ESI - Mathematical Challenges in String Phenomenology - October 8th, 2008

Effective superpotentials for compact D5-brane Calabi-Yau geometries

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arXiv:0808.0761 [hep-th], work in progress

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Introduction & Motivation

c.f. Uranga's review

✓ Semi-realistic type II string model building

- Ingredients: Fluxes, D-branes, O-planes
- N=1 low energy effective description
  - String phenomenology & String cosmology
  - <u>However:</u>
    - Most results only in the large radius limit or for local models
    - Explicit computations often hard or impossible

▶ Quantitative results for the N=1 effective superpotential in type II compactifications

✓ Enumerative geometry in topological string theories with branes

- Tool: Mirror symmetry between the topological A- & B-model
- Compute the partition function of the topological A-model
  - "Count holomorphic curves" in the large radius regime
  - Equivariant invariant in the vicinity of orbifold points
- Disk instanton generated domain wall tensions
- <u>However:</u> Most results for the open sector only for non-compact geometries

▶ Topological disk partition function for compact D5-brane Calabi-Yau geometries

Introduction & Motivation

✓ Mirror symmetry

	Topological B-model		Topological A-model
Bulk geometry (closed sector):	Calabi-Yau Y C.S. moduli space const. maps	closed string ⟷ mirror map	Calabi-Yau X Kähler moduli space holomorphic maps
	Classical geometry		Quantum geometry
	Genus 0 partition function/holomorphic prepotential		
Brane geometry (open sector):	Calabi-Yau Y + hol. vect. bundle (sheave)	open-/closed string ⟷ mirror map	Calabi-Yau X + spec. Lagr. submanifold with flat connection
	Classical obstructions		Quantum obstructions
	Disk partition function/holomorphic superpotential		

✓ Here: Focus on the holomorphic effective superpotential

Outline

1. Effective superpotentials
2. Relative three-form periods for D5-brane Calabi-Yau geometries
3. Explicit example
4. Conclusions

Effective superpotentials

Gukov, Vafa, Witten; Witten; Lerche, Mayr, Warner; Soroush, HJ

✓ Type IIB N=1 effective superpotentials

$$W_{\mathrm{RR}}(z) = \int_Y \Omega(z) \wedge F^{(3)} = N \int_\Gamma \Omega(z) \qquad \Gamma \in H_3(Y,\mathbb{Z})$$

$$W_{\mathrm{D5}} = \int_S \Omega(z) \lrcorner \big(\zeta(u) \wedge \bar\partial \zeta(u)\big) = \hat N \int_{\hat\Gamma(u)} \Omega(z) \qquad \partial\hat\Gamma \subset S$$

$$W(z,u) = \underline{N} \int_{\underline{\Gamma}(u)} \Omega(z) \,, \quad \underline{\Gamma} \in H_3(Y,S,\mathbb{Z})$$

- The quanta $\underline{N}$ specify RR three-form fluxes and wrapped D5-brane cycles (NS three form fluxes and NS5-branes are included via $\underline{N} \rightarrow \underline{N} + \tau\, \underline{M}$)
- 3-form fluxes and 5-branes are often related by geometric transitions

Ooguri, Vafa

✓ Type IIA N=1 effective superpotentials

$$W_{\mathrm{D6}}(t,\hat t) = \sum_{\substack{d,\ell \\ k\geq 1}} \frac{n_{d,\ell}}{k^2} \left(e^{2\pi i t}\right)^{d\,k} \left(e^{2\pi i \hat t}\right)^{\ell\, k}$$

- Disk instanton generated superpotential wrapping classes of $H_2(X,L)$
- t : closed-string Kähler modulus, $\hat t$: open-string modulus (generator of $H_1(L)$)

Periods & Semi Periods

✓ Relative period integrals

$$\underline{\Pi}(z,u) = \int_{\underline{\Gamma}} \Omega(z) - \int_{\partial\underline{\Gamma}} \theta(u) = \int_{\underline{\Gamma}} \underline{\Omega}(z,u)$$

$$\uparrow \quad H^3(Y) \qquad \uparrow \quad H^2_{\mathrm{var}}(S) \qquad \uparrow \quad H^3(Y,S)$$

- Moduli dependence encoded in the relative three form $\underline{\Omega}(z,u)$

✓ Flux three-form periods

- Symplectic A- and B-cycles: $X = \int_A \underline{\Omega}(z,u)\,, \qquad \mathcal{F}_X = \frac{\partial F}{\partial X} = \int_B \underline{\Omega}(z,u)$
- Flat periods: $\Pi = (1\,, t\,, \mathcal{F}_t\,, 2\mathcal{F} - t\mathcal{F}_t)$

✓ Brane semi-periods

- Semi-periods $\hat{\Gamma} = (\hat{\Gamma}_{\hat{X}}\,\hat{\Gamma}_W)\,,\ \partial\hat{\Gamma} \subset S:$ $\quad \hat{X} = \int_{\hat{\Gamma}_{\hat{X}}} \underline{\Omega}(z,u)\,, \qquad W_{\hat{X}} = \int_{\hat{\Gamma}_W} \underline{\Omega}(z,u)$
- Flat semi-periods: $\hat{\Pi} = (\hat{t}\,, W_{\hat{t}}\,, *)$

Variation of Mixed Hodge Structure

Lerche, Mayr, Warner; Soroush, HJ

✓ Relative three forms

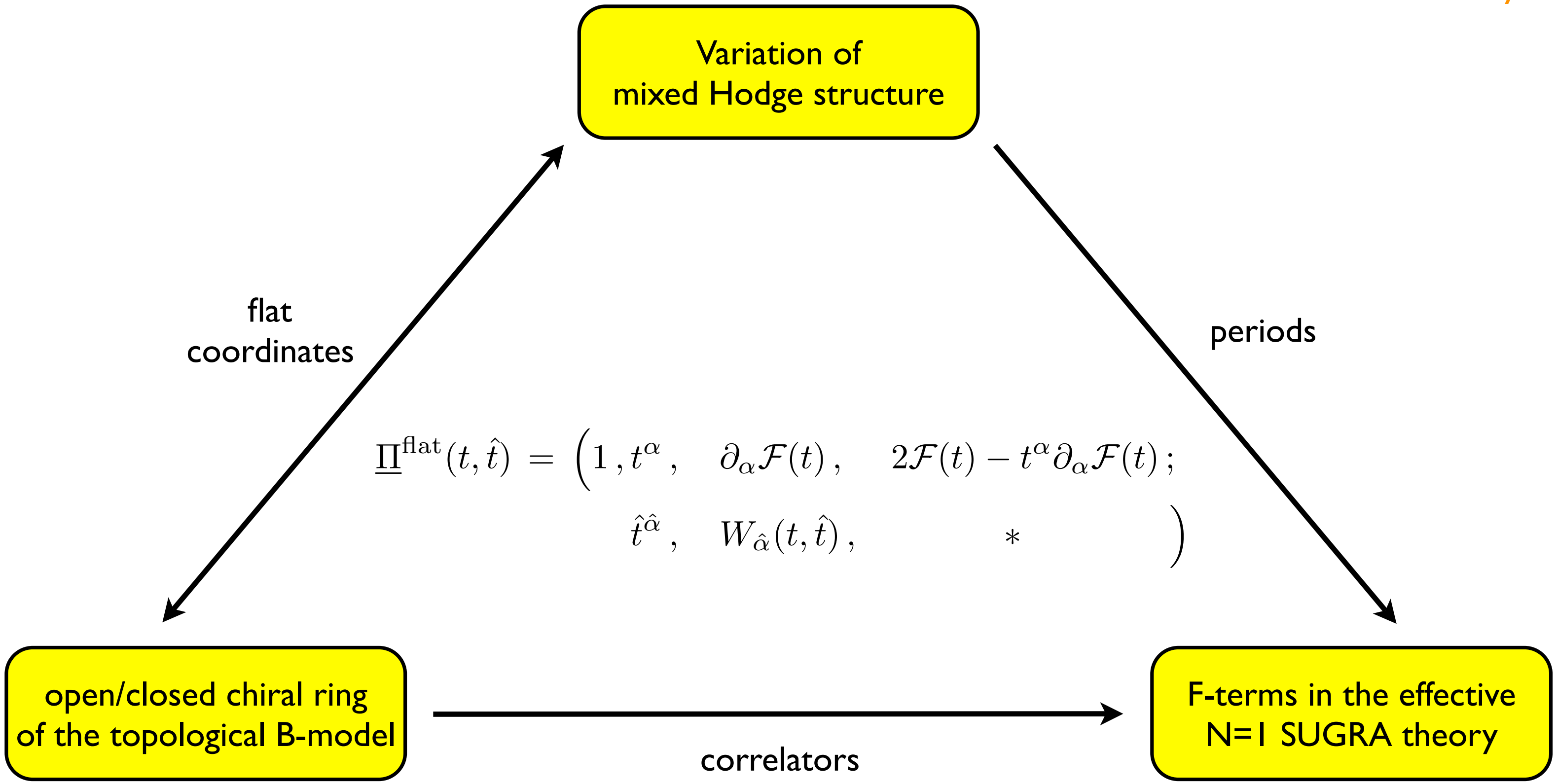
$$H^3(Y,S) \cong H^3(Y) \oplus H^2_{\mathrm{var}}(S)$$

✓ Variation of mixed Hodge structure

$$\begin{array}{ccccccc}
(3,0)_Y & \xrightarrow{\partial_z} & (2,1)_Y & \xrightarrow{\partial_z} & (1,2)_Y & \xrightarrow{\partial_z} & (0,3)_Y \\
& \searrow^{\partial_u} & & \searrow^{\partial_u} & & \searrow^{\partial_u} & \\
& & (2,0)_S & \xrightarrow{\partial_z,\partial_u} & (1,1)_S & \xrightarrow{\partial_z,\partial_u} & (0,2)_S
\end{array}$$

N=1 special geometry

Lerche, Mayr, Warner



$$\underline{\Pi}^{\rm flat}(t,\hat{t}) = \Big(1\,,\, t^\alpha\,, \quad \partial_\alpha \mathcal{F}(t)\,, \quad 2\mathcal{F}(t) - t^\alpha \partial_\alpha \mathcal{F}(t)\,; \quad \hat{t}^{\hat{\alpha}}\,, \quad W_{\hat{\alpha}}(t,\hat{t})\,, \quad * \Big)$$

$$C_{\alpha\beta\gamma}(t) = \partial_\alpha \partial_\beta \partial_\gamma \mathcal{F}(t)$$

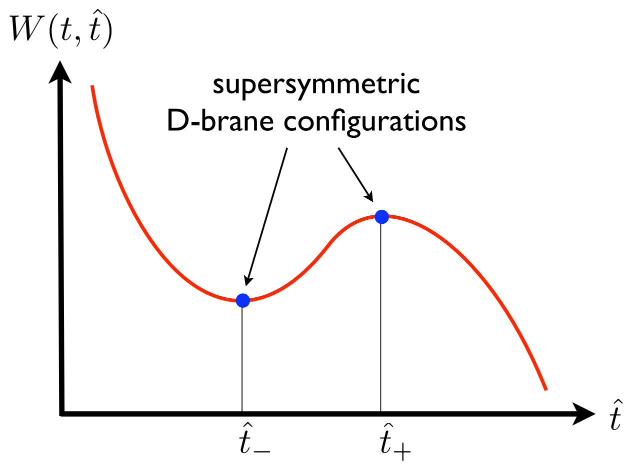
$$\hat{C}_{\alpha\hat{\beta}\hat{\gamma}}(t,\hat{t}) = \partial_\alpha \partial_{\hat{\beta}} W_{\hat{\gamma}}(t,\hat{t})$$

$$\mathcal{L}^{\rm SUGRA}_{\hat{\alpha}} \supset \int d^2\theta\, W_{\hat{\alpha}}(t,\hat{t}) \,+\, {\rm h.c.}$$

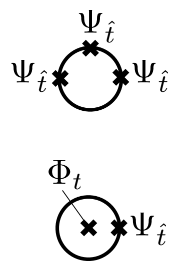
Effective superpotential

c.f. Herbst's review

✓ Variation of mixed Hodge structure



Correlators & obstructions:



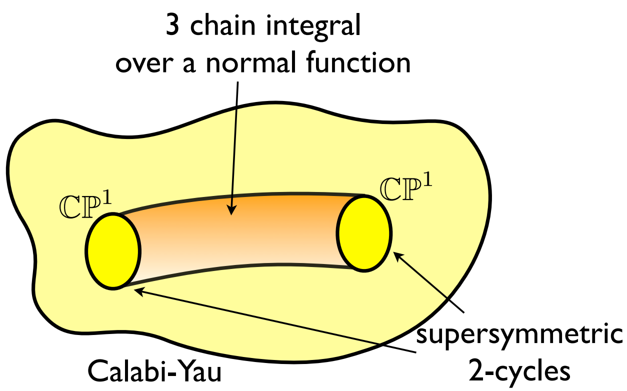
$= \langle \Psi_{\hat{t}} \Psi_{\hat{t}} \Psi_{\hat{t}} \rangle$ open sector

$= \langle \Psi_{\hat{t}} \Phi_t \rangle$ open/closed sector

Walcher; Morrison, Walcher; Krefl, Walcher; Knapp Scheidegger

✓ Domain wall tension

$$\mathcal{T}(t) = W(t,\hat{t}_+) - W(t,\hat{t}_-)$$



▸ Flat coordinates encode "off-shell information" of the superpotential

Calabi-Yau and D5-brane geometry

Griffiths; Lerche, Mayr, Warner

✓ Relative cohomology groups for D5-brane geometries

- Holomorphic 2-cycles only at the critical points of the effective superpotential
- Replace (non-holomorphic) 2-cycles S by holomorphic hypersurfaces/divisor V
- Study the relative cohomology with respect to the divisor V

$$H^3(Y,S) \implies H^3(Y,V)$$

Soroush, HJ

✓ Calabi-Yau and D5-brane geometry

- Calabi-Yau (hypersurface) geometry and complex structure moduli dependence

ambient (weighted) projective space	Calabi-Yau hypersurface
$x \in \mathbb{WP}^4_{(n_1,n_2,n_3,n_4,n_5)}$	$P(x,z) \equiv 0\ ,\ \deg P = n_1 + \ldots + n_5$

- D5-brane divisor V of the Calabi-Yau and its open-string moduli dependence

defining equation	divisor V
$Q = Q(x,u)$	$Q(x,u) \equiv 0\ \&\ P(x,z) \equiv 0$

Explicit example

Walcher; Krefl, Walcher; Knapp, Scheidegger; Soroush, HJ

✓ Bulk & D5-brane geometry

- one complex structure modulus ψ, one open-string modulus ϕ

$$\mathbb{WP}^4_{(1,1,1,1,4)}/(\mathbb{Z}_8)^2 \times \mathbb{Z}_2$$

$$P(\psi) = x_1^8 + x_2^8 + x_3^8 + x_4^8 + x_5^2 - 8\,\psi\, x_1 x_2 x_3 x_4 x_5 \qquad Q(\phi) = x_5 - \phi\, x_1 x_2 x_3 x_4$$

✓ Discrete moduli spaces symmetries

$$\mathbb{Z}_8 \times \mathbb{Z}_2 \,:\, g_1 \,:\, \begin{pmatrix}\psi \\ \phi\end{pmatrix} \mapsto e^{2\pi i/8} \begin{pmatrix}\psi \\ \phi\end{pmatrix} \,,\; g_2 \,:\, \begin{pmatrix}\psi \\ \phi\end{pmatrix} \mapsto \begin{pmatrix}\psi \\ 8\psi - \phi\end{pmatrix}$$

✓ Supersymmetric D5-brane configurations & critical points

$$\phi_+ = 0\,, \qquad C_+ = \{x_5 = 0\,,\, x_1 - \eta\, x_2 = 0\,,\, x_3 - \eta\, x_4 = 0\} \subset Y$$
$$\phi_- = 8\,\psi\,, \quad C_- = \{x_5 - 8\psi x_1 x_2 x_3 x_4 = 0\,,\, x_1 - \eta\, x_2 = 0\,,\, x_3 - \eta\, x_4 = 0\} \subset Y$$

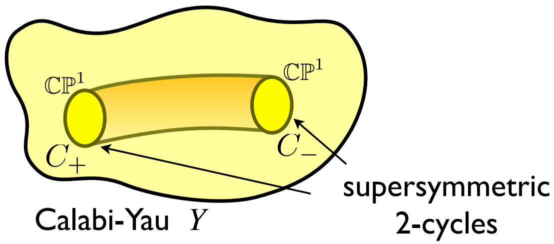
c.f. Herbst's review

✓ RS-branes at the Gepner point & matrix factorizations

$$(L_1, \ldots, L_5) = (3,3,2,1,0)$$

$$Q_+(\psi) = \sum_{\ell=1}^{5} \left(x_\ell^{L_\ell+1}\, \pi_\ell + x_\ell^{n_\ell - L_\ell + 1}\, \bar{\pi}_\ell \right) + x_5\, \pi_5 - 8\psi x_1 x_2 x_3 x_4\, \bar{\pi}_5$$

$$Q_-(\psi) = \sum_{\ell=1}^{5} \left(x_\ell^{L_\ell+1}\, \pi_\ell + x_\ell^{n_\ell - L_\ell + 1}\, \bar{\pi}_\ell \right) - 8\psi x_1 x_2 x_3 x_4\, \pi_5 + x_5\, \bar{\pi}_5$$



Linear Gauss-Manin system

Lerche, Mayr, Warner; Soroush, HJ

✓ Variation of Mixed Hodge Structure

$$\begin{array}{ccccccc} (3,0)_Y & \xrightarrow{\partial_\psi} & (2,1)_Y & \xrightarrow{\partial_\psi} & (1,2)_Y & \xrightarrow{\partial_\psi} & (0,3)_Y \\ & \searrow^{\partial_\phi} & & \searrow^{\partial_\phi} & & \searrow^{\partial_\phi} & \\ & & (2,0)_V & \xrightarrow{\partial_\psi,\partial_\phi} & (1,1)_V & \xrightarrow{\partial_\psi,\partial_\phi} & (0,2)_V \end{array}$$

$$\underline{\Omega}(z,u) \sim \int \frac{\log Q(\phi)}{P(\psi)}\,\Delta$$

✓ Basis of relative three-forms

$$\vec{\pi} \equiv \left(\underline{\Omega},\, \partial_\psi\underline{\Omega}\,,\, \partial_\psi^2\underline{\Omega}\,,\, \partial_\psi^3\underline{\Omega};\;\; \partial_\phi\underline{\Omega}\,,\, \partial_\psi\partial_\phi\underline{\Omega}\,,\, \partial_\psi^2\partial_\phi\underline{\Omega}\right)$$

✓ Linear system

$$\nabla_\psi\vec{\pi} \equiv (\partial_\psi - M_\psi)\vec{\pi} \simeq 0 \qquad\qquad \nabla_\phi\vec{\pi} \equiv (\partial_\phi - M_\phi)\vec{\pi} \simeq 0$$

$$M_\psi = \left(\begin{array}{c|c} \mathrm{B}_{4\times4}(\psi) & \mathrm{C}^{\psi}_{4\times3}(\psi,\phi) \\ \hline 0_{3\times4} & \mathrm{D}^{\psi}_{3\times3}(\psi,\phi) \end{array}\right) \qquad M_\phi = \left(\begin{array}{c|c} 0_{4\times4} & \mathrm{C}^{\phi}_{4\times3}(\psi,\phi) \\ \hline 0_{3\times4} & \mathrm{D}^{\phi}_{3\times3}(\psi,\phi) \end{array}\right)$$

✓ Flatness & integrability

$$[\nabla_\psi\,,\,\nabla_\phi] = \partial_\phi M_\psi - \partial_\psi M_\phi + [M_\psi\,,\,M_\phi] = 0$$

Open/closed Picard-Fuchs equations

✓ Picard-Fuchs equations for the relative periods $\Pi(\psi,\phi)=\int\underline{\Omega}(\psi,\phi)$

$$\mathcal{L}_1\underline{\Pi}(\psi,\phi) = \widetilde{\mathcal{L}}_1\partial_\phi\underline{\Pi}(\psi,\phi) = 0$$

$$\mathcal{L}_2\underline{\Pi}(\psi,\phi) = \widetilde{\mathcal{L}}_2\partial_\phi\underline{\Pi}(\psi,\phi) = 0$$

$$\mathcal{L}_3\underline{\Pi}(\psi,\phi) = \left(\mathcal{L}^{\text{bulk}} + \widetilde{\mathcal{L}}_3\partial_\phi\right)\underline{\Pi}(\psi,\phi)$$

✓ Solutions of the open/closed Picard-Fuchs equations

- Solutions in the vicinity of the $\mathbb{Z}_8$x$\mathbb{Z}_2$-orbifold point

$$\underline{\Pi}^0(\psi) \sim {}_4F_3\left[\begin{matrix}\frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8}\\ & \frac{1}{4} & \frac{2}{4} & \frac{3}{4}\end{matrix}\right]\left((2\psi)^8\right)$$

$$\underline{\Pi}^1(\psi) \sim \psi^2\cdot{}_4F_3\left[\begin{matrix}\frac{3}{8} & \frac{3}{8} & \frac{3}{8} & \frac{3}{8}\\ & \frac{2}{4} & \frac{3}{4} & \frac{5}{4}\end{matrix}\right]\left((2\psi)^8\right)$$

$$\underline{\Pi}^2(\psi) \sim \psi^4\cdot{}_4F_3\left[\begin{matrix}\frac{5}{8} & \frac{5}{8} & \frac{5}{8} & \frac{5}{8}\\ & \frac{3}{4} & \frac{5}{4} & \frac{6}{4}\end{matrix}\right]\left((2\psi)^8\right)$$

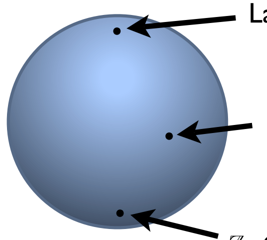
$$\underline{\Pi}^3(\psi) \sim \psi^6\cdot{}_4F_3\left[\begin{matrix}\frac{7}{8} & \frac{7}{8} & \frac{7}{8} & \frac{7}{8}\\ & \frac{5}{4} & \frac{6}{4} & \frac{7}{4}\end{matrix}\right]\left((2\psi)^8\right)$$

$$\underline{\Pi}^4(\psi,\phi) \sim \int {}_3F_2\left[\begin{matrix}\frac{1}{4} & \frac{1}{4} & \frac{1}{4}\\ & \frac{2}{4} & \frac{3}{4}\end{matrix}\right]\left(\frac{\phi^4}{4}(\phi-8\psi)^4\right)d\phi$$

$$\underline{\Pi}^5(\psi,\phi) \sim \int \phi(\phi-8\psi)\cdot{}_3F_2\left[\begin{matrix}\frac{2}{4} & \frac{2}{4} & \frac{2}{4}\\ & \frac{3}{4} & \frac{5}{4}\end{matrix}\right]\left(\frac{\phi^4}{4}(\phi-8\psi)^4\right)d\phi$$

$$\underline{\Pi}^6(\psi,\phi) \sim \int \phi^2(\phi-8\psi)^2\cdot{}_3F_2\left[\begin{matrix}\frac{3}{4} & \frac{3}{4} & \frac{3}{4}\\ & \frac{5}{4} & \frac{6}{4}\end{matrix}\right]\left(\frac{\phi^4}{4}(\phi-8\psi)^4\right)d\phi$$

Flat relative periods



✓ Flat relative periods

$$\underline{\Pi}^{\mathrm{flat}}(t,\hat{t}) = \Big(1\,,\, t\,,\, \partial_t\mathcal{F}(t)\,,\, 2\mathcal{F}(t) - t\partial_t\mathcal{F}(t)\,;\, \hat{t}\,,\, W(t,\hat{t})\,,\, *\Big)$$

✓ Distinguished flat relative periods in the vicinity of the orbifold point

- Regularity in the vicinity of the $\mathbb{Z}_8$x$\mathbb{Z}_2$-orbifold singularity
- Irreducible representations with respect to the $\mathbb{Z}_8$x$\mathbb{Z}_2$-orbifold group

$$\mathbb{Z}_8\times\mathbb{Z}_2\ :\ g_1\ :\ \begin{pmatrix}\psi\\ \phi\end{pmatrix}\mapsto e^{2\pi i/8}\begin{pmatrix}\psi\\ \phi\end{pmatrix}\ ,\ g_2\ :\ \begin{pmatrix}\psi\\ \phi\end{pmatrix}\mapsto\begin{pmatrix}\psi\\ 8\psi-\phi\end{pmatrix}$$

- The prepotential and superpotential arise as appropriate sections of the Hodge bundle

$$\left.\begin{aligned}\underline{\Pi}^0 &\sim 1 + c_0\,\psi^8 + \ldots\\ \underline{\Pi}^1 &\sim \psi^2(1 + c_1\,\psi^8 + \ldots)\\ \underline{\Pi}^2 &\sim \psi^4(1 + c_2\,\psi^8 + \ldots)\\ \underline{\Pi}^3 &\sim \psi^6(1 + c_3\,\psi^8 + \ldots)\end{aligned}\right\}\quad\Rightarrow\quad t = \frac{\underline{\Pi}^1}{\underline{\Pi}^0}\qquad \mathcal{F}(t) = \frac{1}{2}\frac{\underline{\Pi}^3}{\underline{\Pi}^0} + \frac{t}{2}\frac{\underline{\Pi}^2}{\underline{\Pi}^0}$$

$$\left.\begin{aligned}\underline{\Pi}^4 &\sim (\phi - 4\psi)(1 + \ldots)\\ \underline{\Pi}^5 &\sim \psi^2(\phi - 4\psi)(1 + \ldots)\\ \underline{\Pi}^6 &\sim \psi^4(\phi - 4\psi)(1 + \ldots)\end{aligned}\right\}\quad\Rightarrow\quad \hat{t} = \frac{\underline{\Pi}^4}{\underline{\Pi}^0}\qquad W(t,\hat{t}) = \frac{\underline{\Pi}^5}{\underline{\Pi}^0}$$

Mirror symmetry at the orbifold point

Aganagic, Bouchard, Klemm; Bouchard, Klemm, Mariño, Pasquetti; Soroush, HJ

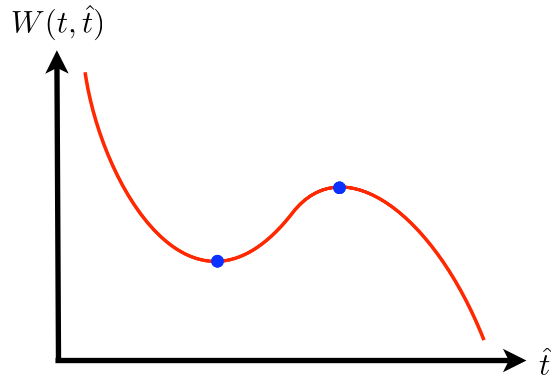
✓ Orbifold invariants of the topological B-model

- Orbifold sphere invariants:

$$\mathcal{F}(t) = \frac{1}{3!}t^3 + \frac{77}{7!\cdot 8}t^7 + \ldots$$

- Orbifold disk invariants:

$$W(t,\hat{t}) = -4\,t\,\hat{t} + \frac{1}{3}\hat{t}^3 - \frac{5}{24}t^5\hat{t} + \ldots$$



- Critical locus/leading order terms in agreement with matrix factorization techniques
- Sub-leading order contains "off-shell" information of the superpotential

✓ Orbifold disk Gromov-Witten invariants in the topological A-model

$$W(t,\hat{t}) = \sum_{k,n} \frac{1}{k!} N^{(0,1)}_{k,n} t^n \hat{t}^k$$

- Equivariant localization techniques for Calabi-Yau with Lagrangian submanifolds
- Fixes overall numerical normalizations
- <u>However</u>: Equivariant A-model computations only performed for local Calabi-Yau geometries

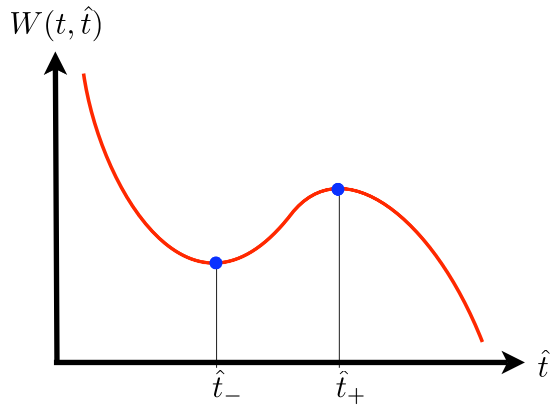
Domain wall tensions

Soroush, HJ

✓ Domain wall tensions in the vicinity of the orbifold point

- Critical loci for supersymmetric D5-brane configurations

$$0 = \partial_{\hat{t}} W(t,\hat{t}) \quad \Rightarrow \quad \phi_+(t,\hat{t}_+) = 0 \ , \ \phi_-(t,\hat{t}_-) = 8\,\psi(t)$$



- Domain wall tension period

$$\tau_{\rm orb}(\psi) = \underline{\Pi}^5(\psi,\phi_+) - \underline{\Pi}^5(\psi,\phi_-)$$

- Inhomogeneous Picard-Fuchs equation

$$\mathcal{L}^{\rm bulk}\tau_{\rm orb}(\psi) \sim \psi^{-1}\sqrt{\psi^8}$$

Walcher; Walcher, Krefl; Knapp, Scheidegger

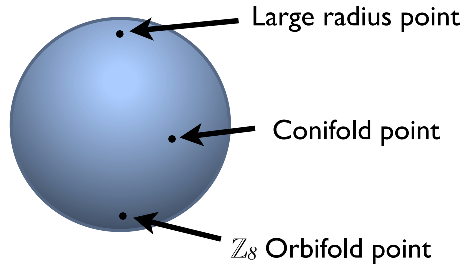
✓ Domain wall tensions period analytically continued to large volume

- Inhomogeneous large volume Picard-Fuchs equation

$$\mathcal{L}^{\rm bulk}\tau_{\rm LV}(z) \sim \sqrt{z} \ , \quad z \sim \psi^{-8}$$

- Large volume domain wall disk instantons

Moduli space of the (mirror) bulk geometry



Conclusions

✓ New techniques to compute N=1 effective superpotentials:

- Open-/closed string Picard-Fuchs equations for compact D5-brane geometries
- N=1 effective superpotentials in flat coordinates beyond the leading order

✓ Conceivable phenomenological applications:

- Insights into the vacuum structure of Type II compactifications with branes and fluxes
- Non-perturbative superpotentials in the context of dynamical supersymmetry breaking

✓ Open mirror symmetry & enumerative quantum geometry

- Techniques to compute disk invariants via open/closed string mirror symmetry
- Mirror symmetry at the orbifold point for compact brane Calabi-Yau geometries
- Domain wall tensions and their instanton expansions from flat effective superpotentials

✓ Further directions and open questions

- Open-/closed-string mirror symmetry in the large volume regime
- Study the structure of the open/closed chiral ring in the context of compact Calabi-Yaus
- Higher genus topological open-string partition functions for compact Calabi-Yaus
- (Equivariant) localization computations in the topological A-model

Residue integrals

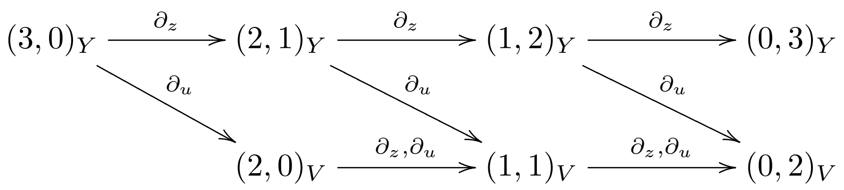
Soroush, HJ

✓ Holomorphic relative three form

$$\underline{\Omega}(z,u) = \int \frac{\log Q(x,u)}{P(x,z)} \Delta\,, \qquad \Delta = \sum_k (-1)^k\, n_k\, x_k\, dx_1 \wedge \ldots \widehat{dx_k} \ldots \wedge dx_5$$

✓ Variation of mixed Hodge structure

$$\int \frac{\log Q}{P}\Delta \qquad \int \frac{p_1 \log Q}{P^2}\Delta \qquad \int \frac{p_2 \log Q}{P^3}\Delta \qquad \int \frac{p_3 \log Q}{P^4}\Delta$$



$$\int \frac{q_{11}}{P\,Q}\Delta \qquad \begin{matrix} \int \frac{q_{21}}{P\,Q^2}\Delta \\ \int \frac{q_{22}}{P^2 Q}\Delta \end{matrix} \qquad \begin{matrix} \int \frac{q_{31}}{P\,Q^3}\Delta \\ \int \frac{q_{32}}{P^2 Q^2}\Delta \\ \int \frac{q_{33}}{P^3 Q}\Delta \end{matrix}$$

✓ Equivalence relations

$$d\underline{\alpha} = \int \left[k\frac{(q\cdot\nabla P)\log Q}{P^{k+1}} - \frac{(\nabla q)\log Q}{P^k} - \frac{q\cdot\nabla Q}{P^k Q}\right]\Delta \qquad d\underline{\beta} = \int \left[k\frac{q\cdot\nabla P}{P^{k+1}Q^\ell} + \ell\frac{q\cdot\nabla Q}{P^k Q^{\ell+1}} - \frac{\nabla q}{P^k Q^\ell}\right]\Delta$$