# Aspects of the stringy instanton calculus: part I

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U.P.O. *"A. Avogadro"* & I.N.F.N. - Alessandria

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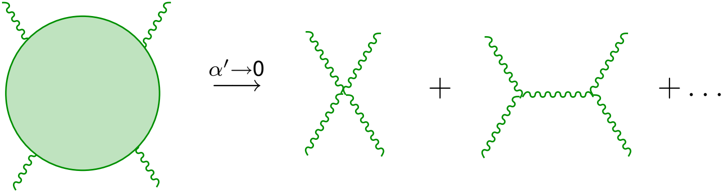
# Plan of this talk

1. Introduction and motivation
2. Branes and instantons in flat space
3. Instanton classical solution from string diagrams
4. The stringy instanton calculus

# Introduction and motivation

String theory is a very powerful tool to analyze field theories, and in particular gauge theories.

Behind this, there is a rather simple and well-known fact: in the field theory limit $\alpha' \to 0$, a single string scattering amplitude reproduces a sum of different Feynman diagrams



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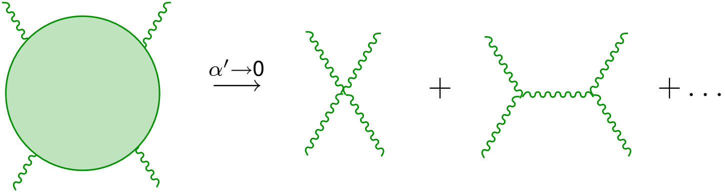


String theory S-matrix elements $\Longrightarrow$ vertices and effective actions in field theory

In general, a $N$-point string amplitude $\mathcal{A}_N$ is given schematically by

$$\mathcal{A}_N = \int_\Sigma \langle V_{\phi_1} \cdots V_{\phi_N} \rangle_\Sigma$$

where

- $V_{\phi_i}$ is the vertex for the emission of the field $\phi_i$: $V_{\phi_i} \equiv \phi_i \mathcal{V}_{\phi_i}$
- $\Sigma$ is a Riemann surface of a given topology
- $\langle \ldots \rangle_\Sigma$ is the v.e.v. with respect to the vacuum defined by $\Sigma$.

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The simplest world-sheets $\Sigma$ are:

**spheres** for **closed strings** and **disks** for **open strings**

- For any **closed string** field $\phi_{\text{closed}}$, one has

$$\left\langle\,\mathcal{V}_{\phi_{\text{closed}}}\,\right\rangle_{\text{sphere}} = 0 \quad \Rightarrow \quad \left\langle\,\phi_{\text{closed}}\right\rangle_{\text{sphere}} = 0$$

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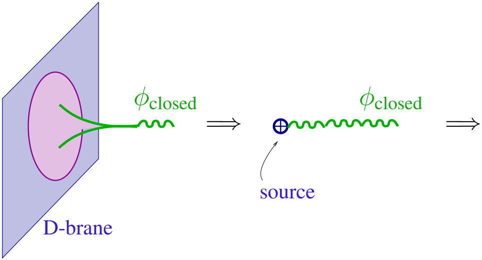
- **spheres** and **disks** describe the trivial vacuum around which ordinary perturbation theory is performed

- **spheres** and **disks** are inadequate to describe non-perturbative backgrounds where fields have non trivial profile!

However, after the discovery of D-branes, the perspective has drastically changed, and nowadays we know that also some non-perturbative properties of field theories can be analyzed using perturbative string theory!

The solitonic brane solutions of SUGRA with RR charge have a perturbative description in terms of closed strings emitted from disks with Dirichlet boundary conditions

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# In this lecture

- We will extend this idea to open strings by introducing “mixed disks” (i.e. disks with mixed boundary conditions) such that

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- We will see that instantons arise as (possibly wrapped) Euclidean branes

- We will show that in addition to the usual field theory effects, this stringy realization of the instanton calculus provides a rationale for explaining the presence of new types of non-perturbative terms in the low energy effection actions of D-brane models.

In phenomenological applications of string theory, instanton effects are important for various reasons, *e.g.*

- they may generate non-perturbative contributions to the effective superpotentials and hence play a crucial rôle for moduli stabilization

- they may generate perturbatively forbidden couplings, like Majorana masses for neutrinos, ...

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Instanton effects in string theory have been studied over the years from various standpoints, mainly exploiting string duality:

Witten, Becker$^2$+Strominger, Harvey+Moore, Beasley+Witten, Antoniadis+Gava+Narain+Taylor, Bachas+Fabre+Kiritsis+Obers+Vanhove, Kiritis+Pioline, Green+Gutperle, + ...

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Only recently concrete tools have been developed to directly compute instanton effects using (perturbative) string theory:

Green+Gutperle, Billò+Frau+Fucito+A.L.+Liccardo+Pesando, Billò+Frau+Fucito+A.L., Blumenhagen,Cvetic+Weigand, Ibañez+Uranga, Akerblom+Blumenhagen+Luest+Plauschinn+Schmidt-Sommerfeld, + ...

# Credits

Florea + Kachru + McGreevy + Saulina, 2006
Bianchi + Kiritsis, 2007
Cvetic + Richter + Weigand, 2007
Argurio + Bertolini + Ferretti + A.L. + Petersson, 2007
Bianchi + Fucito + Morales, 2007
Ibanez + Schellekens + Uranga, 2007
Akerblom + Blumenhagen+ Luest + Schimdt-Sommerfeld, 2007
Antusch + Ibanez + Macri, 2007
Blumenhagen + Cvetic + Luest + Richter + Weigand, 2007
Billó + Di Vecchia + Frau + A.L. + Marotta + Pesando, 2007
Aharony + Kachru, 2007
Camara + Dudas + Maillard + Pradisi, 2007
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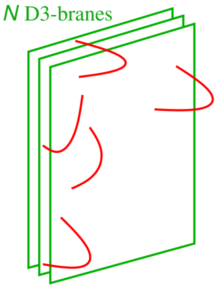
Garcia-Etxebarria + Marchesano + Uranga, 2008
Billò + Ferro + Frau + Fucito + A.L. + Morales, 2008
Uranga, 2008
Amariti + Girardello + Mariotti, 2008

# Branes and instantons in flat space

# String description of SYM theories and their instantons

The effective action of a SYM theory can be given a simple and calculable string theory realization:

- The gauge degrees of freedom are realized by open strings attached to $N$ D3 branes.




- From the disk amplitudes of open string massless fields one recovers the standard Super Yang-Mills action.

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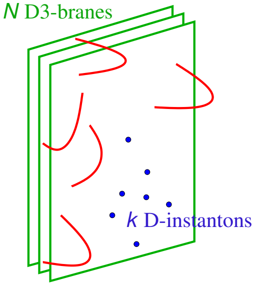


- The instanton sector of charge $k$ is realized by adding $k$ D(–1) branes (D-instantons).

# Instantons and D-instantons

- Consider the effective action for a stack of $N$ D3 branes

$$\text{D. B. I.} + \int_{\text{D3}} \left[ C_4 + \frac{1}{2} C_0 \, \text{Tr}\left( F \wedge F \right) \right]$$

The topological density of an instanton configuration corresponds to a localized source for the R-R scalar $C_0$, i.e., to a distribution of D-instantons inside the D3's.

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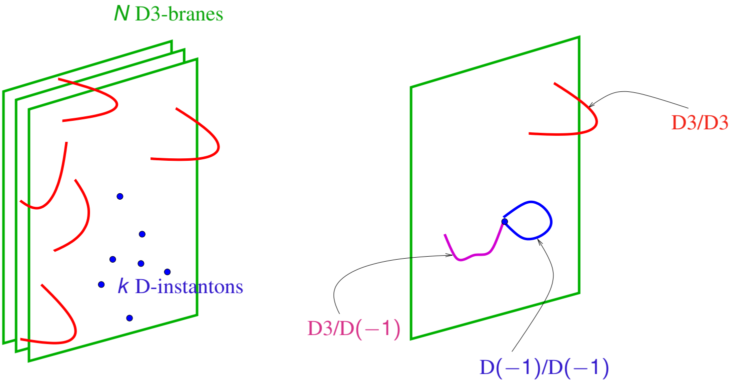
- Instanton solutions of SU($N$) gauge theories with charge $k$ correspond to $k$ D-instantons inside $N$ D3-branes.

[Witten 1995, Douglas 1995, Dorey 1999, ...]

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| D3 | — | — | — | — | $*$ | $*$ | $*$ | $*$ | $*$ | $*$ |
| D($-1$) | $*$ | $*$ | $*$ | $*$ | $*$ | $*$ | $*$ | $*$ | $*$ | $*$ |

# Open string degrees of freedom

In this D-brane system there are different open string sectors




- D3/D3 strings: gauge theory fields
- D(–1)/D(–1) strings: neutral instanton moduli
- D3/D(–1) strings: charged instanton moduli

# Moduli vertices and instanton parameters

Open strings with at least one end on a D(–1) carry no momentum: they are moduli, rather than dynamical fields.

The D(–1)/D(–1) open strings have DD boundary conditions in all directions and the spectrum is:

| | moduli | ADHM Meaning | Vertex | Chan-Paton |
|---|---|---|---|---|
| NS | $a'_\mu$ | *centers* | $\psi^\mu\, \mathrm{e}^{-\varphi}$ | *adj.* $\mathrm{U}(k)$ |
| | $\chi_m$ | *aux.* | $\psi^m\, \mathrm{e}^{-\varphi(z)}$ | ⋮ |
| | $D_c$ | *Lagrange mult.* | $\bar{\eta}^c_{\mu\nu}\psi^\nu\psi^\mu$ | ⋮ |
| R | $M^{\alpha A}$ | *partners* | $S_\alpha S_A\, \mathrm{e}^{-\frac{1}{2}\varphi}$ | ⋮ |
| | $\lambda_{\dot\alpha A}$ | *Lagrange mult.* | $S^{\dot\alpha} S^A\, \mathrm{e}^{-\frac{1}{2}\varphi}$ | ⋮ |

where $\mu, \nu = 0, 1, 2, 3$; $m, n = 4, 5, ..., 9$; $\alpha, \dot\alpha = 1, 2$ and $A = 1, 2, 3, 4$.

In the D3/D(–1) sector the string coordinates $X^\mu$ and $\psi^\mu$ $(\mu = 0, 1, 2, 3)$ satisfy mixed ND or DN boundary conditions $\Rightarrow$ their moding is shifted by 1/2 so that

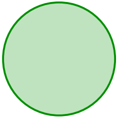
- the lowest state of the NS sector is a bosonic spinor of $SO(4)$
- the lowest state of the R sector is a fermionic scalar of $SO(4)$

| | moduli | ADHM Meaning | Vertex | Chan-Paton |
|---|---|---|---|---|
| NS | $w_{\dot\alpha}$ | *sizes* | $\Delta S^{\dot\alpha}\, \mathrm{e}^{-\varphi}$ | $k \times N$ |
| | $\bar w_{\dot\alpha}$ | *sizes* | $\overline{\Delta} S^{\dot\alpha}\, \mathrm{e}^{-\varphi}$ | $N \times k$ |
| R | $\mu^A$ | *partners* | $\Delta S_A\, \mathrm{e}^{-\frac{1}{2}\varphi}$ | $k \times N$ |
| | $\bar\mu^A$ | ⋮ | $\overline{\Delta} S_A\, \mathrm{e}^{-\frac{1}{2}\varphi}$ | $N \times k$ |

$\Delta$ and $\overline{\Delta}$ are the twist fields whose insertion modify the boundary conditions from D3 to D(–1) type and viceversa.

# Disk amplitudes and effective actions

D3 disks



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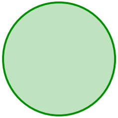
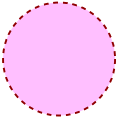


D(–1) disks



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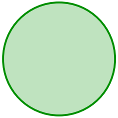

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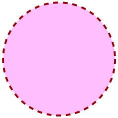
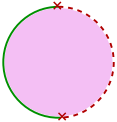


Mixed disks



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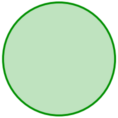

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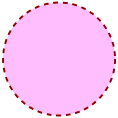

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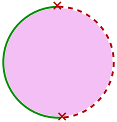
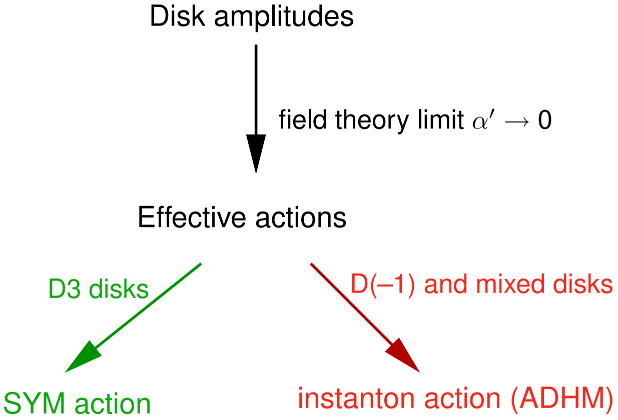
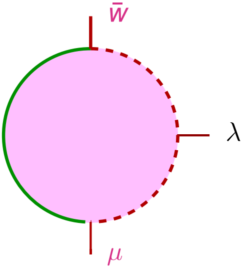




# An example of mixed disk amplitude

Consider the following mixed disk diagram




which corresponds to the following amplitude

$$\left\langle\!\!\left\langle V_\lambda\, V_{\bar{w}}\, V_\mu \right\rangle\!\!\right\rangle \equiv C_0 \int \frac{\prod_i dz_i}{dV_{\mathrm{CKG}}} \times \left\langle V_\lambda(z_1)\, V_{\bar{w}}(z_2)\, V_\mu(z_3) \right\rangle = \ \ldots \ = \mathrm{tr}_k \left\{ \mathrm{i}\, \lambda^{\dot{\alpha}}_A\ \bar{w}_{\dot{\alpha}}\, \mu^A \right\}$$

where $C_0 = 8\pi^2/g^2$ is the disk normalization.

# The instanton moduli action

Collecting all diagrams D(–1) and mixed disk diagrams with insertion of all moduli vertices, we can extract the instanton moduli action

$$S_1 = \mathrm{tr}\Big\{ - [a_\mu, \chi^m]^2 - \frac{i}{4} M^{\alpha A}[\chi_{AB}, M^B_\alpha] + \chi^m \bar{w}_{\dot\alpha} w^{\dot\alpha} \chi_m + \frac{i}{2} \bar{\mu}^A \mu^B \chi_{AB}$$
$$- i D^c \left( \bar{w}^{\dot\alpha} (\tau^c)_{\dot\alpha}^{\ \dot\beta} w_{\dot\beta} + i \bar\eta^c_{\mu\nu} [a^\mu, a^\nu] \right)$$
$$+ i \lambda^{\dot\alpha}_A \left( \bar\mu^A w_{\dot\alpha} + \bar{w}_{\dot\alpha} \mu^A + \sigma^\mu_{\beta\dot\alpha} [M^{\beta A}, a_\mu] \right) \Big\}$$

where $\chi_{AB} = \chi_m (\Sigma^m)_{AB}$.

- $S_1$ is just a gauge theory action dimensionally reduced to $d = 0$ in the ADHM limit.
- The last two lines in $S_1$ correspond to the bosonic and fermionic ADHM constraints.

Take for simplicity $k = 1$ ($\longrightarrow [\ ,\ ] = 0$). The bosonic "equations of motion"

$$w_{u\dot{\alpha}}\,\chi^m = 0 \quad , \quad \bar{w}_{\dot{\alpha}u}(\tau^c)^{\dot{\alpha}\dot{\beta}}w_{u\dot{\beta}} = 0$$

determine the classical vacua.

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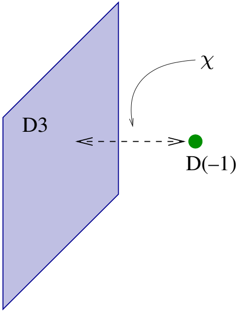

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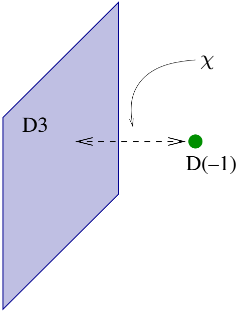
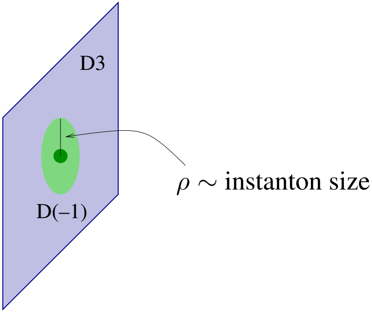



$$\chi^m = 0 \ , \ w_{u\dot\alpha} = \rho \begin{pmatrix} 1_{2\times 2} \\ 0_{(N-2)\times 2} \end{pmatrix}$$



- The neutral zero-modes

$$x^{\mu} = \mathrm{tr}\big(a^{\mu}\big) \quad \text{and} \quad \theta^{\alpha A} = \mathrm{tr}\big(M^{\alpha A}\big)$$

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- The other neutral (anti-chiral) fermionic zero-modes

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- The bosonic charged moduli

$$\bar{w}_{\dot\alpha} \quad , \quad w_{\dot\alpha}$$

describe the instanton size and its orientation (in the $\mathrm{SU}(N)$). They carry dimensions of $(length)$.

To understand better all this,
let us look at the instanton classical solution

# Instanton classical solution

- The mixed disks are the sources for a non-trivial gauge field whose profile is exactly that of the classical instanton!!

[Billó et al. 2002,...]

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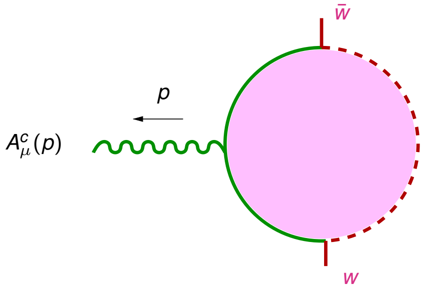
Let us consider the following mixed-disk amplitude:




$$\equiv \left\langle \mathcal{V}_{A^c_\mu(p)} \right\rangle_{\text{mixed disk}}$$

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[Billó et al. 2002,...]

Using the explicit expressions of the vertex operators, for $SU(2)$ with $k = 1$ one finds

$$\left\langle \mathcal{V}_{A^c_\mu(p)} \right\rangle_{\text{mixed disk}} \equiv \left\langle V_{\bar{w}}\, \mathcal{V}_{A^c_\mu}(p)\, V_w \right\rangle$$

$$= -i\, p^\nu\, \bar{\eta}^c_{\mu\nu} \left( \bar{w}^{\dot{\alpha}}\, w_{\dot{\alpha}} \right) \mathrm{e}^{-i\, p\cdot x_0} \equiv A^c_\mu(p; w, x_0)$$

- On this mixed disk the gauge vector field has a non-vanishing tadpole!

- Taking the Fourier transform of $A^c_\mu(p; w, x_0)$, after inserting the free propagator $1/p^2$, we obtain

$$A^c_\mu(x) \equiv \int \frac{d^4p}{(2\pi)^2}\; A^c_\mu(p; w, x_0)\, \frac{1}{p^2}\, \mathrm{e}^{i\,p\cdot x} = 2\,\rho^2\,\bar{\eta}^c_{\mu\nu}\,\frac{(x-x_0)^\nu}{(x-x_0)^4}$$

where we have used the solution of the ADHM constraints so that $\bar{w}^{\dot\alpha}\, w_{\dot\alpha} = 2\,\rho^2$.

- This is the leading term in the large distance expansion of an $\mathrm{SU}(2)$ instanton with size $\rho$ and center $x_0$ in the singular gauge!!

- Taking the Fourier transform of $A^c_\mu(p; w, x_0)$, after inserting the free propagator $1/p^2$, we obtain

$$A^c_\mu(x) \equiv \int \frac{d^4p}{(2\pi)^2}\, A^c_\mu(p; w, x_0)\, \frac{1}{p^2}\, \mathrm{e}^{i\,p\cdot x} = 2\,\rho^2\,\bar\eta^c_{\mu\nu}\,\frac{(x-x_0)^\nu}{(x-x_0)^4}$$

where we have used the solution of the ADHM constraints so that $\bar w^{\dot\alpha}\, w_{\dot\alpha} = 2\,\rho^2$.

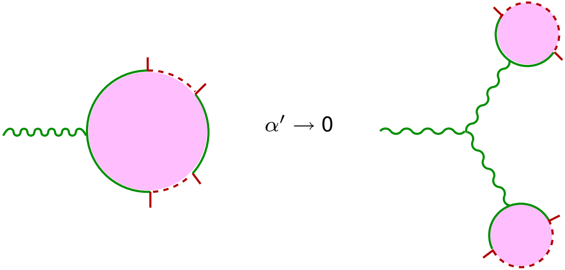
- This is the leading term in the large distance expansion of an $\mathrm{SU}(2)$ instanton with size $\rho$ and center $x_0$ in the singular gauge!!
- In fact

$$A^c_\mu(x)\Big|_{\text{instanton}} = 2\,\rho^2\,\bar\eta^c_{\mu\nu}\,\frac{(x-x_0)^\nu}{(x-x_0)^2\,\big[(x-x_0)^2+\rho^2\big]}$$

$$= 2\,\rho^2\,\bar\eta^c_{\mu\nu}\,\frac{(x-x_0)^\nu}{(x-x_0)^4}\left(1-\frac{\rho^2}{(x-x_0)^2}+\ldots\right)$$

- The subleading terms in the large distance expansion can be obtained from mixed disks with more insertions of moduli.

- For example, at the next-to-leading order we have to consider the following mixed disk which can be easily evaluated for $\alpha' \to 0$

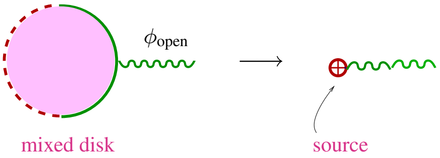



- Its Fourier transform gives precisely the 2nd order in the large distance expansion of the instanton profile

$$A^c_\mu(x)^{(2)} = -2\,\rho^4\,\bar{\eta}^c_{\mu\nu}\,\frac{(x-x_0)^\nu}{(x-x_0)^6}$$

# Summary

- Mixed disks are sources for open strings




$$\longrightarrow \left\langle \mathcal{V}_{\phi_{\text{open}}} \right\rangle_{\text{mixed disk}} \neq 0$$

- The gauge field emitted from mixed disks is precisely that of the classical instanton

$$\left\langle \mathcal{V}_{A_\mu} \right\rangle_{\text{mixed disk}} \quad \Leftrightarrow \quad A_\mu \Big|_{\text{instanton}}$$

- This procedure can be easily generalized to the SUSY partners of the gauge boson.

# The stringy instanton calculus

# The instanton partition function

- The crucial ingredient is the moduli action $S_1$: it is given by (mixed) disk diagrams; the result depends only on the centered moduli $\widehat{\mathcal{M}}_{(k)}$ but not on the center $x^\mu$ nor on its super-partners $\theta^{\alpha A}$

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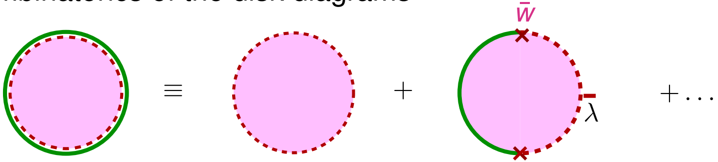
- The combinatorics of the disk diagrams



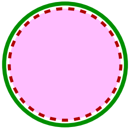

$$\stackrel{\alpha'\to 0}{\simeq} \quad -\,\frac{8\pi^2 k}{g^2} \quad - \quad S_1(\widehat{\mathcal{M}}_{(k)})$$

is such that they exponentiate, leading to the instanton partition function

$$Z^{(k)} \sim \int d^4x\, d^8\theta\, d\widehat{\mathcal{M}}_{(k)}\;\, \mathrm{e}^{-\frac{8\pi^2 k}{g^2} - S_1(\widehat{\mathcal{M}}_{(k)})} \sim \int d^4x\, d^8\theta\;\, \widehat{Z^{(k)}}$$

[Polchinski 1994, ..., Dorey et al. 1999, ...]

- The mixed disk amplitudes giving rise to the moduli action $S_1(\mathcal{M}_{(k)})$, *i.e.*



$$= -\, S_1(\mathcal{M}_{(k)})$$

from the D3 brane point of view represent a “vacuum contribution”.

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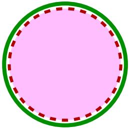
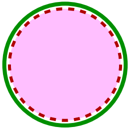
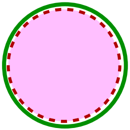
 $= -\, S_1(\mathcal{M}_{(k)})$

  from the D3 brane point of view represent a "vacuum contribution".

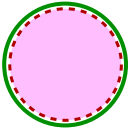
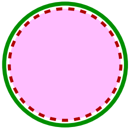
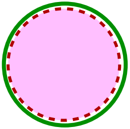
- Since we integrate over the instanton moduli $\mathcal{M}_{(k)}$, also disconnected diagrams must be considered. For example we must take into account also

 $\times$  $= S_1(\mathcal{M}_{(k)})^2$

- Of course one should add more disconnected components and take into account the appropriate symmetry factors

- A careful study of the combinatorics of boundaries leads to the exponentiation of the disk amplitudes, namely to

[Polchinski 1994]

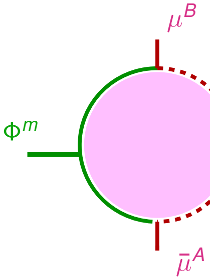
$\Big\{ 1 +$  $+ \frac{1}{2}$  $\times$  $+ \ldots \Big\}$

$$= 1 - S_1(\mathcal{M}_{(k)}) + \frac{1}{2}\, S_1(\mathcal{M}_{(k)})^2 + \ldots$$

$$= \mathrm{e}^{-\,S_1(\mathcal{M}_{(k)})} = \mathrm{e}^{-\,\frac{8\pi^2 k}{g^2} - S_1(\widehat{\mathcal{M}}_{(k)})}$$

# Field dependent moduli action

- Consider correlators of D3/D3 fields, *e.g.* of the scalars $\Phi^m$, in presence of $k$ D-instantons. They are described by disk diagrams with (at least) one insertion of $V_{\Phi^m}$. For example we have



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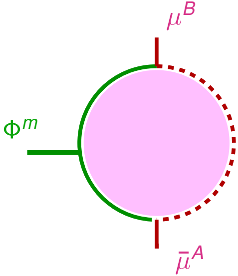


- Considering all such diagrams one obtains the field-dependent moduli action

$$
\begin{aligned}
S_2(\mathcal{M}_{(k)};\Phi) \;=\; & \mathrm{tr}\Big\{\bar{w}_{\dot\alpha}\,\Phi^m\Phi_m\,w^{\dot\alpha} + \frac{i}{2}(\Sigma^m)_{AB}\,\bar{\mu}^A\Phi_m\,\mu^B \\
& + \chi^m\,\bar{w}_{\dot\alpha}\,\Phi_m\,w^{\dot\alpha} + \bar{w}_{\dot\alpha}\,\Phi_m\,w^{\dot\alpha}\chi^m\Big\} + \text{fermion terms}
\end{aligned}
$$

A $k$-instanton contribution is then given by the integral over ALL MODULI

$$\begin{aligned} Z^{(k)} &\propto \int d\mathcal{M}_{(k)}\; e^{-S_1(\mathcal{M}_{(k)})-S_2(\mathcal{M}_{(k)};\Phi)} \\ &\propto \int d\{a,\chi,M,\lambda,D,w,\bar{w},\mu,\bar{\mu}\}\, e^{-S_1-S_2} \end{aligned}$$

Since

$$x^\mu = \mathrm{tr}\big(a^\mu\big) \quad \text{and} \quad \theta^{\alpha A} = \mathrm{tr}\big(M^{\alpha A}\big)$$

are the Goldstone modes of the broken (super)translations and play the role of the superspace coordinates, it is convenient to separate them and write

$$\begin{aligned} Z^{(k)} &\sim \int d^4x\, d^8\theta\, d\widehat{\mathcal{M}}_{(k)}\; \mathrm{e}^{-\frac{8\pi^2 k}{g^2}-S_1(\widehat{\mathcal{M}}_{(k)})-S_2(\widehat{\mathcal{M}}_{(k)};\Phi)} \\ &\sim \int d^4x\, d^8\theta\; \widehat{Z^{(k)}}(\Phi) \end{aligned}$$

# Some simple possibilities:

- D3/D(–1) system on $\mathbb{R}^4 \times \mathbb{C}^3$

  $$\Downarrow$$

  $\mathcal{N} = 4$ SYM + instantons $(A = 1, 2, 3, 4)$

- D3/D(–1) system on $\mathbb{R}^4 \times \mathbb{C} \times \mathbb{C}^2/\mathbb{Z}_2$

  $$\Downarrow$$

  $\mathcal{N} = 2$ SYM + instantons $(A = 1, 2)$

- D3/D(–1) system on $\mathbb{R}^4 \times \mathbb{C}^3/(\mathbb{Z}_2 \times \mathbb{Z}_2)$

  $$\Downarrow$$

  $\mathcal{N} = 1$ SYM + instantons $(A = 1)$

In the case of reduced SUSY, some of the $\theta^{\alpha A}$ will not be present.

- $\mathcal{N} = 2$

$$Z = \int dx^4 \, d\theta^4 \, \mathcal{F} \quad \text{where} \quad \mathcal{F} \propto \int d\widehat{\mathcal{M}}_{(k)} \, e^{-S_1 - S_2}$$

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$$Z = \int dx^4 \, d\theta^2 \, W \quad \text{where} \quad W \propto \int d\widehat{\mathcal{M}}_{(k)} \, e^{-S_1 - S_2}$$

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- The integral over the anti-chiral zero modes $\lambda_{\dot{\alpha} A}$ enforces the fermionic ADHM constraints from $S_1$.
- In general, one must investigate under what conditions these instanton contributions to $\mathcal{F}$ or $W$ are non vanishing, and what is their structure (prepotential, superpotential, ...)

Some explicit examples and applications of this stringy instanton calculus will be discussed in

Marco Billò’s talk

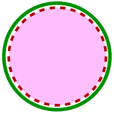
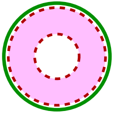
# The instanton partition function (... again)

The $k$-instanton partition function is the "functional" integral over the instanton moduli:

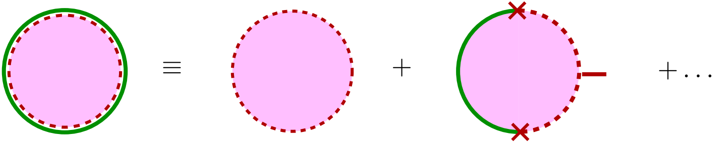
$$Z^{(k)} = \mathcal{C}_k \int d\mathcal{M}_k \, \mathrm{e}^{-S(\mathcal{M}_k)}$$

where

- $\mathcal{C}_k$ is a dimensionful normalization factor which compensates for the dimensions of $d\mathcal{M}_k$
- $S(\mathcal{M}_k)$ is the moduli action which accounts for all interactions among the instanton moduli in the limit $\alpha' \to 0$ at any order of string perturbation theory, *i.e.* on any world-sheet topology.

$-S(\mathcal{M}_k) = \lim_{\alpha' \to 0} \Big\{$  $+$  $+ \ldots \Big\}$

As we have seen before, at the tree-level, we have



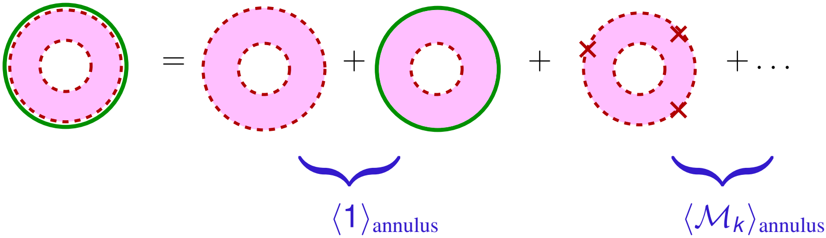
$$= \quad \langle 1 \rangle_{\rm disk} \quad + \quad \langle \mathcal{M}_{(k)} \rangle_{\rm disk}$$

$$\stackrel{\alpha' \to 0}{\simeq} \quad -\frac{8\pi^2 k}{g^2} \quad - \quad S_1(\widehat{\mathcal{M}}_{(k)})$$

Thus,

$$\langle 1 \rangle_{\rm disk} \sim O(g^{-2}) \quad , \quad \langle \mathcal{M}_{(k)} \rangle_{\rm disk} \sim O(g^0)$$

Similarly, at the one-loop level, one can show that




where

$$\langle 1 \rangle_{\text{annulus}} \sim O(g^0) \quad , \quad \langle \mathcal{M}_{(k)} \rangle_{\text{annulus}} \sim O(g^2)$$

Thus, in the semi-classical approximation one has

[Blumenhagen et al., Akerblom et al. 2006]

$$\begin{aligned} Z^{(k)} &= \mathcal{C}_k \int d\mathcal{M}_k \, \mathrm{e}^{-S(\mathcal{M}_k)} \\ &\sim \mathcal{C}_k \int d\mathcal{M}_k \, \mathrm{e}^{\langle 1 \rangle_{\text{disk}} + \langle \mathcal{M}_{(k)} \rangle_{\text{disk}} + \langle 1 \rangle'_{\text{annulus}}} \end{aligned}$$

# The disk vacuum amplitude

The YM action

$$S = \frac{1}{g^2} \int d^4x \ \mathrm{Tr} \left( \frac{1}{2} F_{\mu\nu}^2 \right)$$

- evaluated on a constant gauge field $f$ becomes

$$S(f) = \frac{V_4 f^2}{2 g^2}$$

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$$\frac{1}{g^2} = \frac{S(f)''}{V_4} = \frac{S_{\mathrm{inst}}}{8\pi^2 k}$$

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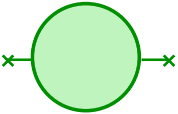
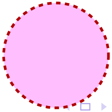


$$\equiv$$



$$= \frac{S_{\mathrm{inst}}}{8\pi^2}$$

# The annulus vacuum amplitude

A similar relation holds also at one-loop:

- in the constant gauge field background we have

$$S(f) + S^{1-\mathrm{loop}}(f) = \frac{V_4\, f^2}{2\, g^2(\mu)}$$

where $g(\mu)$ is the running coupling constant at scale $\mu$

$$\frac{1}{g^2(\mu)} = \frac{1}{g^2} + \frac{b_1}{16\pi^2} \log \frac{\mu^2}{\Lambda_{\mathrm{UV}}^2} + \Delta$$

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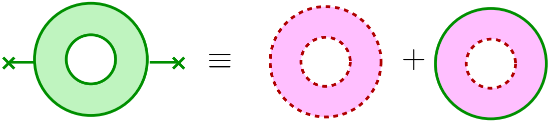
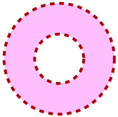


$$= \frac{S^{1-\mathrm{loop}}_{\mathrm{inst}}}{8\pi^2 k}$$

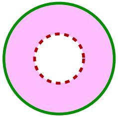
[Abel+Goodsell, Akerblom et al.]

# The rôle of the annulus amplitude

By explicitly computing the annulus diagrams, one finds



$$= 0$$



$$= -8\pi^2 k \left( \frac{1}{16\pi^2}\, b_1 \, \log(\alpha' \mu^2) + \Delta \right)$$

where the $\beta$-function coefficient $b_1$ counts the number of charged (and flavored) ADHM instanton moduli

$$b_1 = n_{\rm bos} - \frac{1}{2} n_{\rm ferm} = \#\left\{w, \bar{w}\right\} - \frac{1}{2}\#\left\{\mu, \bar{\mu}\right\}$$

and $\Delta$ are the threshold corrections.

[Akerblom et al., Billó et al.]

# To conclude:

- The $k$-instanton contributions can be computed from perturbative string diagrams:

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$$Z^{(k)} = \mathcal{C}_k \int d\mathcal{M}_k \, \mathrm{e}^{\langle 1 \rangle_{\text{disk}} + \langle \mathcal{M}_{(k)} \rangle_{\text{disk}} + \langle 1 \rangle'_{\text{annulus}}}$$

$$= \mathcal{C}_k \int d\mathcal{M}_k \, \mathrm{e}^{-\frac{8\pi^2 k}{g^2} - S(\widehat{\mathcal{M}}_{(k)};\Phi) - 8\pi^2 k\, \Delta}$$

where $\mathcal{C}_k \sim (M_s)^{k\, b_1}$. Thus,

$$Z^{(k)} \sim \Lambda^{k\, b_1} \, \mathrm{e}^{-8\pi^2 k\, \Delta} \int d\mathcal{M}_k \, \mathrm{e}^{-S(\widehat{\mathcal{M}}_{(k)};\Phi)}$$

where $\Lambda$ is the dynamically generated scale.

Stay tuned for Marco's talk

Thank you !