An Algorithmic Approach to Heterotic Compactification

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Outline

- Introduction
  - Heterotic Phenomenology (and the necessary mathematics)
  - Why we're interested (and the problems)
- The monad construction
  - The Calabi-Yau Spaces
  - Building vector bundles
  - Particle spectra
  - Bundle stability
- Finding physically relevant bundles - An algorithmic approach
- Future directions - Symmetry breaking, yukawa couplings, moduli stabilization.

Challenge of Heterotic Phenomenology

- How close can string theory get to real world particle physics?? We need unification, symmetries, fermion masses, yukawa couplings, moduli stabilization, etc.
  - How generic is real world physics within string theory? What are the properties of 'realistic' models?
- Heterotic models are promising (gauge unification is automatic, $N = 1$ SUSY, etc.). But mathematical details (algebraic geometry, defining bundles and manifolds) are difficult.
- It's easy to come close to the real world, but very hard to get the details exactly right.
  - Any single heterotic model is likely to fail when confronted with detailed structure of SM physics

    $\Rightarrow$ want to study large numbers of models $\Rightarrow$ an algorithmic approach

A heterotic model

We begin with the $E_8 \times E_8$ Heterotic string in 10-dimensions

- One E_8 gives rise to the "Visible" sector, the other to the "Hidden" sector
- Compactify on a Calabi-Yau 3-fold, X - leads to $\mathcal{N} = 1$ SUSY in $4D$
- Also have a vector bundle V on X (with structure group $G \subset E_8$)

  V breaks E_8 to Low Energy GUT group
- The weakly coupled theory has been studied since the 80's (beginning with the so-called 'Standard Embedding'). The strongly coupled theory is dual to M-Theory on a manifold with boundary - Hořava-Witten Theory -(w/ 5 branes in bulk)

The idea...

- Finding the correct string vacuua to model realistic particle physics is a difficult task. How to choose? How to design the right model?
- A new approach:

  Formulate an algorithmic and systematic search for the correct vacuum using computational algebraic geometry

  1. Produce a computer database of thousands of CY spaces and their topological data
  2. Construct broad, well-defined sets of vector bundles over them
  3. Scan through literally hundreds of billions of potential candidates in the vast landscape of string vacuua for those that are physically relevant

     - How many are close to nature? Study these models...

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General Embedding

A more general choice of vector bundle can be made

- Take $G = SU(n)$, $n = 3, 4, 5$ low energy gauge group
- $4D$ structure group, $H =$Commutant(G, E_8)

$E_8 \to G \times H$	Residual Group Structure
$\mathrm{SU}(3) \times \mathrm{E}_6$	$248 \to (1, 78) \oplus (3, 27) \oplus (\overline{3}, \overline{27}) \oplus (8, 1)$
$\mathrm{SU}(4) \times \mathrm{SO}(10)$	$248 \to (1, 45) \oplus (4, 16) \oplus (\overline{4}, \overline{16}) \oplus (6, 10) \oplus (15, 1)$
$\mathrm{SU}(5) \times \mathrm{SU}(5)$	$248 \to (1, 24) \oplus (5, \overline{10}) \oplus (\overline{5}, 10) \oplus (10, 5) \oplus (\overline{10}, \overline{5}) \oplus (24, 1)$

- We expect "Two-step" Symmetry breaking

1. E_8 breaks to GUT group (E_6,$SO(10)$, or $SU(5)$)
2. Wilson lines break GUT symmetry

Wilson line $\Rightarrow SU(3)_c \times SU(2)_l \times U(1)_Y \times U(1)_{B-L}$ symmetry

The Elements of the construction

- X: A Calabi-Yau 3−fold, X
- V: A holomorphic vector bundle, satisfying the Hermitian YM equations
- G: The structure group of V ($G \subset E_8$)
- H: The low energy $4D$, $\mathcal{N} = 1$ GUT symmetry (H is the commutant of G in E_8)
- V + Wilson line leads to symmetry containing that of the MSSM
- Heterotic vacuua contain $M5$-branes which can wrap a holomorphic effective 2-cycle, W ($\in H_2(X, \mathbb{Z})$), of X. Leads to anomaly cancellation condition

$$c_2(X) - c_2(V) = W_{M5}$$

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Stability

- Hermitian YM equations are a set of wickedly complicated PDE's

  $F_{ab} = F_{\bar{a}\bar{b}} = g^{a\bar{b}} F_{\bar{b}a} = 0$

- We are saved by the Donaldson-Uhlenbeck-Yau Theorem:

  On each stable, holomorphic vector bundle V, there exists a Hermitian YM connection satisfying the HYM equations.

- The slope, $\mu(V)$, of a vector bundle is

  $$\mu(V) \equiv \frac{1}{\mathrm{rk}(V)} \int_X c_1(V) \wedge J^{d-1}$$

  where J is a Kahler form on X.

- V is Stable if for every sub-sheaf, $\mathcal{F}$, of V, $\mu(\mathcal{F}) < \mu(V)$

- Unfortunately, "conservation of misery" $\Rightarrow$ stability still very hard to show!

Spectra and Cohomology

In heterotic models, $4D$ particle spectra is determined by bundle cohomology:

Decomposition	Cohomologies
$\mathrm{SU}(3) \times \mathrm{E}_6$	$n_{27} = h^1(V)$, $n_{\overline{27}} = h^1(V^*) = h^2(V)$, $n_1 = h^1(V \otimes V^*)$
$\mathrm{SU}(4) \times \mathrm{SO}(10)$	$n_{16} = h^1(V)$, $n_{\overline{16}} = h^2(V)$, $n_{10} = h^1(\wedge^2 V)$, $n_1 = h^1(V \otimes V^*)$
$\mathrm{SU}(5) \times \mathrm{SU}(5)$	$n_{10} = h^1(V^*)$, $n_{\overline{10}} = h^1(V)$, $n_5 = h^1(\wedge^2 V)$, $n_{\overline{5}} = h^1(\wedge^2 V^*)$ $n_1 = h^1(V \otimes V^*)$

- We need to compute various bundle cohomologies in order to proceed!

The Plan...

- We need a large class of CY manifolds and a systematic way to construct bundles over them

  ⇒ 7890 CICYs + Monad construction

- We require an explicit construction compatible with "Two-step" symmetry breaking, Wilson lines, etc.

  ⇒ CICYs are the simplest and most explict form of CY construction. Relatively easy to find discrete symmetries, Wilson lines, etc.

- Computerizability. Millions of models cannot be analyzed by hand, need an algorithmic approach that makes good use of existing technology in computational algebraic geometry.

  ⇒ Teach computers how to analyze monads! (C code, Mathematica, Singular, Macaulay)

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  ⇒ Teach computers how to analyze monads! (C code, Mathematica, Singular, Macaulay)

- Need to be able to compute bundle cohomology (Koszul and spectral sequences)
- Need to be able to prove bundle stability (Hoppe's Criterion and generalization)
- Scan millions of bundles for physical suitability!
- How many bundles are there? What distributions? What properties?...

Complete Intersection CYs

- CICYs: A product of projective space $\mathbb{P}^{n_1} \times \ldots \times \mathbb{P}^{n_m}$ and K defining polynomials $\{p_{j=1,\ldots,K}\}$
- The CY 3-fold is described by a configuration matrix (columns $\leftrightarrow$ constraints).

$$\left[\begin{array}{c|cccc} \mathbb{P}^{n_1} & q_1^1 & q_1^2 & \ldots & q_1^K \\ \mathbb{P}^{n_2} & q_2^1 & q_2^2 & \ldots & q_2^K \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbb{P}^{n_m} & q_m^1 & q_m^2 & \ldots & q_m^K \end{array}\right]_{m \times K} \quad (1)$$

- Favorable CICYs: Those for which $h^{1,1}$ = no. of embedding $\mathbb{P}^n$'s (4515 manifolds)

  $\Rightarrow$ The Kahler forms J on the CY descend from those on the ambient space.

  Computationally useful!

- Line bundles on a "favorable" CY: $\mathcal{O}_X(k_1, k_2, \ldots, k_m)$

What is a Monad?

- For this work, we consider monads defined by a short exact sequence of vector bundles (sheaves)

$$0 \to V \xrightarrow{f} B \xrightarrow{g} C \to 0$$

where short exact implies that $ker(g) = im(f)$.

- The vector bundle V is defined as

$$V = ker(g) \text{ with } rk(V) = rk(B) - rk(C)$$

- Where B and C are taken to be direct sums of line bundles

$$B = \bigoplus_{i=1}^{r_B} \mathcal{O}(b_r^i) \ , \qquad C = \bigoplus_{j=1}^{r_C} \mathcal{O}(c_r^j)$$

- The map g can be written as a matrix of polynomials. (e.g. on $\mathbb{P}^n$ the ij-th entry is a homogeneous polynomial of degree $c_i - b_j$)
- The monad construction is a powerful and general way of defining vector bundles. For example, every bundle on $\mathbb{P}^n$ can be written as a monad

Physical Constraints

To begin, we consider the most simple physical constraints:

- $SU(n)$ bundles - (Structure group $SU(n)$, $c_1(V) = 0$)
- Anomaly cancellation condition
- $Ind(V) = 3k$ for $k \in \mathbb{Z}$

  $k > 1 \Rightarrow$ need Wilson lines and discrete symmetries
- Stable bundles
- Monads must define bundles (i.e. defining exact sequences should produce bundles rather than just sheaves)

Classification

The physical and Mathematical constraints can be written as constraints on the integers defining the line bundles of the monad

$$B = \bigoplus_{i=1}^{r_B} \mathcal{O}(b_r^i)\ , \qquad C = \bigoplus_{j=1}^{r_C} \mathcal{O}(c_r^j)$$

where

$$0 \to V \xrightarrow{f} B \xrightarrow{g} C \to 0$$

- Is this a finite class? What are the properties of the bundles defined by these constraints?

Constraints

- $b_r^i \leq c_r^j$ for all $i, j, s \Rightarrow ker(g)$ defines a bundle.
- The map g can be taken to be *generic* so long as exactness of the sequence is maintained.
- $c_1(V) = 0 \Leftrightarrow$
  $\sum_{i=1}^{r_B} b_i^r - \sum_{j=1}^{r_C} c_j^r = 0$
- Anomaly cancellation $\Leftrightarrow$
  $c_2(TX) - c_2(V) = c_2(TX) - \frac{1}{2}(\sum_{i=1}^{r_B} b_s^i b_t^i - \sum_{i=1}^{r_C} c_s^j c_i^t) J^s J^t \geq 0$
- 3 Generations $\Leftrightarrow$
  $c_3(V) = \frac{1}{3}(\sum_{i=1}^{r_B} {b_r}^i {b_s}^i {b_t}^i - \sum_{j=1}^{r_C} {c_r}^j {c_s}^j {c_t}^j) J^r J^s J^t$
  is divisible by 3 (and compatible with the Euler number of X).
- Stability places constraints on the signs of b_r^i and c_r^j

Warming Up...The cyclic CICYs

The mathematical technology of producing bundles and computing their spectra is difficult, so we begin with the most straightforward possible cases...

- There are 5 cyclic ($Pic(X) = \mathbb{Z}$) CICYs
  [4|5], [5|2 4], [5|3 3], [6|3 2 2], [7|2 2 2 2]
- These are the simplest known CYs.
- We can find a complete classification of *all* physical monad bundles on these spaces.
  - Demanding $SU(n)$ and anomaly-free bundles is sufficient to bound the problem
  - A finite class - We find only 37 bundles over these 5 spaces

Cyclic CICYs

- For the cyclic CYs, we find only positive monads to be physical (e.g. those for which $b_i, c_j > 0$)
  - $b_i, c_j \leq 0 \Rightarrow$ unstable
- Can compute the full spectra of these bundles using exact and spectral sequences (and results can be checked using Macaulay, Singular)
- No anti-generations (limits exotics)
- The Higgs content is dependent on the choice of map (where we are in moduli space)
- Using Hoppe's Criterion, we find that all positive monads on CICYs are stable!

Monads on CICYS: An example

- For example consider the monad

  $0 \to V \longrightarrow \mathcal{O}(1)^7 \xrightarrow{g} \mathcal{O}(3) \oplus \mathcal{O}(2)^2 \to 0$

- this is a rank 4 bundle on $[4|5]$

- $$g = \begin{pmatrix} x_i x_j & x_i^2 + \dots & \dots & \dots \\ x_i & x_j \dots & & \\ x_i & x_j \dots & & \end{pmatrix} \tag{2}$$

  - $SU(4)$ bundle, stable, anomaly-free
  - $c_3 = -30 \Rightarrow \mathbb{Z}_5 \times \mathbb{Z}_5$ symmetry for Wilson lines
  - Cohomology calculation gives us $n_{16} = 30$, $n_1 = 112$ (Number of Higgs depends on choice of map)

Rank	$\{b_i\}$	$\{c_i\}$	$c_2(V)/J^2$	$ind(V)$
3	(2, 2, 1, 1, 1)	(4, 3)	7	-60
3	(2, 2, 2, 1, 1)	(5, 3)	10	-105
3	(3, 2, 1, 1, 1)	(4, 4)	8	-75
3	(1, 1, 1, 1, 1, 1)	(2, 2, 2)	3	-15
3	(2, 2, 2, 1, 1, 1)	(3, 3, 3)	6	-45
3	(3, 3, 3, 1, 1, 1)	(4, 4, 4)	9	-90
3	(2, 2, 2, 2, 2, 2, 2, 2)	(4, 3, 3, 3, 3)	10	-90
3	(2, 2, 2, 2, 2, 2, 2, 2, 2)	(3, 3, 3, 3, 3, 3)	9	-75
4	(2, 2, 1, 1, 1, 1)	(4, 4)	10	-90
4	(1, 1, 1, 1, 1, 1, 1)	(3, 2, 2)	5	-30
4	(2, 2, 2, 1, 1, 1, 1)	(4, 3, 3)	9	-75
4	(2, 2, 2, 2, 1, 1, 1, 1)	(3, 3, 3, 3)	8	-60
5	(1, 1, 1, 1, 1, 1, 1, 1)	(3, 3, 2)	7	-45

Positive Monads on favorable CICYs

- To generalize these techniques beyond the cyclic manifolds, we begin by considering positive monads over all favorable CICYs
- How to compute the spectra? Need to develop tools to compute the cohomology of line bundles on CICYs
- For positive line bundles this is easy -

  Kodaira Vanishing theorem: $H^k(X, L) = 0$ for all $k > 0$ if

  $L = O(m_1, m_2, \ldots, m_n)$ with $m_i > 0$.
- But what about general mixed line bundles? e.g. $O(-m_1, -m_2, m_3, \ldots)$. Need new techniques...

Computing line bundle cohomology on CICYs

- Need to combine the techniques of Spectral sequences and the Bott-Borel-Weil theorem
  - General Idea: Use information from the ambient space, A. Define define bundles $\mathcal{L}$ over A and use a Koszul resolution:
    $0 \to \mathcal{L} \otimes \wedge^K N_X^* \to \mathcal{L} \otimes \wedge^{K-1} N_X^* \to \ldots \to \mathcal{L} \otimes N_X^* \to \mathcal{L} \to \mathcal{L}|_X \to 0$
- The Spectral sequence for line bundle cohomology is given by
  $E_1^{j,k}(L) := H^j(A, L \otimes \wedge^k N_X^*),\ k = 0, \ldots, K,\ j = 0, \ldots, \dim(A) = \sum_{i=1}^m n_i.$
- This forms the first term of a spectral sequence- a complex defined by differential maps $d_i : E_i^{j,k} \to E_i^{j-i+1,k-i}$.
- The subsequent terms in the spectral sequence are defined by
  - $E_{i+1}^{j,k}(L) = \frac{ker(d_i : E_i^{j,k}(L) \to E_i^{j-i+1,k-i}(L))}{Im(d_i : E_i^{j+i-1,k+i}(L) \to E_i^{j,k}(L))}$
  - The sequence converges to $E_\infty^{j,k}(L) = E_2^{j,k}(L)$. (For line bundles)
  - $h^q(X, L|_X) = \sum_{m=0}^K \mathrm{rank} E_\infty^{q+m,m}(L)$

Line bundle Cohomology

- By Bott-Borel-Weil, the entries in the tableau $E_1^{j,k}(L) := H^j(A, L \otimes \wedge^k N_X^*)$ can be represented as polynomial spaces (dimensions given by the Bott Formula)
- To compute $E_2^{j,k}(L)$, need to know ranks and kernels of maps $d_i : E_i^{j,k} \longrightarrow E_i^{j-i+1,k-i}$
- Exampe: A co-dimension one tableau: $E_1^{j,k} = \begin{bmatrix} 0 & & 0 \\ E_1^{j,0} & \overset{d_1^1}{\leftarrow} & E_j^{1,1} \\ 0 & & 0 \\ \vdots & & \vdots \end{bmatrix}$
- Need to compute the ranks and kernels of polynomial maps of the type d_1^1.
- Example: For the line bundle $l = O(-k, m)$ on the CICY, $X = \left[\begin{array}{c|c} 1 & 2 \\ 3 & 4 \end{array} \right]$
- Direct computation of the kernel of the map $d_1^1 \Rightarrow$

$$h^q(X, \mathcal{O}_X(-k, m)) = \begin{cases} (k+1)\binom{m}{3} - (k-1)\binom{m+3}{3} & q = 0 \quad k < \frac{(1+2m)(6+m+m^2)}{3(2+3m(1-m))} \\ (k-1)\binom{m+3}{3} - (k+1)\binom{m}{3} & q = 1 \quad k > \frac{(1+2m)(6+m+m^2)}{3(2+3m(1-m))} \\ 0 & \text{otherwise} \end{cases}$$

Positive Monads on favorable CICYs

- For the monads with strictly positive entries ($b^i, c^j > 0$) the $SU(n)$ and anomaly conditions are sufficient to bound the problem. The class is finite and all physical bundles can be classified.
- Over the CICYs we find ~ 7000 bundles and compute their spectra (using new techniques to compute the cohomology of line bundles on CICYs)
- No anti-generations!
- Unfortunately, we are limited by Wilson Lines and discrete symmetries of the Calabi-Yau

  Number of generations $\Leftrightarrow c_3 = 3n$, $n \in \mathbb{Z}$

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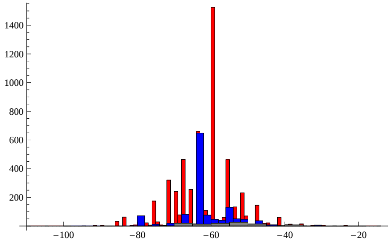
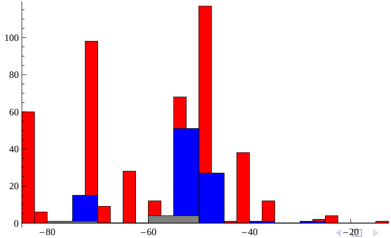
- In order to produce *exactly* 3 generations, need to divide the CY by a discrete symmetry of size n (and $ind(V)$ must divide Euler number of CY)
- If c_3 too large, no symmetries of the right order exist on the CY
- The 37 monads on the cyclic CYs have the smallest $c_3 \Leftrightarrow$ most plausible models for Wilson line symmetry breaking.
- Of the 7000 positive monad bundles found on CICYs there are only 21 models with $ind(V) < 40$

Positive monads and 3-generations





Extending the search...

Since the positive bundles on CICYs are highly restricted, in order to produce a large class for an algorithmic scan, we must extend our search...

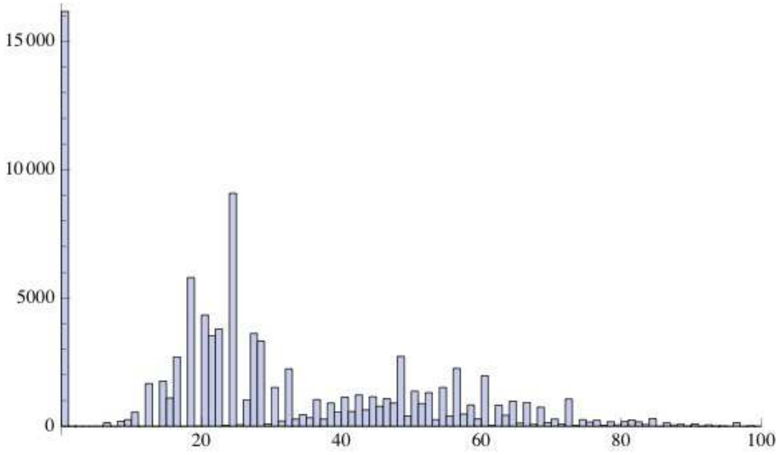
Zero-Entry Monads -"Semi-positive" bundles

For those with entries greater than or equal to zero ($b^i, c^j \geq 0$) the construction is much bigger (and more interesting!)

- Clearly not constrained as before, can produce unbounded sets of bundles

  Example: $B = \mathcal{O}(1,0)^3 \oplus \mathcal{O}(t-3,0)$ and $C = \mathcal{O}(t,0)$ is an anomaly-free bundle for each integer $t > 1$ on $\left[\begin{array}{c|cc} \mathbb{P}^1 & 0 & 2 \\ \mathbb{P}^4 & 4 & 1 \end{array}\right]$
- An infinite class? Isomorphisms?
- Much more physically suitable
  - smaller $c_3(V)$
  - Generically have a Higgs
- Can do checks of stability (scan for $H^{0,3}(X,V) = 0$)
- Can prove stability for some models.

Zero-entry distributions



semi-positives

- An initial scan bounding $b_r^i, c_r^j < 20$ produces $\sim 100,000$ rank 3 bundles on CICYs with $h^{1,1} = 2$
- Number of models with 3-generations and Euler number compatible with CY $= 17,255$
- Number of models with $ind(V) < 20$, 6982
- Initial scans show that many of these are stable

Hoppe's Criterion

Over a projective manifold X with Picard group $Pic(X) \simeq \mathbb{Z}$, let V be a vector bundle. If $H^0(X, [\bigwedge^p V]_{norm}) = 0$ for all $p = 1, 2, \ldots, rk(V) - 1$, then V is stable.

- Those manifolds where $Pic(X) \simeq \mathbb{Z}$ are called *cyclic*
- Where $[V]_{norm} = V(i) := V \otimes O_X(i)$ for a unique i such that $c_1(V(i)) \in [-rk(V) + 1, \ldots, -1, 0]$
- *normalize* V so that the slope $\mu(V)$ is between -1 and 0.
- Hoppe's criterion applies directly to the 5 cyclic CYs
- Need a generalization to arbitrary CICYs?
- We will begin with this criterion and the cyclic manifolds...

The idea of Hoppe's criterion

Proof by contradiction:

- Suppose the bundle is unstable $\Rightarrow$ a de-stabilizing sub-sheaf F (of rank n) w/ $\mu(F) \geq 0$.
- Take wedge powers to form a line bundle, $l = (\wedge^n F)^{**}$
- $c_1(F) \geq 0$ and cyclic space $\Rightarrow$ l has a section
- $\Rightarrow l \subset \wedge^n V$
- $\Rightarrow \wedge^n V$ has a section
- $\Rightarrow H^0(X, \wedge^n V) \neq 0$

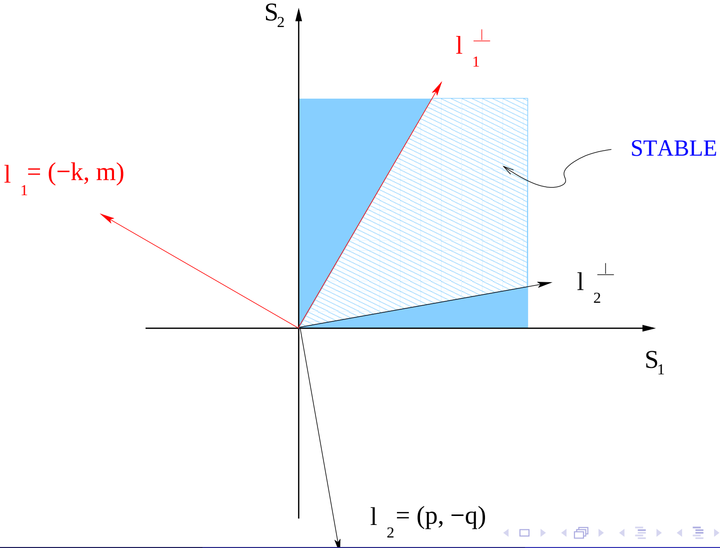
So if $H^0(X, \wedge^n V) = 0$ for $n = 1, 2, ..., rk(V) - 1$ then stable

Generalization to CICYs

A proof of stability for monads on CICYs (to appear...)

- Stable "somewhere" in the Kahler cone?
- Rather than attempt to analyze all possible sub-sheaves, can we indirectly place bounds on them?
- Extend Hoppe's criterion ⇒ Consider potential sub-line bundles of $\wedge^k V$ for $k = 1, \dots (n-1)$.
- Potential sub-line bundles are classified by $c_1(l) = (k, m)$ for $l = O(k, m)$
- Consider all such line bundles (i.e. first chern classes)
- Impose cohomological conditions to bound potential line bundle sub-sheaves, l. e.g. $Hom(l, \wedge^k V) \neq 0$ and $H^0(X, l) = 0$.
- Constrain possible sub-sheaves to the extent that we can show that there exists a well-defined stable region in the Kahler cone

Stability somewhere in the Kahler cone



An algorithmic approach...

- We have successfully constructed a LARGE class of vector bundles and can compute their properties in detail
- Previous attempts have been made to create classes of stable bundles (e.g. Friedman-Morgan-Witten). However, this is the first effort to create an extensive class of stable, physically relevant bundles suitable for algorithmic scans.
- So, far we have scanned for "higher-level" physical constraints - i.e. particle spectra
- We can verify low-energy SUSY (i.e. bundle stability) for a sub-class
- We are now ready to impose more detailed physical constraints - Wilson lines, discrete symmetries, etc.

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Future work

We are only at the beginning of this effort...

- Add Wilson lines, explore realistic 4D models (in progress)
- Extend techniques to the $473, 800, 776$ toric CY manifolds (in progress)
- Generalize these techniques to $U(n)$ bundles
- Compute yukawa couplings? Fermion masses? (in progress)

The End

Finding a Higgs doublet

- Higgs at special points - The "Jumping phenomena".
- Example: Specifically, let is consider the following $SU(4)$ bundle on $[4|5]$:

  $0 \to V \to \mathcal{O}_X^{\oplus 2}(2) \oplus \mathcal{O}_X^{\oplus 4}(1) \xrightarrow{g} \mathcal{O}_X^{\oplus 2}(4) \to 0$

  with ($x_{0,\dots,4}$ are the homogeneous coordinates on $\mathbb{P}^4$)

-

$$g = \begin{pmatrix} 4x_3^2 & 9x_0^2 + x_2^2 & 8x_2^3 & 2x_3^3 & 4x_1^3 & 9x_1^3 \\ x_0^2 + 10x_2^2 & x_1^2 & 9x_2^3 & 7x_3^3 & 9x_1^3 + x_2^3 & x_1^3 + 7x_4^3 \end{pmatrix} . \quad (3)$$

  - We can calculate that

    $n_{16} = h^1(X, V) = 90,\ n_{10} = h^1(X, \wedge^2 V) = 13$

    $n_1 = 277$

So...

We want a stable holomorphic vector bundle V on $X \Rightarrow$

- For stable bundles on a CY 3-fold, X, stability implies that $H^0(X, V) = H^0(X, V^*) = 0$
- In general: V with $SU(n)$

  INDEX THEOREM and particle generations:

  1. $\text{index}(\nabla\!\!\!\!/\,_X) = \sum_{i=0}^{3}(-1)^i h^i(X, V) = \int_X \text{ch}(V)\text{td}(X) = \frac{1}{2}\int_X c_3(V)$
  2. Serre Duality: $h^i(X, V) = h^{3-i}(X, V^* \otimes K\!\!\!/_X)$
- $\Rightarrow$ 3-families: $3 = -h^1(X, V) + h^1(X, V^*)$
- Unfortunately, "conservation of misery" $\Rightarrow$ stability still very hard to show!