow \quad \rho_n &= \frac{1}{w} \int \frac{d^3 \mathbf{p}}{(2\pi)^3} \hat{\mathbf{w}} \cdot \mathbf{p} f(\epsilon_p - \mathbf{p} \cdot \mathbf{w}), \end{aligned} \quad (20)$$

from which we can compute ρ_n (here, $\hat{\mathbf{w}} \equiv \mathbf{w}/w$). In accordance with Landau’s critical velocity, we need to require $w < c$. One can see in the integral that for $w > c$ the occupation number would become negative, which is unphysical. With the z -axis of our coordinate system pointing in the direction of \mathbf{w} and $x = \cos \theta$ with θ being the angle between \mathbf{w} and the momentum \mathbf{p} , the phonon contribution to the normal density is

$$\begin{aligned} \rho_{n,\text{ph}} &= \frac{1}{4\pi^2 w} \int_{-1}^1 dx x \int_0^\infty dp \frac{p^3}{e^{p(c-wx)/T} - 1} = \frac{T^4}{4w\pi^2} \underbrace{\int_{-1}^1 dx \frac{x}{(c-wx)^4}}_{\frac{8cw}{3(c^2-w^2)^3} \text{ for } w < c} \int_0^\infty dy \frac{y^3}{e^y - 1} \\ &= \frac{2\pi^2 T^4}{45c^5} \frac{1}{\left(1 - \frac{w^2}{c^2}\right)^3}. \end{aligned} \quad (21)$$

Notice that in general the normal and superfluid densities are functions of the velocity of the fluid components, i.e., they are not intrinsic properties of the fluid. They are only defined in the presence of a relative velocity. In many applications, however, one is interested in the limit of small relative velocities. In this case, taking the limit $w \rightarrow 0$, we obtain

$$\rho_{n,\text{ph}} = \frac{2\pi^2 T^4}{45c^5} \left[1 + \mathcal{O}\left(\frac{w^2}{c^2}\right) \right] \simeq \frac{s_{\text{ph}} T}{c^2}. \quad (22)$$

For the roton contribution we find for small temperatures

$$\begin{aligned} \rho_{n,\text{rot}} &\simeq \frac{e^{-\Delta/T}}{4\pi^2 w} \int_0^\infty dp p^3 e^{-\frac{(p-p_0)^2}{2mT}} \int_{-1}^1 dx x e^{\frac{pw}{T}} = \frac{T e^{-\Delta/T}}{2\pi^2 w^2} \int_0^\infty dp p^2 e^{-\frac{(p-p_0)^2}{2mT}} \left(\cosh \frac{pw}{T} - \frac{T}{wp} \sinh \frac{pw}{T} \right) \\ &\simeq \frac{T e^{-\Delta/T} p_0^2}{2\pi^2 w^2} \left(\cosh \frac{p_0 w}{T} - \frac{T}{wp_0} \sinh \frac{p_0 w}{T} \right) \int_{-p_0}^\infty dq e^{-\frac{q^2}{2mT}} \\ &\simeq \sqrt{\frac{m}{2\pi^3}} \frac{T^{3/2} e^{-\Delta/T} p_0^2}{w^2} \left(\cosh \frac{p_0 w}{T} - \frac{T}{wp_0} \sinh \frac{p_0 w}{T} \right). \end{aligned} \quad (23)$$

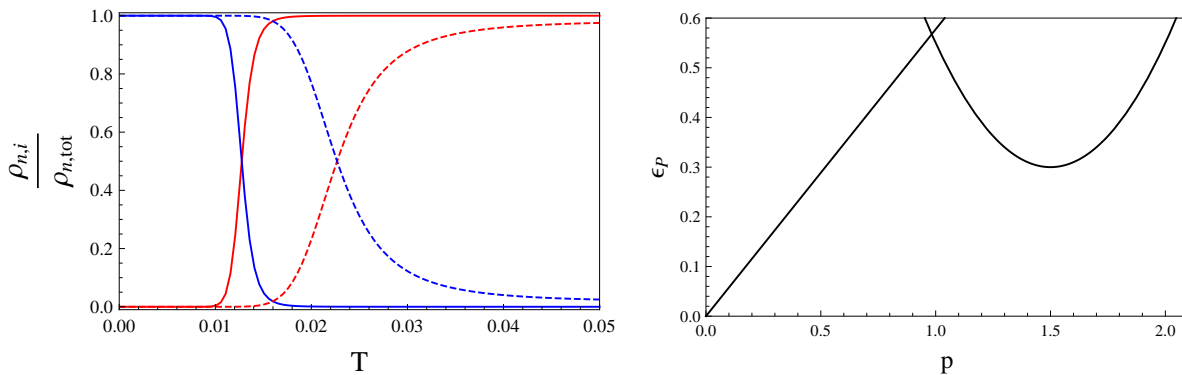


FIG. 3: Left panel: Phonon (red) and roton (blue) fractions of the total normal density $\rho_{n,\text{tot}} = \rho_{n,\text{ph}} + \rho_{n,\text{rot}}$ for vanishing relative velocity, $\mathbf{w} = 0$. The dashed lines are the approximations from Eq. (24). Right panel: dispersions for phonons and rotons with the parameters used in the left plot. we see that even though the temperature scale on the left panel seems small compared to the energy gap of the roton branch, the small-temperature approximation is not applicable for most of this range.

Again taking the limit $w \rightarrow 0$,

$$\rho_{n,\text{rot}} = \sqrt{\frac{m}{2\pi^3}} \frac{e^{-\Delta/T} p_0^4}{3T^{1/2}} \left[1 + \mathcal{O}\left(\frac{p_0^2 w^2}{T^2}\right) \right] \simeq \frac{p_0^2}{3T^2} P_{\text{rot}}. \quad (24)$$

We plot the phonon and roton fractions to the normal density and the energy density for $\mathbf{w} = 0$ as a function of temperature in Fig. 3.

We may also ask how the thermodynamic properties are effected by a nonzero relative velocity. Replacing ϵ_p by $\epsilon_p - \mathbf{p} \cdot \mathbf{w}$ we can repeat the above calculation of the pressure. The details are left as an exercise. The results are

$$P_{\text{ph}}(w) = \frac{P_{\text{ph}}(0)}{\left(1 - \frac{w^2}{c^2}\right)^2}, \quad (25a)$$

$$P_{\text{rot}}(w) \simeq P_{\text{rot}}(0) \frac{T}{p_0 w} \left(\sinh \frac{p_0 w}{T} + \frac{mw}{p_0} \cosh \frac{p_0 w}{T} \right) e^{\frac{mw^2}{2T}} \simeq P_{\text{rot}}(0) \frac{T}{p_0 w} \sinh \frac{p_0 w}{T} \quad (25b)$$

where $P_{\text{ph}}(0)$ and $P_{\text{rot}}(0)$ are the pressures for $w = 0$ from Eqs. (8) and (14). The first expression for the roton contribution neglects higher order terms in T , but is valid for general w ; the second also assumes w to be small. From these two expressions one can now easily compute entropy, specific heat, etc.

D. First and second sound

To discuss sound waves in superfluid ^4He , we need to discuss some hydrodynamics. To this end, we shall start from the relativistic hydrodynamic equations and derive the non-relativistic limit. Here we shall only discuss ideal – i.e., dissipationless – hydrodynamics. We shall see that in a superfluid even the ideal hydrodynamics shows nontrivial physics. The reason is the existence of two fluid components which allow for two sound velocities. The existence of first and second sound is a general property of superfluids (not restricted to the non-relativistic limit). Experimentally, second sound was first observed in ^4He , see for instance Ref. [12].

1. Ideal hydrodynamics of a nonrelativistic, single fluid

The conservation equations for charge and (four-)momentum are in the relativistic form

$$\partial_\mu T^{\mu\nu} = 0, \quad \partial_\mu j^\mu = 0, \quad (26)$$

where

$$j^\mu = n v^\mu \quad (27)$$

is the current associated with a conserved charge and

$$T^{\mu\nu} = (\epsilon + P)v^\mu v^\nu - g^{\mu\nu}P \quad (28)$$

is the stress-energy tensor. Here, n , ϵ , and P are number density, energy density, and pressure in the rest frame of the fluid, $g^{\mu\nu} = \text{diag}(1, -1, -1, -1)$ is the metric tensor, and we have introduced the four-velocity

$$v^\mu = \gamma(1, \mathbf{v}), \quad \gamma = \sqrt{1 - \mathbf{v}^2}. \quad (29)$$

We shall come back to relativistic hydrodynamics later; in particular, we shall discuss current and stress-energy tensor within a field-theoretic model in Sec. III. For now, we shall use the conservation equations (26) to derive the non-relativistic form of these equations. In essence, this amounts to taking the small-velocity limit. First, we note that the definition (28) implies for the components of the stress-energy tensor

$$T^{00} = \frac{\epsilon + Pv^2}{1 - v^2}, \quad T^{0i} = T^{i0} = \frac{\epsilon + P}{1 - v^2} v_i, \quad T^{ij} = \frac{\epsilon + P}{1 - v^2} v_i v_j + \delta_{ij}P. \quad (30)$$

(Here, v_i are the components of the three-velocity \mathbf{v} , not of the four-velocity, which would contain the gamma factor; v^2 is the modulus of the three-velocity.) In particular, the stress-energy tensor is diagonal, and in the rest frame of the fluid, defined by $\mathbf{v} = 0$, i.e., $v^\mu = (1, 0, 0, 0)$, we have the simple form

$$T^{\mu\nu} = \begin{pmatrix} \epsilon & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix}. \quad (31)$$

Since T^{00} is the energy density and T^{ii} the pressure, this confirms that ϵ and P are energy density and pressure in the rest frame of the fluid. As an exercise, one can check, that the general stress-energy tensor (28) can be obtained via a Lorentz transformation from the stress-energy tensor in Eq. (31). In a single-fluid system with uniform fluid velocity it is possible to choose a global frame in which the three-velocity vanishes. This is not possible in a two-fluid system, which describes a superfluid and which we shall discuss below.

In order to take the non-relativistic limit, we have to introduce the non-relativistic energy density in the fluid rest frame by separating the rest energy,

$$\epsilon \rightarrow \rho\sqrt{1 - v^2} + \epsilon, \quad (32)$$

where $\rho = mn$ is the rest mass density in the frame where the fluid moves with velocity \mathbf{v} (remember that n is the 0-th component of the four-current j^μ , i.e., it is a frame-dependent quantity; the mass density in the fluid rest frame is $\rho\sqrt{1 - v^2}$). Neglecting terms of order v^4 , we can thus write

$$T^{00} \simeq \epsilon + (\epsilon + P)v^2 \rightarrow \rho + \epsilon + \left(\frac{\rho}{2} + \epsilon + P\right)v^2 \simeq \rho + \epsilon + \frac{\rho v^2}{2}, \quad (33)$$

where we have neglected terms of order v^4 and where, in the last step, we have used that the non-relativistic energy density and pressure are much smaller than the rest mass density. The 00-component of the non-relativistic stress-energy tensor is then obtained by dropping the rest mass density ρ ,

$$T_{\text{nr}}^{00} = \epsilon + \frac{\rho v^2}{2}. \quad (34)$$

We see that it contains the energy density in the fluid rest frame plus a kinetic term which has the usual non-relativistic form. (Non-relativistically, the mass density ρ is frame independent.) We proceed analogously for the other components. First, we write

$$T^{0i} \simeq (\epsilon + P)(1 + v^2)v_i \rightarrow \rho v_i + \left(\epsilon + \frac{\rho v^2}{2} + P\right)v_i, \quad (35)$$

where, in the $\mathcal{O}(v^3)$ terms, we have neglected $\epsilon + P \ll \rho$. Then, we define the *momentum density* by

$$g_i \equiv T_{\text{nr}}^{0i} = \rho v_i. \quad (36)$$

For the *energy flux* we drop the contribution from the rest mass energy density and define

$$q_i \equiv T_{\text{nr}}^{i0} = \left(\epsilon + \frac{\rho v^2}{2} + P \right) v_i. \quad (37)$$

As a consequence, in the nonrelativistic version, $T_{\text{nr}}^{0i} \neq T_{\text{nr}}^{i0}$. Finally,

$$T^{ij} \simeq (\epsilon + P)v_i v_j + \delta_{ij}P \rightarrow (\rho + \epsilon + P)v_i v_j + \delta_{ij}P \simeq \rho v_i v_j + \delta_{ij}P, \quad (38)$$

and we define the non-relativistic stress tensor

$$\Pi_{ij} \equiv T_{\text{nr}}^{ij} = \rho v_i v_j + \delta_{ij}P. \quad (39)$$

Besides the components of the stress-energy tensor, we also need the current which, from Eq. (27), we can write as $m\mathbf{j}^0 = mn = \rho$, $m\mathbf{j} \simeq \rho\mathbf{v}$.

Now we can formulate the conservation equations in the non-relativistic limit,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{g} = 0, \quad (40a)$$

$$\frac{\partial \varepsilon}{\partial t} + \nabla \cdot \mathbf{q} = 0, \quad (40b)$$

$$\frac{\partial g_i}{\partial t} + \partial_j \Pi_{ji} = 0, \quad (40c)$$

with the energy density

$$\varepsilon = \epsilon + \frac{\rho v^2}{2} \quad (41)$$

(remember that ϵ is the energy density in the rest frame of the fluid). The first equation is (m times) the current conservation $\partial_\mu j^\mu = 0$, the second equation is the $\nu = 0$ component of the stress-energy conservation $\partial_\mu T^{\mu\nu}$, and the third equation is the $\nu = i$ component of the stress-energy conservation. In summary, we repeat the definition of the non-relativistic quantities that appear in these equations,

$$\mathbf{g} = \rho\mathbf{v}, \quad \mathbf{q} = \left(\epsilon + \frac{\rho v^2}{2} + P \right) \mathbf{v}, \quad \Pi_{ij} = \rho v_i v_j + \delta_{ij}P. \quad (42)$$

Inserting these expressions into Eqs. (40), we can bring the hydrodynamic equations in the following form²

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho\mathbf{v}) = 0, \quad (43a)$$

$$\frac{\partial s}{\partial t} + \nabla \cdot (s\mathbf{v}) = 0, \quad (43b)$$

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{\nabla P}{\rho} \quad (\text{Euler equation}). \quad (43c)$$

² If we want to include dissipation, we need to add dissipative terms in the following way,

$$\mathbf{g} = \rho\mathbf{v}, \quad q_i = (\epsilon + P)v_i + v_j \delta \Pi_{ij} + Q_i, \quad \Pi_{ij} = \rho v_i v_j + \delta_{ij}P + \delta \Pi_{ij}.$$

where

$$\mathbf{Q} \equiv -\kappa \nabla T, \quad \delta \Pi_{ij} \equiv -\eta \left(\partial_i v_j - \partial_j v_i - \frac{2}{3} \delta_{ij} \nabla \cdot \mathbf{v} \right) - \zeta \delta_{ij} \nabla \cdot \mathbf{v},$$

with the thermal conductivity κ , the shear viscosity η , and the bulk viscosity ζ . In the presence of dissipation, the entropy current is no longer conserved, i.e., the right-hand side of Eq. (43b) is not zero. With dissipative terms, the Euler equation is known as the Navier-Stokes equation.

The third equation is straightforwardly obtained with the help of the continuity equation (43a). To derive the second equation – which is a continuity equation for the entropy current – start from Eq. (40b) and write

$$\begin{aligned} 0 &= \frac{\partial}{\partial t} \left(\epsilon + \frac{\rho v^2}{2} \right) + \nabla \cdot \left(\epsilon + \frac{\rho v^2}{2} + P \right) \mathbf{v} \\ &= \frac{\partial \epsilon}{\partial t} + \frac{v^2}{2} \frac{\partial \rho}{\partial t} + \rho \mathbf{v} \cdot \frac{\partial \mathbf{v}}{\partial t} + (\mu \rho + Ts) \nabla \cdot \mathbf{v} + \mathbf{v} \cdot \nabla \epsilon + \nabla \cdot \left(\frac{\rho v^2}{2} \mathbf{v} \right) + \mathbf{v} \cdot \nabla P, \end{aligned} \quad (44)$$

where we have used the relation

$$\epsilon + P = \mu \rho + Ts, \quad (45)$$

where s is the entropy density and μ the chemical potential³. Now we remember the thermodynamic relations

$$d\epsilon = \mu d\rho + T ds, \quad (46)$$

and

$$dP = \rho d\mu + s dT. \quad (47)$$

These two thermodynamic relations reflect the fact that ϵ and P are related via two Legendre transforms with respect to the pairs (T, s) and (μ, ρ) . We shall need Eq. (47) later. Here we make use of Eq. (46) which we insert into Eq. (44) to obtain

$$\begin{aligned} 0 &= \mu \frac{\partial \rho}{\partial t} + T \frac{\partial s}{\partial t} + (\mu \rho + Ts) \nabla \cdot \mathbf{v} + \mu \mathbf{v} \cdot \nabla \rho + T \mathbf{v} \cdot \nabla s + \underbrace{\frac{v^2}{2} \frac{\partial \rho}{\partial t} + \rho \mathbf{v} \cdot \frac{\partial \mathbf{v}}{\partial t} + \nabla \cdot \left(\frac{\rho v^2}{2} \mathbf{v} \right) + \mathbf{v} \cdot \nabla P}_{=0} \\ &= T \left[\frac{\partial s}{\partial t} + \nabla \cdot (s \mathbf{v}) \right], \end{aligned} \quad (48)$$

where we have used the continuity equation and have rewritten ∇P with the help of the Euler equation (43c). The result is the entropy conservation (43b). Note that this equation only holds in the absence of dissipation.

[\[End of 3rd lecture, Mar 18th, 2013.\]](#)

2. Two fluids

In a superfluid, we now have to take into account the two fluid components. To this end, we distinguish two reference frames. Imagine a superfluid flowing through a tube. Then, our first reference frame K is the frame where the tube is at rest and where superfluid and normal components have velocities \mathbf{v}_s and \mathbf{v}_n , respectively. The second reference frame K_0 is the one where the superfluid component is at rest, i.e., the tube moves with velocity $-\mathbf{v}_s$ and the superfluid and normal components move with velocities zero and $\mathbf{v}_n - \mathbf{v}_s$, respectively. We denote quantities in K without additional subscript and quantities in K_0 with subscript 0. Then, obviously,

$$\mathbf{g} = \rho_n \mathbf{v}_n + \rho_s \mathbf{v}_s, \quad \mathbf{g}_0 = \rho_n (\mathbf{v}_n - \mathbf{v}_s) \quad (49a)$$

$$\Rightarrow \quad \mathbf{g} = \rho \mathbf{v}_s + \mathbf{g}_0, \quad (49b)$$

where $\rho = \rho_n + \rho_s$ is the total mass density. The stress tensor in the two frames reads

$$\Pi_{ij} = \rho_n v_{ni} v_{nj} + \rho_s v_{si} v_{sj} + \delta_{ij} P, \quad \Pi_{0ij} = \rho_n (v_{ni} - v_{si})(v_{nj} - v_{sj}) + \delta_{ij} P \quad (50a)$$

$$\Rightarrow \quad \Pi_{ij} = \rho v_{si} v_{sj} + v_{si} g_{0j} + v_{sj} g_{0i} + \Pi_{0ij}. \quad (50b)$$

³ To be consistent with the nonrelativistic hydrodynamics literature, we work with the chemical potential per unit mass μ even though this may be a source of confusion since in Sec. III the same symbol denotes the chemical potential per unit charge. These two chemical potential have different units; in this section, μ has the same units as velocity squared (is dimensionless if the speed of light is set to one).

For completeness, although we shall not need this in the following, let us also write down the energy density and the energy density current in the two frames. We have

$$\varepsilon = \epsilon_n + \epsilon_s + \frac{\rho_n v_n^2}{2} + \frac{\rho_s v_s^2}{2}, \quad \varepsilon_0 = \epsilon_n + \epsilon_s + \frac{\rho_n (\mathbf{v}_n - \mathbf{v}_s)^2}{2} \quad (51a)$$

$$\Rightarrow \quad \varepsilon = \frac{\rho v_s^2}{2} + \mathbf{v}_s \cdot \mathbf{g}_0 + \epsilon_0, \quad (51b)$$

where

$$\epsilon_n = -P_n + \mu \rho_n + Ts, \quad \epsilon_s = -P_s + \mu \rho_s \quad (52)$$

are the energy densities of normal and superfluid components in their respective rest frames (analogously for the pressures P_n and P_s). Here we have used (if you wish, defined) that the entropy of the superfluid component is zero. And

$$\mathbf{q} = \left(\epsilon_n + P_n + \frac{\rho_n v_n^2}{2} \right) \mathbf{v}_n + \left(\epsilon_s + P_s + \frac{\rho_s v_s^2}{2} \right) \mathbf{v}_s, \quad \mathbf{q}_0 = \left(\epsilon_n + P_n + \frac{\rho_n (\mathbf{v}_n - \mathbf{v}_s)^2}{2} \right) (\mathbf{v}_n - \mathbf{v}_s) \quad (53a)$$

$$\Rightarrow \quad \mathbf{q} = (\varepsilon + P) \mathbf{v}_s + \left(\mathbf{v}_n \cdot \mathbf{v}_s - \frac{v_s^2}{2} \right) \mathbf{g}_0 + \mathbf{q}_0. \quad (53b)$$

We write the hydrodynamic equations in frame K for the superfluid in the following form,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{g} = 0, \quad (54a)$$

$$\frac{\partial s}{\partial t} + \nabla \cdot (s \mathbf{v}_n) = 0, \quad (54b)$$

$$\frac{\partial \mathbf{g}}{\partial t} + \mathbf{v}_s \nabla \cdot \mathbf{g} + (\mathbf{g} \cdot \nabla) \mathbf{v}_s + \mathbf{g}_0 \nabla \cdot \mathbf{v}_n + (\mathbf{v}_n \cdot \nabla) \mathbf{g}_0 + \nabla P = 0. \quad (54c)$$

The first two equations have the same form as written above for the normal fluid case. The only things to notice are that now the total density ρ and the total momentum density \mathbf{g} appear, and that the total entropy flux $s \mathbf{v}_n$ only receives a contribution from the normal component of the fluid. To derive Eq. (54c) from (40c) one first easily checks that the stress tensor from Eq. (50a) can be written as

$$\Pi_{ij} = v_{sj} g_i + v_{ni} g_{0j} + \delta_{ij} P. \quad (55)$$

(Although not manifest in this form, the stress tensor is of course still symmetric as one can check.) Inserting Eq. (55) into Eq. (40c) immediately yields Eq. (54c).

With some effort, one can derive

$$\frac{\partial \varepsilon}{\partial t} + \nabla \cdot \mathbf{q} = -\rho_s (\mathbf{v}_n - \mathbf{v}_s) \left[\nabla \mu + (\mathbf{v}_s \cdot \nabla) \mathbf{v}_s + \frac{\partial \mathbf{v}_s}{\partial t} \right] + \frac{(\mathbf{v}_n - \mathbf{v}_s)^2}{2} \left[\frac{\partial \rho_s}{\partial t} + \nabla \cdot (\rho_s \mathbf{v}_s) \right]. \quad (56)$$

Since the left-hand side of this equation is zero due to one of the hydrodynamic equations, we read off the following equation for the superfluid velocity,

$$(\mathbf{v}_s \cdot \nabla) \mathbf{v}_s + \frac{\partial \mathbf{v}_s}{\partial t} = -\nabla \mu, \quad (57)$$

where we have neglected the term quadratic in the relative velocities in Eq. (56). We shall need Eq. (57) below in the derivation of the sound velocities.

3. Sound velocities

To discuss sound waves in a superfluid, we first imagine both fluid components to be in rest and the system to be in equilibrium. A sound wave is a (small) oscillation in the thermodynamic quantities like entropy, pressure etc

and in the velocities. We thus imagine adding small deviations from equilibrium to the thermodynamic quantities like $s(\mathbf{x}, t) \rightarrow s_0 + \delta s(\mathbf{x}, t)$, $P(\mathbf{x}, t) \rightarrow P_0 + \delta P(\mathbf{x}, t)$ etc and small deviations (from zero) to the velocities, $\mathbf{v}_n(\mathbf{x}, t) \rightarrow \mathbf{v}_{n,0} + \delta \mathbf{v}_n(\mathbf{x}, t)$ and $\mathbf{v}_s(\mathbf{x}, t) \rightarrow \mathbf{v}_{s,0} + \delta \mathbf{v}_s(\mathbf{x}, t)$. (Here, the subscript 0 denotes thermodynamic equilibrium.) For simplicity we shall consider the situation where $\mathbf{v}_{n,0} = \mathbf{v}_{s,0} = 0$, i.e., we send sound waves through a system at rest, not to a system that flows. In a more complicated setting, one could compute the sound velocities as a function of the angle between the direction of the propagation of the wave and the direction of the relative flow between the two fluid components.

We are interested in small deviations from equilibrium, i.e., we neglect terms quadratic in the deviations, for instance

$$\nabla \cdot \mathbf{g} \simeq \rho_n \nabla \cdot \mathbf{v}_n + \rho_s \nabla \cdot \mathbf{v}_s = \rho_{n,0} \nabla \cdot \delta \mathbf{v}_n + \rho_{s,0} \nabla \cdot \delta \mathbf{v}_s. \quad (58)$$

The linearised hydrodynamic equations (54) thus become

$$\frac{\partial \rho}{\partial t} + \rho_n \nabla \cdot \mathbf{v}_n + \rho_s \nabla \cdot \mathbf{v}_s \simeq 0, \quad (59a)$$

$$\frac{\partial s}{\partial t} + s \nabla \cdot \mathbf{v}_n \simeq 0, \quad (59b)$$

$$\rho_n \frac{\partial \mathbf{v}_n}{\partial t} + \rho_s \frac{\partial \mathbf{v}_s}{\partial t} + \nabla P \simeq 0, \quad (59c)$$

and Eq. (57) simplifies to

$$\frac{\partial \mathbf{v}_s}{\partial t} + \nabla \mu \simeq 0. \quad (60)$$

A usual sound wave is a density oscillation, Therefore, we now seek to extract a wave equation for ρ from the hydrodynamic equations. Due to the presence of two fluid components, we see that we can also extract a wave equation for the entropy. The fact that there are two such wave equations implies that there are two sound waves, corresponding to density and entropy fluctuations.

These equations are

$$\frac{\partial^2 \rho}{\partial t^2} = \Delta P, \quad (61a)$$

$$\frac{\partial^2 S}{\partial t^2} = \frac{S^2 \rho_s}{\rho_n} \Delta T, \quad (61b)$$

where S is the entropy per unit mass, such that $s = \rho S$. Eq. (61a) is immediately obtained from Eqs. (59a) and (59c). Eq. (61b) requires a bit more work. The derivation is as follows. From the thermodynamic relation (47) we obtain $\nabla P = \rho \nabla \mu + s \nabla T$. Now inserting ∇P from Eq. (59c) and $\nabla \mu$ from Eq. (60) and taking the divergence on both sides of the resulting equation yields

$$\rho_n \frac{\partial}{\partial t} \nabla \cdot (\mathbf{v}_n - \mathbf{v}_s) = -s \Delta T. \quad (62)$$

To replace the divergence on the right-hand side of this equation we observe

$$\frac{\partial S}{\partial t} = \frac{1}{\rho} \frac{\partial s}{\partial t} - \frac{S}{\rho} \frac{\partial \rho}{\partial t} = -S \nabla \cdot \mathbf{v}_n + \frac{S}{\rho} (\rho_n \nabla \cdot \mathbf{v}_n + \rho_s \nabla \cdot \mathbf{v}_s) = -\frac{S \rho_s}{\rho} \nabla \cdot (\mathbf{v}_n - \mathbf{v}_s), \quad (63)$$

where, in the second step, we have used Eqs. (59a) and (59b). Now, inserting this result into Eq. (62) and again using the linear approximation yields the second wave equation (61b).

To solve the wave equations we take T and P as independent variables and consider small deviations from thermodynamic equilibrium, such that we can write

$$\delta \rho = \frac{\partial \rho}{\partial P} \delta P + \frac{\partial \rho}{\partial T} \delta T, \quad \delta S = \frac{\partial S}{\partial P} \delta P + \frac{\partial S}{\partial T} \delta T, \quad (64)$$

where all derivatives are evaluated in equilibrium. Inserting this into Eqs. (61a) and (61b) yields

$$\frac{\partial \rho}{\partial P} \frac{\partial^2 \delta P}{\partial t^2} + \frac{\partial \rho}{\partial T} \frac{\partial^2 \delta T}{\partial t^2} = \Delta(\delta P), \quad (65a)$$

$$\frac{\partial S}{\partial P} \frac{\partial^2 \delta P}{\partial t^2} + \frac{\partial S}{\partial T} \frac{\partial^2 \delta T}{\partial t^2} = \frac{S^2 \rho_s}{\rho_n} \Delta(\delta T). \quad (65b)$$

Now we make the ansatz $\delta P(\mathbf{x}, t) = \delta P_0 e^{-i(\omega t - kx)}$, $\delta T(\mathbf{x}, t) = \delta T_0 e^{-i(\omega t - kx)}$ with amplitudes δP_0 , δT_0 that are constant in time and space (without loss of generality, we have chosen the x -direction for the direction of the wave propagation). Then we define

$$u = \frac{\omega}{k}, \quad (66)$$

such that the wave equations become

$$\left(u^2 \frac{\partial \rho}{\partial P} - 1\right) \delta P_0 + u^2 \frac{\partial \rho}{\partial T} \delta T_0 = 0, \quad (67a)$$

$$u^2 \frac{\partial S}{\partial P} \delta P_0 + \left(u^2 \frac{\partial S}{\partial T} - \frac{S^2 \rho_s}{\rho_n}\right) \delta T_0 = 0. \quad (67b)$$

For this system of equations to have nontrivial solutions, we must require the determinant to vanish,

$$u^4 \frac{\partial(S, \rho)}{\partial(T, P)} - u^2 \left(\frac{\partial \rho}{\partial P} \frac{S^2 \rho_s}{\rho_n} + \frac{\partial S}{\partial T} \right) + \frac{S^2 \rho_s}{\rho_n} = 0, \quad (68)$$

where

$$\frac{\partial(S, \rho)}{\partial(T, P)} = \det \begin{pmatrix} \frac{\partial S}{\partial T} & \frac{\partial S}{\partial P} \\ \frac{\partial \rho}{\partial T} & \frac{\partial \rho}{\partial P} \end{pmatrix} \quad (69)$$

is the Jacobian for the function $(S(T, P), \rho(T, P))$. Now we divide the whole equation by the Jacobian and use

$$\frac{\partial \rho}{\partial P} \Big|_T \left[\frac{\partial(S, \rho)}{\partial(T, P)} \right]^{-1} = \frac{\partial T}{\partial S} \Big|_\rho, \quad \frac{\partial S}{\partial T} \Big|_P \left[\frac{\partial(S, \rho)}{\partial(T, P)} \right]^{-1} = \frac{\partial P}{\partial \rho} \Big|_S. \quad (70)$$

Then, with

$$T \frac{\partial S}{\partial T} \Big|_\rho = \frac{c_V}{\rho}, \quad (71)$$

(since c_V is the specific heat per unit volume, c_V/ρ is the specific heat per unit mass) we have

$$u^4 - u^2 \left(\frac{\rho S^2 T \rho_s}{c_V \rho_n} + \frac{\partial P}{\partial \rho} \Big|_S \right) + \left[\frac{\partial(S, \rho)}{\partial(T, P)} \right]^{-1} \frac{S^2 \rho_s}{\rho_n} = 0. \quad (72)$$

Now we rewrite the term constant in u with the help of

$$\frac{\partial S}{\partial T} \Big|_\rho \left[\frac{\partial(S, \rho)}{\partial(P, T)} \right]^{-1} = \frac{\partial P}{\partial \rho} \Big|_T \quad \Rightarrow \quad \left[\frac{\partial(S, \rho)}{\partial(P, T)} \right]^{-1} = \frac{\rho T}{c_V} \frac{\partial P}{\partial \rho} \Big|_T. \quad (73)$$

and approximate

$$\frac{\partial P}{\partial \rho} \Big|_T \simeq \frac{\partial P}{\partial \rho} \Big|_S, \quad (74)$$

which is equivalent to approximating the specific heat at constant pressure c_P by the specific heat at constant volume c_V . This yields

$$u^4 - u^2 \left(\frac{\rho S^2 T \rho_s}{c_V \rho_n} + \frac{\partial P}{\partial \rho} \Big|_S \right) + \frac{\partial P}{\partial \rho} \Big|_S \frac{\rho S^2 T \rho_s}{c_V \rho_n} = 0. \quad (75)$$

This equation has the simple structure $u^4 - u^2(a+b) + ab = 0$ with solutions $u^2 = a, b$. Consequently, the two solutions are

$$u_1 = \sqrt{\frac{\partial P}{\partial \rho}}, \quad u_2 = \sqrt{\frac{\rho S^2 T \rho_s}{c_V \rho_n}} = \sqrt{\frac{s^2 T \rho_s}{\rho c_V \rho_n}}. \quad (76)$$

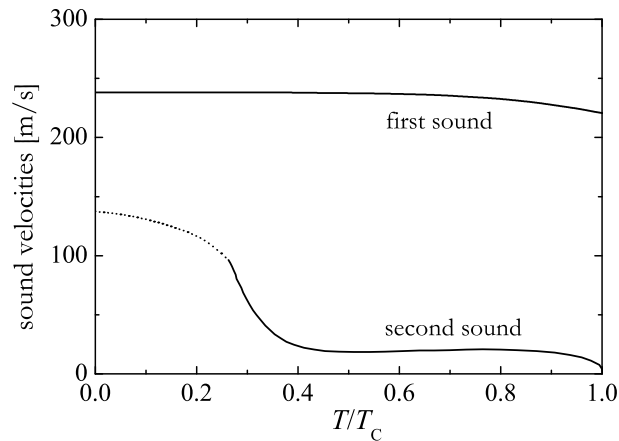


FIG. 4: Sound velocities in ^4He , figure taken from Ref. [13].

These are the velocities of first and second sound. At small temperatures, these velocities are dominated by the phonon contributions. We shall see in the next section that the speed of first sound is equal to the slope of the linear dispersion of the Goldstone mode,

$$\frac{\partial P}{\partial \rho} \simeq c^2. \quad (77)$$

Moreover, with $c_{V,\text{ph}} = 3s_{\text{ph}}$, $\rho_{n,\text{ph}} = s_{\text{ph}}T/c^2$ (see Eqs. (10) and (22), respectively) and approximating $\rho_s \simeq \rho$, we find the zero-temperature limits for the sound velocities,

$$u_1(T=0) = c, \quad u_2(T=0) = \frac{c}{\sqrt{3}}. \quad (78)$$

First sound corresponds to a usual sound wave, where the total density oscillates, i.e., it is a pressure wave with vanishing relative velocity between superfluid and normal components. Second sound, however, is unusual in the sense that the existence of two fluid components, one of them with vanishing entropy, is crucial. It corresponds to temperature oscillations where the total density remains constant at any small region in space, while the relative density of superfluid and normal components varies. See Fig. 4 for a plot of the sound velocities up to the critical temperature. For a nice popular article about second sound and its significance for developments in the theory of superfluid ^4He see Ref. [12].

[End of 4th lecture, Apr 8th, 2013.]

III. BASIC INGREDIENTS OF A SUPERFLUID IN A FIELD-THEORETIC MODEL

The theoretical description of superfluid helium used in the previous section was mostly phenomenological, we have not started from a microscopic theory. We have taken it for granted that there is a condensate and phonon and roton excitations. Given these assumptions, together with the general equations of two-fluid hydrodynamics, we have derived phenomenological properties of a (non-relativistic) superfluid such as the two sound velocities.

In this section we want to discuss superfluidity on a more microscopic level, by starting from a (relativistic) φ^4 model. In this discussion, we shall gain a more fundamental understanding of the characteristic properties of a superfluid. The following general concepts play a central role:

- *$U(1)$ symmetry*: this is the simplest continuous symmetry; in field-theoretical language, its “spontaneous” breakdown is, in all systems we consider, a necessary condition for the phase transition from a non-superfluid to a superfluid state; later we shall also discuss spontaneous breaking of more complicated symmetry groups (e.g., in ^3He , quark matter); in the “standard” superfluid (superfluid ^4He), it is precisely a $U(1)$ that is broken.
- *conserved charge*: a conserved charge (as a consequence of the $U(1)$ symmetry) is essential in all superfluid systems we consider; this charge is carried by the superflow.

- *Bose-Einstein condensation*: essential for all bosonic superfluids we shall discuss; in our context, Bose-Einstein condensation is just another way of saying the the $U(1)$ symmetry is spontaneously broken; the word “condensation” reminds us that there is a phase transition; when we later discuss fermionic superfluids/superconductors Bose-Einstein condensation is also – at least indirectly – involved, namely through condensation of Cooper pairs.
- *spontaneous symmetry breaking*: the “condensate” in a superfluid is not invariant under (part of) the original symmetry of the Lagrangian of the system; this is called spontaneous symmetry breaking.
- *Goldstone mode*: if the spontaneously broken symmetry is global, a massless mode arises for all temperatures below the critical temperature; we shall compute the dispersion of this Goldstone mode explicitly in the φ^4 model; in a superconductor, where a *local* symmetry is spontaneously broken, there is no Goldstone mode, but a Meissner mass for the gauge boson.
- *symmetry restoration, critical temperature*: all superfluids/superconductors we discuss exist at small temperatures; in absolute values, “small” can mean very different things, as mentioned in the introduction; but, superfluidity always ceases to exist above a certain critical temperature where the condensate has melted or, in other words, the ground state is symmetric under the full symmetry of the Lagrangian; in this section, we shall estimate the critical temperature for the φ^4 model.

A. Lagrangian and conservation equations

We start from the following Lagrangian \mathcal{L} for a complex scalar field $\varphi(x)$ depending on space-time, $x \equiv (x_0, \mathbf{x})$,

$$\mathcal{L} = \partial_\mu \varphi^* \partial^\mu \varphi - m^2 |\varphi|^2 - \lambda |\varphi|^4. \quad (79)$$

The Lagrangian describes spin-0 bosons with mass m which interact repulsively with a coupling constant $\lambda > 0$.

We first observe that \mathcal{L} is invariant under $U(1)$ rotations of the field,

$$\varphi \rightarrow e^{-i\alpha} \varphi, \quad (80)$$

with a constant $\alpha \in \mathbb{R}$. Since α is constant one talks about a *global* transformation or a *global* symmetry.

If we want to describe Bose-Einstein condensation, we need to separate the condensate from the fluctuations. Remember from the textbook treatment of (non-relativistic) Bose-Einstein condensation that the condensate describes a macroscopic occupation of the bosons in the ground state of the system, usually the zero-momentum state. Analogously, in field theory, we may Fourier decompose the field and separate the state with zero four-momentum $k = (k_0, \mathbf{k})$. More generally, if we want to allow for a nonzero superflow (giving rise to a superfluid velocity \mathbf{v}_s , as discussed in the previous section), condensation takes place in a state with nonzero four-momentum, say p . We thus decompose

$$\varphi(x) \rightarrow \phi(x) + \varphi(x), \quad (81)$$

where $\phi(x) = \varphi(p) e^{ip \cdot x}$ is the condensate and $\varphi(x) = \sum_{k \neq p} e^{ik \cdot x} \varphi(k)$ are (small) fluctuations. We write the (complex) condensate in terms of its modulus and its phase,

$$\phi(x) = \frac{\rho(x)}{\sqrt{2}} e^{i\psi(x)}. \quad (82)$$

We shall see later that the fluctuations are need to compute the dispersion of the Goldstone mode. As a first step, we shall neglect the fluctuations. In this case, the Lagrangian becomes

$$\mathcal{L}^{(0)} = \frac{1}{2} \partial_\mu \rho \partial^\mu \rho + \frac{\rho^2}{2} (\partial_\mu \psi \partial^\mu \psi - m^2) - \frac{\lambda}{4} \rho^4. \quad (83)$$

Next we write down the equations of motion. From the general form of the Euler-Lagrange equations

$$0 = \frac{\partial \mathcal{L}}{\partial \rho} - \partial_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu \rho)}, \quad (84a)$$

$$0 = \partial_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu \psi)}, \quad (84b)$$

(notice that ψ only appears through its space-time derivative), we compute

$$\square\rho = \rho(\sigma^2 - m^2 - \lambda\rho^2), \quad (85a)$$

$$\partial_\mu(\rho^2\partial^\mu\psi) = 0, \quad (85b)$$

where we have abbreviated

$$\sigma \equiv \sqrt{\partial_\mu\psi\partial^\mu\psi}. \quad (86)$$

The second equation of motion (85b) is nothing but the continuity conservation for the conserved current. We know from Noether's theorem that a system with a continuous symmetry has a conserved current. (It is therefore important that we consider a complex field; the same Lagrangian for a real scalar field has only a discrete \mathbb{Z}_2 symmetry $\varphi \rightarrow -\varphi$ and thus no conserved current.) In general,

Noether's Theorem: If the Lagrangian is invariant under transformations of a continuous global symmetry group there exists a conserved current (= a continuity equation) and vice versa.

Here we have⁴

$$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)} = \rho^2\partial^\mu\psi, \quad (87)$$

such that Eq. (85b) is

$$\partial_\mu j^\mu = 0. \quad (88)$$

This is one of the hydrodynamic equations introduced in Eqs. (26). The second one, namely the conservation of the stress-energy tensor $T^{\mu\nu}$ can be derived as follows. We use the field-theoretic definition

$$T^{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\mathcal{L})}{\delta g_{\mu\nu}} = 2 \frac{\delta\mathcal{L}}{\delta g_{\mu\nu}} - g^{\mu\nu}\mathcal{L}, \quad (89)$$

where $\sqrt{-g} \equiv \sqrt{-\det g^{\mu\nu}}$. This definition ensures that the stress-energy tensor is symmetric⁵. It is applicable to general (space-time dependent) metric tensors $g^{\mu\nu}$; here we are only interested in flat space-time, $g^{\mu\nu} = (1, -1, -1, -1)$. We obtain

$$T^{\mu\nu} = \partial^\mu\rho\partial^\nu\rho + \rho^2\partial^\mu\psi\partial^\nu\psi - g^{\mu\nu}\mathcal{L}. \quad (90)$$

Consequently,

$$\partial_\mu T^{\mu\nu} = \partial_\mu(\partial^\mu\rho\partial^\nu\rho + \rho^2\partial^\mu\psi\partial^\nu\psi) - \partial^\nu\mathcal{L}. \quad (91)$$

We compute

$$\begin{aligned} \partial^\nu\mathcal{L} &= \frac{\partial\mathcal{L}}{\partial\rho}\partial^\nu\rho + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\rho)}\partial^\nu\partial_\mu\rho + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\partial^\nu\partial_\mu\psi \\ &= \partial^\nu\rho\left(\frac{\partial\mathcal{L}}{\partial\rho} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\rho)}\right) - \partial^\nu\psi\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)} + \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\rho)}\partial^\nu\rho + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\partial^\nu\psi\right) \\ &= \partial_\mu(\partial^\mu\rho\partial^\nu\rho + \rho^2\partial^\mu\psi\partial^\nu\psi), \end{aligned} \quad (92)$$

where we have used the equations of motion. We thus have

$$\partial_\mu T^{\mu\nu} = 0. \quad (93)$$

⁴ In terms of the complex field φ , we have $j^\mu = i(\varphi\partial^\mu\varphi^* - \varphi^*\partial^\mu\varphi)$.

⁵ In contrast, the *canonical* stress-energy tensor, defined by

$$T^{\mu\nu} = \partial^\nu\frac{\partial\mathcal{L}}{\partial\partial_\mu\psi} - g^{\mu\nu}\mathcal{L},$$

is conserved too, but in general not symmetric.

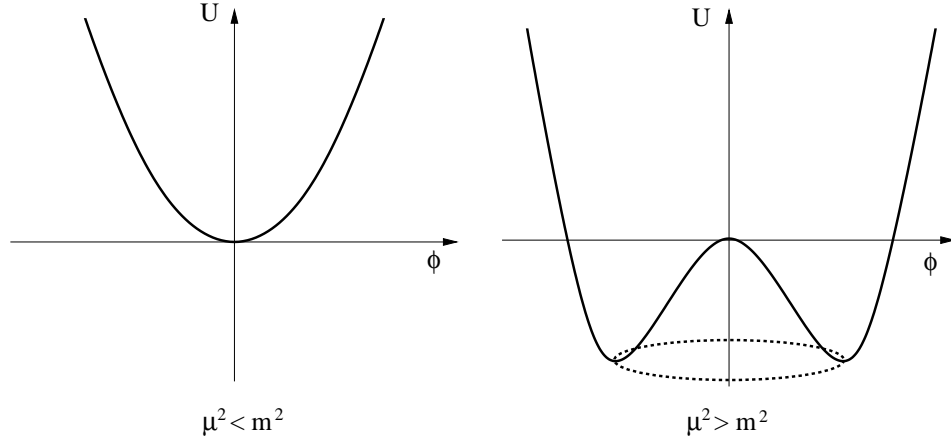


FIG. 5: Illustration of the zero-temperature tree-level potential $U(\phi)$ for $\mu^2 < m^2$ (left) and $\mu^2 > m^2$ (right). In the latter case, the order parameter acquires a nonzero value at a fixed, but arbitrary value on the bottom circle of the potential, thus breaking the $U(1)$ symmetry “spontaneously”. As we shall see, $\sigma = \sqrt{\partial_\mu \psi \partial^\mu \psi}$ plays the role of the chemical potential μ .

B. Spontaneous symmetry breaking

We now assume that ρ and $\partial^\mu \psi$ are constant in space and time. As a consequence, the current also becomes constant in this case and the hydrodynamics appear to be quite boring in this case. However, we shall need this simplification in order to discuss the basic concepts of spontaneous symmetry breaking. In this case, the Lagrangian becomes

$$\mathcal{L}^{(0)} = -U, \quad U = \frac{\rho^2}{2}(\sigma^2 - m^2) + \frac{\lambda}{4}\rho^4, \quad (94)$$

where U is the (tree-level) potential. We see that the second equation of motion (85b) is now trivially fulfilled, and the first one (85a) gives the extremal points of the potential,

$$0 = \frac{\partial U}{\partial \rho} = \rho(\sigma^2 - m^2 - \lambda\rho^2), \quad (95)$$

which has the solutions

$$\rho = 0, \quad \rho^2 = \frac{\sigma^2 - m^2}{\lambda}. \quad (96)$$

Since $\rho \in \mathbb{R}$ we see that we need $\sigma^2 > m^2$ for the nontrivial solution to exist. We expect that Bose-Einstein condensation occurs when the chemical potential is larger than the mass. This suggests that σ plays the role of a chemical potential. This can also be seen from the following argument. Usually, a chemical potential is added to the Lagrangian as if it were the temporal component of a gauge field (see for instance Ref. [1]), i.e.,

$$\mathcal{L} = |(\partial_0 - i\mu)\varphi|^2 - |\nabla\psi|^2 - m^2|\varphi|^2 - \lambda|\varphi|^4. \quad (97)$$

It is easy to check that $|(\partial_0 - i\mu)\varphi|^2$ with a constant phase of φ is identical to $\partial_0\varphi\partial_0\varphi^*$ with a phase $-i\mu t$. Therefore, the chemical potential can be included into the phase of the condensate, $\psi = -\mu t$, such that $\partial_0\psi$ appears to be the chemical potential. But, we have just noticed that $\sigma = \sqrt{\partial_\mu\psi\partial^\mu\psi}$ plays the role of a chemical potential, and σ also involves the spatial components of the four-gradient of the phase ψ . We shall discuss in detail below that σ is the chemical potential in the rest frame of the superfluid, while $\partial_0\psi$ is the chemical potential in the frame where the superfluid moves with a velocity determined by $\nabla\psi$. We illustrate the potential U in Fig. 5. For chemical potentials $|\mu| < m$, the minimum is at $\phi = 0$, i.e., there is no condensation. For condensation, one needs a negative coefficient in front of the ϕ^2 term, i.e., the modulus of the chemical potential must be larger than the mass⁶, $|\mu| > m$. In this case,

⁶ In the textbook discussion of a nonrelativistic, noninteracting Bose gas the chemical potential is always non-positive, and Bose-Einstein condensation (at a given, supercritical density) implies $\mu \rightarrow 0$. (More precisely, this chemical potential is a shifted chemical potential

the potential has a “Mexican hat” or “bottom of a wine bottle” shape (since we consider a repulsive interaction for which $\lambda > 0$ the potential is bounded from below, otherwise the system would be unstable). The $U(1)$ symmetry of the Lagrangian is reflected in the rotationally symmetric wine bottle potential. The nontrivial minimum, at a given angle ψ of the condensate, is not invariant under $U(1)$ (a $U(1)$ transformation would rotate the condensate along the bottom of the wine bottle). However, all possible minima are degenerate – this is obviously another consequence of the $U(1)$ symmetry of the potential. This mechanism, where the Lagrangian has a symmetry which is not respected by the ground state, is called *spontaneous symmetry breaking*. The object that breaks the symmetry and which is zero in the symmetric phase (here the condensate ϕ) is called the *order parameter*.

C. Superfluid velocity and thermodynamics of the superfluid

We have seen that the solution of the equations of motion (96) does not fix the value of σ , i.e., we are free to choose the four-gradient of the phase of the condensate. What is the physical meaning of this phase? In order to answer this question, we discuss the current and stress-energy tensor which we write as

$$j^\mu = n v^\mu, \quad (98a)$$

$$T^{\mu\nu} = (\epsilon + P) v^\mu v^\nu - g^{\mu\nu} P. \quad (98b)$$

(We have already seen this form of the stress-energy tensor in Sec. II D 1.) Here, n , ϵ , and P are charge density, energy density, and pressure in the rest frame of the superfluid. For the superfluid four-velocity v^μ we have by definition $v^\mu v_\mu = 1$, and as for any four-velocity we can thus write

$$v^\mu = \gamma(1, \mathbf{v}_s), \quad \gamma = \frac{1}{\sqrt{1 - \mathbf{v}_s^2}}. \quad (99)$$

Contracting Eq. (98a) with v_μ and j_μ , the two resulting equations yield

$$n = \sqrt{j_\mu j^\mu} = \rho^2 \sigma, \quad (100)$$

and thus

$$v^\mu = \frac{j^\mu}{n} = \frac{\partial^\mu \psi}{\sigma}. \quad (101)$$

This result is important for the interpretation of the phase of the condensate: it shows that the gradient of that phase, divided by σ , is nothing but the four-velocity of the superfluid. As a consequence, the three-velocity is⁷

$$\mathbf{v}_s = -\frac{\nabla \psi}{\partial_0 \psi}. \quad (103)$$

[The minus sign appears because the 3-velocity \mathbf{v}_s corresponds to the spatial components of the contravariant 4-vector v^μ , while the operator ∇ corresponds to the spatial components of the covariant 4-vector ∂_μ , i.e., $\partial^\mu = (\partial_t, -\nabla)$.]

As a result of these “translations” of the microscopic, field-theoretic objects into physical quantities we learn that the phase of the condensate plays a very interesting role; it determines the chemical potential as well as the superflow.

$\mu' = \mu + m$, with μ the chemical potential used in the relativistic context.) The relativistic version thereof is $\mu = m$ for a condensate with positive charge, and $\mu = -m$ for a condensate with negative charge. The fact that $|\mu|$ is allowed to become larger than m is due to the interactions.

⁷ In the non-relativistic context, the superfluid velocity is related to the phase of the condensate by the analogous relation

$$\mathbf{v}_s = -\frac{\nabla \psi}{m}$$

with m being the mass of the constituent particles of the condensate (for instance the mass of the helium atoms in superfluid ^4He). This relation shows that the superfluid flow is curl-free,

$$\nabla \times \mathbf{v}_s = 0. \quad (102)$$

This property is responsible for the formation of vortices in a rotating superfluid since only the vortices (in whose center the condensate is zero) are able to carry an externally imposed angular momentum.

These values are not fixed by the equations of motion, they are rather boundary conditions. Take for instance $\partial_0\psi$: we are free to choose how many times per unit time the condensate rotates around the $U(1)$ circle (the bottom of the Mexican hat). By this choice we fix μ . (Multiple) rotations around the $U(1)$ circle are called topological modes because closed paths on the circle can be topologically classified by their winding number; a closed path with winding number n cannot be continuously deformed into a closed path with winding number $m \neq n$. These modes are different from small (harmonic) oscillations of the phase and the modulus of the condensate, which exist on top of the topological modes and which determine the collective excitations of the system such as the Goldstone mode, see next subsection. Analogously to the winding per time, we are also free to choose how many times per unit length the phase winds around. This determines the superflow or, more precisely, the three-velocity \mathbf{v}_s . Consequently, one can picture a superflow microscopically as a spiral whose axis points in the direction of the superflow and whose windings are in the internal $U(1)$ space. In this picture, denser windings correspond to larger flow velocities.

As we have discussed above, the time derivative of the phase corresponds to the temporal component of a gauge field, and we could have started with a Lagrangian containing this field with the same effect. The same holds for the spatial components of course. It is left as an exercise to start from a Lagrangian with a covariant derivative $D^\mu = \partial^\mu - iA^\mu$ and show that this Lagrangian leads to the same results with $\partial^\mu\psi$ replaced by A^μ .

After determining n in terms of the microscopic parameters, we may also compute ϵ and P . To this end, we first notice that they can be obtained by the following contractions from the stress-energy tensor,

$$\epsilon = v_\mu v_\nu T^{\mu\nu}, \quad P = -\frac{1}{3}(g_{\mu\nu} - v_\mu v_\nu)T^{\mu\nu}, \quad (104)$$

where we have used $g_{\mu\nu}g^{\mu\nu} = g_\mu{}^\mu = 4$. Using the microscopic result for $T^{\mu\nu}$ from Eq. (90) (with a constant ρ), this yields

$$\epsilon = \rho^2 \sigma^2 - \mathcal{L} = \sigma n - \mathcal{L}, \quad P = \mathcal{L}. \quad (105)$$

Now, with the thermodynamic relation $\epsilon + P = \mu n$, we confirm that σ can be interpreted as the chemical potential measured in the superfluid rest frame, as mentioned above,

$$\mu = \sigma. \quad (106)$$

With the definition of the three-velocity (103) we can write $\mu = \partial_0\psi\sqrt{1-\mathbf{v}_s^2}$ which suggests that $\partial_0\psi$ is the chemical potential in the frame in which the superfluid has velocity \mathbf{v}_s .

Finally, we compute energy density and pressure explicitly,

$$\epsilon = \frac{(3\sigma^2 + m^2)(\sigma^2 - m^2)}{4\lambda}, \quad P = \frac{(\sigma^2 - m^2)^2}{4\lambda}. \quad (107)$$

We can write

$$\epsilon = 3P + \frac{2m^2\sqrt{P}}{\sqrt{\lambda}}, \quad (108)$$

which implies

$$\frac{\partial\epsilon}{\partial P} = \frac{3\sigma^2 - m^2}{\sigma^2 - m^2}. \quad (109)$$

As we shall see in the next subsection, and as we have already assumed in Sec. IID 3, the inverse of this expression is the square of the slope of the dispersion of the Goldstone mode.

[End of 5th lecture, Apr 15th, 2013.]

D. Goldstone mode

We are now interested in the excitations of the Bose-condensed system. To this end, we need to include the fluctuations which we have neglected in the previous subsection. It is convenient to introduce the transformed fluctuation field $\varphi'(x)$ via

$$\varphi'(x) = \varphi(x)e^{-i\psi(x)}, \quad (110)$$

where $\varphi(x)$ is the fluctuation field from Eq. (81). We also introduce real and imaginary part of the fluctuations,

$$\varphi'(x) = \frac{1}{\sqrt{2}}[\varphi'_1(x) + i\varphi'_2(x)]. \quad (111)$$

With these notations, the Lagrangian (79) becomes

$$\mathcal{L} = \mathcal{L}^{(0)} + \mathcal{L}^{(1)} + \mathcal{L}^{(2)} + \mathcal{L}^{(3)} + \mathcal{L}^{(4)}, \quad (112)$$

with $\mathcal{L}^{(0)}$ from Eq. (83) and the fluctuation terms, listed by their order in the fluctuation from linear to quartic,

$$\mathcal{L}^{(1)} = \varphi'_1 \left[\rho(\partial_\mu \psi \partial^\mu \psi - m^2 - \lambda \rho^2) - \square \rho \right] - \frac{\varphi'_2}{\rho} \partial_\mu (\rho^2 \partial^\mu \psi) + \partial_\mu (\varphi'_1 \partial^\mu \rho + \varphi'_2 \rho \partial^\mu \psi), \quad (113a)$$

$$\begin{aligned} \mathcal{L}^{(2)} = & \frac{1}{2} \left[\partial_\mu \varphi'_1 \partial^\mu \varphi'_1 + \partial_\mu \varphi'_2 \partial^\mu \varphi'_2 + (\varphi_1'^2 + \varphi_2'^2)(\partial_\mu \psi \partial^\mu \psi - m^2) \right. \\ & \left. + 2\partial_\mu \psi (\varphi'_1 \partial^\mu \varphi'_2 - \varphi'_2 \partial^\mu \varphi'_1) - \lambda \rho^2 (3\varphi_1'^2 + \varphi_2'^2) \right], \end{aligned} \quad (113b)$$

$$\mathcal{L}^{(3)} = -\lambda \rho \varphi'_1 (\varphi_1'^2 + \varphi_2'^2), \quad (113c)$$

$$\mathcal{L}^{(4)} = -\frac{\lambda}{4} (\varphi_1'^2 + \varphi_2'^2)^2. \quad (113d)$$

Here we are only interested in the terms quadratic in the fluctuations. They define the propagator whose poles determine the excitation energies we are interested in. We shall use the formalism of thermal field theory and introduce the Fourier transforms of the fields as

$$\varphi'_i(x) = \frac{1}{\sqrt{TV}} \sum_k e^{-ik \cdot x} \varphi'_i(k), \quad (114)$$

where T is the temperature, V the volume, and $\varphi'_i(k)$ the (dimensionless) Fourier transform of $\varphi'_i(x)$. The scalar product in the exponential is formally taken with the Minkowski metric $g^{\mu\nu} = (1, -1, -1, -1)$, i.e., $k \cdot x = k_0 x_0 - \mathbf{k} \cdot \mathbf{x}$. However, this is merely a convenient convention. Since we work at finite temperature, we have $x_0 = -i\tau$ with “imaginary time”⁸ $\tau \in [0, 1/T]$ with the temperature T and $k_0 = -i\omega_n$ with the bosonic Matsubara frequencies $\omega_n = 2\pi nT$, $n \in \mathbb{Z}$. Hence, $k \cdot x = -(\omega_n \tau + \mathbf{k} \cdot \mathbf{x})$ essentially is a Euclidean scalar product.

Using the Fourier transform (114), one can now compute the contribution to the action from the quadratic terms

$$S^{(2)} = \int_0^{1/T} d\tau \int d^3x \mathcal{L}^{(2)} = -\frac{1}{2} \sum_k [\varphi'_1(-k), \varphi'_2(-k)] \frac{D_0^{-1}(k)}{T^2} \begin{pmatrix} \varphi'_1(k) \\ \varphi'_2(k) \end{pmatrix}. \quad (115)$$

where we have denoted

$$D_0^{-1}(k) = \begin{pmatrix} -k^2 + m^2 + 3\lambda \rho^2 - \sigma^2 & -2ik_\mu \partial^\mu \psi \\ 2ik_\mu \partial^\mu \psi & -k^2 + m^2 + \lambda \rho^2 - \sigma^2 \end{pmatrix}. \quad (116)$$

(The subscript 0 indicates that we have so far neglected the cubic and quartic terms; note however that D_0 does know about the interaction because of the condensate which appears together with the coupling constant λ .)

Now we can compute the thermodynamic potential which is defined as

$$\Omega = -T \ln Z, \quad (117)$$

with the partition function

$$Z = N \int \mathcal{D}\varphi'_1 \mathcal{D}\varphi'_2 e^S, \quad (118)$$

⁸ Remember that this imaginary time arises in thermal field theory due to the formal equivalence between the statistical partition function $Z = \text{Tr} e^{-\beta \hat{H}}$ and a sum over transition amplitudes with identical initial and final states, $\langle \varphi | e^{-i\hat{H}t} | \varphi \rangle$.

where N is an (infinite) constant which is irrelevant since we are interested in $\ln Z$ and are free to renormalize the thermodynamic potential by any constant value. Inserting Eq. (115) into Eq. (118), performing the functional integral, and adding the result from the tree-level potential $U = -\mathcal{L}^{(0)}$ yields

$$\frac{\Omega}{V} = U + \frac{1}{2} \frac{T}{V} \text{Tr} \ln \frac{D_0^{-1}(k)}{T^2}, \quad (119)$$

where the trace is taken over the internal 2×2 space and over momentum space. For the explicit evaluation of this expression we assume $\nabla\psi = 0$, i.e., we do not include a superflow. The reason is that we need to compute the zeros of the determinant of the inverse propagator D_0^{-1} (which are the poles of the propagator). In the presence of a superflow, this determinant is a quartic equation for k_0 with very lengthy solutions. It is left as an exercise to compute these solutions for instance with Mathematica and show by expanding for small three-momenta that one of the solutions corresponds to the Goldstone mode. Here we proceed with the simpler case without superflow. In this case, there are two zeros of the determinant of the inverse propagator which we denote by ϵ_k^\pm , i.e., $\det D_0^{-1} = [k_0^2 - (\epsilon_k^+)^2][k_0^2 - (\epsilon_k^-)^2]$. Then we compute

$$\begin{aligned} \frac{1}{2} \frac{T}{V} \text{Tr} \ln \frac{D_0^{-1}(k)}{T^2} &= \frac{1}{2} \frac{T}{V} \ln \det \frac{D_0^{-1}(k)}{T^2} = \frac{1}{2} \frac{T}{V} \ln \prod_k \frac{[(\epsilon_k^+)^2 - k_0^2][(\epsilon_k^-)^2 - k_0^2]}{T^4} \\ &= \frac{1}{2} \frac{T}{V} \sum_k \left[\ln \frac{(\epsilon_k^+)^2 - k_0^2}{T^2} + \ln \frac{(\epsilon_k^-)^2 - k_0^2}{T^2} \right] = \sum_{e=\pm} \int \frac{d^3\mathbf{k}}{(2\pi)^3} \left[\frac{\epsilon_k^e}{2} + T \ln \left(1 - e^{-\epsilon_k^e/T} \right) \right], \end{aligned} \quad (120)$$

where, in the last step, we have performed the summation over Matsubara frequencies (dropping a constant, infinite contribution) and have taken the thermodynamic limit, $\frac{1}{V} \sum_{\mathbf{k}} \rightarrow \int \frac{d^3\mathbf{k}}{(2\pi)^3}$. We see that we have re-derived the familiar expression from statistical mechanics for the thermodynamic potential within thermal field theory (with an additional, divergent, contribution which has to be renormalized). In particular, we see that the poles of the propagator indeed are the excitation energies of the system. Before discussing the potential, we are interested in the structure of the excitation energies. They are given by⁹

$$\epsilon_k^\pm = \sqrt{k^2 + m^2 + 2\lambda\rho^2 + \mu^2 \mp \sqrt{4\mu^2(k^2 + m^2 + 2\lambda\rho^2) + \lambda^2\rho^4}}, \quad (121)$$

where $k \equiv |\mathbf{k}|$ (not to be confused with the four-vector k from above), and where we have identified $\partial_0\psi = \mu$. If we set the condensate to zero, $\rho = 0$, we reproduce the dispersions of free particles,

$$\rho = 0 : \quad \epsilon_k^\pm = \sqrt{k^2 + m^2} \mp \mu, \quad (122)$$

where the upper sign corresponds to particles (with $U(1)$ charge $+1$) and the lower sign to antiparticles (with charge -1). With the $T = 0$ result for the condensate (96) the dispersions become

$$\epsilon_k^\pm = \sqrt{k^2 + (3\mu^2 - m^2) \mp \sqrt{4\mu^2k^2 + (3\mu^2 - m^2)^2}}. \quad (123)$$

We see that ϵ_k^+ becomes gapless, i.e., $\epsilon_{k=0}^+ = 0$, as shown in Fig. 6. We can expand this mode for small momenta to obtain

$$\epsilon_k^+ \simeq \sqrt{\frac{\mu^2 - m^2}{3\mu^2 - m^2}} k + \mathcal{O}(k^3). \quad (124)$$

Remember that this result only holds in the condensed phase since we have already inserted the nontrivial value for the condensate; therefore, we have $\mu^2 > m^2$ and the argument of the square root is positive. We see that the slope of the low-momentum, zero-temperature dispersion of the Goldstone mode is identical to $(\partial P / \partial \epsilon)^{1/2}$, see Eq. (109). [Strictly speaking, we have shown this only for vanishing superflow $\nabla\psi$, but one can show that in the general case,

⁹ As an exercise, you may re-derive the excitation energies in the following alternative way. Take the equations of motion (85) and insert an ansatz for modulus and phase of the condensate that contains fluctuations, i.e., $\rho \rightarrow \rho + \delta\rho e^{i(\omega t - \mathbf{k} \cdot \mathbf{x})}$, $\psi \rightarrow \psi + \delta\psi e^{i(\omega t - \mathbf{k} \cdot \mathbf{x})}$. After linearizing the equations in the fluctuations, one finds a condition for ω which yields the excitation energies.

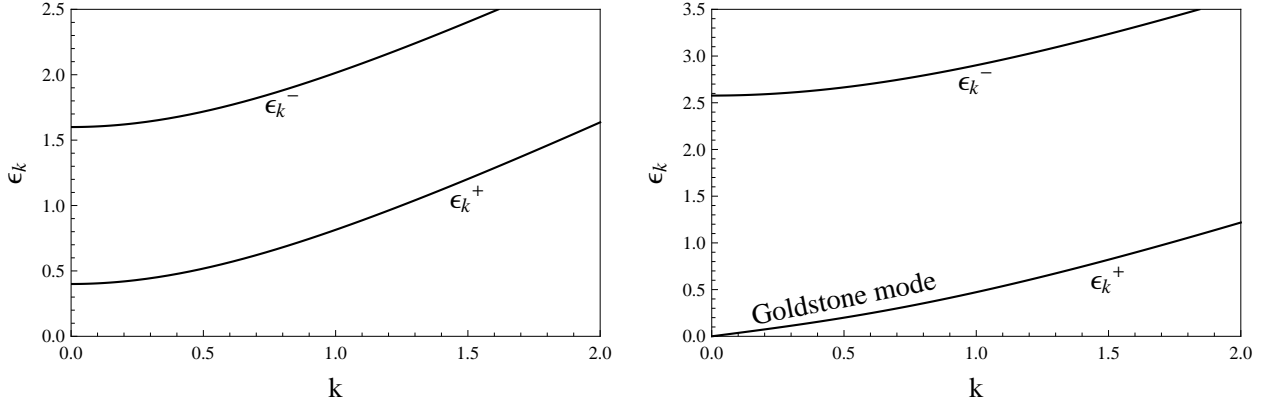


FIG. 6: Dispersions ϵ_k without (left, $\mu < m$) and with (right, $\mu > m$) Bose condensation. In the latter case, one of the modes becomes gapless with a linear behavior for small momenta k , see Eqs. (123) and (124). Such a “Goldstone mode” appears always for a spontaneously broken global symmetry.

the slope of the dispersion relation indeed assumes the form of Eq. (109).] This, in turn, is the speed of first sound, as derived in Sec. IID 3. Consequently, for low temperatures, the speed of first sound is the slope of the Goldstone dispersion¹⁰.

The second mode does have an energy gap and behaves quadratically for small k ,

$$\epsilon_k^- = \sqrt{2} \sqrt{3\mu^2 - m^2} + \frac{1}{2\sqrt{2}} \frac{5\mu^2 - m^2}{(3\mu^2 - m^2)^{3/2}} k^2. \quad (125)$$

The gaplessness of one of the modes is a very general phenomenon for spontaneously broken *global* symmetries. Its existence is predicted by the Goldstone Theorem:

Goldstone Theorem: If a continuous global symmetry group of the Lagrangian is spontaneously broken there exists a gapless mode. This mode is called Goldstone mode.

We shall not present a general proof of this theorem. The proof can be found in most textbooks about quantum field theory, see for instance Ref. [14].

In Lorentz invariant systems, one can also make a precise statement about the number of Goldstone modes: If the global symmetry group G of the Lagrangian is spontaneously broken to a subgroup $H \subset G$, there exist $\dim G/H$ many gapless modes. (For discussions about deviations from this rule in systems with broken Lorentz symmetry see for instance Refs. [15, 16]; in particular, Sec. 5 of Ref. [16] discusses the counting of Goldstone modes in detail.)

In our case, $G = U(1)$, $H = \mathbf{1}$, such that there is $\dim G/H = 1$ Goldstone mode. Less formally speaking, there are as many Goldstone modes as broken generators. There are many examples of Goldstone modes, some of which are

- superfluid ^4He : in terms of symmetries, this case corresponds to our φ^4 model; the Goldstone mode is usually called “phonon”.
- ferromagnetism: here the order parameter is magnetization and breaks rotational symmetry $SO(3) \rightarrow U(1)$; there is one corresponding Goldstone mode (with quadratic dispersion), called “magnon” or “spin wave”. This is an example of a system with broken Lorentz symmetry where the number of Goldstone modes is less than the number of broken generators. In an antiferromagnet, in contrast, there are two magnon degrees of freedom (both with linear dispersion).
- chiral symmetry breaking in QCD: here the symmetry breaking pattern is $SU(3)_L \times SU(3)_R \rightarrow SU(3)_{R+L}$ and there are eight Goldstone modes, the meson octet $\pi^0, \pi^\pm, K^0, \bar{K}^0, K^\pm, \eta$; since the chiral symmetry is not an

¹⁰ This is no longer true at large temperatures. At temperatures slightly below the critical temperature for superfluidity, the slope of the Goldstone dispersion rather approaches the speed of *second* sound. The Goldstone excitation itself is sometimes called *fourth sound*.

exact symmetry to begin with, these Goldstone modes have small masses and are therefore sometimes called pseudo-Goldstone modes¹¹.

We emphasize again that for the existence of the gapless mode it is crucial that the broken symmetry is global. We shall discuss later what changes in the case of a spontaneously broken local symmetry, see Sec. V.

It is instructive to write down the expression for the charge density of the condensed system. With the usual thermodynamic definition and Eq. (120) we compute

$$n = -\frac{\partial(\Omega/V)}{\partial\mu} = -\sum_{e=\pm} \int \frac{d^3\mathbf{k}}{(2\pi)^3} \left[\frac{1}{2} + f(\epsilon_k^e) \right] \frac{\partial\epsilon_k^e}{\partial\mu}. \quad (126)$$

Let us ignore the temperature-independent (divergent) first term. For the uncondensed case, we have $\partial\epsilon_k^e/\partial\mu = -e$, and the charge density has the expected form: one particle and one antiparticle contribute one unit of charge each, with opposite signs. This is different in the condensed case, where particles and antiparticles are replaced by the Goldstone and the massive mode. These “quasiparticles” do not have a well-defined charge (they are no eigenstates of the charge operator). This is reflected in the momentum-dependent factor $\partial\epsilon_k^e/\partial\mu$ which becomes complicated in the presence of a condensate. One can say that the quasiparticles are combinations of the original modes, and the coefficients of these combinations are functions of momentum. We will encounter an analogous situation in a fermionic system with Cooper pairing, see next section. In that case, the corresponding coefficients are called Bogoliubov coefficients and determine the mixing between fermions and fermion holes, see Eq. (193).

[End of 6th lecture, Apr 22nd, 2013.]

E. Symmetry restoration at finite temperature

Let us now include the effect of a nonzero temperature. From Eqs. (119) and (120) we know the tree-level thermodynamic potential,

$$\frac{\Omega}{V} = U + T \sum_{e=\pm} \int \frac{d^3\mathbf{k}}{(2\pi)^3} \ln \left(1 - e^{-\epsilon_k^e/T} \right), \quad (127)$$

where we have subtracted the divergent $T = 0$ part. For a simple estimate of the temperature-dependent term we shall make use of the high-temperature expansion of the corresponding term of a noninteracting Bose gas (see for instance Appendix A in Ref. [18]),

$$T \sum_{e=\pm} \int \frac{d^3\mathbf{k}}{(2\pi)^3} \ln \left[1 - e^{-(\sqrt{k^2+M^2}-e\mu)/T} \right] = -\frac{\pi^2 T^4}{45} + \frac{(M^2 - 2\mu^2)T^2}{12} - \frac{(M^2 - \mu^2)^{3/2}T}{6\pi} + \dots \quad (128)$$

Now, unfortunately, our dispersion relations are more complicated. We thus apply the approximation

$$\epsilon_k^\pm \simeq \sqrt{k^2 + m^2 + 2\lambda\rho^2} \mp \mu, \quad (129)$$

which results from neglecting the term $\lambda^2\rho^4$ in Eq. (121), i.e., we assume the condensate to be small. In this approximation we lose the gaplessness of our Goldstone mode. Since we are interested in large temperatures, however, this is unproblematic.

With this approximation and the help of Eq. (128) we can compute the potential,

$$\frac{\Omega(\rho^2)}{V} \simeq \left(-\frac{\mu^2 - m^2}{2} + \frac{\lambda T^2}{6} \right) \rho^2 + \frac{\lambda}{4} \rho^4 + \text{const}, \quad (130)$$

where “const” denotes the terms quadratic and quartic in T which do not depend on ρ . We see that the effect of the temperature is to make the quadratic term in ρ less negative and eventually positive. Then, there is no nontrivial

¹¹ Although far beyond the scope of these lectures, Ref. [17] is an interesting work about the similarities between magnons in an antiferromagnet and pions, which illustrates the general value of the Goldstone Theorem, spanning completely different physical systems.

minimum for ρ and the $U(1)$ symmetry is restored. This is the phase transition to the “normal” phase were there is no condensate. The critical temperature is the temperature where the prefactor of the quadratic term is zero,

$$T_c^2 = \frac{3(\mu^2 - m^2)}{\lambda}. \quad (131)$$

We may use Eq. (130) to compute the temperature-dependent condensate,

$$\rho^2(T) = \frac{\mu^2 - m^2}{\lambda} - \frac{T^2}{3} = \rho_0^2 \left(1 - \frac{T^2}{T_c^2}\right), \quad (132)$$

with $\rho_0^2 = \rho^2(T=0)$. We see that the condensate “melts” when T approaches T_c . At $T = T_c$, the condensate vanishes and remains zero for $T > T_c$ (i.e., the above expression is only valid for $T < T_c$). Although there is nothing obviously wrong with our temperature-dependent condensate, we need to keep in mind that we have employed a very crude finite-temperature approximation. The crudeness becomes obvious when we consider the dispersion relation of the Goldstone mode. Inserting Eq. (132) into Eq. (121) and expanding for small T shows that $\epsilon_{k=0}^+$ becomes imaginary,

$$(\epsilon_{k=0}^+)^2 = -\frac{\lambda T^2}{3} \frac{\mu^2 - m^2}{3\mu^2 - m^2}. \quad (133)$$

This is obviously an unphysical result. The reason is that we have ignored loop corrections which yield further λT^2 terms. Here we were simply interested in a rough picture of how the condensate melts and will not go into further details; see Ref. [1] where it is explained how the one-loop self-energy cures this problem.

IV. COOPER PAIRING IN A FERMIONIC SYSTEM

Not only bosonic, but also fermionic systems can become superfluid. In order for a fermionic system to develop a condensate, it has to form bosonic states that can Bose-condense. The mechanism that provides such a condensation is Cooper pairing, which we discuss in this section. As for the bosonic systems of the previous two sections, we shall start with a rather general calculation, in this case the derivation of the gap equation. As in the φ^4 model, we shall use a relativistic, field-theoretical description which then, as a second step, shall be applied to different physical systems.

To put the following technical calculation in a physical context, let us first discuss Cooper pairing in simple physical terms. Consider a system of fermions at zero temperature with chemical potential μ and thermodynamical potential

$$\Omega = E - \mu N. \quad (134)$$

Now first suppose the fermions are non-interacting. Then, adding a fermion with energy μ , i.e., at the Fermi surface, leaves the free energy Ω obviously unchanged: the energy E is increased by μ , but the second term subtracts the same amount since we add $N = 1$ fermion. Now let us switch on an arbitrarily small attractive interaction between the fermions. Then, by adding two fermions at the Fermi surface, we can actually lower the free energy because the attractive interaction will lead to an energy gain from the binding energy. Therefore, the Fermi surface we have started with is unstable. A new ground state is formed in which pairs of fermions are created at the Fermi surface. Viewing two fermions formally as a boson, these fermion pairs will form a Bose condensate. This formation of a condensate of fermion pairs due to an arbitrarily small interaction is called *Cooper’s Theorem* and the fermion pairs are called *Cooper pairs*. This argument already shows the similarity to the bosonic superfluids discussed above. Simply speaking, fermionic systems undergo an intermediate step, Cooper pairing, before they are “ready” to become a superfluid (or superconductor).

A. Derivation of the gap equation

The contents of this subsection is based on Appendix A of Ref. [19]. It is the relativistic version of the famous Bardeen-Cooper-Schrieffer (BCS) theory [20] for an (electronic) superconductor at weak coupling. We consider a theory that contains fermions and bosons and their interactions. Our Lagrangian thus assumes the following form,

$$\mathcal{L} = \mathcal{L}_{\text{fermions}} + \mathcal{L}_{\text{bosons}} + \mathcal{L}_{\text{interactions}}, \quad (135)$$

Here,

$$\mathcal{L}_{\text{fermions}} = \bar{\psi}(i\gamma^\mu \partial_\mu + \gamma^0 \mu - m)\psi \quad (136)$$

is the free fermionic part with the four-spinor ψ , $\bar{\psi} = \psi^\dagger \gamma^0$, the chemical potential μ and the mass m . The bosonic Lagrangian for a real, scalar boson with mass M is

$$\mathcal{L}_{\text{bosons}} = \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi - \frac{1}{2} M^2 \varphi^2, \quad (137)$$

and the interaction is given by

$$\mathcal{L}_{\text{interactions}} = -g \bar{\psi} \psi \varphi, \quad (138)$$

where $g > 0$ is the coupling constant. We have kept the structure of this term as simple as possible, using a Yukawa-type interaction, but our main arguments will also hold for more complicated interactions. For instance in QCD, the fermions (in this case quarks) interact via the exchange of gluons and the interaction has a richer structure,

$$-g \bar{\psi} \psi \varphi \rightarrow -g \bar{\psi}_\alpha \gamma^\mu T_a^{\alpha\beta} \psi_\beta A_\mu^a, \quad (139)$$

with the gluon fields A_μ^a , the Gell-Mann matrices T_a ($a = 1, \dots, 8$), and color indices $1 \leq \alpha, \beta \leq 3$.

The partition function is

$$\mathcal{Z} = \int \mathcal{D}\bar{\psi} \mathcal{D}\psi \mathcal{D}\varphi e^S, \quad (140)$$

with the action

$$S = \int_x \mathcal{L} = \int_{x,y} \left[\bar{\psi}(x) G_0^{-1}(x,y) \psi(y) - \frac{1}{2} \varphi(x) D^{-1}(x,y) \varphi(y) \right] - g \int_x \bar{\psi}(x) \psi(x) \varphi(x), \quad (141)$$

where

$$G_0^{-1}(x,y) = \delta(x-y) (i\gamma^\mu \partial_\mu + \gamma^0 \mu - m) \quad (142)$$

is the inverse fermionic tree-level propagator and $D^{-1}(x,y) = \delta(x-y) (\square + M^2)$ the free bosonic propagator (whose form we shall not need in the following).

The first step is to integrate out the bosonic fields. To this end, we use

$$\frac{1}{2} \xi A^{-1} \xi + \xi B = -\frac{1}{2} B A B + \frac{1}{2} \xi' A^{-1} \xi', \quad (143)$$

where $\xi' = \xi + AB$. This relation is useful since it transforms the terms linear and quadratic in the original bosonic fields into a term constant and quadratic in the new, shifted bosonic fields. Applying this to the last two terms of Eq. (141), i.e., identifying $\xi \rightarrow \varphi(x)$, $A \rightarrow D(x,y)$, $B \rightarrow g \bar{\psi}(x) \psi(x)$, and integrating over the shifted fields we obtain

$$\mathcal{Z} = \mathcal{Z}_{\text{bosons}} \int \mathcal{D}\bar{\psi} \mathcal{D}\psi e^{S'}, \quad (144)$$

with a bosonic partition function $\mathcal{Z}_{\text{bosons}}$ that is irrelevant for our purpose, and the new fermionic action

$$S' = \int_{x,y} \left[\bar{\psi}(x) G_0^{-1}(x,y) \psi(y) + \frac{g^2}{2} \bar{\psi}(x) \psi(x) D(x,y) \bar{\psi}(y) \psi(y) \right]. \quad (145)$$

Next, we want to apply an approximation for the interaction term. The goal is to write the product of two fermion spinors as its expectation value (later to be determined in the gap equation) plus fluctuations around this value which are assumed to be small. The expectation value of the difermion will then correspond to a condensate of fermion pairs. In the present formalism, we have two options. First, there might be a condensate of fermion-antifermion pairs. In that case, one may proceed rather straightforwardly since the scalar $\bar{\psi}(x) \psi(x)$ is the relevant object. However, this is not the case we are interested in. Cooper pairs are not fermion-antifermion pairs, but fermion-fermion pairs. In this case, the object that “wants” to obtain an expectation value cannot be simply written as $\psi(x) \psi(x)$ (think of ψ as a column vector and $\bar{\psi}$ as a row vector in Dirac space, then $\bar{\psi} \psi$ is a scalar, $\psi \psi$ a 4×4 matrix, but the products $\psi \psi$ and $\bar{\psi} \bar{\psi}$ are not defined). In other words, we would like to have a fermion which is described by a row vector. This is done by introducing the so-called charge-conjugate spinor ψ_C , such that a Cooper pair of fermions can be written as $\psi \bar{\psi}_C$ (which is a 4×4 matrix) and a Cooper pair of anti-fermions (which will play no role in the physical

systems we discuss but which is convenient to introduce) as $\psi_C \bar{\psi}$. The details of this procedure are as follows. With the charge-conjugation matrix $C = i\gamma^2\gamma^0$ we define

$$\psi_C \equiv C\bar{\psi}^T, \quad (146)$$

which implies $\bar{\psi}_C = \psi^T C$, $\psi = C\bar{\psi}_C^T$, $\bar{\psi} = \psi_C^T C$. Here, $\bar{\psi}_C$ is understood as first charge-conjugating, then taking the Hermitian conjugate and multiplying by γ^0 . For instance, the first relation is obtained as $\bar{\psi}_C = (C\bar{\psi}^T)^\dagger \gamma^0 = (C\gamma^0\psi^*)^\dagger \gamma^0 = \psi^T \gamma^0 C^\dagger \gamma^0 = -\psi^T \gamma^0 C \gamma^0 = \psi^T C$.¹² Since

$$\bar{\psi}_C \psi_C = \psi^T C C \bar{\psi}^T = -\psi^T \bar{\psi}^T = (\bar{\psi}\psi)^T = \bar{\psi}\psi, \quad (147)$$

where $C = -C^{-1}$ and the antisymmetry of the fermion fields (Grassmann variables!) have been used, we can write

$$\begin{aligned} \bar{\psi}(x)\psi(x)\bar{\psi}(y)\psi(y) &= \frac{1}{2}[\bar{\psi}_C(x)\psi_C(x)\bar{\psi}(y)\psi(y) + \bar{\psi}(x)\psi(x)\bar{\psi}_C(y)\psi_C(y)] \\ &= -\frac{1}{2}\text{Tr}[\psi_C(x)\bar{\psi}(y)\psi(y)\bar{\psi}_C(x) + \psi(x)\bar{\psi}_C(y)\psi_C(y)\bar{\psi}(x)], \end{aligned} \quad (148)$$

where the trace is taken over Dirac space, and the minus sign arises from the Grassmann property of the spinors. Now we are prepared to employ the approximation explained before. We write

$$\psi_C(x)\bar{\psi}(y) = \langle\psi_C(x)\bar{\psi}(y)\rangle - [\langle\psi_C(x)\bar{\psi}(y)\rangle - \psi_C(x)\bar{\psi}(y)], \quad (149a)$$

$$\psi(y)\bar{\psi}_C(x) = \langle\psi(y)\bar{\psi}_C(x)\rangle - [\langle\psi(y)\bar{\psi}_C(x)\rangle - \psi(y)\bar{\psi}_C(x)], \quad (149b)$$

and consider the difference in the square brackets as a small fluctuation. Neglecting terms quadratic in these fluctuations, we derive (the few lines of algebra for the derivation is left as an exercise)

$$\begin{aligned} \int_{x,y} D(x,y)\bar{\psi}(x)\psi(x)\bar{\psi}(y)\psi(y) &= \frac{1}{2} \int_{x,y} D(x,y)\text{Tr}[\langle\psi_C(x)\bar{\psi}(y)\rangle\langle\psi(y)\bar{\psi}_C(x)\rangle] \\ &\quad - \int_{x,y} D(x,y)\text{Tr}[\langle\psi_C(x)\bar{\psi}(y)\rangle\psi(y)\bar{\psi}_C(x) + \langle\psi(y)\bar{\psi}_C(x)\rangle\psi_C(y)\bar{\psi}(x)]. \end{aligned} \quad (150)$$

where we have assumed that the boson propagator is symmetric under $x \leftrightarrow y$. The first term does not depend on the fermion fields and thus we can pull it out of the functional integral,

$$\mathcal{Z} = \mathcal{Z}_{\text{bosons}} \mathcal{Z}_0 \int \mathcal{D}\bar{\psi} \mathcal{D}\psi e^{S''}, \quad (151)$$

where we have denoted

$$\mathcal{Z}_0 \equiv \exp \left\{ \frac{g^2}{4} \int_{x,y} D(x,y)\text{Tr}[\langle\psi_C(x)\bar{\psi}(y)\rangle\langle\psi(y)\bar{\psi}_C(x)\rangle] \right\}. \quad (152)$$

For the derivation of the gap equation we shall not need this term. However, it is important for the free energy which is obtained from $\ln \mathcal{Z}$. Therefore, in an alternative – equivalent – derivation of the gap equation, where the free energy is minimized with respect to the gap, this term *is* needed. We shall come back to the free energy of a fermionic superfluid in Sec. VII B.

¹² Here and in the following we need a few properties of the Dirac matrices, such as $(\gamma^0)^\dagger = \gamma^0$, $(\gamma^i)^\dagger = -\gamma^i$, and $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$ (consequently, γ^0 anti-commutes with γ^i). Moreover, $(\gamma^0)^2 = 1$, $(\gamma^i)^2 = -1$. In the Dirac representation, we have

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad \gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

with the Pauli matrices σ^i which are defined as

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In Eq. (151) we have used the abbreviation

$$S'' = \int_{x,y} \left\{ \bar{\psi}(x) G_0^{-1}(x,y) \psi(y) + \frac{1}{2} [\bar{\psi}_C(x) \Phi^+(x,y) \psi(y) + \bar{\psi}(x) \Phi^-(x,y) \psi_C(y)] \right\}, \quad (153)$$

where we have defined

$$\Phi^+(x,y) \equiv g^2 D(x,y) \langle \psi_C(x) \bar{\psi}(y) \rangle, \quad (154a)$$

$$\Phi^-(x,y) \equiv g^2 D(x,y) \langle \psi(x) \bar{\psi}_C(y) \rangle. \quad (154b)$$

It is easy to check that Φ^+ and Φ^- are related via

$$\Phi^-(y,x) = \gamma^0 [\Phi^+(x,y)]^\dagger \gamma^0. \quad (155)$$

Within our approximation we could absorb all effects of the interaction into Φ^\pm , which have to be determined self-consistently from the gap equation, see below. Our new effective action (153) is quadratic in the fields and one can in principle perform the functional integral. To this end, let us go to momentum space. We introduce the Fourier transforms of the fields

$$\psi(x) = \frac{1}{\sqrt{V}} \sum_k e^{-ik \cdot x} \psi(k), \quad \bar{\psi}(x) = \frac{1}{\sqrt{V}} \sum_k e^{ik \cdot x} \bar{\psi}(k), \quad (156a)$$

$$\psi_C(x) = \frac{1}{\sqrt{V}} \sum_k e^{-ik \cdot x} \psi_C(k), \quad \bar{\psi}_C(x) = \frac{1}{\sqrt{V}} \sum_k e^{ik \cdot x} \bar{\psi}_C(k). \quad (156b)$$

The normalization factor is chosen such that the Fourier transformed fields are dimensionless ($\psi(x)$ has mass dimensions 3/2). There is some freedom in the choice of the signs in the exponentials of the charge-conjugate fields. They are chosen deliberately to lead to an action diagonal in momentum space, see Eq. (159). One easily sees [for instance by charge-conjugating both sides of the first relation in Eq. (156a) and comparing the result with the first relation in Eq. (156b)] that the given choice implies $\psi_C(k) = C \bar{\psi}^T(-k)$ and, analogously, $\bar{\psi}_C(k) = \psi^T(-k) C$, which differs from the charge conjugation of the spinor in position space, see Eq. (146): while in position space we have $\psi_C(x) = C \bar{\psi}^T(x)$, in momentum space charge conjugation includes a sign flip of the four-momentum. For the Fourier transform of Φ^\pm we assume translational invariance, $\Phi^\pm(x,y) = \Phi^\pm(x-y)$, to write

$$\Phi^\pm(x-y) = \frac{T}{V} \sum_k e^{-ik \cdot (x-y)} \Phi^\pm(k), \quad (157)$$

(while $\Phi^\pm(x,y)$ has mass dimensions 5, $\Phi^\pm(k)$ has mass dimensions 1), such that with Eq. (155) we have $\Phi^-(k) = \gamma^0 [\Phi^+(k)]^\dagger \gamma^0$. Inserting the Fourier transforms into the interaction terms of the action (153) and using

$$\int_x e^{-ik \cdot x} = \frac{V}{T} \delta_{k,0}, \quad (158)$$

yields

$$\begin{aligned} \int_{x,y} \bar{\psi}_C(x) \Phi^+(x-y) \psi(y) &= \frac{T}{V^2} \int_{x,y} \sum_{k_1, k_2, k_3} e^{-i(k_2 - k_1) \cdot x} e^{-i(k_3 - k_2) \cdot y} \bar{\psi}_C(k_1) \Phi^+(k_2) \psi(k_3) \\ &= \frac{1}{T} \sum_k \bar{\psi}_C(k) \Phi^+(k) \psi(k). \end{aligned} \quad (159)$$

[End of 7th lecture, May 6th, 2013.]

Eventually, we want to consider $\bar{\psi}_C(k)$, $\psi_C(k)$ integration variables independent of $\bar{\psi}(k)$, $\psi(k)$. We can rewrite the integration in terms of an integral over all four variables by restricting ourselves to momenta in one half of the full momentum space,

$$\mathcal{D}\bar{\psi} \mathcal{D}\psi = \prod_k d\bar{\psi}(k) d\psi(k) = \prod_{k>0} d\bar{\psi}(k) d\bar{\psi}(-k) d\psi(k) d\psi(-k) = \mathcal{N} \prod_{k>0} d\bar{\psi}(k) d\psi_C(k) d\psi(k) d\bar{\psi}_C(k), \quad (160)$$

with an irrelevant constant \mathcal{N} which arises from the change of integration variables. To be left with a sum over $k > 0$ in the integrand we divide the sum over k in Eq. (159) into a sum over $k > 0$ and a sum over $k < 0$ and rewrite the latter as

$$\begin{aligned}
\sum_{k<0} \bar{\psi}_C(k) \Phi^+(k) \psi(k) &= \sum_{k>0} \bar{\psi}_C(-k) \Phi^+(-k) \psi(-k) \\
&= \sum_{k>0} \psi^T(k) C \Phi^+(-k) C \bar{\psi}_C^T(k) \\
&= \sum_{k>0} [\psi^T(k) C \Phi^+(-k) C \bar{\psi}_C^T(k)]^T \\
&= - \sum_{k>0} \bar{\psi}_C(k) C^T [\Phi^+(-k)]^T C^T \psi(k) \\
&= \sum_{k>0} \bar{\psi}_C(k) \Phi^+(k) \psi(k), \tag{161}
\end{aligned}$$

where, in the last step, we have used $C^T = -C$ and $C[\Phi^+(-k)]^T C = -\Phi^+(k)$. The latter can be seen as follows. In position space we have

$$\int_{x,y} \bar{\psi}_C(x) \Phi^+(x-y) \psi(y) = - \int_{x,y} \psi^T(y) [\Phi^+(x-y)]^T \bar{\psi}_C^T(x) = \int_{x,y} \bar{\psi}_C(x) C^{-1} [\Phi^+(y-x)]^T C \psi(y), \tag{162}$$

i.e., $\Phi^+(x-y) = C^{-1} [\Phi^+(y-x)]^T C$. With the Fourier transform (157) and $C^{-1} = -C$ this yields $C[\Phi^+(-k)]^T C = -\Phi^+(k)$.

Using the result (161) in Eq. (159) and doing the analogous calculation for $\bar{\psi}(x) \Phi^-(x, y) \psi_C(y)$ yields the interaction part of the action in momentum space

$$\frac{1}{2} \int_{x,y} [\bar{\psi}_C(x) \Phi^+(x, y) \psi(y) + \bar{\psi}(x) \Phi^-(x, y) \psi_C(y)] = \frac{1}{T} \sum_{k>0} [\bar{\psi}_C(k) \Phi^+(k) \psi(k) + \bar{\psi}(k) \Phi^-(k) \psi_C(k)]. \tag{163}$$

The tree-level contribution becomes, using the definition of the tree-level propagator in position space (142) and the Fourier transformed fields from Eq. (156a),

$$\int_{x,y} \bar{\psi}(x) G_0^{-1}(x, y) \psi(y) = \frac{1}{T} \sum_k \bar{\psi}(k) (\gamma^\mu k_\mu + \mu \gamma^0 - m) \psi(k) \tag{164}$$

Again we need to divide the sum over k into two sums, one over $k > 0$ and one over $k < 0$ and rewrite the latter as

$$\begin{aligned}
\sum_{k<0} \bar{\psi}(k) (\gamma^\mu k_\mu + \mu \gamma^0 - m) \psi(k) &= \sum_{k>0} \bar{\psi}(-k) (-\gamma^\mu k_\mu + \mu \gamma^0 - m) \psi(-k) \\
&= \sum_{k>0} \psi_C^T(k) C (-\gamma^\mu k_\mu + \mu \gamma^0 - m) C \bar{\psi}_C^T(k) \\
&= \sum_{k>0} [\psi_C^T(k) C (-\gamma^\mu k_\mu + \mu \gamma^0 - m) C \bar{\psi}_C^T(k)]^T \\
&= - \sum_{k>0} \bar{\psi}_C(k) C^T (-\gamma_\mu^T k^\mu + \mu \gamma^0 - m) C^T \psi_C(k) \\
&= \sum_{k>0} \bar{\psi}_C(k) (\gamma^\mu k_\mu - \mu \gamma^0 - m) \psi_C(k), \tag{165}
\end{aligned}$$

where we used $C^T = -C$ and $C \gamma_\mu^T C = \gamma_\mu$. Consequently,

$$\int_{x,y} \bar{\psi}(x) G_0^{-1}(x, y) \psi(y) = \frac{1}{T} \sum_{k>0} \{ \bar{\psi}(k) [G_0^+(k)]^{-1} \psi(k) + \bar{\psi}_C(k) [G_0^-(k)]^{-1} \psi_C(k) \}, \tag{166}$$

with the propagators for fermions and charge-conjugate fermions,

$$[G_0^\pm(k)]^{-1} = \gamma^\mu k_\mu \pm \gamma_0 \mu - m. \tag{167}$$

With the new integration measure (160), the interaction part (163) and the tree-level part (166) the partition function (151) can be written in the following compact way,

$$\mathcal{Z} = \mathcal{N} \mathcal{Z}_{\text{bosons}} \mathcal{Z}_0 \int \mathcal{D}\bar{\Psi} \mathcal{D}\Psi \exp \left[\sum_{k>0} \bar{\Psi}(k) \frac{\mathcal{S}^{-1}}{T} \Psi(k) \right]. \quad (168)$$

Here we have abbreviated the integration measure

$$\mathcal{D}\bar{\Psi} \mathcal{D}\Psi \equiv \prod_{k>0} d\bar{\psi}(k) d\psi_C(k) d\psi(k) d\bar{\psi}_C(k), \quad (169)$$

introduced the new spinors

$$\Psi \equiv \begin{pmatrix} \psi \\ \psi_C \end{pmatrix}, \quad \bar{\Psi} \equiv (\bar{\psi}, \bar{\psi}_C), \quad (170)$$

and the inverse propagator

$$\mathcal{S}^{-1} = \begin{pmatrix} [G_0^+]^{-1} & \Phi^- \\ \Phi^+ & [G_0^-]^{-1} \end{pmatrix}. \quad (171)$$

The two-dimensional space that has emerged from the introduction of charge-conjugate spinors is called *Nambu-Gorkov space*. Together with the 4×4 structure of Dirac space, \mathcal{S}^{-1} is an 8×8 matrix. We can write Eq. (171) in the form of a Schwinger-Dyson equation where the inverse propagator is decomposed into a non-interacting part \mathcal{S}_0^{-1} and a self-energy Σ ,

$$\mathcal{S}^{-1} = \mathcal{S}_0^{-1} + \Sigma, \quad (172)$$

with

$$\mathcal{S}_0^{-1} = \begin{pmatrix} [G_0^+]^{-1} & 0 \\ 0 & [G_0^-]^{-1} \end{pmatrix}, \quad \Sigma = \begin{pmatrix} 0 & \Phi^- \\ \Phi^+ & 0 \end{pmatrix}. \quad (173)$$

We can compute the propagator in Nambu-Gorkov space from inverting the matrix (171),

$$\mathcal{S} = \begin{pmatrix} G^+ & F^- \\ F^+ & G^- \end{pmatrix}, \quad (174)$$

with

$$G^\pm \equiv ([G_0^\pm]^{-1} - \Phi^\mp G_0^\mp \Phi^\pm)^{-1}, \quad (175a)$$

$$F^\pm \equiv -G_0^\mp \Phi^\pm G^\pm. \quad (175b)$$

It is easy to check that $\mathcal{S}^{-1}\mathcal{S} = 1$. The off-diagonal elements F^\pm of the full propagator are called *anomalous propagators*. They describe the propagation of a fermion that is converted into a charge-conjugate fermion or vice versa. This is possible through the Cooper pair condensate which can be thought of as a reservoir of fermions and fermion-holes. Therefore, charge conservation is spontaneously broken, in a very similar manner compared to Bose-Einstein condensation. More formally speaking, Φ^\pm is not invariant under $U(1)$ rotations: with the transformation of the fermion spinor $\psi \rightarrow e^{-i\alpha}\psi$ we have with Eqs. (154)

$$\Phi^\pm \rightarrow e^{\pm 2i\alpha} \Phi^\pm. \quad (176)$$

Since Φ^\pm can be considered as the order parameter (just like the Bose condensate in our discussion of Bose-Einstein condensation), $U(1)$ is spontaneously broken down to \mathbb{Z}_2 (the order parameter is still invariant under multiplying both fermion spinors with -1 ; this residual group is thus only present in the condensation of Cooper pairs, not in the condensation of individual bosons).

To derive the gap equation we notice that the fermionic propagator is, on the one hand, given by Eq. (174) and, on the other hand, the usual definition, extended to Nambu-Gorkov space, yields

$$\mathcal{S}(x, y) = -\langle \Psi(x) \bar{\Psi}(y) \rangle = - \begin{pmatrix} \langle \psi(x) \bar{\psi}(y) \rangle & \langle \psi(x) \bar{\psi}_C(y) \rangle \\ \langle \psi_C(x) \bar{\psi}(y) \rangle & \langle \psi_C(x) \bar{\psi}_C(y) \rangle \end{pmatrix}. \quad (177)$$

In particular, for the lower left component in Nambu-Gorkov space we have $F^+(x, y) = -\langle \psi_C(x) \bar{\psi}(y) \rangle$ (we could also consider the upper right component, the resulting gap equation would be equivalent). Inserting this into Eq. (154a) yields $\Phi^+(x, y) = -g^2 D(x, y) F^+(x, y)$, which becomes in Fourier space (again assuming translational invariance, i.e., all two-point functions only depend on the difference $x - y$),

$$\begin{aligned} \frac{T}{V} \sum_p e^{-ip \cdot (x-y)} \Phi^+(p) &= -g^2 \frac{T^2}{V^2} \sum_{q,k} e^{-i(q+k) \cdot (x-y)} D(q) F^+(k) \\ &= -g^2 \frac{T^2}{V^2} \sum_{p,k} e^{-ip \cdot (x-y)} D(p-k) F^+(k), \end{aligned} \quad (178)$$

where, in the second step, we have introduced the new summation variable $p = q + k$. We can now compare the coefficients of the Fourier series in p to obtain

$$\Phi^+(p) = -g^2 \frac{T}{V} \sum_k D(p-k) F^+(k). \quad (179)$$

This is the gap equation. Since $F^+(k)$ contains the gap function $\Phi^+(k)$, the gap equation is an integral equation for the gap function.

[End of 8th lecture, May 13th, 2013.]

B. Quasiparticle excitations

Next we need to compute the various components of the propagator in Nambu-Gorkov space explicitly. This is necessary for solving the gap equation, but even before doing so we shall also learn something about the structure of the fermionic quasiparticle dispersion relations. Here “quasiparticle” refers to an excitation of the system in which the interaction in the above mean-field approximation is included. It is an important property of weakly coupled many-body systems that after including the effect of the interaction, the system again looks like being composed of single particles, but now with different properties compared to the non-interacting case.

In the following, we shall restrict ourselves for simplicity to massless (= ultrarelativistic) fermions, $m = 0$. Including a mass makes the calculation more complicated, but the essential physics will be captured already in the massless case. It is convenient to express the inverse tree-level propagators for massless fermions in terms of energy projectors,

$$[G_0^\pm]^{-1} = \gamma^\mu K_\mu \pm \gamma^0 \mu = \sum_{e=\pm} [k_0 \pm (\mu - ek)] \gamma^0 \Lambda_k^{\pm e}, \quad (180)$$

with

$$\Lambda_k^e \equiv \frac{1}{2} \left(1 + e \gamma^0 \boldsymbol{\gamma} \cdot \hat{\mathbf{k}} \right). \quad (181)$$

It is easy to check that Λ_k^+ and Λ_k^- form a set of orthogonal projectors, i.e.,

$$\Lambda_k^+ + \Lambda_k^- = 1, \quad \Lambda_k^+ \Lambda_k^- = 0, \quad \Lambda_k^e \Lambda_k^e = \Lambda_k^e. \quad (182)$$

The benefit of writing the inverse propagator in terms of projectors is that inversion is very simple. A matrix of the form $A = \sum_i a_i \mathcal{P}_i$ with a set of complete, orthogonal projectors \mathcal{P}_i and scalars a_i , has obviously the inverse $A^{-1} = \sum_i a_i^{-1} \mathcal{P}_i$. The only subtlety in our case is the additional matrix γ^0 . But, as it turns out, all one needs to do is to invert the order of γ^0 and the energy projector, such that the tree-level propagators are,

$$G_0^\pm = \sum_e \frac{\gamma^0 \Lambda_k^{\mp e}}{k_0 \pm (\mu - ek)}. \quad (183)$$

Since $\gamma^0 \Lambda_k^e = \Lambda_k^{-e} \gamma^0$, one easily confirms $G_0^\pm [G_0^\pm]^{-1} = 1$. Next, we use the following ansatz for the gap matrix,

$$\Phi^\pm(K) = \pm \Delta(K) \gamma^5, \quad (184)$$

with a gap function $\Delta(K)$ which is assumed to be real. (Remember that Φ^+ and Φ^- are related via $\Phi^- = \gamma^0 (\Phi^+)^\dagger \gamma^0$, i.e., once we make the ansatz $\Phi^+ = \Delta \gamma^5$, the expression for Φ^- follows.) This ansatz corresponds to Cooper pairing

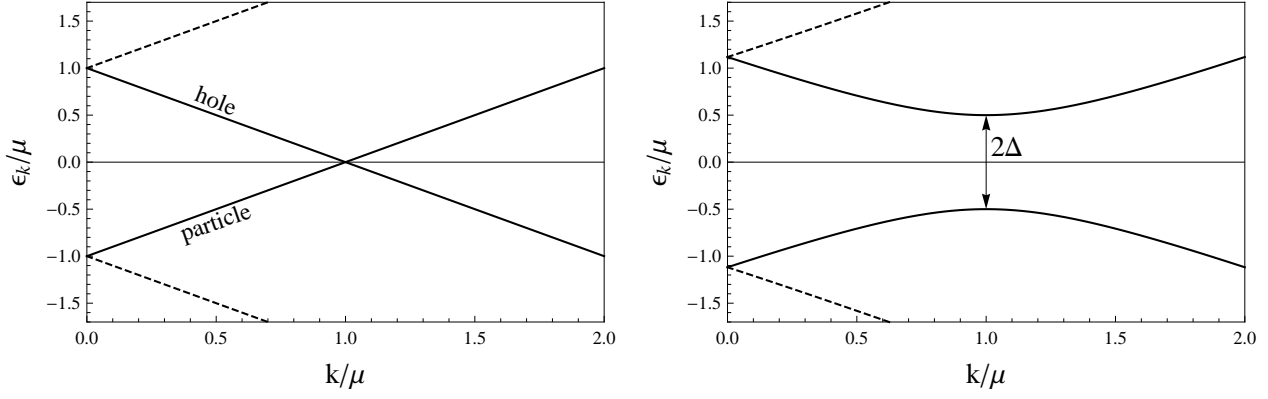


FIG. 7: Left panel: particle and hole dispersions in a non-interacting fermionic system with finite chemical potential (solid) and corresponding antiparticle dispersions (dashed). Right panel: Cooper pairing introduces an energy gap of size 2Δ in the spectrum. It also leads to a mixing of particle and hole states.

in the even parity channel, with constituent fermions in a Cooper pair of the same chirality and Cooper pairs with total spin zero. We shall compute the value of the gap Δ with the help of the gap equation in the next subsection (for the case where $\Delta(k)$ is constant). First we discuss some properties of the superfluid system for which the precise value of Δ is not relevant.

Inserting Eqs. (180), (183), (184) into the expression of the full propagators (175a) and using that γ^5 anti-commutes with the other Dirac matrices, $\{\gamma^5, \gamma^\mu\} = 0$, as well as $(\gamma^5)^2 = 1$, yields

$$\begin{aligned} G^\pm(K) &= \left\{ \sum_e \left[k_0 \pm (\mu - ek) - \frac{\Delta^2}{k_0 \mp (\mu - ek)} \right] \gamma^0 \Lambda_k^{\pm e} \right\}^{-1} \\ &= \sum_e \frac{k_0 \mp (\mu - ek)}{k_0^2 - (\epsilon_k^e)^2} \gamma^0 \Lambda_k^{\mp e} = [G_0^\mp]^{-1} \sum_e \frac{\Lambda_k^{\mp e}}{k_0^2 - (\epsilon_k^e)^2}, \end{aligned} \quad (185)$$

with

$$\epsilon_k^e \equiv \sqrt{(\mu - ek)^2 + \Delta^2}. \quad (186)$$

The anomalous propagators become

$$F^\pm(K) = \pm \sum_e \frac{\Delta(K) \gamma^5 \Lambda_k^{\mp e}}{k_0^2 - (\epsilon_k^e)^2}. \quad (187)$$

We see that all components of the Nambu-Gorkov propagator have the same poles, $k_0 = \pm \epsilon_k^e$. These are the excitation energies for quasi-particles ($e = +$) and quasi-antiparticles ($e = -$) (both for the upper sign) and quasi-holes ($e = +$) and quasi-anti-holes ($e = -$) (both for the lower sign). We now see that Δ is indeed an energy gap in the quasiparticle spectrum, see Fig. 7.

With the anomalous propagator being computed, we are prepared to evaluate the gap equation. Before doing so let us elaborate on the structure of the excitations in the superfluid. We may gain some physical insight by computing the occupation numbers. To this end, we start from the pressure P , which is defined as

$$P = \frac{T}{V} \ln \mathcal{Z}. \quad (188)$$

With the partition function \mathcal{Z} from Eq. (168) we have

$$P = \frac{1}{2} \frac{T}{V} \sum_K \text{Tr} \ln \mathcal{S}^{-1} + \frac{T}{V} \ln \mathcal{Z}_0 + \frac{T}{V} \ln \mathcal{Z}_{\text{bosons}} + \text{const} \quad (189)$$

where the trace in the first term is taken over Nambu-Gorkov and Dirac space. We have evaluated the functional integral in the partition function, which is formally the same as for non-interacting fermions (see for instance Sec.

VI of Ref. [1]). Remember that the additional degree of freedom from the charge-conjugate fermions had resulted in restricting the momentum sum to one half space, $K > 0$. In equation (189) we sum over all K , but have taken care of this overcounting by multiplying by $\frac{1}{2}$. For the charge density and the fermionic occupation number we are not interested in the contribution of \mathcal{Z}_0 (which does not depend explicitly on the chemical potential) and the bosonic part. Hence we only need the $\text{Tr} \ln$ term. The charge density becomes

$$n = \frac{\partial P}{\partial \mu} = \frac{1}{2} \frac{T}{V} \sum_K \text{Tr} \left[\mathcal{S}(K) \frac{\partial \mathcal{S}^{-1}(K)}{\partial \mu} \right] = \frac{1}{2} \frac{T}{V} \sum_K \text{Tr} [\gamma^0 (G^+(K) - G^-(K))], \quad (190)$$

where, in the last step, the trace over Nambu-Gorkov space has been performed. Inserting the propagators from Eq. (185) and using $\text{Tr}[\Lambda_k^e] = 2$ (since $\text{Tr}[\gamma^0 \gamma^i] = 0$) yields

$$n = -2 \frac{T}{V} \sum_K \sum_e \frac{\mu - ek}{k_0^2 - (\epsilon_k^e)^2} = 2 \sum_e \int \frac{d^3 \mathbf{k}}{(2\pi)^3} \frac{\mu - ek}{2\epsilon_k^e} \tanh \frac{\epsilon_k^e}{2T}, \quad (191)$$

where, in the second step, we have performed the sum over fermionic Matsubara frequencies, $K = (k_0, \mathbf{k})$ with $k_0 = -i\omega_n$, $\omega_n = (2n+1)\pi T$,

$$T \sum_{k_0} \frac{1}{k_0^2 - a^2} = -\frac{1}{2a} \tanh \frac{a}{2T}, \quad (192)$$

and taken the thermodynamic limit $\frac{1}{V} \sum_{\mathbf{k}} \rightarrow \int \frac{d^3 \mathbf{k}}{(2\pi)^3}$. Now, using $\tanh \frac{\epsilon_k^e}{2T} = 1 - 2f(\epsilon_k^e)$ with the Fermi distribution $f(x) = [e^{x/T} + 1]^{-1}$, and subtracting the vacuum contribution $\mu = T = \Delta = 0$, we can write the result as

$$\begin{aligned} n &= 2 \sum_e e \int \frac{d^3 \mathbf{k}}{(2\pi)^3} \left[\frac{1}{2} \left(1 - \frac{k - e\mu}{\epsilon_k^e} \right) + \frac{k - e\mu}{\epsilon_k^e} f(\epsilon_k^e) \right] \\ &= 2 \sum_e e \int \frac{d^3 \mathbf{k}}{(2\pi)^3} \left\{ |u_k^e|^2 f(\epsilon_k^e) + |v_k^e|^2 [1 - f(\epsilon_k^e)] \right\}, \end{aligned} \quad (193)$$

with

$$|u_k^e|^2 \equiv \frac{1}{2} \left(1 + \frac{k - e\mu}{\epsilon_k^e} \right), \quad |v_k^e|^2 \equiv \frac{1}{2} \left(1 - \frac{k - e\mu}{\epsilon_k^e} \right). \quad (194)$$

The second line of Eq. (193) shows that the quasiparticles are mixtures of fermions and fermion-holes, where the mixing coefficients are the so-called *Bogoliubov coefficients* $|u_k^e|^2$ and $|v_k^e|^2$ with $|u_k^e|^2 + |v_k^e|^2 = 1$.

We may first check that this expression reduces to the usual charge density of free fermions if we set the gap to zero. In this case, $\epsilon_k^e = |k - e\mu|$ and we find

$$\begin{aligned} n_{\Delta=0} &= 2 \sum_e e \int \frac{d^3 \mathbf{k}}{(2\pi)^3} [\Theta(e\mu - k) + \text{sgn}(k - e\mu) f(|k - e\mu|)] \\ &= 2 \sum_e e \int \frac{d^3 \mathbf{k}}{(2\pi)^3} f(k - e\mu), \end{aligned} \quad (195)$$

as expected. [For the second step, divide the integral into integrals over the intervals $[0, e\mu]$ and $[e\mu, \infty]$ and use $1 - f(x) = f(-x)$].

To illustrate the effect of the gap, we consider the zero-temperature limit of Eq. (193). Since $\epsilon_k^e > 0$, we have $f(\epsilon_k^e) \rightarrow \Theta(-\epsilon_k^e) = 0$ at zero temperature, and thus, neglecting the contribution of the antiparticles,

$$n_{T=0} \simeq 2 \int \frac{d^3 \mathbf{k}}{(2\pi)^3} \frac{1}{2} \left[1 - \frac{k - \mu}{\sqrt{(k - \mu)^2 + \Delta^2}} \right]. \quad (196)$$

We plot the integrand, i.e., the occupation number in the presence of a gap, in Fig. 8. We see that the gap has a similar effect as a nonzero temperature: the sharp Fermi surface of the non-interacting system becomes a smeared surface in the superfluid system.

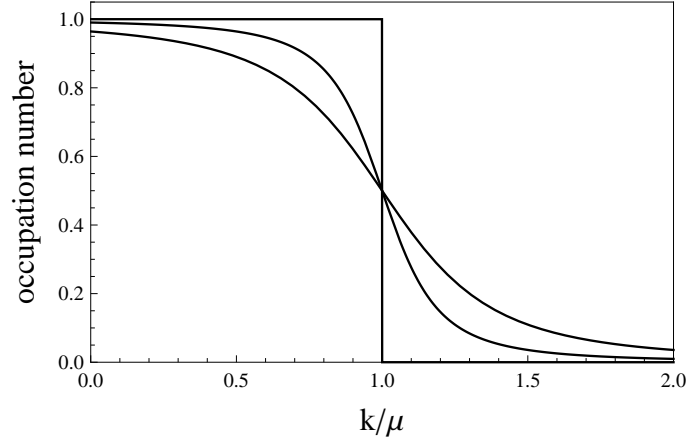


FIG. 8: Fermion occupation number at $T = 0$ from Eq. (196) for $\Delta = 0$, $\Delta = 0.2\mu$, $\Delta = 0.4\mu$.

C. Solving the gap equation

Inserting the ansatz for the gap matrix (184) and the anomalous propagator (187) into the gap equation (179), we obtain

$$\Delta(P)\gamma^5 = -g^2 \frac{T}{V} \sum_K D(P-K) \frac{\Delta(K)\gamma^5\Lambda_k^-}{k_0^2 - \epsilon_k^2}. \quad (197)$$

where we have neglected the antiparticle contribution and abbreviated $\epsilon_k \equiv \epsilon_k^+$. To get rid of the matrix structure, we multiply both sides of the equation with $\gamma^5\Lambda_p^+$ from the right and take the trace over Dirac space,

$$\Delta(P) = -\frac{g^2 T}{2V} \sum_K D(K-P) \frac{\Delta(K)}{k_0^2 - \epsilon_k^2} (1 - \hat{\mathbf{k}} \cdot \hat{\mathbf{p}}), \quad (198)$$

where we have used

$$\text{Tr}[\gamma^5\Lambda_k^-\gamma^5\Lambda_p^+] = 1 - \hat{\mathbf{k}} \cdot \hat{\mathbf{p}}. \quad (199)$$

Now let us assume that the interaction between the fermions is point-like, i.e., the boson propagator is approximated by the simple expression $D^{-1}(Q) \simeq -Q^2 + M^2 \simeq M^2$ where M is the mass of the exchanged boson. In this case, $\Delta(P)$ becomes independent of P and after performing the Matsubara sum we obtain

$$\Delta = G \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{\Delta}{2\epsilon_k} \tanh \frac{\epsilon_k}{2T}. \quad (200)$$

with the effective coupling constant $G = \frac{g^2}{2M^2}$ (which has mass dimensions -2). The term proportional to $\hat{\mathbf{k}} \cdot \hat{\mathbf{p}}$ in the integrand has vanished because of the angular integral (in the given ansatz Δ and thus also ϵ_k are isotropic).

1. Zero-temperature gap

Let us first discuss the solution of this equation for zero temperature, where $\tanh \frac{\epsilon_k}{2T} = 1$. We also assume that the interaction is only nonzero for fermions in a small vicinity around the Fermi surface $[\mu - \delta, \mu + \delta]$ with

$$\Delta_0 \ll \delta \ll \mu, \quad (201)$$

where $\Delta_0 \equiv \Delta(T = 0)$. This assumption corresponds to the weak coupling limit. In this case, the gap equation becomes

$$\Delta_0 \simeq \frac{\mu^2 G}{2\pi^2} \int_0^\delta d\xi \frac{\Delta_0}{\sqrt{\xi^2 + \Delta_0^2}}, \quad (202)$$

where we have approximated the measure of the integral by $dk k^2 \simeq dk \mu^2$ and pulled out the factor μ^2 . We have also used the symmetry of the integrand with respect to $\xi \rightarrow -\xi$, such that we can restrict ourselves to the interval $\xi \in [0, \delta]$ and multiply the result by 2. Obviously, $\Delta_0 = 0$ is one solution of the equation. To find the nontrivial solution, we divide both sides of the equation by Δ_0 . Then we see that there must be a nonzero Δ_0 for any coupling $G > 0$, no matter how small: if Δ_0 were zero, the integral would diverge logarithmically at the lower boundary (which corresponds to the Fermi surface). This is the manifestation of Cooper's Theorem that we have explained in physical terms at the beginning of the chapter. The reason for this infrared divergence is that the integral has essentially become one-dimensional due to the restriction to momenta within a small vicinity around the Fermi surface. Therefore, a formal way of saying why the Fermi surface is unstable with respect to the formation of a Cooper pair condensate is that there is an effective dimensional reduction of the dynamics of the system from 3+1 to 1+1 dimensions.

With

$$\int \frac{d\xi}{\sqrt{\xi^2 + \Delta_0^2}} = \ln[2(\xi + \sqrt{\xi^2 + \Delta_0^2})], \quad (203)$$

we find

$$\Delta_0 \simeq 2\delta \exp \left[-\frac{2\pi^2}{G\mu^2} \right]. \quad (204)$$

This is the famous result for the BCS gap. It shows the dependence of the energy gap on the coupling: at weak coupling (only in this regime is our mean-field approximation valid, and only in this regime are we allowed to restrict ourselves to a small vicinity of the Fermi surface) the energy gap is exponentially suppressed, with the fermion-boson coupling g appearing quadratically in the exponential. This result is non-perturbative in the sense that there is no Taylor expansion around $G = 0$.

[End of 9th lecture, May 17th, 2013.]

2. Critical temperature

We may also use the gap equation to compute the critical temperature T_c for the superconducting phase transition. By definition, T_c is the point where the gap approaches zero. In BCS theory, one indeed has a second order phase transition, such that the gap vanishes continuously, not by a jump. Therefore, we may use the gap equation and imagine we are sitting at a point just below the critical temperature. At this point, the gap is still nonzero, and we may divide Eq. (200) by Δ . Then we can take the limit $\Delta \rightarrow 0$ in the resulting equation to obtain an equation for the critical temperature,

$$1 = \frac{G\mu^2}{2\pi^2} \int_0^\delta \frac{d\xi}{\xi} \tanh \frac{\xi}{2T_c}. \quad (205)$$

With the new integration variable $z = \xi/(2T_c)$ and after integration by parts we obtain

$$\begin{aligned} \frac{2\pi^2}{G\mu^2} &= \ln z \tanh z \Big|_0^{\delta/(2T_c)} - \int_0^{\delta/(2T_c)} dz \frac{\ln z}{\cosh^2 z} \\ &\simeq \ln \frac{\delta}{2T_c} - \underbrace{\int_0^\infty dz \frac{\ln z}{\cosh^2 z}}_{-\gamma + \ln \frac{\pi}{4}}. \end{aligned} \quad (206)$$

In the second step we have used that $\delta \gg T_c$. This approximation is consistent with the above assumption (201) because T_c will turn out to be of the same order as Δ_0 . We have written the result of the integral in terms of the Euler-Mascheroni constant $\gamma \simeq 0.577$. Solving the resulting equation for T_c and using Eq. (204) yields

$$T_c = \frac{e^\gamma}{\pi} \Delta_0 \simeq 0.57 \Delta_0, \quad (207)$$

i.e., the critical temperature of a BCS superfluid/superconductor is about half the zero-temperature gap. As an exercise, you may solve the gap equation numerically for all temperatures below T_c .

D. Examples

To which physical systems does the above apply? Although our formalism has been relativistic we can easily transfer the main arguments to a variety of physical systems. One reason is that the Cooper theorem is very general. All one needs is a Fermi energy that is sufficiently large compared to the temperature and an arbitrarily weak attractive interaction. In particular, for the main arguments it doesn't matter whether one starts with relativistic fermions with dispersion $\epsilon_k = \sqrt{k^2 + m^2}$ or from nonrelativistic fermions with dispersion $\epsilon_k = \frac{k^2}{2m}$. The reason is that (weak-coupling) Cooper pairing is a Fermi surface phenomenon and *at the Fermi surface* both dispersions are linear. We can expand the dispersion for momenta close to the Fermi momentum

$$\epsilon_k \simeq \epsilon_{k_F} + v_F(k - k_F), \quad (208)$$

with the Fermi energy $\epsilon_{k_F} = \mu$, and where we have defined the *Fermi velocity*

$$v_F \equiv \left. \frac{\partial \epsilon_k}{\partial k} \right|_{k=k_F} = \begin{cases} \frac{k_F}{m} & \text{non-relativistic, } k_F = \sqrt{2\mu m} \\ \frac{k_F}{\mu} & \text{relativistic, } k_F = \sqrt{\mu^2 - m^2} \\ 1 & \text{ultra-relativistic, } k_F = \mu \end{cases}, \quad (209)$$

Consequently, only the slope of the linear dispersion at the Fermi surface is different.

Before we discuss the application of the gap equation to various system, let us give a brief overview over the history of Cooper pairing. The first and most important system where Cooper pairing takes place is an electronic superconductor. Superconductivity was first observed in mercury by H. Kamerlingh Onnes in 1911. He observed a vanishing resistivity below a temperature of $T_c = 4.2$ K (by the way, he cooled his system with ^4He). In 1933, Meissner and Ochsenfeld discovered that a superconductor expels an applied magnetic field, the Meissner effect. Although some properties of superconductors could be described with phenomenological models (for instance the Ginzburg-Landau model, 1950), it took almost 50 years until the microscopic BCS theory and thus Cooper pairing of electrons was formulated. The original paper appeared in 1957, and Bardeen, Cooper, and Schrieffer were awarded the Nobel Prize in 1972. One obstacle in the understanding of electronic superconductors is of course the origin of the attractive interaction. This is provided by the lattice of ions, more precisely by phonon exchange. Although two electrons have the same charge and thus seem to repel each other, the exchange of phonons gives rise to a net attractive interaction. It is crucial that the repulsive force is screened in the crystal by the positively charged ions. Important progress had been made in the years before BCS regarding the quantitative understanding of the electron-phonon interaction, in particular by Fröhlich in 1950. The next major discovery in the field of electronic superconductors was the discovery of high-temperature superconductors in 1986 by Bednorz and Müller (Nobel Prize 1987). They found a material that shows superconducting properties for temperatures below $T \simeq 35$ K. By now, superconductors with critical temperatures as large as $T_c \simeq 130$ K have been observed. The mechanism behind high- T_c superconductivity cannot be explained with BCS theory and is still a matter of current research.

Once the mechanism of Cooper pairing had been understood, the question arose which other systems may show this effect. The most prominent one is maybe superfluid ^3He . In contrast to ^4He , ^3He atoms are fermions, i.e., they cannot show superfluidity by simply forming a Bose condensate. They first have to form Cooper pairs. Since there is no lattice of ions in liquid helium, the attractive force must come from the inter-atomic van-der-Waals interaction. Since the atoms repel each other strongly at very small distances, Cooper pairs in ^3He form in the p -wave channel. In this case, the fermions of the Cooper pair “avoid” the region of vanishing relative distance and “make use” of the attractive interaction at intermediate distance. The superfluid phase transition of ^3He was established experimentally by D. Lee, P. Osheroff, and R. Richardson in 1972. In fact, two different transitions at 2.6 mK and 1.8 mK were observed, owing to the two different superfluid phases of ^3He , called “*A phase*” and “*B phase*”.

Eventually, Cooper pairing has been observed in ultra-cold Fermi gases. Superfluidity was directly observed by vortex formation in a gas of ^6Li atoms in 2005 by W. Ketterle and his group. In this case, the critical temperature is about 200 nK. We will have to say more about these atomic systems in Sec. VI.

There are other systems where Cooper pairing is expected, but experimental evidence is hard to establish. For instance, it is assumed that neutrons are superfluid and protons superconducting in the interior of neutron stars. Indications that this is indeed the case come from various astrophysical observations, for example pulsar glitches (sudden jumps in the rotation frequency of the star which is attributed to a sudden “un-pinning” of superfluid vortices in the inner crust of the star) and cooling (recent measurements that show an unexpectedly rapid cooling over about 10 years are attributed to a superfluid phase transition, suggesting a critical temperature for neutron superfluidity of $T_c \simeq 5.5 \cdot 10^8$ K). Moreover, quarks may form a so-called “*color superconductor*” in the deconfined phase of QCD, also in the interior of neutron stars, if the density in these objects is sufficiently large. The possibility

of Cooper pairing of quarks was already mentioned in 1969, before the theory of QCD was even established. This phenomenon was termed “color superconductivity” by Bailin and Love in 1984. Only much later, in 1998, it was realized that the energy gap in quark matter may be large enough to be important for the phenomenology of compact stars. Color superconductors are expected to have critical temperatures in the range of about 10 keV – 100 MeV (corresponding to $10^8 - 10^{12}$ K), depending on the specific phase (since quarks have more quantum numbers than electrons, a multitude of phases with quark Cooper pairing is conceivable).

1. Electronic superconductor

In an electronic superconductor, we first have to replace our quasiparticle dispersion by the nonrelativistic version,

$$\epsilon_k = \sqrt{\xi_k^2 + \Delta^2}, \quad \xi_k \equiv \frac{k^2}{2m} - \mu. \quad (210)$$

The structure of the gap equation (200) remains unchanged, and the cutoff δ is naturally given by the Debye frequency ω_D . Consequently, we can write at zero temperature

$$\begin{aligned} 1 &= G \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{1}{2\sqrt{\xi_k^2 + \Delta_0^2}} \\ &= G \int_{-\omega_D}^{\omega_D} d\xi \int \frac{d^3\mathbf{k}}{(2\pi)^3} \delta(\xi - \xi_k) \frac{1}{2\sqrt{\xi_k^2 + \Delta_0^2}} \\ &= G \int_{-\omega_D}^{\omega_D} d\xi \frac{N(\xi)}{2\sqrt{\xi^2 + \Delta_0^2}}, \end{aligned} \quad (211)$$

with the density of states

$$N(\xi) \equiv \frac{k^2}{2\pi^2} \left(\frac{\partial \xi_k}{\partial k} \right)_{k=k(\xi)}^{-1}, \quad (212)$$

where $k(\xi)$ is the solution of $\xi = \xi_k$. Since the integral is dominated by a small vicinity around the Fermi surface, we have

$$1 \simeq G N(0) \int_0^{\omega_D} \frac{d\xi}{\sqrt{\xi^2 + \Delta_0^2}}, \quad N(0) = \frac{k_F m}{2\pi^2}, \quad (213)$$

with the Fermi momentum $k_F = \sqrt{2m\mu}$. Thus, in complete analogy to above,

$$\Delta_0 = 2\omega_D \exp \left[-\frac{1}{G N(0)} \right], \quad (214)$$

We now also realize that the factor $\mu^2/(2\pi^2)$ which appears in the exponential of the relativistic version, Eq. (204), is nothing but the density of states at the Fermi surface. In both non-relativistic and ultra-relativistic cases we can express the density of states in the universal form $k_F^2/(2\pi^2 v_F)$, where we have used the expressions for the Fermi velocity of Eq. (209). This means that the smaller the Fermi velocity the larger the density of states at the Fermi surface and thus the larger the superconducting gap.

2. Anisotropic superfluid, e.g., ^3He

In systems where the total angular momentum of the Cooper pair is nonzero, the gap may become anisotropic. This occurs for instance in superfluid ^3He . Let us, instead of Eq. (184), use the following ansatz for the gap matrix,

$$\Phi^\pm = \Delta(k) \hat{k}_3, \quad (215)$$

i.e., there is one preferred direction in momentum space, related to the total angular momentum of the Cooper pair, here chosen to be the 3-direction. This is not unlike the order parameter of ^3He . However, it is only a simplification

since in ^3He the order parameter is actually a 3×3 matrix in the space of spin and angular momentum (which are separately conserved in a nonrelativistic system). In this sense, our ansatz is not quite realistic. In particular, in the given approach there is no reason why the ground state should be anisotropic, i.e., Eq. (215) is a solution to the gap equation, but if we were to compute the free energy of the corresponding phase we would find it to be larger than the one of the isotropic phase. Nevertheless, we shall compute the relation between the critical temperature and the zero-temperature gap and will see that it is modified compared to the standard BCS relation (207), as it is in the anisotropic phases of ^3He .

The gap equation (179) becomes (already dropping antiparticles)

$$\Delta(p)\hat{p}_3 = g^2 \frac{T}{V} \sum_k D(p-k) \frac{\hat{k}_3 \Delta(k) \Lambda_k^-}{k_0^2 - \epsilon_k^2}, \quad (216)$$

with

$$\epsilon_k = \sqrt{(k - \mu)^2 + \hat{k}_3^2 \Delta^2}. \quad (217)$$

This dispersion shows that there are directions in momentum space where the quasifermions are ungapped. More precisely, the gap function $\hat{k}_3 \Delta$ has a nodal line at the equator of the Fermi sphere. To deal with the angular dependence, we multiply both sides with $\hat{p}_3 \Lambda_p^+$, take the trace over Dirac space and take the angular average with respect to the direction of the external vector \mathbf{p} ,

$$\langle \hat{p}_3^2 \rangle_p = -G \int \frac{d^3 \mathbf{k}}{(2\pi)^3} \frac{\hat{k}_3}{2\epsilon_k} \langle \hat{p}_3 (1 - \hat{\mathbf{k}} \cdot \hat{\mathbf{p}}) \rangle_p, \quad (218)$$

where $\langle - \rangle_p \equiv \int \frac{d\Omega_p}{4\pi}$. We have also performed the Matsubara frequency, taken the zero-temperature limit, and divided both sides of the equation by Δ (assuming that the whole angular dependence of the gap function is contained in \hat{p}_3).

Evaluating this equation at $T = 0$ and at the critical point yields a modified relation between Δ_0 and T_c ,

$$\begin{aligned} T_c &= \frac{e^\gamma}{\pi} \Delta_0 \exp \left[\frac{1}{2} \frac{\langle \hat{k}_3^2 \ln \hat{k}_3^2 \rangle}{\langle \hat{k}_3^2 \rangle} \right] \\ &= \frac{e^\gamma}{\pi} \Delta_0 e^{-\frac{1}{3}} \simeq 0.717 \frac{e^\gamma}{\pi} \Delta_0. \end{aligned} \quad (219)$$

As an exercise, determine Δ_0 and T_c for an anisotropic gap with the ansatz $\Phi^+ = \Delta \gamma_{\perp,3}$ with $\gamma_{\perp,i} \equiv \gamma_i - \hat{k}_i \boldsymbol{\gamma} \cdot \hat{\mathbf{k}}$.

3. Quark matter

Cooper pairing in quark matter has several complications compared to the cases discussed so far. The two main differences are (i) the larger number of fermionic degrees of freedom (color, flavor) and (ii) the specific form of the gluonic interaction. The general gap equation reads, in analogy to Eq. (179),

$$\Phi^+(P) = g^2 \frac{T}{V} \sum_K \gamma^\mu T_a^T F^+(K) \gamma^\nu T_b D_{\mu\nu}^{ab}(P-K), \quad (220)$$

where g is the QCD coupling constant, T_a the Gell-Mann matrices, and $D_{\mu\nu}^{ab}$ the gluon propagator. Now, regarding point (i), Φ^+ is not only a matrix in Dirac space but also in color and flavor space. Therefore, with an ansatz for the gap matrix one not only has to “guess” which spin, parity etc. the Cooper pair has, but also which colors and flavors participate in pairing and which of the quarks pairs with which other quark. It is beyond the scope of this course to get deeper into these questions. Suffices to say that it is favorable to form Cooper pairs of quarks with different colors and different flavors. Moreover, in the most favorable (and symmetric) pairing pattern *all* quarks participate in pairing. This specific color superconductor¹³ is called the color-flavor locked (CFL) state. Ignoring all

¹³ A quark-quark Cooper pair carries color charge and thus breaks the color gauge group spontaneously. Hence the name color superconductor, in analogy to the electronic superconductor which breaks the electromagnetic gauge group.

these complications, we are still left with point (ii), the effect of the fundamental QCD interaction. Again omitting all details, one arrives at a gap equation which, instead of Eq. (202), can be written as

$$\Delta_p \simeq \frac{g^2}{36\pi} \int_0^\delta d(k - \mu) \frac{\Delta_k}{\epsilon_k} \frac{1}{2} \ln \frac{b^2 \mu^2}{|\epsilon_k^2 - \epsilon_p^2|}, \quad (221)$$

where the gap depends on momentum, and where $b \equiv 256\pi^4 [2/(N_f g^2)]^{5/2}$. This different structure, arising from the specific form of the gluon propagator $D_{\mu\nu}^{ab}(P - K)$ (more physically speaking, from the long-range interaction mediated by magnetic gluons), has a crucial consequence for the dependence of the gap on the coupling. One finds for the value of the gap at the Fermi surface (see for instance Ref. [21])

$$\Delta_0 = 2b\mu \exp\left(-\frac{3\pi^2}{\sqrt{2}g}\right). \quad (222)$$

Consequently, the QCD gap is parametrically larger ($e^{-\text{const}/g}$ vs. $e^{-\text{const}/g^2}$) than the BCS gap.

[End of 10th lecture, Jun 3rd, 2013.]

V. MEISSNER EFFECT

Superconductors expel externally applied magnetic fields. This is called the Meissner effect. More precisely, the magnetic field is screened in the superconductor like $B \propto e^{-x/\lambda}$ with the *penetration depth* λ . In field-theoretical terms, the photon (more generally, the gauge boson that “sees” the charge of the Cooper pair; e.g., gluons in a color superconductor) acquires a magnetic mass, called the *Meissner mass* m_M . This mass is nothing but the inverse penetration depth, $m_M = \lambda^{-1}$. In the following we want to compute this Meissner mass within the field-theoretical framework that we have developed above. Before doing so we explain the occurrence of a massive gauge boson in a spontaneously broken symmetry in general terms.

A. Massive gauge boson

We have discussed in Sec. III that a spontaneously broken global symmetry results in a massless mode, the Goldstone mode. We have explicitly constructed the dispersion for this mode in a φ^4 model with a chemical potential. Let us revisit this problem in a slightly simpler formulation. Consider the Lagrangian

$$\mathcal{L} = \partial_\mu \varphi^* \partial^\mu \varphi + \eta^2 |\varphi|^2 - \lambda |\varphi|^4. \quad (223)$$

Instead of introducing a mass m and a chemical potential μ we have taken into account the possibility of Bose condensation with the new parameter η^2 . It plays, roughly speaking, the role of $\mu^2 - m^2$, such that we expect Bose condensation for $\eta^2 > 0$. Since η is taken to be real, a nonzero expectation value for φ will occur at sufficiently small temperatures. This Lagrangian is invariant under the global $U(1)$ symmetry

$$\varphi \rightarrow e^{i\alpha} \varphi. \quad (224)$$

Let us introduce polar coordinates,

$$\varphi = \frac{\rho}{\sqrt{2}} e^{i\psi}. \quad (225)$$

In this parametrization, the Lagrangian becomes

$$\mathcal{L} = \frac{1}{2} \partial_\mu \rho \partial^\mu \rho + \frac{\rho^2}{2} \partial_\mu \psi \partial^\mu \psi + \frac{\eta^2}{2} \rho^2 - \frac{\lambda}{4} \rho^4. \quad (226)$$

Now we separate the condensate v from the fluctuations and assume the condensate to be constant in space and time, $\rho(X) = v + \chi(X)$, where

$$v^2 = \frac{\eta^2}{\lambda}. \quad (227)$$

(In analogy to Sec. III, where the condensate squared was $(\mu^2 - m^2)/\lambda$.) This yields

$$\mathcal{L} = \frac{1}{2}\partial_\mu\chi\partial^\mu\chi + \frac{(v+\chi)^2}{2}\partial_\mu\psi\partial^\mu\psi - \eta^2\chi^2 - \sqrt{\lambda}\eta\chi^3 - \frac{\lambda}{4}\chi^4 + \frac{\eta^4}{4\lambda}. \quad (228)$$

This shows, in a quick way, that there is a massive mode χ with mass term $-\eta^2\chi^2$, and a massless mode ψ . This is the Goldstone mode. All other terms in the Lagrangian are interaction terms between ψ and χ or self-interactions of χ (plus one constant term that is independent of all fields).

Now let's make the symmetry local, i.e., we now require the Lagrangian to be invariant under space-time dependent transformations $e^{i\alpha(X)}$. To this end, it is necessary to introduce a covariant derivative, i.e., a gauge field A_μ ,

$$\mathcal{L} = (D_\mu\varphi)^*D^\mu\varphi + \eta^2|\varphi|^2 - \lambda|\varphi|^4 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \quad (229)$$

with the covariant derivative $D_\mu = \partial_\mu - igA_\mu$ and the field strength tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Now the Lagrangian is invariant under *local* $U(1)$ transformations

$$\varphi \rightarrow e^{i\alpha(X)}\varphi, \quad A_\mu \rightarrow A_\mu + \frac{1}{g}\partial_\mu\alpha. \quad (230)$$

With the parametrization of Eq. (225) we obtain

$$\mathcal{L} = \frac{1}{2}\partial_\mu\rho\partial^\mu\rho + \frac{g^2\rho^2}{2}\left(A_\mu - \frac{1}{g}\partial_\mu\psi\right)\left(A^\mu - \frac{1}{g}\partial^\mu\psi\right) + \frac{\eta^2}{2}\rho^2 - \frac{\lambda}{4}\rho^4 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}. \quad (231)$$

There are certain terms that mix the gauge field A_μ with the Goldstone field ψ . We may however define the gauge invariant combination

$$B_\mu \equiv A_\mu - \frac{1}{g}\partial_\mu\psi, \quad (232)$$

as our new gauge field (notice that the phase ψ transforms as $\psi \rightarrow \psi + \alpha$). Then, we obtain with the same replacement $\rho = v + \chi$ as above,

$$\mathcal{L} = \frac{1}{2}\partial_\mu\chi\partial^\mu\chi + \frac{g^2v^2}{2}B_\mu B^\mu + g^2v\chi B_\mu B^\mu + \frac{g^2}{2}\chi^2 B_\mu B^\mu - \eta^2\chi^2 - \sqrt{\lambda}\eta\chi^3 - \frac{\lambda}{4}\chi^4 + \frac{\eta^4}{4\lambda} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}. \quad (233)$$

(The term with the field strength tensors is the same in terms of the new gauge fields, $\partial_\mu A_\nu - \partial_\nu A_\mu = \partial_\mu B_\nu - \partial_\nu B_\mu$.) This result has to be compared to Eq. (228): the Goldstone mode has disappeared! It has been “eaten up” by the gauge field, which has acquired a mass term with mass gv . In our context, this corresponds to the Meissner mass.

It is instructive to count degrees of freedom in both cases.

- Global $U(1)$ symmetry: we start with 2 degrees of freedom, represented by the complex field φ . After spontaneous symmetry breaking, we obtain 1 massive mode χ and 1 massless mode ψ .
- Local $U(1)$ symmetry: here we start with 2 degrees of freedom from the complex field φ plus 2 degrees of freedom of the massless gauge field A_μ . Spontaneous symmetry breaking leads to 1 massive mode χ plus 3 degrees of freedom of the now massive gauge field B_μ . So we end up with the same number of degrees of freedom, $2 + 2 = 1 + 3$, as it should be. There is no Goldstone mode.

This mechanism is very general and applies also to more complicated gauge groups, for instance in the context of electroweak symmetry breaking¹⁴.

¹⁴ In this case, the symmetry breaking pattern is $SU(2) \times U(1) \rightarrow U(1)$, i.e., there are 3+1 massless gauge fields to start with. Together with a complex doublet, the Higgs doublet, there are $2 \times (3 + 1) + 4 = 12$ degrees of freedom. After symmetry breaking, 3 gauge fields have become massive (eating up the 3 would-be Goldstone modes) and 1 remains massless. The massive gauge fields correspond to the W^\pm and the Z bosons, while the massless gauge field is the photon. One massive scalar is also left, i.e., there are $3 \times 3 + 2 \times 1 + 1 = 12$ degrees of freedom.

B. Meissner mass from the one-loop polarization tensor

For the existence of a Meissner mass it is important that the condensate is charged under a gauge symmetry. It is not important whether the microscopic degrees of freedom are bosonic or fermionic. In other words, we need a superconductor, not a superfluid, but it is irrelevant whether we have Bose-Einstein condensation or Cooper pairing.

To compute the Meissner mass explicitly we now return to our fermionic formalism from the previous subsections. This formalism did not have a gauge boson which we shall now add. The boson that we had introduced to account for the attractive interaction between the fermions does not have to be identical with the boson that becomes massive. And in our case it isn't since we simply introduced a scalar boson, not a gauge boson. Also in the case of a usual electronic superconductor, these are indeed two different bosons, the phonon and the photon. In a color superconductor, the gluons that provide the interaction between the quarks are also the gauge bosons that acquire a Meissner mass.

The Meissner mass (squared) is defined as

$$m_M^2 = \frac{1}{2} \lim_{p \rightarrow 0} (\delta_{ij} - \hat{p}_i \hat{p}_j) \Pi^{ij}(0, \mathbf{p}), \quad (234)$$

where $\Pi^{\mu\nu}(P)$ is the photon polarization tensor¹⁵. At one loop level, it is computed from

$$\Pi^{\mu\nu}(P) = \frac{1}{2} \frac{T}{V} \sum_K \text{Tr}[\Gamma^\mu \mathcal{S}(K) \Gamma^\nu \mathcal{S}(K - P)], \quad (235)$$

where the trace runs over Nambu-Gorkov and Dirac space, and where the photon-fermion vertex in Nambu-Gorkov space is

$$\Gamma^\mu = \begin{pmatrix} e\gamma^\mu & 0 \\ 0 & -e\gamma^\mu \end{pmatrix}. \quad (236)$$

Let us for the following abbreviate

$$Q \equiv K - P. \quad (237)$$

With the Nambu-Gorkov propagator (174) the trace over Nambu-Gorkov space yields

$$\Pi^{\mu\nu}(P) = \frac{e^2 T}{2 V} \sum_K \text{Tr}[\gamma^\mu G^+(K) \gamma^\nu G^+(Q) + \gamma^\mu G^-(K) \gamma^\nu G^-(Q) - \gamma^\mu F^-(K) \gamma^\nu F^+(Q) - \gamma^\mu F^+(K) \gamma^\nu F^-(Q)]. \quad (238)$$

Let us now abbreviate

$$\xi_k^e \equiv k - e\mu, \quad (239)$$

such that the propagators from Eqs. (185) and (187) become

$$G^\pm(K) = \sum_e \frac{k_0 \pm e\xi_k^e}{k_0^2 - (\xi_k^e)^2} \gamma^0 \Lambda_k^{\mp e}, \quad F^\pm(K) = \pm \sum_e \frac{\Delta}{k_0^2 - (\xi_k^e)^2} \gamma^5 \Lambda_k^{\mp e}. \quad (240)$$

For the spatial components $\mu = i, \nu = j$ – which are the only ones that we are interested in – we need the Dirac traces

$$\text{Tr}[\gamma^i \gamma^0 \Lambda_k^{-e_1} \gamma^j \gamma^0 \Lambda_q^{-e_2}] = \text{Tr}[\gamma^i \gamma^5 \Lambda_k^{e_1} \gamma^j \gamma^5 \Lambda_q^{-e_2}] = \delta^{ij} (1 - e_1 e_2 \hat{\mathbf{k}} \cdot \hat{\mathbf{q}}) + e_1 e_2 (\hat{k}^i \hat{q}^j + \hat{k}^j \hat{q}^i), \quad (241)$$

where we have used

$$\text{Tr}[\gamma^i \gamma^j] = -4\delta^{ij}, \quad \text{Tr}[\gamma^i \gamma^j \gamma^k \gamma^\ell] = 4(\delta^{ij} \delta^{k\ell} + \delta^{i\ell} \delta^{jk} - \delta^{ik} \delta^{j\ell}). \quad (242)$$

¹⁵ The Meissner mass is the *magnetic* screening mass and thus related to the spatial components of the polarization tensor. In contrast, the *electric* screening mass is related to the temporal components. It is called the Debye mass and defined as

$$m_D^2 = - \lim_{p \rightarrow 0} \Pi^{00}(0, \mathbf{p}).$$

It is nonvanishing even without superconductivity and thus not of special interest in our context. It is left as an exercise to compute m_D .

Consequently,

$$\Pi^{ij}(P) = e^2 \frac{T}{V} \sum_{e_1 e_2} \sum_K [\delta^{ij} (1 - e_1 e_2 \hat{\mathbf{k}} \cdot \hat{\mathbf{q}}) + e_1 e_2 (\hat{k}^i \hat{q}^j + \hat{k}^j \hat{q}^i)] \frac{k_0 q_0 + e_1 e_2 \xi_k^{e_1} \xi_q^{e_2} + \Delta^2}{[k_0^2 - (\epsilon_k^{e_1})^2][q_0^2 - (\epsilon_q^{e_2})^2]}. \quad (243)$$

We now perform the Matsubara sum over fermionic Matsubara frequencies $k_0 = -(2n+1)i\pi T$ and use that $p_0 = -2mi\pi T$ from the external four-momentum is a bosonic Matsubara frequency. This yields (the explicit calculation in terms of a contour integral in the complex k_0 plane is left as an exercise)

$$\begin{aligned} \Pi^{ij}(P) &= \frac{e^2}{4} \sum_{e_1 e_2} \int \frac{d^3 \mathbf{k}}{(2\pi)^3} [\delta^{ij} (1 - e_1 e_2 \hat{\mathbf{k}} \cdot \hat{\mathbf{q}}) + e_1 e_2 (\hat{k}^i \hat{q}^j + \hat{k}^j \hat{q}^i)] \\ &\times \left\{ \frac{\epsilon_k^{e_1} \epsilon_q^{e_2} + e_1 e_2 \xi_k^{e_1} \xi_q^{e_2} + \Delta^2}{\epsilon_k^{e_1} \epsilon_q^{e_2}} \left(\frac{1}{p_0 + \epsilon_k^{e_1} - \epsilon_q^{e_2}} - \frac{1}{p_0 - \epsilon_k^{e_1} + \epsilon_q^{e_2}} \right) [f(\epsilon_k^{e_1}) - f(\epsilon_q^{e_2})] \right. \\ &\left. + \frac{\epsilon_k^{e_1} \epsilon_q^{e_2} - e_1 e_2 \xi_k^{e_1} \xi_q^{e_2} - \Delta^2}{\epsilon_k^{e_1} \epsilon_q^{e_2}} \left(\frac{1}{p_0 - \epsilon_k^{e_1} - \epsilon_q^{e_2}} - \frac{1}{p_0 + \epsilon_k^{e_1} + \epsilon_q^{e_2}} \right) [1 - f(\epsilon_k^{e_1}) - f(\epsilon_q^{e_2})] \right\}. \quad (244) \end{aligned}$$

This looks complicated, but will simplify immediately since we are not interested in the full energy and momentum dependence of the polarization tensor. According to the definition of the Meissner mass, we now set $p_0 = 0$ and take the limit $\mathbf{p} \rightarrow 0$ which corresponds to $\mathbf{q} \rightarrow \mathbf{k}$. First, with $p_0 = 0$ we have

$$\begin{aligned} \Pi^{ij}(0, \mathbf{p}) &= \frac{e^2}{2} \sum_{e_1 e_2} \int \frac{d^3 \mathbf{k}}{(2\pi)^3} [\delta^{ij} (1 - e_1 e_2 \hat{\mathbf{k}} \cdot \hat{\mathbf{q}}) + e_1 e_2 (\hat{k}^i \hat{q}^j + \hat{k}^j \hat{q}^i)] \\ &\times \left(\frac{\epsilon_k^{e_1} \epsilon_q^{e_2} + e_1 e_2 \xi_k^{e_1} \xi_q^{e_2} + \Delta^2}{\epsilon_k^{e_1} \epsilon_q^{e_2}} \frac{f(\epsilon_k^{e_1}) - f(\epsilon_q^{e_2})}{\epsilon_k^{e_1} - \epsilon_q^{e_2}} - \frac{\epsilon_k^{e_1} \epsilon_q^{e_2} - e_1 e_2 \xi_k^{e_1} \xi_q^{e_2} - \Delta^2}{\epsilon_k^{e_1} \epsilon_q^{e_2}} \frac{1 - f(\epsilon_k^{e_1}) - f(\epsilon_q^{e_2})}{\epsilon_k^{e_1} + \epsilon_q^{e_2}} \right) \quad (245) \end{aligned}$$

Then, with $\mathbf{q} \rightarrow \mathbf{k}$ and the angular integral

$$\int \frac{d\Omega}{4\pi} \hat{k}^i \hat{k}^j = \frac{\delta^{ij}}{3}, \quad (246)$$

we may replace

$$\delta^{ij} (1 - e_1 e_2 \hat{\mathbf{k}} \cdot \hat{\mathbf{q}}) + e_1 e_2 (\hat{k}^i \hat{q}^j + \hat{k}^j \hat{q}^i) \rightarrow \begin{cases} \frac{2}{3} \delta^{ij} & \text{for } e_1 = e_2 \\ \frac{4}{3} \delta^{ij} & \text{for } e_1 \neq e_2 \end{cases}. \quad (247)$$

Consequently, we find

$$\begin{aligned} \Pi^{ij}(0, \mathbf{p} \rightarrow 0) &= \frac{2\delta^{ij} e^2}{3\pi^2} \int_0^\infty dk k^2 \left[\frac{df(\epsilon_k^+)}{d\epsilon_k^+} + \frac{df(\epsilon_k^-)}{d\epsilon_k^-} + 2 \frac{\epsilon_k^+ \epsilon_k^- - \xi_k^+ \xi_k^- + \Delta^2}{\epsilon_k^+ \epsilon_k^-} \frac{f(\epsilon_k^+) - f(\epsilon_k^-)}{\epsilon_k^+ - \epsilon_k^-} \right. \\ &\quad \left. - 2 \frac{\epsilon_k^+ \epsilon_k^- + \xi_k^+ \xi_k^- - \Delta^2}{\epsilon_k^+ \epsilon_k^-} \frac{1 - f(\epsilon_k^+) - f(\epsilon_k^-)}{\epsilon_k^+ + \epsilon_k^-} \right]. \quad (248) \end{aligned}$$

At zero temperature this becomes

$$\Pi^{ij}(0, \mathbf{p} \rightarrow 0) = -\frac{4\delta^{ij} e^2}{3\pi^2} \int_0^\infty dk k^2 \frac{\epsilon_k^+ \epsilon_k^- + \xi_k^+ \xi_k^- - \Delta^2}{\epsilon_k^+ \epsilon_k^- (\epsilon_k^+ + \epsilon_k^-)}. \quad (249)$$

This expression has to be renormalized. Usually, one has to subtract the vacuum contribution $\mu = T = 0$. We have already taken the zero-temperature limit, so we are left with $\mu = 0$. Since the gap vanishes for $\mu = 0$ this implies that one expects to set $\Delta = 0$ in the vacuum subtraction. However, here we need to take into account that we have ignored any momentum dependence of the gap. Such a full dependence would allow us to subtract the vacuum contribution

straightforwardly. Our simplified treatment of the gap now requires that we subtract the $\mu = T = 0$ contribution *with nonzero gap* Δ .

The integral in Eq. (249) can be performed exactly,

$$\int dk k^2 \frac{\epsilon_k^+ \epsilon_k^- + \xi_k^+ \xi_k^- - \Delta^2}{\epsilon_k^+ \epsilon_k^- (\epsilon_k^+ + \epsilon_k^-)} = \frac{1}{4} (\xi_k^+ \epsilon_k^- + \xi_k^- \epsilon_k^+) - \frac{\Delta^2}{2\mu} (\epsilon_k^+ - \epsilon_k^-) - \frac{3\Delta^2}{4} (\ln[2(\xi_k^+ + \epsilon_k^+)] + \ln[2(\xi_k^- + \epsilon_k^-)]) . \quad (250)$$

With the renormalization prescription just explained we obtain

$$\Pi_{\text{ren}}^{ij}(0, \mathbf{p} \rightarrow 0) = \frac{2\delta^{ij} e^2 \mu^2}{3\pi^2} , \quad (251)$$

and thus the Meissner mass (squared) is

$$m_M^2 = \frac{2e^2 \mu^2}{3\pi^2} . \quad (252)$$

Note that this result is independent of the gap Δ . This does not contradict the fact that the Meissner mass vanishes for vanishing gap, as it should be. The reason is that we have taken the limit $\mathbf{p} \rightarrow 0$ for a fixed nonzero Δ . If we first take $\Delta \rightarrow 0$ we find indeed $m_M = 0$. This calculation is left as an exercise.

[End of 11th lecture, Jun 10th, 2013.]

VI. BCS-BEC CROSSOVER

We have discussed bosonic and fermionic superfluids. The underlying mechanisms were, on the one hand, Bose-Einstein condensation (BEC) and, on the other hand, Cooper pairing according to BCS theory. We have seen that both cases are described in terms of spontaneous symmetry breaking, in the simplest cases breaking of a $U(1)$. We have also mentioned that fermionic superfluidity is, in some sense, also a form of BEC because a Cooper pair can be considered as a boson. In this section we want to make this connection more precise. We shall see that BEC and BCS are continuously connected. This is a very general statement and in principle may apply to a variety of physical systems. Theoretical works indicating that BEC is a limit of the very general BCS theory have been pioneered for instance by Eagles in 1969 and Nozieres/Schmitt-Rink in 1984. Despite its theoretical generality, we shall put the following in the context of ultra-cold atomic gases, where the BEC-BCS crossover has first been demonstrated experimentally and which since then has remained an extremely active research field.

A. Ultra-cold atomic gases

Experiments with ultra-cold fermionic gases have been based on the experience gained from similar experiments with ultra-cold bosonic gases, which, as we have already mentioned, have led to the first direct observation of Bose-Einstein condensation in 1995. Interest in the fermionic counterparts has begun in the 1990's and around 2003 several groups had established the creation of ultra-cold Fermi gases, most notably groups at Rice University in Houston (R. Hulet), in Boulder, Colorado (D. Jin), at MIT (W. Ketterle), at Ecole Normale Supérieure in Paris (C. Salomon), at Duke University, North Carolina (J. Thomas), and in Innsbruck (R. Grimm). The fermions used in these experiments are usually ^{40}K or ^6Li . If you are interested in the details of these experiments I recommend Ref. [5], where also large parts of the theory are laid out. Another nice review with more emphasis on the theory is Ref. [22]. The main characteristics of the systems created in all these experiments are the low temperature and the diluteness. After several stages of different cooling techniques, the fermionic gases are brought down to temperatures $T \sim 50 \text{ nK}$ at densities of $n \sim 5 \times 10^{12} \text{ cm}^{-3}$. They are cooled in a so-called *optical trap*, typically $N \sim 10^7$ atoms in a trap with diameter $d \sim 100 \mu\text{m}$ (the trap is not isotropic, but rather cigar shaped). The low temperature and diluteness are crucial for the properties of the gas. In this regime, both the thermal wavelength $\lambda = \sqrt{2\pi/mT}$, where m is the mass of a single atom, and the mean inter-particle distance $n^{-1/3} \sim 10^4 a_0$ are much larger than the spatial range of the inter-atomic potential $R_0 \sim 50 a_0$ with the Bohr radius a_0 . As a consequence, the complicated details of the short-range interaction potential become unimportant and the interaction is basically characterized by one single quantity, the scattering length a . This scattering length is under experimental control and can be varied at will through a magnetic field,

$$a(B) = a_{\text{bg}} \left(1 - \frac{\Delta B}{B_0 - B} \right) , \quad (253)$$

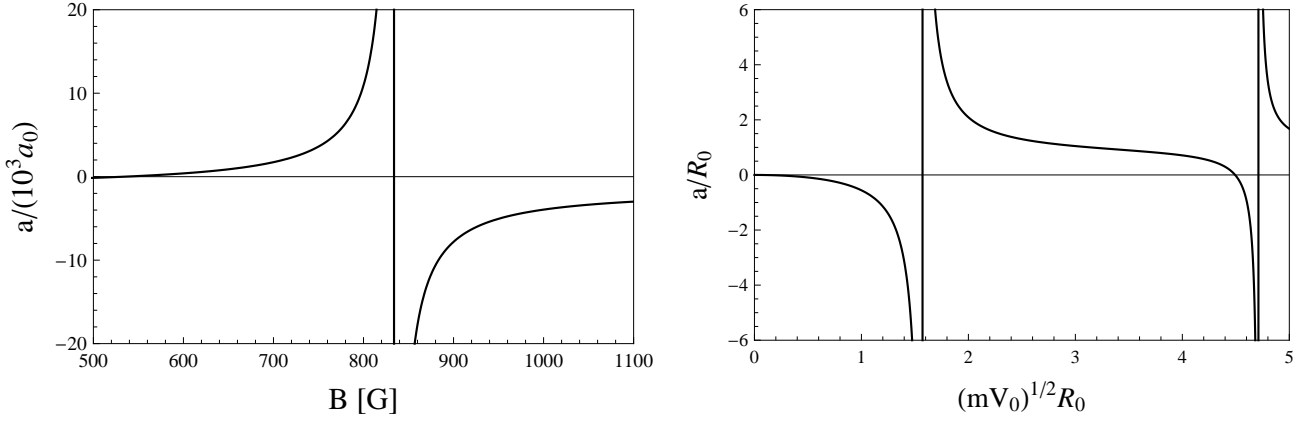


FIG. 9: Left panel: Feshbach resonance. Scattering length in units of 10^3 times the Bohr radius a_0 as a function of the applied magnetic field in Gauss according to the parametrization (253) with the numerical values for ${}^6\text{Li}$, $B_0 = 834.15$ G, $\Delta B = 300$ G, $a_{\text{bg}} = -1405 a_0$. Right panel: scattering length for a square-well potential (259) according to Eq. (260) as a function of the potential depth. The scattering length diverges whenever a new bound state enters the system.

see left panel of Fig. 9. This parametrization describes the so-called *Feshbach resonance* at $B = B_0$ with a width ΔB and a background scattering length a_{bg} far away from the resonance. At $B = B_0$ the scattering length is infinite. This is called the *unitary limit*. The unitary limit is particularly interesting since the length scale of the scattering length drops out. Therefore the unitary limit is of very general significance. For instance, in the very dense nuclear matter inside a neutron star, neutrons have a scattering length larger than their mean inter-particle distance, and parallels to the unitary limit in ultra-cold atoms may help to understand this otherwise barely accessible system.

Although we are not aiming for a complete understanding of the atomic physics involved in the experiments we give a short reminder of how the scattering length is defined (you may consult Ref. [23] to remind yourself of the basics of scattering theory): the Schrödinger equation for scattering of two particles with masses m_1 , m_2 and reduced mass $m_r \equiv m_1 m_2 / (m_1 + m_2)$ can be written in the center-of-mass frame in terms of the scattering potential $V(\mathbf{r})$,

$$\left[-\frac{\nabla^2}{m} + V(\mathbf{r}) \right] \psi_k(\mathbf{r}) = E \psi_k(\mathbf{r}), \quad (254)$$

with $m \equiv 2m_r$ being the mass of a single atom in the case $m_1 = m_2$. If the incoming particle moves along the z -axis and the angle between the z -axis and the scattered particle is denoted by θ , the solution of the Schrödinger equation at large distances is written as

$$\psi_k(\mathbf{r}) \simeq e^{ikz} + f_k(\theta) \frac{e^{ikr}}{r}, \quad (255)$$

with the *scattering amplitude* $f_k(\theta)$, which determines the differential cross section $d\sigma = |f_k(\theta)|^2 d\Omega$. From the general expression

$$f_k(\theta) = \frac{1}{2ik} \sum_{\ell=0}^{\infty} (2\ell+1) [e^{2i\delta_\ell(k)} - 1] P_\ell(\cos \theta), \quad (256)$$

where $\delta_\ell(k)$ is the *phase shift* of the collision and P_ℓ the Legendre polynomials, we are only interested in the *s*-wave scattering amplitude

$$f_k(\theta) \simeq f_s(k) = \frac{1}{2ik} [e^{2i\delta_s(k)} - 1] = \frac{1}{k \cot \delta_s(k) - ik} \simeq \frac{1}{-\frac{1}{a} + R_0 \frac{k^2}{2} - ik}. \quad (257)$$

Here we have introduced the scattering length a and the effective range of the potential R_0 which appear as coefficients in the low-momentum expansion of $k \cot \delta_s(k)$, i.e., we may define the scattering length as

$$a = - \lim_{k \ll R_0^{-1}} \frac{\tan \delta_s(k)}{k}. \quad (258)$$

In our context we have $R_0 \ll k^{-1}$, as discussed above. Therefore, for very small scattering lengths, $k|a| \ll 1$, the scattering amplitude is $f_s \simeq -a$ while for large scattering lengths, i.e., in the unitary limit $k|a| \gg 1$, we have $f_s \simeq i/k$.

To illustrate the meaning of the scattering length, it is useful to consider an attractive square-well scattering potential

$$V(\mathbf{r}) = -V_0 \Theta(R_0 - r). \quad (259)$$

(Below, we shall rather work with the point-like $V(\mathbf{r}) = V_0 \delta(\mathbf{r})$.) In this case, one computes the scattering length [23]

$$a = R_0 \left[1 - \frac{\tan(R_0 \sqrt{mV_0})}{R_0 \sqrt{mV_0}} \right]. \quad (260)$$

As shown in the right panel of Fig. 9, at very shallow potentials the scattering length starts off with small negative values. With increasing depth of the potential, it becomes more and more negative, until it diverges at $R_0 \sqrt{mV_0} = \pi/2$. This is the point where the first bound state develops. Then the scattering length is large and positive until the next bound state approaches etc.

Before we come to the discussion of superfluidity in the atomic system, let us compute the energy E of a shallow bound state. By “shallow” we mean $E = -\frac{\kappa^2}{m}$ with $\kappa \ll R_0^{-1}$. From the Schrödinger equation (254) we obtain after Fourier transformation [and denoting the Fourier transform of $V(\mathbf{r})$ by $v(\mathbf{q})$]

$$\begin{aligned} \int \frac{d^3 \mathbf{q}}{(2\pi)^3} (q^2 + \kappa^2) e^{i\mathbf{q} \cdot \mathbf{r}} \psi_\kappa(\mathbf{q}) &= -m \int \frac{d^3 \mathbf{p}}{(2\pi)^3} \int \frac{d^3 \mathbf{k}}{(2\pi)^3} e^{i(\mathbf{p}+\mathbf{k}) \cdot \mathbf{r}} v(\mathbf{p}) \psi_\kappa(\mathbf{k}) \\ &= -m \int \frac{d^3 \mathbf{q}}{(2\pi)^3} \int \frac{d^3 \mathbf{k}}{(2\pi)^3} e^{i\mathbf{q} \cdot \mathbf{r}} v(\mathbf{q} - \mathbf{k}) \psi_\kappa(\mathbf{k}), \end{aligned} \quad (261)$$

and thus

$$\begin{aligned} \psi_\kappa(\mathbf{q}) &= -\frac{m}{q^2 + \kappa^2} \int \frac{d^3 \mathbf{k}}{(2\pi)^3} v(\mathbf{q} - \mathbf{k}) \psi_\kappa(\mathbf{k}) \\ &\simeq -\frac{mv_0}{q^2 + \kappa^2} \int \frac{d^3 \mathbf{k}}{(2\pi)^3} \psi_\kappa(\mathbf{k}), \end{aligned} \quad (262)$$

where we have used that, for small momenta, the scattering potential is approximated by a δ -function, $V(\mathbf{r}) \propto \delta(\mathbf{r})$ in position space and thus by a constant v_0 in momentum space. Integrating both sides over \mathbf{q} then yields

$$-\frac{1}{v_0} = m \int \frac{d^3 \mathbf{q}}{(2\pi)^3} \frac{1}{q^2 + \kappa^2}. \quad (263)$$

The integral on the right-hand side is divergent. This is due to our use of the point-like potential, where we did not care about large momenta. The physical potential is not constant in momentum space for all momenta. We thus need to renormalize our potential which can be done by the prescription

$$\frac{1}{v_0} = \frac{m}{4\pi a} - \int \frac{d^3 \mathbf{q}}{(2\pi)^3} \frac{m}{q^2}. \quad (264)$$

This can be viewed as going from the bare coupling v_0 to a physical coupling given by the scattering length a : if we “switch on” the (divergent) second term on the right-hand side we need to adjust the bare coupling in order to keep the physical coupling fixed. (To derive the relation between the potential and the scattering length one derives a relation between the scattering amplitude and the potential, see for instance Ref. [5].) With this prescription we obtain

$$-\frac{m}{4\pi a} = \frac{m}{2\pi^2} \underbrace{\int_0^\infty dq \left(\frac{q^2}{q^2 + \kappa^2} - 1 \right)}_{-\kappa\pi/2} = -\frac{\kappa m}{4\pi}. \quad (265)$$

Consequently, $\kappa = a^{-1}$, and we need a to be positive for the bound state to exist. The energy of the bound state is

$$E = -\frac{1}{ma^2}. \quad (266)$$

We shall come back to this result later in the interpretation of the BCS-BEC crossover.

B. Crossover in the mean-field approximation

At sufficiently small temperatures, the atoms in the optical trap become superfluid. In this subsection we are interested in their behavior as a function of the scattering length a . The product $k_F a$ with the Fermi momentum k_F will play the role of an effective, dimensionless coupling constant. In this way we generalize the weak-coupling BCS theory from Sec. IV to arbitrary values of the coupling. Since we keep using the mean-field approximation, the results will have to be taken with some care. Especially at nonzero temperature, we shall see that our approach does not provide a complete description of the system. For zero temperature, however, the mean-field approximation is, at least qualitatively, correct.

For an effective four-point coupling between the fermions we can write our gap equation (179) as

$$\Phi^+ = -v_0 \frac{T}{V} \sum_K F^+(K). \quad (267)$$

Now v_0 plays the role of the (bare) coupling strength, instead of G^2 in Sec. IV. Note that both quantities have mass dimensions -2 . We can easily transfer our relativistic formalism to a nonrelativistic situation. Instead of fermions with four degrees of freedom we consider fermions with two degrees of freedom. This can be spin, such that we simply neglect the antiparticle degrees of freedom compared to the relativistic case. Or, more generally, we can speak of two fermion species. They may have different masses and chemical potentials. Here we shall restrict ourselves to fermions which are only distinguished by being of species 1 or 2, but having identical masses and chemical potentials (for Cooper pairing of fermions with different chemical potentials see Sec. VII). The distinction of two species (or two spin degrees of freedom) is necessary since the Cooper pair wave function has to be antisymmetric. This can only be achieved with at least one quantum number that distinguishes the constituents of a Cooper pair. In the experimental setup of ultra-cold atoms, the two species are provided by two hyperfine states of the respective fermionic atom or by two different atom species.

Assuming equal masses and chemical potentials, the tree-level propagator is proportional to the unit matrix in this internal “spin space”,

$$[G_0^\pm(K)]^{-1} = (k_0 \mp \xi_k) \cdot \mathbf{1}_{\text{spin}}, \quad \xi_k \equiv \frac{k^2}{2m} - \mu, \quad (268)$$

where we have replaced the ultrarelativistic dispersion of the previous sections by a nonrelativistic one, cf. Eq. (167) for the relativistic version of the tree-level propagator. Our ansatz for the gap matrix is

$$\Phi^+ = \Delta \sigma_2, \quad (269)$$

where the second Pauli matrix σ_2 takes into account that fermions of different species form Cooper pairs. Then one can easily compute the components of the Nambu-Gorkov propagator,

$$G^\pm(K) = ([G_0^\pm]^{-1} - \Phi^\mp G_0^\mp \Phi^\pm)^{-1} = \frac{k_0 \pm \xi_k}{k_0^2 - \epsilon_k^2}, \quad (270a)$$

$$F^\pm(K) = -G_0^\mp \Phi^\pm G^\pm = -\frac{\Delta \sigma_2}{k_0^2 - \epsilon_k^2}, \quad (270b)$$

with

$$\epsilon_k = \sqrt{\xi_k^2 + \Delta^2}. \quad (271)$$

Consequently, the gap equation becomes [using the Matsubara sum from Eq. (192)],

$$-\frac{1}{v_0} = \int \frac{d^3 \mathbf{k}}{(2\pi)^3} \frac{\tanh \frac{\epsilon_k}{2T}}{2\epsilon_k}. \quad (272)$$

With the renormalization given in Eq. (264) we obtain

$$-\frac{m}{4\pi a} = \int \frac{d^3 \mathbf{k}}{(2\pi)^3} \left(\frac{\tanh \frac{\epsilon_k}{2T}}{2\epsilon_k} - \frac{m}{k^2} \right). \quad (273)$$

(For comparison, remember that in an electronic superconductor it is the Debye frequency which provides a natural cut off.)

It is convenient to express the gap equation in terms of the Fermi momentum and the Fermi energy

$$k_F = (3\pi^2 n)^{1/3}, \quad E_F = \frac{k_F^2}{2m} = \frac{(3\pi^2 n)^{2/3}}{2m}. \quad (274)$$

They are defined in terms of the density n , not the chemical potential since, in the context of cold atoms, the density is fixed. Then, taking the zero-temperature limit and changing the integration variable in the gap equation from k to $x = k/\sqrt{2m\Delta}$, we can write the gap equation as

$$-\frac{1}{k_F a} = \frac{2}{\pi} \sqrt{\frac{\Delta}{E_F}} I_1\left(\frac{\mu}{\Delta}\right), \quad (275)$$

with the abbreviation

$$I_1(z) \equiv \int_0^\infty dx x^2 \left(\frac{1}{\sqrt{(x^2 - z)^2 + 1}} - \frac{1}{x^2} \right). \quad (276)$$

The dimensionless combination $k_F a$ plays the role of a dimensionless coupling constant.

[End of 12th lecture, Jun 17th, 2013.]

Besides the gap equation we have a second equation that arises from fixing the number density, and we need to solve both equations for Δ and μ . For the second equation we compute the number density from Eq. (190),

$$\begin{aligned} n &= \frac{1}{2V} \sum_K \text{Tr} \left[\mathcal{S}(K) \frac{\partial \mathcal{S}^{-1}(K)}{\partial \mu} \right] = \frac{1}{2V} \sum_K \text{Tr} [G^+(K) - G^-(K)] \\ &= 2 \frac{T}{V} \sum_K \frac{\xi_k}{k_0^2 - \epsilon_k^2} = - \int \frac{d^3 \mathbf{k}}{(2\pi)^3} \frac{\xi_k}{\epsilon_k} \tanh \frac{\epsilon_k}{2T}. \end{aligned} \quad (277)$$

At zero temperature and after subtracting the vacuum contribution $\mu = T = \Delta = 0$ this becomes

$$n = \int \frac{d^3 \mathbf{k}}{(2\pi)^3} \left(1 - \frac{\xi_k}{\epsilon_k} \right), \quad (278)$$

in agreement to Eq. (196). Analogously to the gap equation, we rewrite this equation as

$$1 = \frac{3}{2} \left(\frac{\Delta}{E_F} \right)^{3/2} I_2\left(\frac{\mu}{\Delta}\right), \quad (279)$$

with

$$I_2(z) = \int_0^\infty dx x^2 \left(1 - \frac{x^2 - z}{\sqrt{(x^2 - z)^2 + 1}} \right). \quad (280)$$

We now have to solve Eqs. (275) and (279) for μ and Δ for given E_F and $k_F a$. The equations can be decoupled by solving Eq. (279) for Δ/E_F and inserting the result into Eq. (275),

$$-\frac{1}{k_F a} = \frac{2}{\pi} \left[\frac{2}{3 I_2\left(\frac{\mu}{\Delta}\right)} \right]^{1/3} I_1\left(\frac{\mu}{\Delta}\right), \quad (281a)$$

$$\frac{\Delta}{E_F} = \left[\frac{2}{3 I_2\left(\frac{\mu}{\Delta}\right)} \right]^{2/3}. \quad (281b)$$

In this form, the first equation only depends on the ratio μ/Δ . We may (numerically) solve this equation for μ/Δ and then insert the result into the second equation to obtain Δ . The numerical evaluation yields the results shown in Fig. 10.

From the numerical results we read off that

$$\frac{\mu}{\Delta} \rightarrow \mp \infty \quad \text{for} \quad \frac{1}{k_F a} \rightarrow \pm \infty, \quad (282)$$

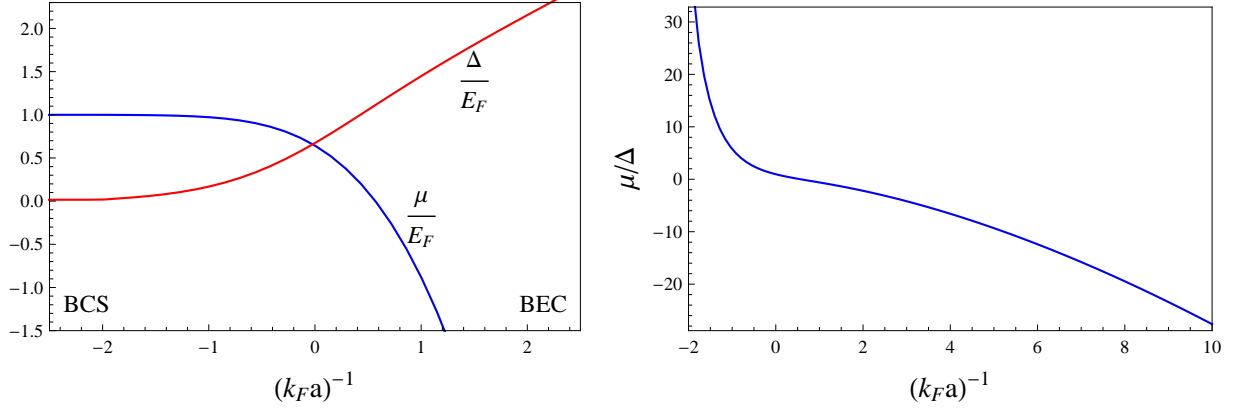


FIG. 10: Left panel: Δ and μ in units of the Fermi energy throughout the crossover, computed from Eqs. (281). Right panel: ratio μ/Δ .

see right panel of Fig. 10. Therefore, to obtain analytical approximations for these two limit cases, we need the asymptotic values of I_1 and I_2 ,

$$I_1(z) \rightarrow \begin{cases} \sqrt{z}(\ln 8z - 2) & \text{for } z \rightarrow +\infty \\ -\frac{\pi}{2}\sqrt{|z|} & \text{for } z \rightarrow -\infty \end{cases}, \quad (283a)$$

$$I_2(z) \rightarrow \begin{cases} \frac{2}{3}z^{3/2} & \text{for } z \rightarrow +\infty \\ \frac{\pi}{8}|z|^{-1/2} & \text{for } z \rightarrow -\infty \end{cases}. \quad (283b)$$

For $(k_F a)^{-1} \rightarrow -\infty$ we thus find from Eq. (281a)

$$-\frac{1}{k_F a} \simeq \frac{2}{\pi}(\ln 8z - 2) \Rightarrow z \simeq \frac{e^2}{8} e^{\frac{\pi}{2k_F |a|}}, \quad (284)$$

where e is Euler's constant. Eq. (281b) yields

$$\mu \simeq E_F, \quad (285)$$

such that we obtain the gap

$$\Delta \simeq \frac{8E_F}{e^2} e^{-\frac{\pi}{2k_F |a|}}. \quad (286)$$

For the opposite limit $(k_F a)^{-1} \rightarrow +\infty$ Eq. (281a) yields

$$|z| \simeq \sqrt{\frac{3\pi}{16}} \frac{1}{(k_F a)^{3/2}}, \quad (287)$$

Then, from inserting this result into Eq. (281b) we obtain

$$\Delta \simeq \sqrt{\frac{16}{3\pi}} \frac{E_F}{\sqrt{k_F a}}, \quad (288)$$

from which we get

$$\mu \simeq -\frac{E_F}{(k_F a)^2} = -\frac{1}{2ma^2}. \quad (289)$$

What do we learn from these results? Firstly, as expected, we recover the BCS results from Sec. IV C for small negative values of $k_F a$: the gap is exponentially suppressed by the effective coupling $k_F |a|$, and the chemical potential

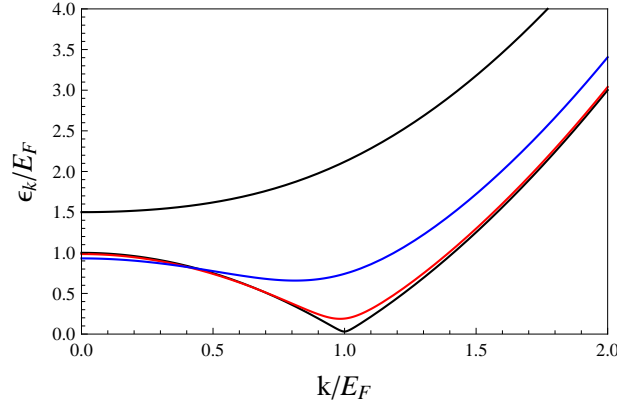


FIG. 11: Single-fermion excitation energies for $(k_F a)^{-1} = -1.85, -0.93, -0.02, +0.89$ and $m = E_F/2$.

equals the Fermi energy. At small positive values, on the other side of the Feshbach resonance, we find that the chemical potential is one half of the energy of the bound state E from Eq. (266), i.e., by adding a single fermion one gains half of the binding energy. This suggests that the fermions are all bound in molecules of two fermions. In other words, the system has effectively become bosonic. Therefore, we may call this regime the BEC regime. Since the BCS and BEC regimes are not separated by a phase transition we may speak of the BCS-BEC crossover.

In Fig. 11 we plot the dispersion throughout the crossover. We know that in the BCS regime, Δ corresponds to the energy gap. More precisely, the single-particle dispersion has a minimum at a certain nonzero momentum, and one needs the energy 2Δ to excite single fermions in the system with this momentum. Now our formalism goes beyond this situation since Δ becomes large while μ becomes negative. Fig. 11 shows that the minimum at a finite momentum disappears and in the BEC regime the minimum occurs at $k = 0$. In this case, the gap is not Δ , but

$$\sqrt{\mu^2 + \Delta^2} = \frac{1}{2ma^2} \sqrt{1 + \frac{16}{3\pi}(k_F a)^3}. \quad (290)$$

The physical picture of the BCS-BEC crossover is thus as follows. Without interactions there is a well-defined Fermi surface at $\mu = E_F$. Now we switch on a weak interaction. Weakly coupled Cooper pairs start to form due to the usual BCS mechanism. This is a pure Fermi surface phenomenon, i.e., everything happens in a small vicinity of the Fermi surface. But, in a sense, the Fermi surface is gone (since by definition the Fermi surface is where quasifermions can be excited with infinitesimally small energy; this is not possible after pairing because now the energy 2Δ is needed). Now we increase the interaction strength. The point is that we can understand the physics qualitatively by starting from our BCS picture: the Cooper pairs get bound stronger and stronger while at the same time we can no longer speak of a Fermi surface phenomenon because the strong interaction is able to “dig” into the Fermi sphere. As a consequence, more fermions participate in Cooper pairing (the ones that, at weak coupling, were just sitting in the Fermi sphere, not contributing in any dynamics). Eventually, a bound state in the strict sense appears at the point where the scattering length diverges, and, going further to the regime where the scattering length goes to zero again, this time from above, *all* fermions become paired in bosonic molecules (the particle number of the two species has to be identical in order for all fermions to find a partner). Now, as Eq. (290) shows, one needs, to lowest order in $k_F a$, half of the binding energy of a molecule to excite a single fermion.

For some further insight into the physics of the BCS-BEC crossover, it is instructive to consider nonzero temperatures, although our mean-field treatment becomes more questionable in this case. Nevertheless, let us try to determine the critical temperature T_c from our gap equation and number equation. As before, we define T_c as the temperature where Δ becomes zero. From Eqs. (273) and (277) we find in this case

$$-\frac{m}{4\pi a} = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \left(\frac{\tanh \frac{\xi_k}{2T_c}}{2\xi_k} - \frac{m}{k^2} \right), \quad (291a)$$

$$n = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \left(1 - \tanh \frac{\xi_k}{2T_c} \right), \quad (291b)$$

since $\epsilon_k = |\xi_k|$ for $\Delta = 0$. Analogously to the $T = 0$ case we can rewrite these equations as

$$-\frac{1}{k_F a} = \frac{2}{\pi} \left[\frac{2}{3J_2\left(\frac{\mu}{T_c}\right)} \right]^{1/3} J_1\left(\frac{\mu}{T_c}\right), \quad (292a)$$

$$\frac{T_c}{E_F} = \left[\frac{2}{3J_2\left(\frac{\mu}{T_c}\right)} \right]^{2/3}, \quad (292b)$$

with

$$J_1(z) \equiv \int_0^\infty dx x^2 \left(\frac{\tanh \frac{x^2 - z}{2}}{x^2 - z} - \frac{1}{x^2} \right), \quad (293a)$$

$$J_2(z) \equiv \int_0^\infty dx x^2 \left(1 - \tanh \frac{x^2 - z}{2} \right), \quad (293b)$$

$$(293c)$$

The numerical evaluation is shown in Fig. 12. The behavior in the BCS regime is as expected, one finds the BCS relation (207) between the critical temperature and the zero-temperature gap. In the BEC regime, the critical temperature seems to increase without boundary. Is this expected? If the picture of bosonic molecules is correct, one might think that the critical temperature is given by the critical temperature of Bose-Einstein condensation. For non-interacting bosons this temperature is

$$T_c^{\text{BEC}} = \frac{\pi}{m} \left[\frac{n}{2\zeta(3/2)} \right]^{2/3}, \quad (294)$$

where we evaluated the general critical temperature for bosons with density $n/2$ (half the fermionic density since there are two fermions for each boson) and mass $2m$ (for the same reason), see for instance Ref. [1]. We might thus expect that the critical temperature saturates at this value (possibly slightly corrected by interactions between the molecules). The reason for the discrepancy between T_c and T_c^{BEC} is that they indeed describe two different transitions. In general, there is one temperature where fermions start forming pairs and one where the fermion pairs start forming a Bose condensate. In the weak-coupling BCS theory these temperatures are identical. In general they are different, i.e., coming from high temperatures, the system first forms Cooper pairs before, at some smaller temperature, these Cooper pairs form a condensate, see right panel of Fig. 12.

In a nutshell, the important points to take away from this discussion are:

- At zero temperature, there is no phase transition between the BCS state at weak coupling – where loosely bound Cooper pairs are formed in a small vicinity around the Fermi surface – and the BEC state at strong coupling – where strongly bound difermions form a Bose condensate. This implies that there is no qualitative difference between these two states.
- This so-called BCS-BEC crossover can be realized experimentally in ultra-cold atoms in an optical trap. This is one of the few systems where the interaction strength is under complete experimental control and can be varied at will. Because of the very mild dependence on the details of the inter-atomic interaction, these experiments can give insight into a large class of physical systems.

[End of 13th lecture, Jun 24th, 2013. (End of semester)]

VII. COOPER PAIRING WITH MISMATCHED FERMI MOMENTA

In the previous sections we have assumed that the two Fermi momenta of the fermions that form a Cooper pair, say fermions A and fermions B, are identical. This is the simplest and standard form of Cooper pairing. It is an interesting question what happens if we release this constraint. The general expectation is that it becomes more difficult for the fermions to form Cooper pairs because Cooper pairing, at least at weak coupling, occurs in a small

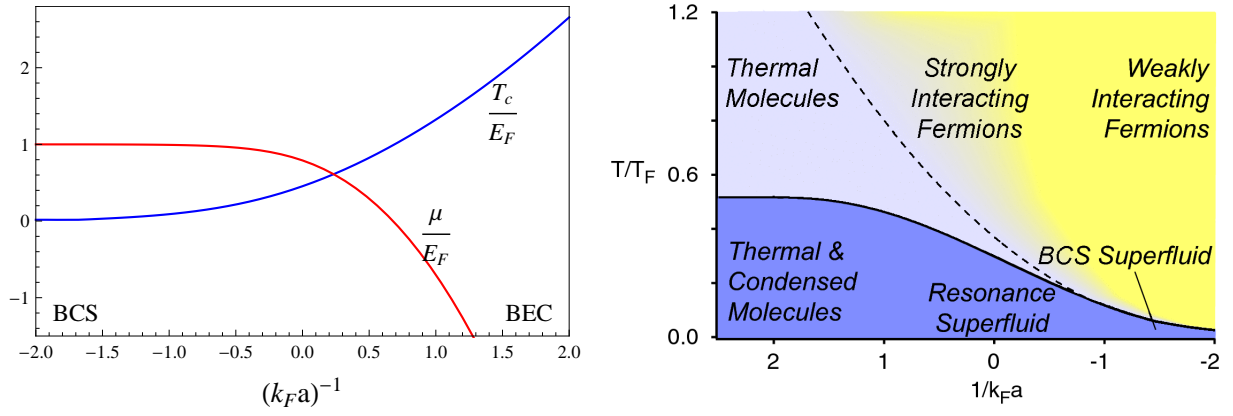


FIG. 12: Left panel: temperature T_c at which $\Delta = 0$ and chemical potential at T_c within the mean-field calculation, i.e., from Eqs. (292). Right panel: phase diagram taken from Ref. [5], where the critical temperature from the left panel is shown to be the temperature where molecules start to form, to be distinguished from the temperature where these molecules condense, Only in the BCS limit, these two temperatures coincide. (Note that the scale on the horizontal axis is inversed in the right panel.)

vicinity of the Fermi surface, as we have seen. Now, when there are two different Fermi surfaces, can the fermions from Fermi surface A pair with fermions from Fermi surface B? If they do so, it seems that they would have to form Cooper pairs with nonzero momentum (in the standard Cooper pairing, fermions on opposite sides of the same Fermi sphere pair, such that the total momentum of a Cooper pair vanishes). This would then necessarily lead to an anisotropic system. Although this is indeed one possibility, we shall see that, if the mismatch in Fermi momenta is sufficiently small, usual isotropic Cooper pairing is still possible.

The question of mismatched Cooper pairing was first discussed for an electronic superconductor. In this case, the two fermion “species” are simply distinguished by their spin. In this case, a mismatch in Fermi momenta can be created by Zeeman splitting in an external magnetic field. This situation has already been discussed a few years after the development of BCS theory in 1962 by Clogston and Chandrasekhar [24]. In recent years, the question of mismatched Cooper pairing has regained interest for example in the context of quark matter and ultra-cold atoms. In dense quark matter inside a compact star, the Fermi momenta of the quarks of different flavors are necessarily different. This is explained in detail for unpaired quark matter for instance in Ref. [21]. Due to the three different flavors and three colors of quark matter, the situation is considerably more complicated than in a two-species system. Therefore, we shall not discuss the details of this system. For ultra-cold atoms, the particle number can be controlled experimentally. Thus, one can perform experiments with a given number of atoms of species A and a given number of atoms of species B, i.e., also for the question of mismatched pairing, ultra-cold atoms are an ideal playground towards understanding more complicated systems.

A. Quasiparticle excitations

Let us first go back to our formalism of ultrarelativistic fermions from Sec. IV. In addition to the four-dimensional Dirac space, we introduce a two-dimensional space for two fermion species (think of two different atoms or two “frozen” spin states, or two quark flavors etc.) For convenience we refer to these two species as different “flavors”. As a consequence, our propagator will now be a 16×16 matrix: $2 \times 2 \times 4 = 16$ because of the different fermion species (2), fermions/charge-conjugate fermions (2), spin and particle/antiparticle degrees of freedom (4)¹⁶. Since we want fermions of different species to form Cooper pairs, our ansatz for the gap matrix is

$$\Phi^\pm = \pm \Delta \sigma_2 \gamma_5. \quad (295)$$

¹⁶ In three-flavor quark matter, we have instead $3 \times 3 \times 2 \times 4 = 72$ due to color and flavor degrees of freedom.

We also have to promote the inverse tree-level propagator from Eq. (180) to a matrix in flavor space,

$$[G_0^\pm]^{-1} = \begin{pmatrix} \gamma^\mu K_\mu \pm \mu_1 \gamma^0 & 0 \\ 0 & \gamma^\mu K_\mu \pm \mu_2 \gamma^0 \end{pmatrix} = \sum_{e=\pm} \gamma^0 \Lambda_k^{\pm e} \begin{pmatrix} k_0 \pm (\mu_1 - ek) & 0 \\ 0 & k_0 \pm (\mu_2 - ek) \end{pmatrix}, \quad (296)$$

where we have introduced different chemical potentials μ_1 and μ_2 for the two flavors. Since we work with vanishing fermion masses, the chemical potentials are identical to the Fermi momenta, $\mu_i = k_{F,i}$. We easily obtain the tree-level propagator,

$$G_0^\pm = \sum_{e=\pm} \gamma^0 \Lambda_k^{\mp e} \begin{pmatrix} \frac{1}{k_0 \pm (\mu_1 - ek)} & 0 \\ 0 & \frac{1}{k_0 \pm (\mu_2 - ek)} \end{pmatrix}. \quad (297)$$

Our first goal is to compute the quasiparticle excitations. To this end we need to compute the full propagator from Eq. (175a),

$$G^\pm \equiv ([G_0^\pm]^{-1} - \Phi^\mp G_0^\mp \Phi^\pm)^{-1}. \quad (298)$$

With

$$\begin{aligned} \Phi^\mp G_0^\mp \Phi^\pm &= - \sum_e \Delta^2 \gamma^5 \gamma^0 \Lambda_k^{\pm e} \gamma^5 \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{k_0 \mp (\mu_1 - ek)} & 0 \\ 0 & \frac{1}{k_0 \mp (\mu_2 - ek)} \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\ &= \sum_e \Delta^2 \gamma^0 \Lambda_k^{\pm e} \begin{pmatrix} \frac{1}{k_0 \mp (\mu_2 - ek)} & 0 \\ 0 & \frac{1}{k_0 \mp (\mu_1 - ek)} \end{pmatrix} \end{aligned} \quad (299)$$

(note the flip of the chemical potentials due to the matrix multiplication in flavor space) we find

$$\begin{aligned} G^\pm &= \left[\sum_e \gamma^0 \Lambda_k^{\pm e} \begin{pmatrix} k_0 \pm (\mu_1 - ek) - \frac{\Delta^2}{k_0 \mp (\mu_2 - ek)} & 0 \\ 0 & k_0 \pm (\mu_2 - ek) - \frac{\Delta^2}{k_0 \mp (\mu_1 - ek)} \end{pmatrix} \right]^{-1} \\ &= \sum_e \gamma^0 \Lambda_k^{\mp e} \begin{pmatrix} \frac{k_0 \mp (\mu_2 - ek)}{(k_0 \pm \delta\mu)^2 - (\epsilon_k^e)^2} & 0 \\ 0 & \frac{k_0 \mp (\mu_1 - ek)}{(k_0 \mp \delta\mu)^2 - (\epsilon_k^e)^2} \end{pmatrix}, \end{aligned} \quad (300)$$

with

$$\epsilon_k^e \equiv \sqrt{(\bar{\mu} - ek)^2 + \Delta^2}. \quad (301)$$

Here we have introduced

$$\bar{\mu} \equiv \frac{\mu_1 + \mu_2}{2}, \quad \delta\mu \equiv \frac{\mu_1 - \mu_2}{2}, \quad (302)$$

i.e., $\bar{\mu}$ is the average chemical potential and $\delta\mu$ the difference between the chemical potentials (divided by 2). Without loss of generality we may assume $\mu_1 > \mu_2$, such that $\delta\mu > 0$. In the derivation of Eq. (300) we have used the relation

$$[k_0 \pm (\mu_1 - ek)][k_0 \mp (\mu_2 - ek)] = (k_0 \pm \delta\mu)^2 - (\bar{\mu} - ek)^2, \quad (303)$$

which is useful for the following calculations.

We see that the poles of the propagator are $\epsilon_k^e + \delta\mu$, $\epsilon_k^e - \delta\mu$, $-\epsilon_k^e + \delta\mu$, $-\epsilon_k^e - \delta\mu$. This is illustrated in Fig. 13.

B. Free energy and Clogston limit

Next we are interested in the free energy of the system. In Sec. IV we have assumed without explicit proof that the paired state has lower free energy than the unpaired state. We shall now confirm this expectation for vanishing mismatch and discuss the case of nonzero $\delta\mu$. We expect that for small $\delta\mu$ the standard BCS state survives but that it cannot sustain arbitrarily high values of $\delta\mu$, such that there will be a transition to the unpaired state. One of the results of the following calculation will be the critical value of $\delta\mu$ at which this transition happens.

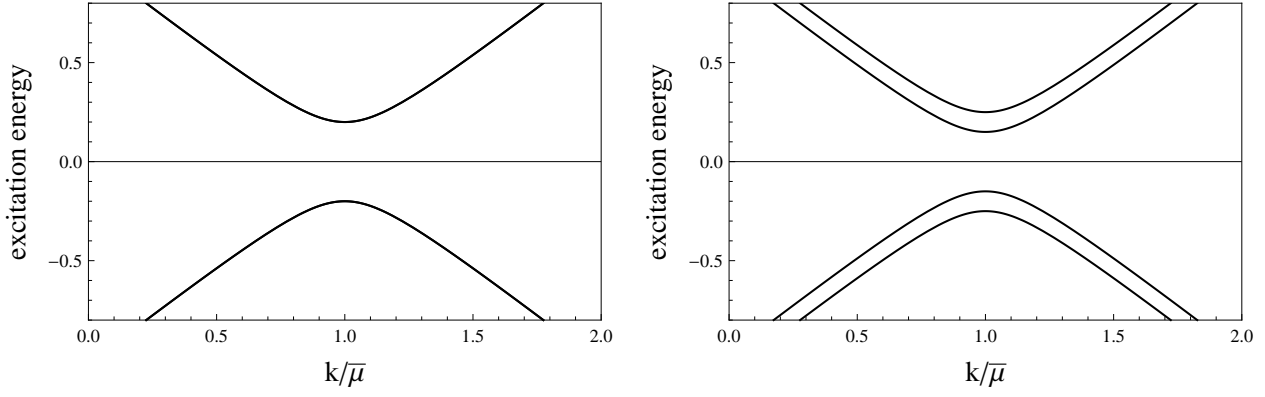


FIG. 13: Quasiparticle and quasihole excitations in an ultrarelativistic superfluid without (left) and with (right) mismatch in Fermi momenta. Without mismatch, the two quasiparticle excitations (from the two “flavors”) are degenerate. This degeneracy is broken by the mismatch, such that there are four different excitations. As a consequence, the gap is effectively reduced to $\Delta - \delta\mu$. In principle, the dispersions can even become “gapless” for $\delta\mu > \Delta$, although there is still Cooper pairing. It turns out, however, that this straightforward extrapolation of our simple dispersion relations leads to an unstable state, and we shall not discuss this scenario further.

We start from the free energy

$$\Omega = -\frac{1}{2}\text{Tr} \ln \mathcal{S}^{-1} + \frac{1}{4}\text{Tr}[1 - \mathcal{S}_0^{-1}\mathcal{S}]. \quad (304)$$

In the case of a superfluid, we have an additional term besides the well-known $\text{Tr} \ln$ term that appears also in a noninteracting system. This term is proportional to Δ^2 , see comment below Eq. (152). Here we write this term in a form that arises from a more elaborate treatment, within the so-called Cornwall-Jackiw-Tomboulis (CJT) formalism¹⁷. This is a self-consistent formalism which is particularly suited for systems with spontaneously broken symmetry. We shall not go into the details of this formalism, but rather make use of the form of the thermodynamic potential obtained within this formalism. If you are interested in the details, have a look at Sec. IV in Ref. [6] and references therein.

For the $\text{Tr} \ln$ term we perform the trace in Nambu-Gorkov space with the help of $\text{Tr} \ln = \ln \det$ and

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(AD - BD^{-1}CD). \quad (305)$$

Then, with the full inverse propagator from Eq. (171),

$$\mathcal{S}^{-1} = \begin{pmatrix} [G_0^+]^{-1} & \Phi^- \\ \Phi^+ & [G_0^-]^{-1} \end{pmatrix}, \quad (306)$$

we find

$$\text{Tr} \ln \mathcal{S}^{-1} = \text{Tr} \ln ([G_0^+]^{-1}[G_0^-]^{-1} - \Phi^- G_0^- \Phi^+ [G_0^-]^{-1}). \quad (307)$$

With the help of Eq. (296) we compute

$$[G_0^+]^{-1}[G_0^-]^{-1} = \sum_e \Lambda_k^{-e} \begin{pmatrix} k_0^2 - (\mu_1 - ek)^2 & 0 \\ 0 & k_0^2 - (\mu_2 - ek)^2 \end{pmatrix}, \quad (308)$$

¹⁷ To be more precise, in the CJT formalism one has an effective action

$$\Gamma[\mathcal{S}] = \frac{1}{2}\text{Tr} \ln \mathcal{S}^{-1} - \frac{1}{2}\text{Tr}[1 - \mathcal{S}_0^{-1}\mathcal{S}] + \Gamma_2[\mathcal{S}],$$

which is a functional of the full propagator \mathcal{S} , and the gap equation results from minimizing this action with respect to the propagator; Γ_2 denotes all two-particle irreducible diagrams. At the stationary point, i.e., after solving (formally) the gap equation and inserting back the solution into the effective action, one obtains the free energy (304).

and, with the help of Eqs. (295), (296), (297),

$$\Phi^- G_0^- \Phi^+ [G_0^-]^{-1} = \Delta^2 \sum_e \Lambda_k^{-e} \begin{pmatrix} \frac{k_0 - (\mu_1 - ek)}{k_0 - (\mu_2 - ek)} & 0 \\ 0 & \frac{k_0 - (\mu_2 - ek)}{k_0 - (\mu_1 - ek)} \end{pmatrix}, \quad (309)$$

such that

$$[G_0^+]^{-1} [G_0^-]^{-1} - \Phi^- G_0^- \Phi^+ [G_0^-]^{-1} = \sum_e \Lambda_k^{-e} \begin{pmatrix} \frac{k_0 - (\mu_1 - ek)}{k_0 - (\mu_2 - ek)} [(k_0 + \delta\mu)^2 - (\epsilon_k^e)^2] & 0 \\ 0 & \frac{k_0 - (\mu_2 - ek)}{k_0 - (\mu_1 - ek)} [(k_0 - \delta\mu)^2 - (\epsilon_k^e)^2] \end{pmatrix}. \quad (310)$$

To compute the $\text{Tr} \ln$ of this expression, notice that in Dirac space it has the form $a_+ P_+ + a_- P_-$ with complete, orthogonal projectors P_\pm . Since the trace does not depend on the basis in which we express the matrix we may diagonalize $a_+ P_+ + a_- P_-$ which is simply a diagonal matrix with $\text{Tr}[P_+]$ many entries a_+ and $\text{Tr}[P_-]$ many entries a_- . ($\text{Tr}[P_\pm]$ is the degeneracy of the eigenvalue a_\pm .) Therefore, we have $\text{Tr} \ln(a_+ P_+ + a_- P_-) = \text{Tr}[P_+] \ln a_+ + \text{Tr}[P_-] \ln a_-$. With $\text{Tr}[\Lambda_k^{-e}] = 2$ and treating the flavor space analogously (which factorizes from Dirac space and where the matrix already is diagonal), we obtain

$$\begin{aligned} -\frac{1}{2} \text{Tr} \ln \mathcal{S}^{-1} &= -\frac{1}{2} \frac{T}{V} \sum_K \sum_e \text{Tr}[\Lambda_k^{-e}] \left[\ln \frac{k_0 - (\mu_1 - ek)}{k_0 - (\mu_2 - ek)} \frac{(k_0 + \delta\mu)^2 - (\epsilon_k^e)^2}{T^2} + \ln \frac{k_0 - (\mu_2 - ek)}{k_0 - (\mu_1 - ek)} \frac{(k_0 - \delta\mu)^2 - (\epsilon_k^e)^2}{T^2} \right] \\ &= -\sum_e \int \frac{d^3 \mathbf{k}}{(2\pi)^3} T \sum_{k_0} \left[\ln \frac{(k_0 + \delta\mu)^2 - (\epsilon_k^e)^2}{T^2} + \ln \frac{(k_0 - \delta\mu)^2 - (\epsilon_k^e)^2}{T^2} \right]. \end{aligned} \quad (311)$$

For the Matsubara sum we use

$$T \sum_{k_0} \left[\ln \frac{(k_0 + \delta\mu)^2 - (\epsilon_k^e)^2}{T^2} + \ln \frac{(k_0 - \delta\mu)^2 - (\epsilon_k^e)^2}{T^2} \right] = 2 \left[\epsilon_k^e + T \ln \left(1 + e^{-\frac{\epsilon_k^e - \delta\mu}{T}} \right) + T \ln \left(1 + e^{-\frac{\epsilon_k^e + \delta\mu}{T}} \right) \right], \quad (312)$$

(the proof is left as an exercise) to find

$$-\frac{1}{2} \text{Tr} \ln \mathcal{S}^{-1} = 2 \sum_e \int \frac{d^3 \mathbf{k}}{(2\pi)^3} \left[\epsilon_k^e + T \ln \left(1 + e^{-\frac{\epsilon_k^e - \delta\mu}{T}} \right) + T \ln \left(1 + e^{-\frac{\epsilon_k^e + \delta\mu}{T}} \right) \right]. \quad (313)$$

For the second term on the right-hand side of Eq. (304) we first find after performing the trace over Nambu-Gorkov space

$$\text{Tr}[1 - \mathcal{S}_0^{-1} \mathcal{S}] = \text{Tr} [2 - [G_0^+]^{-1} G^+ - [G_0^-]^{-1} G^-]. \quad (314)$$

Therefore, with

$$[G_0^\pm]^{-1} G^\pm = \sum_e \Lambda_k^{-e} \begin{pmatrix} 1 + \frac{\Delta^2}{(k_0 \pm \delta\mu)^2 - (\epsilon_k^e)^2} & 0 \\ 0 & 1 + \frac{\Delta^2}{(k_0 \mp \delta\mu)^2 - (\epsilon_k^e)^2} \end{pmatrix} \quad (315)$$

we find after performing the traces over Dirac and flavor space,

$$\frac{1}{4} \text{Tr}[1 - \mathcal{S}_0^{-1} \mathcal{S}] = -\sum_e \int \frac{d^3 \mathbf{k}}{(2\pi)^3} T \sum_{k_0} \left[\frac{\Delta^2}{(k_0 + \delta\mu)^2 - (\epsilon_k^e)^2} + \frac{\Delta^2}{(k_0 - \delta\mu)^2 - (\epsilon_k^e)^2} \right]. \quad (316)$$

Now the Matsubara sum

$$T \sum_{k_0} \frac{1}{(k_0 \pm \delta\mu)^2 - (\epsilon_k^e)^2} = -\frac{1}{2\epsilon_k^e} [1 - f(\epsilon_k \mp \delta\mu) - f(\epsilon_k \pm \delta\mu)], \quad (317)$$

yields

$$\frac{1}{4} \text{Tr}(1 - \mathcal{S}_0^{-1} \mathcal{S}) = \sum_e \int \frac{d^3 \mathbf{k}}{(2\pi)^3} \frac{\Delta^2}{\epsilon_k^e} [1 - f(\epsilon_k + \delta\mu) - f(\epsilon_k - \delta\mu)]. \quad (318)$$

This confirms that the additional term in the potential is proportional to Δ^2 , as discussed above. Inserting the results (313) and (318) into the thermodynamic potential (304) yields

$$\begin{aligned}\Omega &= -2 \sum_e \int \frac{d^3 \mathbf{k}}{(2\pi)^3} \left\{ \epsilon_k^e + T \ln \left(1 + e^{-\frac{\epsilon_k^e - \delta\mu}{T}} \right) + T \ln \left(1 + e^{-\frac{\epsilon_k^e + \delta\mu}{T}} \right) - \frac{\Delta^2}{2\epsilon_k} [1 - f(\epsilon_k^e - \delta\mu) - f(\epsilon_k^e + \delta\mu)] \right\} \\ &\rightarrow -2 \sum_e \int \frac{d^3 \mathbf{k}}{(2\pi)^3} \left[\epsilon_k^e + (\delta\mu - \epsilon_k^e) \Theta(\delta\mu - \epsilon_k^e) - \frac{\Delta^2}{2\epsilon_k^e} \Theta(\epsilon_k^e - \delta\mu) \right],\end{aligned}\quad (319)$$

where, in the second step, we have taken the zero temperature limit, $T \ln(1 + e^{x/T}) \rightarrow x \Theta(x)$, $f(x) \rightarrow \Theta(-x)$. We have also used that $\delta\mu > 0$. [For an alternative, maybe more physical, way of writing this use $\epsilon + (\delta\mu - \epsilon) \Theta(\delta\mu - \epsilon) = \frac{1}{2}(|\epsilon + \delta\mu| + |\epsilon - \delta\mu|)$; in this form, the absolute value of the quasiparticle dispersions plotted in Fig. 13 appear in the integrand.]

As in our treatment of the gap equation in Sec. IV C we assume the gap to vanish everywhere except for the interval $[\bar{\mu} - \delta, \bar{\mu} + \delta]$ where it is assumed to be constant with $\Delta \ll \delta \ll \bar{\mu}$. We shall also assume that $\delta\mu$ is smaller than or at most of the order of the gap Δ , such that $\delta\mu \ll \delta$. With I_Δ^e abbreviating the integrand in the free energy (319) we rewrite the integration over k as

$$\sum_e \int_0^\infty I_\Delta^e = \int_0^\infty I_\Delta^+ + \int_0^\infty I_0^- = \int_0^\infty (I_0^+ + I_0^-) + \int_{\bar{\mu}-\delta}^{\bar{\mu}+\delta} (I_\Delta^+ - I_0^+) = \Omega_0 + \int_{\bar{\mu}-\delta}^{\bar{\mu}+\delta} (I_\Delta^+ - I_0^+), \quad (320)$$

where, in the first step, we have set the antiparticle gap to zero (antiparticles do not contribute to the free energy anyway) and, in the second step, used that the gap is only nonvanishing in the interval $[\bar{\mu} - \delta, \bar{\mu} + \delta]$. We have denoted the free energy of the non-superfluid state $\Delta = 0$ by Ω_0 . Consequently, we can write

$$\begin{aligned}\Omega &= \Omega_0 - \frac{1}{\pi^2} \int_{\bar{\mu}-\delta}^{\bar{\mu}+\delta} dk k^2 \left\{ \epsilon_k^+ - \frac{\Delta^2}{2\epsilon_k^+} + \left(\delta\mu - \epsilon_k^+ + \frac{\Delta^2}{2\epsilon_k^+} \right) \Theta(\delta\mu - \epsilon_k^+) \right. \\ &\quad \left. - \left[|k - \bar{\mu}| + (\delta\mu - |k - \bar{\mu}|) \Theta(\delta\mu - |k - \bar{\mu}|) \right] \right\}.\end{aligned}\quad (321)$$

The results for the three contributions, computed separately, are as follows. The first contribution is

$$\begin{aligned}\int_{\bar{\mu}-\delta}^{\bar{\mu}+\delta} dk k^2 \left(\epsilon_k^+ - \frac{\Delta^2}{2\epsilon_k^+} \right) &= \delta \sqrt{\delta^2 + \Delta^2} \left(\bar{\mu}^2 + \frac{\delta^2}{2} - \frac{\Delta^2}{4} \right) + \frac{\Delta^4}{8} \ln \frac{\sqrt{\delta^2 + \Delta^2} + \delta}{\sqrt{\delta^2 + \Delta^2} - \delta} \\ &= \bar{\mu}^2 \delta^2 + \frac{\bar{\mu}^2 \Delta^2}{2} + \frac{\delta^4}{2} + \mathcal{O}(\Delta^4).\end{aligned}\quad (322)$$

where $\Delta \ll \delta \ll \bar{\mu}$ has been used. The second contribution is

$$\begin{aligned}\int_{\bar{\mu}-\delta}^{\bar{\mu}+\delta} dk k^2 \left(\delta\mu - \epsilon_k^+ + \frac{\Delta^2}{2\epsilon_k^+} \right) \Theta(\delta\mu - \epsilon_k^+) &= \Theta(\delta\mu - \Delta) \left[\delta\mu \sqrt{\delta\mu^2 - \Delta^2} \left(\bar{\mu}^2 + \frac{\delta^2}{6} + \frac{\Delta^2}{12} \right) - \frac{\Delta^4}{8} \ln \frac{\delta\mu + \sqrt{\delta\mu^2 - \Delta^2}}{\delta\mu - \sqrt{\delta\mu^2 - \Delta^2}} \right] \\ &= \Theta(\delta\mu - \Delta) \delta\mu \bar{\mu}^2 \sqrt{\delta\mu^2 - \Delta^2} + \mathcal{O}(\delta\mu^4, \Delta^4),\end{aligned}\quad (323)$$

where we have used $\delta\mu < \delta$. Finally,

$$\int_{\bar{\mu}-\delta}^{\bar{\mu}+\delta} dk k^2 \left[|k - \bar{\mu}| + (\delta\mu - |k - \bar{\mu}|) \Theta(\delta\mu - |k - \bar{\mu}|) \right] = \bar{\mu}^2 \delta^2 + \frac{\delta^4}{2} + \frac{(\bar{\mu} + \delta\mu)^4}{12} + \frac{(\bar{\mu} - \delta\mu)^4}{12} - \frac{\bar{\mu}^4}{6}, \quad (324)$$

again using $\delta\mu < \delta$ for the evaluation of the step function.

Putting everything together yields (note the cancellation of all δ terms)

$$\Omega \simeq \Omega_0 + \frac{\bar{\mu}^2 \delta\mu^2}{\pi^2} - \frac{\bar{\mu}^2 \Delta^2}{2\pi^2} - \Theta(\delta\mu - \Delta) \frac{\bar{\mu}^2 \delta\mu \sqrt{\delta\mu^2 - \Delta^2}}{\pi^2}. \quad (325)$$

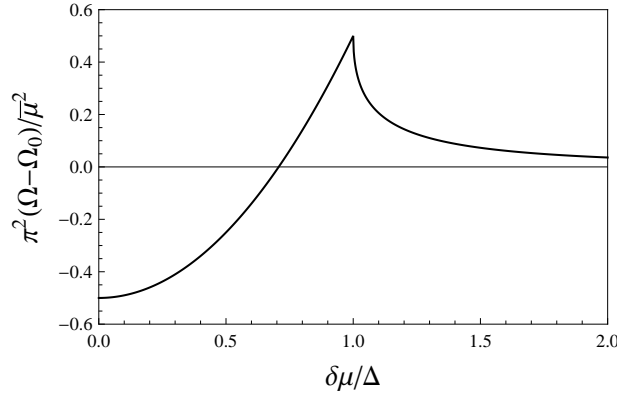


FIG. 14: Difference in free energies between superfluid and normal states as a function of the mismatch $\delta\mu$ from Eq. (325). If the difference is smaller than zero, the superfluid state is preferred.

First we check that we reproduce the free energy of the non-superfluid case: upon setting $\Delta = 0$ we find $\Omega = \Omega_0$. Evaluating this non-superfluid free energy gives

$$\begin{aligned}\Omega &= \Omega_0 = 2 \int \frac{d^3\mathbf{k}}{(2\pi)^3} (k - \mu_1) \Theta(\mu_1 - k) + 2 \int \frac{d^3\mathbf{k}}{(2\pi)^3} (k - \mu_2) \Theta(\mu_2 - k) \\ &= -\frac{\mu_1^4}{12\pi^2} - \frac{\mu_2^4}{12\pi^2}\end{aligned}\quad (326)$$

where we have subtracted the infinite vacuum contribution $\Omega(T = \mu_1 = \mu_2 = \Delta = 0)$. (Because the single-particle excitations take the form $|k - \bar{\mu}|$, not $k - \bar{\mu}$, it takes a few lines of algebra to obtain this result; this is left as an exercise.) The first line shows that the free energy has the form $\Omega = \epsilon - \mu n$ with the energy density ϵ and the number density n .

Next we consider the case $\delta\mu = 0$. In this case, the free energy of the superfluid state is ($\mu_1 = \mu_2 = \bar{\mu} \equiv \mu$)

$$\Omega \simeq \Omega_0 - \frac{\mu^2 \Delta^2}{2\pi^2}. \quad (327)$$

This shows that a nonzero gap lowers the free energy compared to the non-superfluid state: Cooper pairing occurs. The term $\frac{\mu^2 \Delta^2}{2\pi^2}$ is often called *condensation energy* (more precisely, condensation energy density).

Next we consider the case $0 < \delta\mu < \Delta$ where we have

$$\Omega \simeq \Omega_0 + \frac{\bar{\mu}^2 \delta\mu^2}{\pi^2} - \frac{\bar{\mu}^2 \Delta^2}{2\pi^2}. \quad (328)$$

Now we see that there is an additional, positive, contribution to the free energy: you not only gain energy from pairing but in the presence of a mismatch you also have to pay a price in energy. The superfluid state is now only preferred over the non-superfluid state for $\Delta > \sqrt{2} \delta\mu$. This is called the *Clogston limit*, derived in the papers by Clogston and Chandrasekhar mentioned above: beyond this limit, i.e., for mismatches larger than $\Delta/\sqrt{2}$, the superfluid state breaks down. It depends on the specific system under consideration whether Δ or $\delta\mu$ is larger. For instance in the experiments with ultra-cold atoms both quantities can be more or less controlled independently, since Δ is basically a measure of the interaction strength while $\delta\mu$ (or rather δn in this case) can be tuned directly. Therefore, the whole phase diagram in the Δ - δn plane can be explored. We discuss this phase diagram briefly in Sec. VII D. In QCD, on the other hand, both Δ and $\delta\mu$ (in this case rather several Δ 's and $\delta\mu$'s) are functions of a single parameter, the quark chemical potential.

We can also read off a very nice picture for mismatched Cooper pairing from Eq. (328). To this end, let us first write the free energy of the non-superfluid state in terms of $\delta\mu$ and $\bar{\mu}$, neglecting terms of order $\delta\mu^4$,

$$-\frac{\mu_1^4}{12\pi^2} - \frac{\mu_2^4}{12\pi^2} + \frac{\bar{\mu}^2 \delta\mu^2}{\pi^2} = -2 \frac{\bar{\mu}^4}{12\pi^2} + \mathcal{O}(\delta\mu^4). \quad (329)$$

We may thus write the free energy (328) as

$$\Omega \simeq -2 \frac{\bar{\mu}^4}{12\pi^2} - \frac{\bar{\mu}^2 \Delta^2}{2\pi^2}. \quad (330)$$

But this is just the free energy of a usual superfluid (i.e., without mismatch), where both fermion species have the common Fermi momentum $\bar{\mu}$! Therefore, the free energy of the superconducting state *with* mismatch can be understood by first creating a (fictitious) state where both flavors have one common Fermi surface – paying a cost in free energy $\propto \bar{\mu}^2 \delta\mu^2$ – and then by forming Cooper pairs in the usual BCS way at this common Fermi surface – which yields an energy gain $\propto \bar{\mu}^2 \Delta^2$. If the gain exceeds the cost, pairs will form.

We plot the difference between the free energies of the superfluid and normal states in Fig. 14. Here we have included the term with the step function, i.e., we have allowed for $\delta\mu > \Delta$. As expected, however, once the Clogston limit is surpassed, the non-superfluid state is always preferred over the superfluid state.

C. Occupation numbers and number densities

So far we have asked whether the superfluid state is favorable in the presence of a given mismatch in Fermi momenta of the fermions that pair. How does this translate into the number of fermions of the two species? A related question is whether and when the superfluid state is favorable for a given mismatch in the number density. This is the more relevant question in the context of cold atoms, where, as mentioned above, experiments with different number densities of the two “flavors” can be performed. In quark matter, the question is a bit more complicated. In this case, one has different quarks with different electric charges (and color charges) and one requires the system to be neutral (if one is interested in realistic quark matter inside a neutron star). Also in this case, one asks for a superfluid state that accommodates different number densities of the different species. However, in this case these number densities are not externally given but adjust themselves dynamically under the neutrality constraint. In any case, we see that the following general question is of interest.

- Can a fermionic superfluid accommodate different number densities of the fermion species that form Cooper pairs? And if yes, how?

Put differently: if I impose a difference in number densities, will the system be superfluid?

We can compute the number densities of the two flavors with the general expression already used in Eq. (190),

$$n_i = -\frac{1}{2} \text{Tr} \left[\frac{\partial \mathcal{S}_0^{-1}}{\partial \mu_i} \mathcal{S} \right] = -\frac{1}{2} \text{Tr} \left[\frac{\partial [G_0^+]^{-1}}{\partial \mu_i} G^+ + \frac{\partial [G_0^-]^{-1}}{\partial \mu_i} G^- \right], \quad (331)$$

where, in the second step, we have performed the trace in Nambu-Gorkov space. For notational convenience we now focus on n_1 ; n_2 is then easily obtained from the final result upon exchanging the indices 1 and 2. With the propagators from Eqs. (296) and (300) we obtain after performing the trace in flavor and Dirac space,

$$n_1 = -\frac{T}{V} \sum_K \sum_e \left[\frac{k_0 - (\mu_2 - ek)}{(k_0 + \delta\mu)^2 - (\epsilon_k^e)^2} - \frac{k_0 + (\mu_2 - ek)}{(k_0 - \delta\mu)^2 - (\epsilon_k^e)^2} \right]. \quad (332)$$

After a bit of algebra we rewrite

$$\frac{k_0 - (\mu_2 - ek)}{(k_0 + \delta\mu)^2 - (\epsilon_k^e)^2} - \frac{k_0 + (\mu_2 - ek)}{(k_0 - \delta\mu)^2 - (\epsilon_k^e)^2} = \frac{-2k_0^2(\mu_1 - ek) + 2(\mu_2 - ek)(\epsilon_k^2 - \delta\mu^2)}{[(k_0 + \delta\mu)^2 - (\epsilon_k^e)^2][(k_0 - \delta\mu)^2 - (\epsilon_k^e)^2]}. \quad (333)$$

Now we use the general Matsubara sum

$$T \sum_{k_0} \frac{\alpha k_0^2 + \beta}{[(k_0 + \delta\mu)^2 - \epsilon^2][(k_0 - \delta\mu)^2 - \epsilon^2]} = \frac{\alpha(\epsilon - \delta\mu)^2 + \beta}{8\epsilon\delta\mu(\epsilon - \delta\mu)} \tanh \frac{\epsilon - \delta\mu}{2T} - \frac{\alpha(\epsilon + \delta\mu)^2 + \beta}{8\epsilon\delta\mu(\epsilon + \delta\mu)} \tanh \frac{\epsilon + \delta\mu}{2T}. \quad (334)$$

With the obvious identifications of α and β one finds, again after some algebra,

$$\frac{\alpha(\epsilon \pm \delta\mu)^2 + \beta}{8\epsilon(\epsilon \pm \delta\mu)\delta\mu} = -\frac{1}{2} \left(1 \mp e \frac{\xi_k^e}{\epsilon_k^e} \right), \quad (335)$$

where

$$\xi_k^e \equiv k - e\bar{\mu}. \quad (336)$$

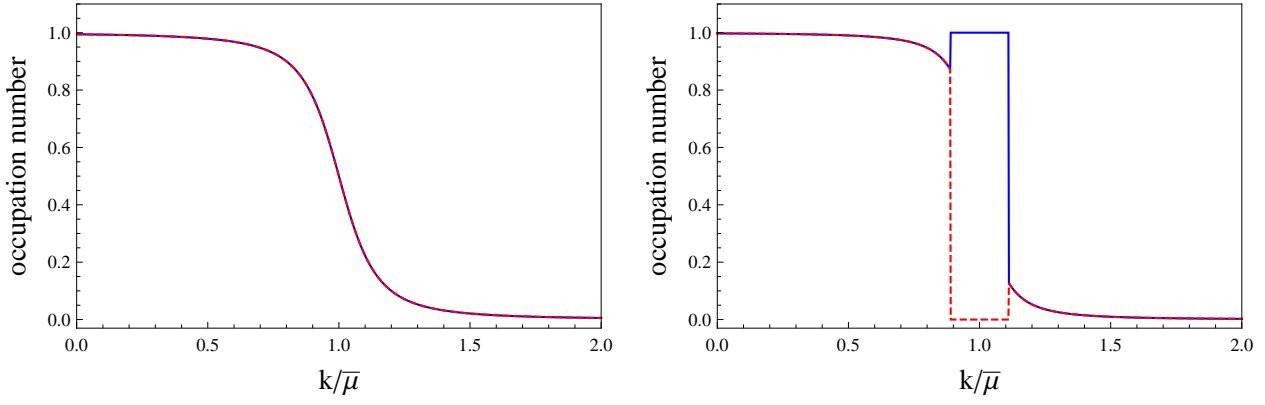


FIG. 15: Zero-temperature occupation number ($e = +$) for the two particle species that form Cooper pairs in the presence of a nonzero mismatch $\delta\mu$. Left panel: for $\delta\mu < \Delta$, the occupation numbers are identical and independent of $\delta\mu$. Right panel: for $\delta\mu > \Delta$, there is a region in momentum space where only particles of the majority species reside, sometimes called “breach”, here shown for $\delta\mu/\bar{\mu} = 0.15$ and $\Delta = 0.1$.

Inserting the result for the Matsubara sum into Eq. (332) yields

$$\begin{aligned}
 n_1 &= - \sum_e \int \frac{d^3\mathbf{k}}{(2\pi)^3} \left[\frac{1}{2} \left(1 + e^{\frac{\xi_k^e}{\epsilon_k^e}} \right) \tanh \frac{\epsilon_k^e - \delta\mu}{2T} - \frac{1}{2} \left(1 - e^{\frac{\xi_k^e}{\epsilon_k^e}} \right) \tanh \frac{\epsilon_k^e + \delta\mu}{2T} \right] \\
 &\rightarrow 2 \sum_e \int \frac{d^3\mathbf{k}}{(2\pi)^3} \left\{ \frac{1}{2} \left(1 - e^{\frac{\xi_k^e}{\epsilon_k^e}} \right) [1 - f(\epsilon_k^e + \delta\mu)] + \frac{1}{2} \left(1 + e^{\frac{\xi_k^e}{\epsilon_k^e}} \right) f(\epsilon_k^e - \delta\mu) \right\}, \quad (337)
 \end{aligned}$$

where, in the second step, we have subtracted the vacuum contribution $\mu = T = \Delta = 0$. It is easy to check that for $\delta\mu = 0$ we reproduce the result (193). Here we are interested in the $T = 0$ results. In this case (using $\delta\mu > 0$),

$$n_1 = 2 \sum_e \int \frac{d^3\mathbf{k}}{(2\pi)^3} \left[\frac{1}{2} \left(1 - e^{\frac{\xi_k^e}{\epsilon_k^e}} \right) + \frac{1}{2} \left(1 + e^{\frac{\xi_k^e}{\epsilon_k^e}} \right) \Theta(\delta\mu - \epsilon_k^e) \right], \quad (338a)$$

$$n_2 = 2 \sum_e \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{1}{2} \left(1 - e^{\frac{\xi_k^e}{\epsilon_k^e}} \right) \Theta(\epsilon_k^e - \delta\mu), \quad (338b)$$

where n_2 is obtained from exchanging $1 \leftrightarrow 2$ in Eq. (337). The integrands of these expressions give the occupation numbers, which are plotted in Fig. 15. We see that for $\delta\mu < \Delta$ the two species are “locked” together, i.e., they have the same occupation numbers and thus also the same number densities¹⁸. Moreover, the number densities are given by the same expression as in the case without mismatch with the chemical potential replaced by the average chemical potential. This supports the observation we have made in the context of the free energy: as long as $\delta\mu < \Delta$, the system seems to behave like a usual superfluid with common Fermi momentum $\bar{\mu}$. However, even though not obvious from the densities, there are properties of the system which depend on the mismatch $\delta\mu$, even if $\delta\mu < \Delta$. For instance, as we have seen in Fig. 13, the energy gap is effectively reduced by $\delta\mu$.

For $\delta\mu > \Delta$ the step functions in Eq. (338) become nonzero,

$$\int_0^\infty \Theta(\delta\mu - \epsilon_k^e) = \Theta(\delta\mu - \Delta) \int_{k_-^e}^{k_+^e}, \quad (339)$$

¹⁸ In the relativistic context, we should rather talk about charge densities instead of number densities. However, since only particles, not antiparticles, matter due to $\mu \gg T$, this terminology is justified. Note however that if one discusses the BCS-BEC crossover in a relativistic framework – and fixing charge densities instead of chemical potentials as in Sec. VI – the chemical potential will approach zero in the BEC regime and antiparticles play an important role.

where

$$k_{\pm}^e \equiv e\bar{\mu} \pm \sqrt{\delta\mu^2 - \Delta^2}. \quad (340)$$

In this case, there is a region in momentum space, namely between k_-^e and k_+^e , which is only populated by fermions of the majority species. This is the way the superfluid allows for a difference in particle numbers.

Our results show that, if one wants to accommodate different particle numbers $n_1 \neq n_2$ one needs $\delta\mu > \Delta$, and, at least in the weak-coupling case, the system puts the excess particles into a spherical shell in momentum space with width $k_+^e - k_-^e$. This looks like an interesting state, but our calculation of the free energy suggests that this state is not preferred over the normal state, see Fig. 14. In other words, if we want more particles of one flavor than of another flavor, the system will choose not to be superfluid. In general, the answer to the question of superfluid states with mismatched fermion numbers is more complicated and is a much debated question in recent research.¹⁹ Although the breached state has not been found to be a viable option to the ground state in any case (be it quark matter or cold atoms or nuclear matter etc.), it helps us to understand possible other superfluid states that can accommodate excess particles of one fermion species: instead of an isotropic breach in momentum space, there are phases in which the excess, unpaired, particles sit in “caps” at the north and/or south pole of the Fermi sphere (see for instance Ref. [25], in particular Fig. 2 in this reference). In general, these phases will exhibit counter-propagating currents and a crystalline structure of the order parameter, i.e., a periodically varying gap function $\Delta(\mathbf{r})$. They have been originally suggested in the context of solid state physics by Fulde, Ferrell [26], and Larkin, Ovchinnikov [27] (hence they are termed FFLO or LOFF). Their possible variants are discussed for instance in dense quark matter and ultra-cold atoms. The details of these states are beyond the scope of this course.

Another – maybe less spectacular – option for combining a superfluid with a mismatch in number densities is phase separation, i.e., a state in which certain regions in position space are filled with a usual superfluid, while others contain the non-superfluid state where excess particles of one species can easily be accommodated.

D. Phase diagrams of spin-polarized ultra-cold atoms

We conclude this chapter with a brief discussion of the phase diagrams with mismatch in chemical potential and number densities for the case of cold atoms. In particular, we are interested in how the BCS-BEC crossover is affected by a mismatch. Before we come to these diagrams, we note that the nonrelativistic analogues of the number densities (338) are

$$n_1 = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \left[\frac{1}{2} \left(1 - \frac{\xi_k}{\epsilon_k} \right) + \frac{1}{2} \left(1 + \frac{\xi_k}{\epsilon_k} \right) \Theta(\delta\mu - \epsilon_k) \right], \quad (341a)$$

$$n_2 = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{1}{2} \left(1 - e^{\frac{\xi_k}{\epsilon_k}} \right) \Theta(\epsilon_k - \delta\mu), \quad (341b)$$

with

$$\epsilon_k = \sqrt{\xi_k^2 + \Delta^2}, \quad \xi_k = \frac{k^2}{2m} - \bar{\mu}. \quad (342)$$

Instead of Eq. (340) we now have

$$k_{\pm}^2 = 2m(\bar{\mu} \pm \sqrt{\delta\mu^2 - \Delta^2}). \quad (343)$$

Now remember that in the BCS-BEC crossover, μ (here: $\bar{\mu}$) is no longer large compared to Δ . This opens up a third qualitatively different situation in addition to the two shown in Fig. 15. Namely, if $\bar{\mu} < \sqrt{\delta\mu^2 - \Delta^2}$ there is

¹⁹ In fact, there are situations where the free energy comparison *does* indicate the breached phase to be preferred (for instance in dense quark matter under the neutrality constraint). In this case, however, one finds a more subtle instability: if one computes the Meissner mass as in Sec. VB with a nonzero $\delta\mu$, one finds $m_M^2 < 0$ for $\delta\mu > \Delta$ which is unphysical. (As a – rather long – exercise, you may repeat the calculation of Sec. VB with a nonzero $\delta\mu$ to verify this statement.) The result indicates an instability. In other words, suppose that, among the phases we have considered (normal and superfluid) the breached superfluid *is* preferred. Then the imaginary Meissner mass indicates that our ansatz does not include the ground state: there must be a state with even lower free energy which we have not considered so far. The solution to this problem is a superfluid state that breaks rotational and/or translational invariance.

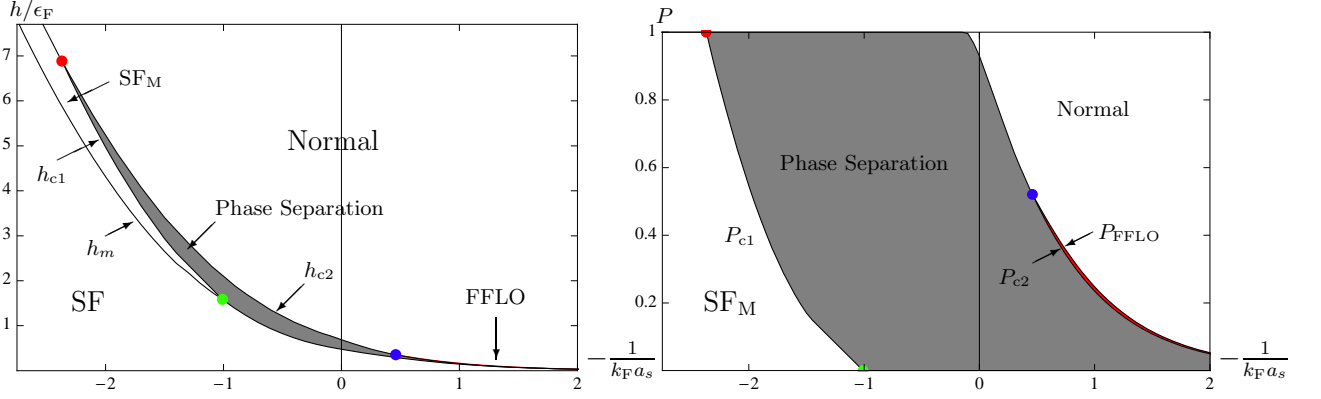


FIG. 16: Phase diagrams from Ref. [28] in the plane of the effective coupling strength $(k_F a)^{-1}$ and a mismatch in chemical potentials [left panel, the h corresponds to our $\delta\mu$] and number densities [right panel, the “polarization” P is defined as $(n_1 - n_2)/(n_1 + n_2)$]. Note that the horizontal scale is $-(k_F a)^{-1}$, in contrast to Figs. 10 and 12, where the scale is $(k_F a)^{-1}$.

no k_- (it formally becomes imaginary), i.e., what is a shell in momentum space in the right panel of Fig. 15 now becomes a sphere with radius k_+ . This superfluid state is only possible in the strong coupling regime and is called SF_M (magnetized superfluid) in Fig. 16. This figure shows a theoretical prediction for the phase structure in the plane of coupling strength and mismatch. We see that phase transitions occur, i.e., there is no BCS-BEC crossover anymore. The usual superfluid state SF is only possible in the left diagram since it locks the densities of the two fermion species together. Therefore, as soon as a nonzero “polarization” $n_1 - n_2$ is imposed, this state cannot exist. The Clogston limit discussed in VII B applies to the lower right corner of the left panel: in the BCS regime the gap is exponentially suppressed. Hence the mismatch needed to break the superfluid is also very small compared to the mismatch needed in the strong coupling regime. Besides the usual superfluid, there is the non-superfluid “normal” state and three “unusual” superfluids, the phase separated superfluid, the magnetized superfluid, and the FFLO state. The later is only present in a tiny region of the phase diagram. It is therefore very difficult to detect experimentally. Despite some effort it has not been seen so far in the experiment.

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