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Chapter 1

Topology

If we are interested in global aspects of geometry then concepts like distances or even smoothness are not important. What we investige in topology is just a very basic set of structures that allow us to identify global data of a space like the number of holes of a surface, but also local properties like the dimension of a manifold (in fact, the very definition of a manifold is in terms of notions that belong to topology). Like differentiable manifolds are identified by what functions on them we declare to be differentiable, we might define the topological structure of a set, whose elements we call points, by choosing a consistent set of (real) functions on the set that we declare to be continuous.

A somewhat more suggestive approach is to start with what we call a neighborhood of a point. As we will see the standard definition of a topological space, which uses the concept of open sets, is closely related to this, but is slightly more efficient as it only requires us to declare what subsets of a space X are open instead of declaring for each point $x \in X$ what subsets of X we call neighorhoods of x.

After introducing basic concepts like compactness and connectedness we then find some data that are topological invariants, so that they help us to decide whether two topological spaces are isomorphic (i.e. 'essentially the same'). Such data often can be equipped with an algebraic structure, like a group or a ring; then we enter realm of algebraic topology. Next we will turn to our main interest, which is manifolds, and show how algebraic calculations allow us to classify the topology of surfaces. In the last section we consider homology groups and their dimensions, the Betti numbers of a manifold.

The mathematical symbols that we use include '$\wedge$' for the logical *and*, '$A - B$' for set-theoretic *complement* of B in A, and 'iff' for 'if and only if'.

1.1 Definitions

A family $\mathcal{T}$ of subsets of a set X is called a **topology** on X if it contains X and the empty set, as well as finite intersections and arbitrary unions of elements of $\mathcal{T}$:

$$\textbf{Top1: } \emptyset \in \mathcal{T},\ X \in \mathcal{T}, \qquad \textbf{Top2: } \mathcal{O}_1 \cap \mathcal{O}_2 \in \mathcal{T}, \qquad \textbf{Top3: } \bigcup_{i\in I} \mathcal{O}_i \in \mathcal{T} \quad \forall\, \mathcal{O}_i \in \mathcal{T}. \tag{1.1}$$

The sets $\mathcal{O}_i$ are called **open** and their complements $\mathcal{A}_i = X - \mathcal{O}_i$ are the **closed sets** of the topological space $(X, \mathcal{T})$. Finite unions and arbitrary intersections of closed sets are closed. $\mathcal{P}(X)$ and $\{\emptyset, X\}$ are called discrete and indiscrete topology, respectively.

Two topological spaces $(X_1, \mathcal{T}_1)$ and $(X_2, \mathcal{T}_2)$ are called homeomorphic iff there exists a bijection $f : X_1 \to X_2$ that induces a bijective map $f : \mathcal{T}_1 \to \mathcal{T}_2$ of the topologies. The map f is called a **homeomorphism**. A map $g : X \to X'$ is called **continuous** iff the sets $g^{-1}(\mathcal{O}') \in \mathcal{T}$ are open in $(X, \mathcal{T})$ for all open sets $\mathcal{O}' \in \mathcal{T}'$. Equivalently, g is continuous iff all inverse images of closed sets are closed. A bijective map h is a homeomorphism iff it is bi-continuous, i.e. iff h and h^{-1} are both continuous. For each subset $M \subseteq X$ we can define the **induced topology**, which is given by the set of intersections $\mathcal{O}_i \cap M$ with $\mathcal{O}_i \in \mathcal{T}$.

The smallest closed subset of X that contains $M \subset X$ is called the **closure** $\overline{M}$ of M in X. The **interior** M^0 of M is defined to be the largest open set contained in M. ($\overline{M}$ and M^0 are given by the intersection/union of all closed/open sets containing/contained in M.) A subset $M \subset X$ is called **dense** in X iff $X = \overline{M}$. The difference $\partial M := \overline{M} - M^0$ is called the **boundary** of M. A subset $\mathcal{U} \subset X$ is called a **neighborhood** of $x \in X$ iff x is contained in an open subset of $\mathcal{U}$. Note that a subset $\mathcal{S} \subseteq X$ is open iff it is a neighborhood of all of it points.[1]

A less abstract way to define neighborhoods exists in metric spaces: A **metric** on a set X of points is a non-negative function $d : X \times X \to \mathbb{R}$ satisfying

$$d(x, y) = d(y, x), \qquad d(x, y) > 0 \Leftrightarrow x \neq y, \qquad d(x, y) + d(y, z) \geq d(x, z). \tag{1.2}$$

$d(x, y)$ is called the distance between x and y. With every metric space there comes a natural topology, where the neighborhoods of points x are the sets that contain open discs/balls $U_\varepsilon(x) = \{y \in X :\ |y - x| < \varepsilon\}$ for some $\varepsilon \in \mathbb{R}_+$. The open sets are the unions of open balls. Unless stated differently we will assume that $\mathbb{R}^n$ is equipped with its **natural topology**.

A family of subsets $M_i \subseteq X$ is called a **covering** of X if $X = \bigcup M_i$. A topological space $(X, \mathcal{T})$ is called **Hausdorff space** iff any two points can be separated by neighborhoods (i.e. iff $\forall x \neq y \in X$ there exist $\mathcal{U}, \mathcal{V} \in \mathcal{T}$ with $x \in \mathcal{U} \wedge y \in \mathcal{V} \wedge \mathcal{U} \cap \mathcal{V} = \emptyset$). A subset $\mathcal{A} \subseteq X$ is

[1] Using this correspondence topological spaces may equivalently be defined by a set of axioms for neighborhoods, with open sets being a derived concept.

called quasi-compact if each covering of $\mathcal{A}$ by open sets $\mathcal{O}_i$ contains a finite subcovering, it is called **compact** if it is quasi-compact and Hausdorff, and it is called **locally compact** if each point $x \in X$ has a compact neighborhood.

Example: Let $A^n = \mathbb{C}[x_1, \ldots, x_n]$ be the complex polynomial ring in n varibles and regard x_i as coordinates of $\mathbb{C}^n$. For any subset $T \subseteq A^n$ of polynomials the vanishing set is defined as

$$Z(T) = \{x \in \mathbb{C}^n | P(x) = 0 \ \ \forall P \in T\} \tag{1.3}$$

A subset of $\mathbb{C}^n$ is called algebraic if it is equal to $Z(T)$ for some set of polynomials. In algebraic geometry the **Zariski topology** $\mathcal{T}_Z$ is used, which consists of all complements of algebraic sets. It can be shown that any algebraic set is the zero set of a finite set of polynomials.

Exercise: Show that $\mathcal{T}_Z$ is a topology. Is $\mathbb{C}^n$ compact with respect to the Zariski topology?

Theorems: • For Hausdorff spaces all compact subsets $\mathcal{A} \subseteq X$ are closed. In turn,
• closed subsets of compact spaces are compact.
• Continuous images and closed subspaces of compact spaces are compact.
• Metric spaces M are compact iff every sequence in M contains a convergent subsequence.
• A subset of $\mathbb{R}^n$ is compact iff it is bounded and closed [Heine-Borel].

A compact topological space $(\tilde{X}, \tilde{\mathcal{T}})$ is called a **compactification** of $(X, \mathcal{T})$ if X is (homeomorphic to) a dense subset of $\tilde{X}$ and if $\mathcal{T}$ is the topology that is induced on X by $\tilde{\mathcal{T}}$. All locally compact spaces can be compactified in the following way by adding only a single point [Alexandroff]: Define $\tilde{X} = X \cup \{\omega\}$ with $\mathcal{O} \notin X$ and let $\tilde{\mathcal{T}}$ contain all open sets $\mathcal{O} \in \mathcal{T}$ and, in addition, all subsets $\tilde{\mathcal{O}} \subseteq \tilde{X}$ that contain ω and whose complement $\tilde{X} - \tilde{\mathcal{O}}$ is a compact subset of X. It can be checked that $\tilde{\mathcal{T}}$ is a topology on $\tilde{X}$ and that the resulting topological space is compact. (X is dense in $\tilde{X}$ if X is not compact. The one point compactification is unique up to homeomorphisms.)

Examples: The one point compactification of $\mathbb{R}^n$ is homeomorphic to the n-dimensional sphere S^n (consider, for example, the stereographic projection). An alternative compactification of $\mathbb{R}^n$ is the **projective space** $\mathbb{P}^n$. It is the set of equivalence classes $[v] = \{\lambda v : v \in \mathbb{R}^{n+1} - \{0\} \wedge \lambda \in \mathbb{R}^*\}$ of non-vanishing vectors $v \in \mathbb{R}^{n+1}$ modulo scaling by non-vanishing real numbers λ. Taking a vector of length $|v| = 1$ as a representative of the class it is easy to see that $\mathbb{P}^n$ is homeomorphic to the sphere S^n modulo the $\mathbb{Z}_2$ identification $v \to -v$.[2]
Exercise: Show that S^n and $\mathbb{P}^n$ are compactifications of $\mathbb{R}^n$ (the topology of $\mathbb{P}^n$ is defined via its covering by $n+1$ coordinate patches $\mathcal{U}_i$ with the coordinate v_i set equal to 1).

A topological space is called disconnected iff X can be decomposed into two disjoint open

[2] Alternatively, we can describe $\mathbb{P}^n$ as the space of 1-dimensional linear subspaces of $\mathbb{R}^{n+1}$. An generalization of this are the spaces of s-dimesional linear subspaces of $\mathbb{R}^{n+s}$, which are called **Grassmannian manifolds**.

sets $\mathcal{O}_i \in \mathcal{T}$, $X = \mathcal{O}_1 \cup \mathcal{O}_2$, $\mathcal{O}_1 \cap \mathcal{O}_2 = \emptyset$ with $\mathcal{O}_i \neq \emptyset$ and $\mathcal{O}_i \neq X$. If such sets don't exist then $(X, \mathcal{T})$ is called **connected**. Since the closed sets are the complements of open sets we could replace all open sets by closed sets in this definition. There is a related notion, which is equivalent to connectedness except for 'pathological' situations. For this we first need the concept of a path in a topological space: A continuous map $f : I \to X$ from the closed interval $I = [0, 1]$ to X is called a **path** from $a \in X$ to $b \in X$ if $f(0) = a$ and $f(1) = b$. A topological space is called **arcwise connected** if any two points in X can be connected by a path. One can show that all arcwise connected spaces are connected.[3] A closed path, i.e. the case where $a = b$, is called a **loop** with base point a in X.

1.2 Homotopy

Continuous images of (arcwise) connected spaces are (arcwise) connected. Connectedness is therefore a **topological invariant**, i.e. a property that is invariant under homemorphisms (which are bi-continuous). One important task of topology is to find useful topological invariants that describe the topological properties of manifolds. These invariants may themselves have an algebraic structure (for example a group or a ring structure). [4] Then we are in the realm of **algebraic topology** [MA91, BO82, CR78]. As a first example in this direction we will now consider the homotopy groups.

We are mainly interested in manifolds, which locally look like $\mathbb{R}^n$. Hence, the local topology is fixed and the interesting things happen globally, and should therefore be independent of deformations. We call two continous maps $f : X \to Y$ and $g : X \to Y$ **homotopic** if there exists a continuous map $F : X \times I \to Y$ with $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$ for all $x \in X$. (The 'product topology' on $X \times I$ is given by the unions of set-theoretic products of open sets in X and I, respectively.) For a pair of topological spaces X and Y we can now consider homotopy classes of continuos maps. Of particular interest for the construction of topological invariants is the case where $X = S^n$ is a sphere of some dimension n.

[3] As a counterexample for the inverse direction consider the graph A of the function $y = \sin 1/x$ from $(0, 1)$ to $[-1, 1]$. The closure $\overline{A} = A \cup \{(0, y) : -1 \leq y \leq 1\}$ of A in $\mathbb{R}^2$ is connected, but not arcwise connected.

[4] An algebra is a vector space (or more generally a module) with a compatible ring structure, i.e. you can add an multiply its elements, and also multiply them with 'scalars' that belong to a ring. A homomorphism of algebras $\mathcal{A}$ and $\mathcal{B}$ is a map $f : \mathcal{A} \to \mathcal{B}$ that commutes with the algebraic operations (i.e. a linear map f that is compatible with the ring structure). f is called endomorphism if $\mathcal{A} = \mathcal{B}$. An epimorphism is a surjective homoporphism, i.e. 'onto'; a monoporhism is injective, i.e. one-to-one. An isomorphism is epi and mono; an automorphism is iso and endo. A presentation of an algebra (or, more generally, of a module over a ring) is a realization in terms of generators and relations (a module is defined like a vector space except that the scalars belong to a ring that need not be a field). A finitely generated abelian group, for example, always is isomorphic to $(\mathbb{Z})^r \times \mathbb{Z}_{p_1} \times \ldots \times \mathbb{Z}_{p_l}$. r is called the rank of the group and the last l generators are constrained by the relations $p_i\, g_i = 0$. If $l = 0$ then the group is called free abelian; the last l factors are called the torsion subgroup.

We first consider the most important case $n = 1$: We denote by $\pi_1(X, x_0)$ the set of homotopy classes $[a]$ of loops $a : I \to X$ in X with base point $x_0 = a(0) = a(1)$. We can define a product on this space in the following way:

$$[a] * [b] := [a * b], \qquad (a * b)(t) := \begin{cases} a(2t) & \text{if } 0 \le t \le \frac{1}{2} \\ b(2t-1) & \text{if } \frac{1}{2} \le t \le 1 \end{cases}. \tag{1.4}$$

This operation is associative (modulo homotopy!), the class of contractible loops (the homotopy class of the constant loop) acts as unit element, and an inverse is given by $a^{-1}(t) := a(1-t)$. Hence $\pi_1(X, x_0)$ is a group, which we call the **fundamental group** of X (with base point x_0). This group is, in general, not commutative. If X is arcwise connected it is easy to see that π_1 is independent of the base point. A topological space is called **simply connected** if it is connected and the fundamental group is trival (i.e. all loops are contractible).

A covering space of a topological space $(X, \mathcal{T})$ is a connected and locally connected [5] space $(\tilde{X}, \tilde{\mathcal{T}})$ together with a surjective map $f : \tilde{X} \to X$ with the property that for every point $x \in X$ there exists a neighborhood $U(x)$ such that f is a homeomorphism from $\tilde{U}$ to $U(x)$ for every connected component $\tilde{U}$ of $f^{-1}U(x)$ (loosely speaking, $\tilde{X}$ locally looks like X). It is an important theorem that for every connected and locally connected topological space X there exists a covering space that is simply connected. This space is unique up to homeomorphisms and is therefore called **universal covering space**. It can be constructed as the space of homotopy classes of paths with a fixed starting point (we thus get $|\pi_1|$ copies of each point).

The higher homotopy groups are defined in a similar way: $\pi_n(X, x_0)$ denotes the space of homotopy classes of maps $f : S^n \to X$ with $x_0 \in f(S^n)$. Loosely speaking, multiplication is defined by moving some parts of the spheres together and then taking away some common, topologically trivial n-dimensional piece of surface that has the base point at its boundary to form a bigger sphere. It can be shown that this operation[6] defines a product on $\pi_n(X, x_0)$ that satisfies the group axioms and that is abelian if $n > 1$ [NA90] (as for the case $n = 1$ it is independent of x_0 if X is arcwise connected). These groups are useful to capture more of the topology of X. For a chunk of Emmentaler cheese, for example, the fundamental group would be trivial, whereas π_2 tells us something about the holes. A more important application is instanton physics [eg80,NA90]: Instantons are topologically non-trivial gauge field configurations of minimal Euclidian action. They contribute to non-perturbative phenomena like tunneling and quantum mechanical violation of conservation laws.

[5] $(X, \mathcal{T})$ is called locally connected if every point has a connected neighborhood.

[6] For a precise definition we may use the fact that the sphere S^n is homeomorphic to $I^n/\partial I^n$, i.e. to a hypercube with the boundary identified with one point [since $\tan \pi(x - \frac{1}{2})$ is bi-continuous on the interior $(0, 1)$ of I, this open interval is homeomorphic to $\mathbb{R}$; the same is true for a product of n such factors, and we already know that the one point compactification of $\mathbb{R}^n$ is S^n]. Continuous functions from $I^n/\partial I^n$ to Y are called n-loops with base point equal to the image of the boundary. The product of two such objects can be defined by attaching two hypercubes along a face before identifying the boundary with the base point of the sphere.

We conclude with a list of homotopy groups of spheres:

$$\pi_{k<n}(S^n) = 0, \qquad \pi_n(S^n) = \mathbb{Z} \;\; [n \geq 1], \qquad \pi_{n+1}(S^n) = \mathbb{Z}_2 \;\; [n \geq 3], \tag{1.5}$$

$$\pi_{n>1}(S^1) = 0, \qquad \pi_3(S^2) = \mathbb{Z}, \qquad \pi_4(S^2) = \mathbb{Z}_2, \qquad \pi_5(S^2) = \pi_5(S^3) = \mathbb{Z}_2. \tag{1.6}$$

Further results are $\pi_6(S^2) = \pi_6(S^3) = \mathbb{Z}_{12}$; a general formula for $\pi_k(S^n)$ is not known.

Since $S^0 = \{-1, 1\}$ the elements of π_0 correspond to the arcwise connected components of X (the continuous maps from $S^0 \to X$ coorespond to the pairs of points (x_0, x_1) in X; keeping x_0 point fixed we get one homotopy class for each component of X). π_0 cannot be given a natural group structure.[7]

1.3 Manifolds

A (topological) **manifold** of dimension n is a Hausdorff space such that every point has an open neighborhood homeomorphic to an open subset of $\mathbb{R}^n$. An open subset U of some topological space M together with a homeomorphism φ from U to an open subset V of $\mathbb{R}^n$ is called a (coordinate) **chart** of M. The coordinates $(x^1, \ldots, x^n)$ of the image $\varphi(x) \in \mathbb{R}^n$ of a point $x \in U$ are called the **coordinates** of x in the chart (U, φ). Two charts (U_1, φ_1) and (U_2, φ_2) are called C^r **compatible** if $V = U_1 \cap U_2 = \emptyset$ or if the homeomorphisms $\varphi_2 \circ \varphi_1^{-1}$ is an r times continuously differentiable map from $\varphi_1(V) \subset \mathbb{R}^n$ onto $\varphi_2(V) \subset \mathbb{R}^n$. A C^r **atlas** on a manifold X is a set of C^r compatible coordinate charts such that the domains of the charts cover X. Two atlases are called equivalent iff their union is again a C^r atlas. A C^r manifold is a manifold together with an equivalence class of C^r atlases. A **differentiable manifold** is a C^∞ manifold. Similarly, we can define an **analytic manifold** and a **complex manifold** by requiring that all functions $\varphi_2 \circ \varphi_1^{-1}$ are analytic or holomophic, respectively. In the latter case the φ_i should be complex coordinates, i.e. maps from the neighborhoods U_i to subsets of $\mathbb{C}^n$ and n is the complex dimension of the manifold (its real dimension is $2n$). A subset S of a manifold M is called a **submanifold** of dimension s if for every point in $x \in S$ there exists a chart (U_x, φ) of M containing x such that $U_x \cap S$ is identical to the subset of $U_x \cap M$ for which the last $n - s$ coordinates vanish.

A **Lie group** G is a group that is also a differentiable manifold such that the operation $f : G \times G \to G$ with $f(x, y) = xy^{-1}$ is differentiable. A **left/right group action on a manifold** is a differentiable map $\sigma : G \times M \to M$ such that $\sigma_g \circ \sigma_h = \sigma_{gh}$ or $\sigma_g \circ \sigma_h = \sigma_{hg}$, respectively, where $\sigma_g(x) := \sigma(g, x)$. We say that the action of G on M is

[7] Any group structure would require choices for the interpretation of connected components that cannot be made on the basis of purely topological data. The situation is different, for example, for topological groups: The Lorentz group has 4 connected components (which can be reached from the identity by parity and time reversal) and $\pi_0 = (\mathbb{Z}_2)^2$ would be a natural identification.

- **effective** if $\sigma_g(x) = x\ \forall x \in M \Rightarrow g = e$ (i.e. only the identity e acts trivially),
- **free** if $g \neq e \Rightarrow \sigma_g(x) \neq x\ \forall x \in M$ (i.e. only σ_e has fixed points),
- **transitive** if $\forall x, y \in M$ there exists a $g \in G$ such that $\sigma_g(x) = y$.

The **isotropy group** (also called **little group** or **stabilizer**) of a point $x \in M$ is the subgroup $H(x) = \{g \in G | \sigma_g(x) = x\}$ of G consisting of the group elements that have x as a fixed point.

Examples: The following subgroups of the matrix groups $GL(n,\mathbb{R})$ and $GL(n,\mathbb{C})$ of invertible real/complex matrices (the name means 'general linear') are called the **classical Lie groups**:
$SL(n,\mathbb{R})$ is the group of 'special linear' matrices, i.e. matrices with determinant 1.
$SO(n,\mathbb{R})$ is the group of orthogonal matrices with det $= 1$ (i.e. the connected component of $O(n,\mathbb{R})$. Orthogonal matrices leave the metric $g_{mn} = \delta_{mn}$ of Euclidean space invariant. Consider an antisymmetric matrix $\omega_{mn} = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. The $2n \times 2n$ matrices
$Sp(2n,\mathbb{R})$ that leave the n-fold tensorproduct of ω_{mn} invariant are called symplectic matrices.
$SU(n)$ is the group of special unitary matrices, i.e. complex unitary matrices with det $= 1$.

Cartan showed that in addition to these inifinite series of *classical* Lie groups there is only a finite number of *exceptional Lie groups* with 'simple Lie Algebra'. They are called E_6, E_7, E_8, F_4 and G_2. In his classification $SL(n+1,\mathbb{R})$ and $SU(n+1)$, which are related by analytic continuation, are denoted by A_n, B_n corresponds to $SO(2n+1)$, C_n corresponds to $Sp(2n)$, and D_n corresponds to $SO(2n)$. Actually the truth is a little more complicated: For each of the groups $A_1, \ldots, G_2$ there is a unique choice of the real form of the Lie algebra such that the Lie Group becomes compact. And there is, in addition, a choice in the global structure, which corresponds to dividing by a subgroup of the center of the universal covering group.[8]

In quantum mechanics the groups SO_3 and SU_2 are of particular importance. The group manifold of SU_2 is homeomophic to S^3, since its elements are of the form $\left(\begin{smallmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{smallmatrix}\right)$ with $\alpha^2 + \beta^2 = 1$. It is the double covering of $SO_3 \cong \mathbb{P}^3$ (this can be illustrated with the *belt trick:* A twisted belt with its ends kept parallel corresponds to a closed path in SO_3. A twist by 4π can be undone without twisting the ends). The correspondence between SO_3 and SU_2 can be constructed in the following way: Vectors in $\mathbb{R}^3$ can be identified with traceless Hermitian matrices via $H_x := \vec{x}\vec{\sigma}$, where σ_i are the Pauli matrices. Since $|\vec{x}|^2 = \operatorname{tr} H_x^2$ the 'adjoint' action $H \to A^{-1}HA$ of SU_2 matrices A on Hermitian matrices H_x leaves lengths invariant and hence defines an SO_3 action on $\mathbb{R}^3$. This map from SU_2 to SO_3 is two-to-one since A and $-A$ define the same transformation.[9]

Returning to general manifolds, we will next consider homology, which can be thought of

[8] The center of a group is the (abelian) subgroup that consists of all elements that commute with all other group elements.

[9] $\pm\mathbf{1}$ are the only SU_2 transformations that are mapped to the identity, as can be shown using Schur's lemma.

as the study of topological properties of integration domains. Here we use approximations of manifolds by simplices, the higher dimensional analogues of triangles. But before entering the general definition of homology groups we first consider the use of triangulations in the classification of surfaces (i.e. 2-dimensional manifolds).

1.4 Surfaces

A subset of a surface X is called a (topological) triangle if it is homeomorphic to some triangle in $\mathbb{R}^2$. A finite collection of triangles T_i is called a **triangulation** of X if $X = \bigcup T_i$ and if any non-empty intersection $T_i \cap T_j$ is either a common vertex or a common edge of T_i and T_j for $i \neq j$. We can give an orientation to a triangle by choosing an order for its vertices up to cyclic permutations. This induces a direction for the edges of the triangle. Let a be the edge from P to Q. Then we denote the edge from Q to P by a^{-1}. We say that a triangulation is oriented if we assign orientations in such a way that common edges of two triangles are always oriented in reverse directions. A surface is orientable if it admits an oriented triangulation.

Let us denote a triangle by the symbol abc if it has vertices PQR and is bounded by the oriented edges $P \xrightarrow{a} Q \xrightarrow{b} R \xrightarrow{c} P$. To any triangulation with n triangles we can assign a polygon with $n+2$ edges by joining the triangles along edges c_i and c_i^{-1} (in a first step we only join new triangles along 1 edge). We thus obtain a topological model of the surface by sewing the polygon along its bounding edges, i.e. by identifying the segments of the boundary according to their symbols. If the original surface X has no boundary then each edge occurs exactly twice, and the orientations must be inverse if X is orientable. By cutting and pasting such polygons in an appropriate way we can bring them into a normal form and thereby classify the topologies of triangulable surfaces.

Theorem: The normal form of a connected and oriented compact triangulable surface is either aa^{-1} or $\prod_{i=1}^{g} (a_i b_i a_i^{-1} b_i^{-1})$, where all vertices of the polygon correspond to a single point on the surface in the latter case. g is called the genus of the surface, with $g := 0$ if the normal form is aa^{-1}.

Proof: We proceed in three steps [FA80]. First we show that we can choose all vertices to correspond to the same point. Then we convince ourselves that each of the edges a of the resulting polygon is linked to some other edge b, i.e. that the symbol of the polygon is of the form $a \cdots b \cdots a^{-1} \cdots b^{-1} \cdots$. Eventually we bring all linked edges together.

1. We single out some vertex P which we want to become the base point of all edges. Assuming that there are more than 2 edges and that we remove any factor aa^{-1}, the symbol representing the polygon must have the form $P \xrightarrow{a} Q \xrightarrow{b} R \cdots R \xrightarrow{b^{-1}} Q \cdots$. If $Q \neq P$ we now cut the polygon

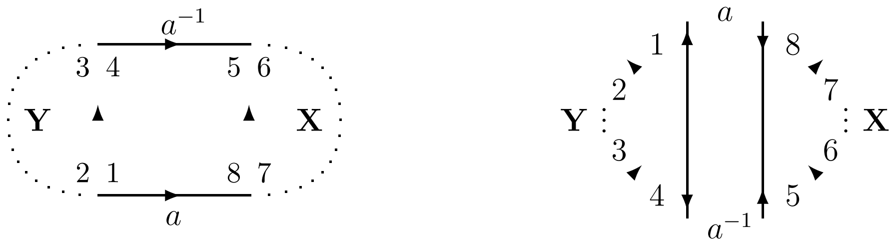

Fig. 1: Trying to paste a neighborhood of P for a non-linked edge a.

from P to R along a line c and glue the resulting triangle abc^{-1} back to the polygon along b. Now the polygon has the form $P \xrightarrow{c} R \cdots R \xrightarrow{c^{-1}} P \xrightarrow{a} Q \cdots$, i.e. we removed one vertex Q and replaced it by a vertex P at a later position along the polygon. Since we did not change the number of edges, the iteration of this process must lead to a polygon with all vertices identical to P in a finite number of steps (unless we obtain aa^{-1}, the normal form for $g = 0$).

2. Next we show for a polygon with only one vertex P that every edge a must be linked with some other edge. Assuming that this is not the case we consider a polygon of the form $aXa^{-1}Y$ where the segments X and Y of the boundary have no common edges. Gluing together all edges in X and all edges in Y we obtain a surface which only has the two bounding edges, namely a and a^{-1} as shown in Fig. 1. Sewing along a would have to produce a surface without boundary. But this is not possible without ignoring the orientation of a, as we observe by checking the neighborhood of P, which ought to look like some small disk in $\mathbb{R}^2$.

3. Considering some fixed linked pair a, b of a polygon $aXbYa^{-1}Zb^{-1}W$. we now make a cut c from the end point of a to the beginning of a^{-1} and glue the resulting polygons along b. This gives us a polygon whose symbol is of the form $aca^{-1}ZYc^{-1}XW$ (with c homotopic to XbY on the original surface). Next we make a cut d from the beginning of c^{-1} to the beginning of a^{-1} and glue along a. This results in a symbol of the form $dcd^{-1}c^{-1}XWZY$ (with $[d^{-1}] = [a^{-1}ZY]$ or $[a] = [ZYd]$). Note that we never made any modifications within a segment of the boundary that is abbreviated by a capital letter so that linked edges that already are in normal form stay together. This completes the proof. □

Strictly speaking we have not yet shown that surfaces with different genus are topologically distinct. This can be done by finding a topological invariant that distinguishes between different genera. This is the job of the **Euler characteristic** of a 2-surface, which is defined by $\chi := v - e + f$, where v, e and f are the numbers of vertices, edges and faces of a triangulation, respectively. For an orientable surface

$$\chi = v - e + f = 2 - 2g - b, \tag{1.7}$$

where g is the genus and b is the number of components of the boundary.

Exercises:

- Show that the Euler characteristic is independent of the triangulation. (A triangulation

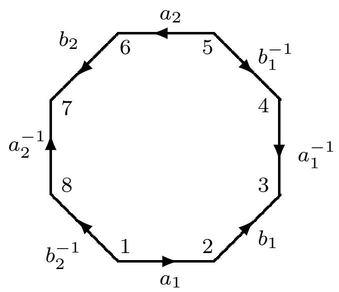

Fig. 2: Normal form for genus 2.

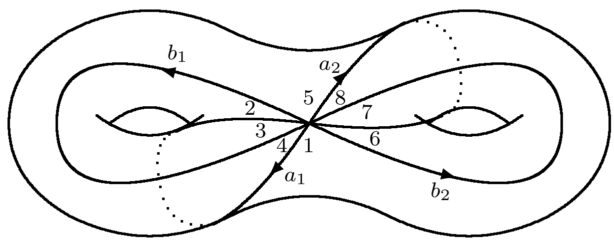

Fig. 3: Decomposition of a genus 2 surface.

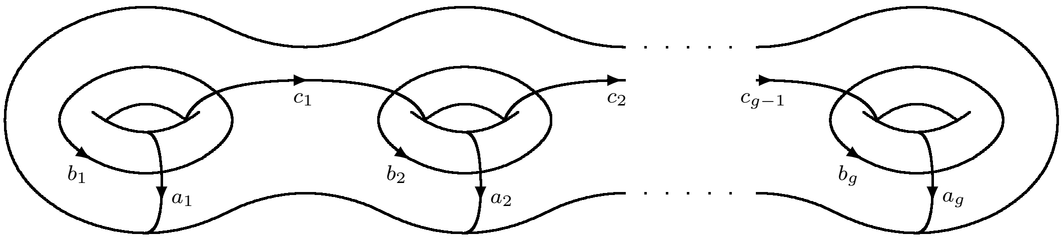

Fig. 4: a_i and b_i are a basis of homology cycles for a compact orientable genus g surface.

can be refined by successively adding new vertices, either on edges or in the interior of faces. Any two triangulations admit – up to homotopy – a common refinement.)

- Show that the formula for the Euler characteristic can also be used for subdivisions of the surface into polygons with more than 3 edges.
- Show that the Euler characteristic for a compact surface of genus g is $\chi = 2 - 2g$. (Use the normal form or induction by gluing tori along common triangles[10]; the formula (1.7) for the case with boundaries can be obtained by taking away the faces of b triangles.)

Groups can be specified by a **presentation**, i.e. a set of generators g_i and relations $\mathcal{R}_I(g_i)$ (a group element is then an equivalence class of 'words' in $g_i^{\pm n}$ with $n \in \mathbb{N}$ modulo all computation rules that can be derived from the defining relations). The normal forms of compact orientable surfaces also give us a presentaion of their **fundamental groups:** It can be shown that π_1 of a genus g surface is generated by a_i and b_i with the single relation $\prod_{i=1}^{g} a_i b_i a_i^{-1} b_i^{-1} = 1$ (i.e. the boundary of the polygon model Fig. 2 of the surface is contractible).

There are homeomorphisms of these surfaces that cannot be continuously deformed to the identity. The group of homotopy classes of such homeomorphisms is the **mapping class group** (MCG), which is generated by **Dehn twists**, i.e. twists by 2π around non-contractible loops. The twists around a_i, b_i and c_i in Fig. 4 provide a non-minimal set of generators of a subgroup

[10] The gluing of two surfaces S_i along some common triangle is called connected sum $S_1\#S_2$. It is easy to see that $\chi(S_1\#S_2) = \chi(S_1) + \chi(S_2) - 2$. All compact surfaces can be obtained as connected sums of tori and projective planes [MA91].

of the MCG, called the **(Siegel) modular group** (MG), which is isomorphic to $Sp(2g,\mathbb{Z})$.[11] The quotient MCG/MG is called Torelli group, which is non-trivial for $g>1$ (it is generated by Dehn twists around homologically trivial cycles and thus leaves intersection numbers of homology cycles invariant).

The **non-orientable case** can be treated in a similar way [MA91]. Here the normal form of a compact surface is $a_1a_1\cdots a_qa_q$ with all vertices identified, so that the Euler characteristic is $\chi=2-q$. Note that a non-orientable surface has an orientable double cover (take two copies of either orientation for each triangle of a triangulation and glue the copies along edges whose orientations match): The case $q=1$ is the projective space $\mathbb{P}^2$, whose double cover is the sphere with $g=0$. The double cover of the Klein bottle $q=2$ is the torus $g=1$. Going from a non-orientable X_q to the double cover X_g with genus $g=q-1$ the Euler characteristic doubles, as it must be. The orientable surfaces can be obtained from the sphere by attaching g handles, whereas the non-orientable ones are obtained by attaching q Möbius strips (whose boundaries only have one component).

Exercise:

• Show that $\mathbb{P}^2=D\cup M$ and that $M=D\#\mathbb{P}^2$ is an annulus (or cylinder) closed by a "crosscap", where D is a disk and M is a Möbius strip.

• An annulus closed by two crosscaps gives a Klein bottle.

• For a surface obtained from a sphere by attatching h handles, b boundaries and c crosscaps the Euler number is $\chi=2-2h-b-c$. Enumerate all surfaces with nonnegative χ.

In higher dimensions the situation is more complicated: In 1958 A.A.Markov showed that there exists no algorithm for a classification of compact triangulable 4-manifolds; Poincaré's conjecture from the beginning of the century that S^3 is the only simply connected compact 3-manifold is still an open problem (in 4-dimensions manifolds like $S^2\times S^2$ have these properties but are different from S^4). Triangulability of surfaces has been shown in 1925 by T.Radó [AH60], who pointed out the necessity of assuming a countable basis for the topology. In 1952 E.Moise proved triangulability of 3-manifolds; recently A.Casson and M.Freedman showed that some 4-manifolds cannot be triangulated [MA91].

[11] These transformations leave the intersection matrix of a canonical homology basis $a_i\cap b_j=-b_j\cap a_i=\delta_{ij}$ invariant. Since $D_{a_1}b_1=a_1+b_1$, $D_{c_1}b_1=b_1+c_1$ and $D_{c_1}b_2=b_2-c_1$ with $c_1=a_1a_2^{-1}$ their action on (a_1,b_1,a_2,b_2) is described by

$$D_{a_1}=\begin{pmatrix}1&0&0&0\\1&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix},\quad D_{b_1}=\begin{pmatrix}1&-1&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix},\quad D_{c_1}=\begin{pmatrix}1&0&0&0\\1&1&-1&0\\0&0&1&0\\-1&0&1&1\end{pmatrix}. \tag{1.8}$$

1.5 Homology

If we are interested in contour integrals then the order in which the parts of a closed path is passed through is irrelevant. This suggests to modify the concept of homotopy groups by allowing to split a loop into several elementary loops, which may be deformed individually and which may be added formally with integral (or real) coefficients, with the result being called a *cycle*. Since integrals of total derivatives are equal to integrals over the boundary of the integration domain (see below) the appropriate notion of equivalence for domains of dimension r is that a domain is in the class of 0 if it is the boundary of some $(r+1)$-dimensional domain. Integration domains that differ by boundaries are called **homologous**. The essential fact we will use is that a boundary has no boundary (see below). Therefore integrals of total derivatives over boundaries are zero.

A formal sum of r-dimensional (integration) domains is called r**-chain**. Since $\partial B = 0$ if $B = \partial M$ is the boundary of a chain M, we can write $\partial^2 = 0$ for the boundary operator ∂, which maps r-chains to $(r-1)$-chains. A chain is called a **cycle** Z if it satisfies $\partial Z = 0$. Of course it is not true in general that a cycle must be a boundary. The **homology group** $H_r(X) = Z_r/B_r$ is defined as the quotient of the group $Z_r(X)$ of r-cycles by the group $B_r(X)$ of r-boundaries.[12] The additive group structure is given by the formal sums of cycles, hence H_r is an abelian group. If we insist on integral or real coefficients then it is safer to write $H_r(X, \mathbb{Z})$ or $H_r(X, \mathbb{R})$. Being pedantic, we may insist that we have, in fact, different boundary operators $\partial_r : C_r \to B_{r-1} \subseteq C_{r-1}$ that act on r-dimensional chains. Thus we have a finite sequence[13]

$$0 \to C_n \xrightarrow{\partial_n} C_{n-1} \dots C_r \xrightarrow{\partial_r} C_{r-1} \dots C_0 \xrightarrow{\partial_0} 0 \tag{1.9}$$

of homomorphisms ∂_r of abelian groups C_n with $\partial_{r-1} \circ \partial_r = 0$. Such a structure is called a **complex** and the sequence is called **exact** if $\operatorname{Ker} h_{n-1} = \operatorname{Im} h_n$. The dimensions $b_r(X)$ of $H_r(X, \mathbb{R})$ are called **Betti numbers** of the manifold.

To be more precise about what we mean by r-dimensional integration domains we use triangulations of the surface, i.e. decompositions into simplexes: A **simplex** $\sigma_k = \langle p_0 p_1 \dots p_k \rangle := \{x \in \mathbb{R}^n \mid x = \sum_{i=0}^{k} c_i p_i,\ c_i \geq 0,\ \sum_{i=0}^{k} c_i = 1\}$ in $\mathbb{R}^n$ is the **convex hull** of $k+1$ geometrically independent points (i.e. the points p_i should span a k-dimensional affine subspace of $\mathbb{R}^n$; the $k+1$ unique numbers c_i are called **barycentric coordinates** of $x \in \sigma_k$). The convex hull of $q+1$ vertices of σ_k is called a face of σ_k. It is a q simplex which is called a proper face of σ_k if $q < k$ and a facet if $q = k-1$. A finite set K of simplexes σ_i in $\mathbb{R}^n$ is called a **simplicial**

[12]Besides *homotopy* and *homology* groups there is also the **holonomy** group, which is the subgroup of linear transformations of tangent vectors that can be achieved by parallel transport along loops. This is, however, not a topological but a geometrical concept, since we need more structure (namely a connection) to define parallel transport on manifolds.

[13]The quotion group $\operatorname{CoKer}(h_n) = A_{n-1}/h_n(A_n)$ is called *cokernel* of h_n.

complex if each face of a simplex in K belongs to K and if the intersection $\sigma_i \cap \sigma_j$ of any two simplexes in K is either empty or a face of both simplexes, σ_i and σ_j. The union $|K| = \bigcup_{\sigma_i \in K} \sigma_i$ is called the polyhedron of K. A topological space is said to be triangulable if it is homeomorphic to $|K|$ for some simplicial complex K.

In order to define simplicial homology we also need to consider orientations. Let $\sigma = (p_0 \ldots p_r) = (p_1 \ldots p_r p_0) = -(p_1 p_0 p_2 \ldots p_r)$ denote an **oriented simplex**, which is given by an ordered set of vertices up to even permutations; odd permutations reverse the orientation. The boundary of an oriented simplex σ_r is a formal sum of the oriented facets of the simplex: $\partial_r(p_0 \ldots p_r) = (p_1 p_2 \ldots p_r) - (p_0 p_2 \ldots p_r) + \ldots + (-1)^r (p_0 p_1 \ldots p_{r-1})$. In particular, the boundary of an oriented line is its end point minus its initial point. It is easy to check that the boundary of a boundary is zero $\partial_{r-1} \circ \partial_r = 0$. We can now go on to define chains and cycles as above. The r-**chain group** $C_r(K)$ of a simplicial complex is the free abelian group generated by the r-simplexes of K, i.e. it consists of formal sums of σ_r's with integral coefficients. The **cycle group** Z_r is the kernel of ∂_r and the **boundary group** B_r is the image of ∂_{r+1}. Hence Z_r consists of chains without boundaries and B_r consists of boundaries. Two cycles are homologous if their difference is a boundary.

Since $\partial^2 = 0$ all boundaries are cycles and we define the **simplicial homology groups** (with integral coefficients) $H_r(K) = Z_r(K)/B_r(K)$ as the group of homology classes of cycles. Note that the homology group is not always a free abelian group. It is easy to see that any finitely generated abelian group is isomorphic to a direct sum of cyclic groups $\mathbb{Z}_r$ and $\mathbb{Z}$, i.e. $G \equiv G_{free} \oplus G_{tor}$ with $G_{free} = \mathbb{Z} \oplus \ldots \oplus \mathbb{Z}$ and $G_{tor} = \mathbb{Z}_{n_1} \oplus \ldots \oplus \mathbb{Z}_{n_l}$. The number of $\mathbb{Z}$'s in the free abelian factor G_{free} is called the rank of the group G and G_{tor} is its **torsion** subgroup (a cyclic group has one generator g and $\mathbb{Z}_n$ is defined by the relation $g^n = 1$ in a multiplicative notation). An example of a homology group with torsion is $\mathbb{P}_2$, for which $H_1 = \mathbb{Z}_2$ (if we take twice the non-contractible loop, which generates H_1, then we obtain a boundary). Loosely speaking, the torsion subgroup of $H_r(K)$ describes the 'twisting' of the complex K [NA83]. We can get rid of the torsion subgroup by allowing rational or real coefficients of chains, because then we can divide the equation $na = \partial b$ for a generator $a \in G_{tor}$ of order n by n. Hence $H_r(K, \mathbb{R}) = \mathbb{R}^{b_k}$ where $b_k = \mathrm{rank}(H_r)$ are the Betti numbers.

Euler Poincaré theorem: Let K be an n-dimensional simplicial complex and let I_r be the number of r-simplexes in K. Then the Euler characteristic is related to the Betti numbers as follows:

$$\chi(K) := \sum_{r=0}^{n} (-1)^r I_r = \sum_{r=0}^{n} (-1)^r b_r. \tag{1.10}$$

Exercises:

- Find a *proof* for the Euler Poincaré theorem. (Hint: use that $b_r = \dim Z_r - \dim B_r$ and convince yourself that $I_r = \dim C_r = \dim(\ker \partial_r) + \dim(\mathrm{im}\, \partial_r) = \dim Z_r + \dim B_{r-1}$).

- Compute the Betti numbers for a genus g surface.

For an arcwise connected topological space it can be shown that the first homotopy group is isomorphic to the abelianization of the fundamental group [MA91] (the abelianization can be obtained by adding commutativity to the other relations of a presentation of π_1).

Chapter 2

Differentiable manifolds

In differential geometry [CH77, eg80, GO62, KO63, NA90] we are interested in spaces that locally look like $\mathbb{R}^n$ for some dimension n: A **differentiable manifold** $\mathcal{M}$ is a (topological) space that can be covered by a collection of **local coordinate charts** $\mathcal{U}_I \subseteq \mathcal{M}$ with $\bigcup \mathcal{U}_I = \mathcal{M}$ and local coordinates defined on $\mathcal{U}_I$. These coordinates $x_i^{(I)}$ have to be compatible in the sense that $x_i^{(I)}$ and $x_j^{(J)}$ are related by differentiable transition functions on $\mathcal{U}_I \cap \mathcal{U}_J$ whenever that intersection is non-empty. To give a meaning to the word *local* we first need a **topology** on $\mathcal{M}$, which tells us about the neighborhoods of points. The charts $\mathcal{U}_I$ should be open sets with respect to this topology, and, in turn, the open sets of $\mathcal{U}_I$ (in the topology inherited from $\mathbb{R}^n$ via the local coordinates) should provide a basis for the topology of $\mathcal{M}$.

The local coordinates tell us how to differentiate functions and allow us to define the **tangent space** and infinitesimal operations, as well as cotangent vectors and general tensors (tangent vectors have upper/contravariant indices). Spaces of vector and tensor fields may themselves be considered as manifolds, which leads us to the notion of fiber **bundles**. The coordinate independent **exterior derivative** can be defined on differential forms, which correspond to anti-symmetric tensor fields with lower indices. These are useful in integration theory and in formulating structure equations and integrability conditions.

2.1 Tangent space and tensors

Differentiable (as well as topological) manifolds can always be embedded in Euclidean spaces of higher dimensions. We could therefore think of tangent vectors as vectors in that embedding space. It does not make much sense, however, to deal with all the ambiguous additional structure if we are only interested in intrisic geometrical properties of a manifold. It is thus more useful to work with a more abstract definition: Tangent vector fields of a smooth manifold are derivations on the algebra of smooth functions. The idea behind this definition is that we can identify a direction on a manifold with a directional derivative. But first we have to spell out the details of this definition.

Smooth functions on a manifold form an algebra, because they form a ring (they can be added and multiplied) as well as a vector space over the field $\mathbb{R}$ (linear combinations again are smooth functions). A **tangent vector** $v_x : C^\infty(\mathcal{M}) \to \mathbb{R}$ at a point $x \in X$ is a linear map

$$v_x(\alpha f + \beta g) = \alpha v_x(f) + \beta v_x(g) \tag{2.1}$$

that satisfies the **Leibniz rule**

$$v_x(fg) = f(x)v_x(g) + g(x)v_x(f) \tag{2.2}$$

(here it would be sufficient to consider the algebra of smooth functions that are defined in some neighborhood of x, since v_x only depends on the functions in an arbitrarily small neighborhood). From this definition it follows that $v_x = v^m \partial_m$ is a linear combination of partial derivatives with respect to any coordinates x^m. This must be true for any choice of coordinates, so the chain rule $\frac{\partial}{\partial x^n} = \frac{\partial \hat{x}^m}{\partial x^n}\frac{\partial}{\partial \hat{x}^m}$ implies that the local coordinates (or components) of a vector v transform contravariantly $\hat{v}^m = \frac{\partial \hat{x}^m}{\partial x^n} v^n$ under a diffeomorphism $x \to \hat{x}(x)$. The tangent vectors v_x at a point x form a vector space of the dimension of the manifold, the tangent space $T_x(\mathcal{M})$
Exercise: Use Taylor expansion to show that the vectors ∂_n form a basis of $T_x(\mathcal{M})$.

More general tensors can be obtained as duals and tensor products of tangent vectors. The **dual** $V^* = \mathrm{Hom}(V, \mathbb{R})$ of a **vector space** is the linear space of *linear forms* (or functionals) on V, i.e. $w \in V^*$ is a linear map (or homomorphism) $w : V \to \mathbb{R}$. The value of this map evaluated on a vector v is denoted by the bracket $\langle w, v\rangle := w(v)$, which is also called duality **pairing**. If we have a basis E_i of vectors then there is a natural dual basis, the **co-basis** e^j that is defined by the following action of the co-vectors e^i on the basis of V:

$$\langle e^j, E_i\rangle = \delta^j{}_i \qquad \Rightarrow \qquad \langle w, v\rangle := w(v) = w_i v^i \quad \text{with} \quad w = w_j e^j,\ v = v^i E_i. \tag{2.3}$$

The dual of $T_x\mathcal{M}$ is the space $T_x^*\mathcal{M}$ of cotangent vectors at a point $x \in \mathcal{M}$. General **tensor**s with m upper and n lower indices are elements of the tensor product[1] of m copies of $T_x\mathcal{M}$ and n copies of $T_x^*\mathcal{M}$. The coordinates (components) of such a tensor t are defined by $t = t^{i_1 \ldots i_m}{}_{j_1 \ldots j_n} E_{i_1} \otimes \ldots \otimes E_{i_m} \otimes e^{j_1} \otimes \ldots \otimes e^{j_n}$ (possibly with some other ordering of the indices).

The basis dual to the partial derivatives ∂_i that come with a coordinate system is denoted by dx^j. Since vectors $v \in T_x\mathcal{M}$ and $w \in T_x^*\mathcal{M}$, as well as the duality pairing $\langle w, v\rangle$ are

[1] The tensor product $V \otimes W$ of two vector spaces is given by the set of all linear combinations of tensor products $v \otimes w$ of vectors $v \in V,\ w \in W$. A basis of the product space is given by all tensor products $E_i \otimes F_j$ of basis elements $E_i \in V,\ F_j \in W$, so that its dimension is the product of the dimensions of the factors. The *direct sum* $V \oplus W$, on the other hand, consists of pairs (v, w) with componentwise vector operations, so that $(E_i, 0)$ and $(0, F_j)$ provide a basis and the dimensions are added.

independent of the choice of coordinates, we find the following transformations of indices under diffeomorphism $x \to \hat{x}(x)$:

$$\partial_i = (D\hat{x})_i{}^j \hat{\partial}_j, \qquad d\hat{x}^j = dx^i (D\hat{x})_i{}^j, \qquad \hat{v}^j = v^i (D\hat{x})_i{}^j, \qquad w_i = (D\hat{x})_i{}^j \hat{w}_j, \tag{2.4}$$

with the **jacobian matrix** $(D\hat{x})_i{}^j := \frac{\partial \hat{x}^j(x)}{\partial x^i}$ (the determinant of $D\hat{x}_i{}^j$, which will be important in integration theory, is called *jacobian*). For general tensors we get a jacobian matrix or its inverse for each index: The upper/lower indices of contravariant/covariant tensors transform inverse/identical to the basis ∂_i of tangent vectors. (Tensors could also be defined via a collection of component functions that transforms in this way under diffeomorphism, i.e. "a tensor is an object that transforms like a tensor".)

The formulas (2.4), with $\hat{x}(x)$ replaced by $f(x)$, are also valid for a more general smooth map $f : X \to Y$ from a manifold X to a manifold Y. The mapping among (co)vectors is then well-defined as long as we do not have to invert the (in general rectangular) jacobian matrix Df. This suggests that there should be natural maps $f_* : T_xX \to T_yY$ and $f^* : T^*_yY \to T^*_xX$ with $y = f(x)$. Indeed, this is easy to see in a coordinate independent way: For any function g of Y we can define the **pull back** $f^*g := g \circ f$, which is a function on X. But this map $f^* : C^\infty(Y) \to C^\infty(X)$ can be used to define for each tangent vector $v_x : C^\infty(X) \to \mathbb{R}$ a vector $v_y = f_*v_x$ by $(f_*v_x)(g) := v_x(f^*g)$. The operation f_* is called **push forward** or **differential map** and it is defined for any contravariant tensor in the obvious way. It maps, in particular, tangent vectors of a curve to the tangent vectors of the image curve. In turn, cotangent vectors (and covariant tensors) can again be pulled back in a natural way: For an element $w_y \in T^*_yY$ the **pull back** $w_x = f^*w_y$ is defined by $f^*w_y(v_x) := w_y(f_*v_x)$. If f is a diffeomorphism then we can, of course, push and pull arbitrary tensors in either direction along f.

Exercise: Show that f_*v_x is an element of T_yY and write down the pull back (push forward) for covariant (contravariant) tensors in terms of components.

In differential geometry we are interested in **tensor fields** on manifolds rather than in tensors at a single point. These can be defined via (smooth/continuous) coordinate dependent component functions. For vector fields it is easy to write down a coordinate independent version of this definition: A smooth vector field is a **derivation** on the algebra $C^\infty(\mathcal{M})$, i.e. a linear map $v : C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M})$ that satisfies the Leibniz rule. This is so because a vector v_x assigns a number to a function, so that a vector field (a vector at each point) assigns a number at each point (a function) to a function.

2.2 Fiber bundles

The tangent space of a manifold can itself be regarded as a manifold, which may have an interesting and non-trivial topological structure. There exists, for example, no smooth non-vanishing tangent vector field on a sphere ('you cannot comb the hair on a sphere'). This is the motivation for defining vector bundles and more general fiber bundles.

A **bundle** is a triple (E, B, π) consisting of two topological spaces E and B and a continuous surjective map $\pi : E \to B$. B is called **base** space, π is the **projection** of the total space E to the base space, and $F_x = \pi^{-1}(x)$ is the **fiber** at x. Usually we will be interested in the situation where all fibers F_x are homeomorphic, in which case we call $F \cong F_x$ the typical fiber. In the case of vector bundles the typical fiber is a vector space. Locally the tangent bundle looks like a product space $\mathcal{M} \times V$ with $V \cong T_x\mathcal{M}$, but globally there may be a twist: For a trivial bundle $S^2 \times V$ over the sphere, for example, any constant vector would provide a non-vanishing 'vector field'. Similarly, the Möbius band is a twisted line bundle over S^1, whose topology differs from the trivial bundle $S^1 \times \mathbb{R}$. The twist is generated by the way the fibers are glued together globally, but locally these bundles look like a product. This leads to the following definition of *fiber bundles*, which are also called *twisted products*.

A **fiber bundle** (E, B, π, F) is a bundle (E, B, π) with typical fiber F and a covering of B by a family of open sets $\{U_i\}$ such that there are homeomorphisms $\varphi_i : \pi^{-1}(U_i) \to U_i \times F$ with $\pi = pr_1 \circ \varphi_i$, where pr_1 projects to the first component of $(x, f) \in U_i \times F$ (i.e. locally the bundle is a trivial bundle; the maps φ_i are called **local trivializations**).
The transition functions $g_{ij} = \varphi_i \circ \varphi_j^{-1}$, which provide a homeomorphism of the fiber onto itself for each point in $x \in U_i \cap U_j$, satisfy the *cocycle condition* $g_{ij} \circ g_{jk} = g_{ik}$ on the overlap of three coordinate patches. In turn, the **reconstruction theorem** asserts that for a covering $\{U_i\}$ of the base space B and a set of transition functions g_{ij} satisfying the cocycle condition we can always construct a fiber bundle with the prescribed transition functions.

If E and M are both C^k manifolds then we may consider C^k fiber bundles, in which case we require that π is a C^k map and that φ_i are C^k diffeomorphisms. In the following we will consider C^∞ bundles. A **vector bundle** is a fiber bundle whose fiber is a vector space V and whose local trivializations φ_i act as linear isomorphims on the fibers (i.e. $\varphi_i : \pi^{-1}(x) \to V$ is a linear isomorphism for $x \in U_i$). If all automorphisms $g_{ij}(x) : V \to V$ belong to a subgroup $G \subseteq GL(n)$ then we call G the **structure group** of the vector bundle. A **line bundle** is a vector bundle whose fiber is 1-dimensional.

A **principal bundle** $P(M, G)$ is a fiber bundle (P, M, π, G), with P and M being smooth manifolds and with a Lie group G as its typical fiber, if the following conditions are met:
• G acts freely and within fibers from the right on the total space, i.e. $\rho : P \times G \to G$ with

$(x, g) \mapsto x \cdot g := \rho_g(x)$ satisfies $\rho_g(u) \in \pi^{-1}(\pi(u))\ \forall u \in P$, and

- the group action is compatible with the local trivializations, i.e. $\varphi_i(u \cdot g) = \varphi_i(u) \cdot g$.

Given a principal G-bundle and a manifold V with a left action $\lambda : G \times V \to V$ of the group we can define the **associated bundle** by the following construction: G acts on $P \times V$ by $(u, v) \mapsto (\rho_g(u), \lambda_{g^{-1}}(v))$. The orbit space $P \times V/G$ is the bundle associated to P and V with group action λ. Usually V will be a vector space with a linear action of G.

For a vector bundle over M we may construct a principal bundle by using the structure group as the typical fiber and pointwise right-multiplication with the inverse of g_{ij} as transition functions for the reconstruction process. It can be shown that the result of this construction is the (up to isomorphism) unique principal bundle whose associated bundle for V with its left action of G is the vector bundle we started with. In turn, principal bundles encode all topological information on the gluing of the fibers and thus classify the vector bundles.

The fibers of the **frame bundle** of a manifold M are the bases of the tangent space. Since a change of basis corresponds to a $GL(n)$ transformation, a frame bundle is a principal bundle with structure group $GL(n)$. A global **section** of a bundle is a map $f : B \to E$ with $\pi \circ f = \mathbf{1}$ (a local section is a map that satisfies the same condition, but is only defined on some subset of the base manifold). A tensor field can be defined to be a (smooth) section of a vector bundle. It is easy to show that a principal bundle that admits a (smooth) global section is trivial (the frame bundle of S^2, for example, is non-trivial and does not admit a global section).

Exercise: The Hopf fibration of an odd-dimensional sphere S^{2n+1} is given by the projection $(x_0, \ldots, x_n) \mapsto (x_0 : \ldots : x_n)$ onto $\mathbb{CP}^n$ with fiber S^1. Relate the Hopf fibration of S^3 to the principal bundle for the tangent bundle of S^2 and show that this bundle has no global section.

2.3 Lie derivatives

A vector field $\xi = \xi^m \partial_m$ defines a curve through any given point on the manifold: We may think of moving along the direction $\xi^m(x)$ at each point with coordinates x^m on a manifold with a velocity that is proportional to the length of ξ. This suggests to consider the system of first order differential equations $\frac{d}{dt}x^m(t) = \xi^m(x(t))$, whose solutions $x(t)$ we call **integral curves** of the vector field ξ. The integral curves of the basis vector fields ∂_i are the coordinate lines themselves.

Integral curves exist at least locally and they can be used to define a t-dependent map of points on the manifold, i.e. a *diffeomorphism* of the manifold onto itself. Such a diffeomorphism is often called an *active* transformations, to which we can associate a *coordinate transformation* (or *passive* transformation) by assigning the old coordinates of the target points as new coordinates to the original points. For infinitesimal transformations, i.e. for small t, we obtain the map $\phi_t : x^m \to \tilde{x}^m = x^m + t\xi^m + O(t^2)$ in either case (only the interpretation of $\tilde{x}^m$ is different).

The **Lie derivative** $\mathcal{L}_\xi t$ of a tensor field t is its variation induced by the infinitesimal transformation that comes with the vector field ξ. It is thus defined to be the term of leading order for $t \to 0$ in the difference $\phi_t^* T - T$ between a tensor field and its pull back along ϕ_t,

$$\mathcal{L}_\xi T := \left(\tfrac{d}{dt}(\phi_t^* T)\right)\Big|_{t=0}, \qquad \phi_t(x) = x + t\xi + O(t^2). \tag{2.5}$$

This is easy to compute for a function: $\phi_t^* f = f + t\xi(f) + O(t^2)$ so that $\mathcal{L}_\xi f = \xi(f) = \xi^m \partial_m f$. For the component functions of tensor fields, however, we have additional terms from the jacobian matrices and their inverses [see eq. (2.4)]. The difference between tensors of the same type is again a tensor, and so is the Lie derivative of a tensor. Putting the pieces together we find its components

$$\mathcal{L}_\xi T_i{}^p = \xi^m \partial_m T_i{}^p + \partial_i \xi^m T_m{}^p - \partial_n \xi^p T_i{}^n \tag{2.6}$$

with additional positive/negative terms for additional lower/upper indices.

In computations with tensor fields whose rank we do not want to specify it is often useful to write the part of the Lie derivative that comes from the jacobian factor in a more symbolic form. For this purpose we introduce the symbol $\Delta_i{}^j$ for infinitesimal $GL(n)$ transformation, which acts on (co)vectors by $\Delta_i{}^j v^u = \delta_i{}^u v^j$ and $\Delta_i{}^j v_l = -\delta_l{}^j v_i$ and which is extended to general tensors by the Leibniz rule and by linearity (i.e. $\Delta_i{}^j$ is a derivation on the tensor algebra; on a tensor with m upper and n lower indices it acts by a collection of m positive and n negative terms containing $\delta_i{}^u$ and $\delta_l{}^j$, respectively). Then

$$\mathcal{L}_\xi = \xi^l \partial_l - \partial_i \xi^k \Delta_k{}^i, \qquad \Delta_k{}^i v^n = \delta_k{}^n v^i, \qquad \Delta_k{}^i v_m = -\delta_m{}^i v_k; \tag{2.7}$$

the first term $\xi^l \partial_l$ of $\mathcal{L}_\xi$ is called *shift term* for obvious reasons.
Exercise: Show that $[\Delta_i{}^j, \Delta_k{}^l] = \delta_i{}^l \Delta_k{}^j - \delta_k{}^j \Delta_i{}^l$ is the commutator of GL_n-transformations (since both sides are derivations it is sufficient to check this on covariant and contravariant vectors, which generate the tensor algebra).

Symmetry transformations act simultaneously on each factor of a tensor product. If we consider infinitesimal transformations, then we are collecting all terms that are linear in a small parameter. We thus get one term for each factor in a product, so that infinitesimal transformations act as derivations on algebras. Non-commutativities of finite symmetry transformations manifest themselves in non-vanishing commutators of infinitesimal transformations. It is easy to check that the commutator of two derivations is again a derivation, and thus corresponds to another infinitesimal transformation. In the case of diffeomorphisms inifinitesimal transformations correspond to (Lie derivatives along) vector fields. This is the motivation for defining the

Lie bracket of two vector fields as

$$[v,w] := v\circ w - w\circ v = (v^i\partial_i w^l - w^i\partial_i v^l)\partial_l = \mathcal{L}_v(w) = -\mathcal{L}_w(v), \tag{2.8}$$

which again is a vector field.

Note that all Lie brackets among the natural basis vector fields ∂_i vanish (partial derivatives commute). A set of n pointwise linearly independent vector fields v_i defines a basis of tangent space. It can be shown that their integral curves define local coordinates (of a submanifold if n is smaller than the number of dimensions) iff $[v_i, v_j] = 0\ \forall i, j$ (such a basis v_i of TM is called holonomous). Geometrically a non-vanishing Lie bracket $[v,w] \neq 0$ means that the flows along the integrals curves of v and w do not commute.

It is easy to check that the Lie bracket is anti-symmetric and satisfies the Jacobi identity,

$$[v,w] = -[w,v], \qquad \sum_{i,j,k}[v_i,[v_j,v_k]] := [v_i,[v_j,v_k]] + [v_j,[v_k,v_i]] + [v_k,[v_i,v_j]] = 0. \tag{2.9}$$

It commutes with the push forward $f_*[v,w] = [f_*v, f_*w]$ and satisfies the following identities

$$\mathcal{L}_{fv}(w) = f[v,w] - w(f)v, \quad \mathcal{L}_v(fw) = f[v,w] + v(f)w, \qquad [\mathcal{L}_v,\mathcal{L}_w] = \mathcal{L}_{[v,w]} \tag{2.10}$$

for $f \in C^\infty(\mathcal{M})$; the last identity gives the commutator of Lie derivatives in terms of the Lie brackets of vector fields.

2.4 Differential forms

The total differential $df = dx^i\partial_i f$, which maps functions to cotangent vectors, is an intrinsic notion on a manifold that does not require any additional structure. It can be extended to a coordinate independent (graded) derivation on the algebra of anti-symmetric covariant tensors, the exterior algebra: We denote by $\Lambda_x^p(\mathcal{M})$ the space of antisymmetric tensors of rank $(0,p)$, i.e. with no upper and p lower indices, at $x \in \mathcal{M}$. A basis of $\Lambda_x^p(\mathcal{M})$ is provided by the anti-symmetrized tensor product of p elements of a cobasis,

$$dx^{i_1}\wedge dx^{i_2}\wedge\ldots\wedge dx^{i_p} := \sum_\pi \mathrm{sign}(\pi)\ dx^{i_{\pi(1)}}\otimes dx^{i_{\pi(2)}}\otimes\ldots\otimes dx^{i_{\pi(p)}} \tag{2.11}$$

which is the sum over all permutations π of the p indices i_k with a minus sign for odd permutations. An element $\omega \in \Lambda^p$, which we call **differential form** of degree p, or **p-form**, can therefore be written as

$$\omega = \tfrac{1}{p!}\ dx^{i_1}\wedge\ldots\wedge dx^{i_p}\ \omega_{i_1\ldots i_p}. \tag{2.12}$$

$\Lambda_x^p(\mathcal{M})$ has dimension $\binom{n}{p}$ for n-dimensional manifold $\mathcal{M}$. Out of these spaces we can make the 2^n–dimensional **exterior (Grassmann) algebra** $\Lambda_x = \bigoplus_{p=0}^{n} \Lambda_x^p$ by defining the **wedge product** (or exterior/Grassmann product) of anti-symmetric tensors as the associative and multilinear product

$$\alpha \wedge \beta := \frac{1}{p!q!}\, dx^{i_1} \wedge \ldots \wedge dx^{i_{p+q}}\; \alpha_{i_1\ldots i_p}\beta_{i_{p+1}\ldots i_{p+q}} \;\in\; \Lambda^{p+q} \qquad \forall \alpha \in \Lambda^p,\; \beta \in \Lambda^q; \tag{2.13}$$

if $p + q > n$ then $\alpha \wedge \beta := 0 \in \Lambda$. With this definition the indices get totally anti-symmetrized

$$(\alpha \wedge \beta)_{i_1\ldots i_p j_1\ldots j_q} = \tfrac{(p+q)!}{p!q!}\alpha_{[i_1\ldots i_p}\beta_{j_1\ldots j_q]}, \qquad t_{[i_1\ldots i_p]} := \tfrac{1}{p!}\sum_\pi \mathrm{sign}(\pi) t_{i_{\pi(1)}\ldots i_{\pi(p)}}. \tag{2.14}$$

The wedge product is super-commutative in the sense that we have a $\mathbb{Z}_2$ grading, defined by the degree of the form modulo 2, and $\alpha_p \wedge \beta_q = (-1)^{pq}\beta_q \wedge \alpha_p$.

Physicists are used to calculations with anti-commuting ('fermionic') objects, so we will often omit the wedge symbol and just remember that we get a negative sign whenever we exchange objects with an odd grading. In principle it would be consistent to keep separate gradings for differentials and fermionic fields (or even for fermions of different types) and to multiply the resulting sign factors. It is, however, equivalent and much more efficient to have a single $\mathbb{Z}_2$ grading that just counts the total number of anti-commuting objects.

On the exterior algebra $\Lambda(\mathcal{M})$ of anti-symmetric tensor fields we can define the **exterior derivative** d, which is an **anti-derivation** (or odd graded derivation), i.e.

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-)^{|\alpha|}\alpha \wedge d\beta, \qquad |\omega_p| \equiv p \bmod 2, \tag{2.15}$$

that is nilpotent $d^2 = 0$ and coincides with the total differential on functions; in particular $d(x^i) = dx^i$ and $d(dx^i) = 0$. With our rule to consider dx^i as an odd element of the algebra and d as an odd (graded) derivation we could simply use $d = dx^i\partial_i$ as the definition of d.

Exercise: Show that $(d\omega)(v_0, \ldots, v_p) = \sum_0^p (-)^l v_l(\omega(v_0, \ldots, v_p)) + \sum_{i<j} (-)^{i+j}\omega([v_i, v_j], v_o, \ldots, v_p)$ with the obvious omissions of the vectorfields v_l and v_i, v_j in the arguments of ω; in coordinates $(d\omega)_{i_0\ldots i_p} = \sum_{k=0}^p \partial_{i_k}\omega_{i_1\ldots i_p]} = (p+1)\partial_{[i_0}\omega_{i_1\ldots i_p]}$.
Show that d corresponds to gradient, curl, and divergence on functions, 1-forms, and 2-forms in $\mathbb{R}^3$, respectively.

The **interior product** i_v is the anti-derivation on Λ whose action on 1-forms is given by $i_v\, w := \langle w, v\rangle = w(v)$. This implies $i_v f := 0$, $i_v dx^m := dx^m(v) = v^m$ On Λ, i.e. on p-forms without additional tensor indices, we find

$$i_v^2 = 0, \qquad \{d, i_v\} = \mathcal{L}_v, \qquad [\mathcal{L}_v, i_w] = i_{[v,w]}, \qquad (i_v\omega)_{n_2\ldots n_p} = v^l\omega_{l n_2\ldots n_p} \tag{2.16}$$

The interior product is sometimes also called *interior derivative.*

Exercise: Show that the Lie derivative of 'tensor-valued p forms' is $\mathcal{L}_\xi = \{d, i_\xi\} - \partial_l\xi^n{\Delta_n}^l$, where ${\Delta_n}^l$ acts on the free indices (which are not contracted with coordinate differentials).

One of the main applications of differential forms is integration theory: Under a change of coordinates the component functions of an n-form transform with an inverse Jacobian because we get an inverse Jacobi matrix for each lower index and the antisymmetrization in all indices generates the determinant. This just compensates the Jacobian that we get for a change of variables in an n-dimensional integral. Therefore, if a domain can be covered by a single coordinate patch then

$$\int_C \omega := \int_C dx^1 \ldots dx^n \omega_{1\ldots n} \tag{2.17}$$

is a geometrical construction that does not depend on the coordinates chosen for the evaluation of the integral, except possibly for a sign. More general integration domains have to be decomposed into patches. To take care of the sign ambiguity we require that the manifold is orientable, i.e. that we can choose an atlas in such a way that all Jacobians for changes of coordinates are positive on the overlap of the respective patches. Then we allow only coordinates with positive orientation and coordinate independence of (2.17) implies that the result is independent of the decomposition of the complete integration domain.[2]

The same construction can be used if ω is a p form and if $C \subset M$ is some p-dimensional submanifold (with smooth boundary). Recall that we always can pull back covariant tensors along differentiable maps; we can thus regard the integral of a p form as being defined via the integral of the pull back of that form over a p-dimensional coordinate domain that parametrizes the (sub)manifold. By linearity we can extend the definition of integration to integrals of p forms over arbitrary p-cycles. Using, for example, a decomposition with rectanguar coordinate domains it can be shown that integrals of $d\omega$ over cycles are related to integrals over boundaries:

$$\int_{C_r} d\omega_{r-1} = \int_{\partial C_r} \omega_{r-1} \tag{2.18}$$

This is **Stokes' theorem**, which generalizes Gauß' and Stokes' theorems in $\mathbb{R}^3$.

Since $d^2 = 0$ we can define the **de Rham complex**

$$0 \to \Lambda^0 M \xrightarrow{d_0} \Lambda^1 M \xrightarrow{d_1} \ldots \xrightarrow{d_{n-2}} \Lambda^{n-1} M \xrightarrow{d_{n-1}} \Lambda^n M \xrightarrow{\partial_n} 0 \tag{2.19}$$

in analogy to the chain complex for the bounday operator. A p form ω is called **closed**, or a **cocycle**, if $d\omega = 0$. It is called **exact**, or a **coboundary**, if $\omega = d\omega'$ for some $p-1$ form ω'. The **de Rham cohomology** groups $H^p = Z^p/B^p = \operatorname{Ker} d_p / \operatorname{Im} d_{p-1}$ are the additive groups of closed p forms modulo exact p forms, where Z^p and B^p are the cocycle and coboundary groups, respectively. As the names suggest, these groups are dual to the homology groups, at least under certain conditions. This is the content of the

[2] For rigorous definitions and proofs one often uses partitions of unity $1 = \sum_i f_i$ with $0 \le f_i \le 1$ that are compatible with a locally finite covering U_i (i.e. $f_i \in C^\infty(M)$ vanishes outside U_i). One can use simplicial or, equivalently but more convenient for integration, rectangular coordinate domains.

de Rham theorem: The bilinear map

$$([C_r],[\omega_r]) \to \int_{C_r} \omega_r \in \mathbb{R}, \qquad C_r \in Z_r,\ \omega_r \in Z^r \tag{2.20}$$

is non-degenerate. (It is easy to see that this map is independent of the representatives C_r, ω_r of the (co)homology classes $[C_r] \in H_r$, $[\omega_r] \in H^r$ because of Stokes's theorem.) If M is compact then H_r and H^r are finite-dimensional and the dimensions $b^r = \dim H^r$ coincide with the Betti numbers $b_r = \dim H_r$.

The integral $(C,\omega) := \int_C \omega$ is called the **period** of the closed r-form ω over the r-cycle C. De Rham's theorem implies that, given a basis C_i of $H_r(M,\mathbb{R})$, a closed r-form is exact iff its periods over all C_i vanish. In turn, for any set of r real numbers there exists a closed form whose periods are given by these numbers. The de Rham theorem thus provides an amazing link between topology (counting faces in triangulations) and analysis (global existence of solutions to differential equations). In particular we obtain the

Poincaré lemma: If M is contractible then $H^r = 0$ for $r > 0$, i.e. all closed forms are exact. This applies, in particular, to contractible coordinate neighborhoods.

Proof (for $\mathbb{R}^n$): We use the 'homotopy operator'

$$K : \Omega(x) \to K(\Omega(x)) = i_x \int_0^1 \frac{dt}{t}\, \Omega_t(x), \qquad \Omega_t(x) := \Omega(tx) = \frac{t^p}{p!} \Omega_{i_1 \ldots i_p}(tx) dx^{i_1} \ldots dx^{i_p}, \tag{2.21}$$

on Λ^p, where i_x is the interior product with the vector field x^m. K satisfies

$$Kd + dK = 1 \tag{2.22}$$

on p-forms, which can be seen as follows: With $f : x \to tx$ we observe $f_*(x^m) = tx^m$, $f^*(\Omega) = \Omega_t$ and $i_x f^* = f^* i_{tx} = t f^* i_x$, so that

$$K(\Omega) = \int_0^1 dt f^*(i_x\Omega), \qquad \{K,d\}\Omega = \int_0^1 dt\, f^*\{i_x, d\}\Omega = \int_0^1 dt\, f^*(\mathcal{L}_x\Omega). \tag{2.23}$$

The Lie derivative $\mathcal{L}_x\Omega$ can be written as $\mathcal{L}_x\Omega(x) = \frac{d}{d\varepsilon}\Omega_{(1+\varepsilon)}(x)_{|\varepsilon=0} = \frac{d}{dt}\Omega_t(x)_{|t=1}$ and we find

$$\{K,d\}\Omega(x) = \int_0^1 dt \frac{d}{dt}\Omega_t(x) = \Omega_1(x) - \Omega_0(x) = \Omega(x). \tag{2.24}$$

Putting the pieces together we find $\Omega = d\omega$ with $\omega = K\Omega$ for any closed p-form Ω because $d\omega = \{K,d\}\Omega = \Omega$. Note that our construction yields a 'potential' ω that satisfies the 'radial gauge' condition $i_x\omega = 0$ (which makes ω unique because $d(\delta\omega) = i_x(\delta\omega) = 0 \Rightarrow \mathcal{L}_x(\delta\omega) = 0$).

Chapter 3

Riemannian geometry

So far we only considered differentiable manifolds without additional structure. Such structure is needed if we want to introduce geometrical concepts like parallel transport of vectors, distances and curvature. To this end Riemannian geometry introduces a metric and a connection, which are closely related by a compatibility condition. First we add a **connection**, which tells us how to relate tangent spaces at different points and how to do parallel transport of vectors along curves on the manifold. The result of parallel transport in general depends on the curve, which leads to the notion of (intrinsic) **curvature** (and torsion). In many applications it is also important to be able to measure **distances** and **angles** with the parallel transport being compatible in the sense that it conserves scalar products. In order to describe spinors in curved space it is necessary to introduce the vielbein and the **spin connection**. A metric on tangent space will allow the definition of the **Hodge dual** and of an inner product on the exterior algebra.

3.1 Covariant derivatives and connections

The simplest situation is the embedding of a manifold into some Euclidian space E of higher dimension with coordinates X^μ. Then tangent vectors can be identified with vectors in E, with a basis of T_xM provided by $\partial_m X^\mu(x)$. The components of the induced metric are $g_{mn} = \partial_m X^\mu \partial_n X^\nu \delta_{\mu\nu}$. Partial differentition of tangent vectors $V^\mu = V^m \partial_m X^\mu$, however, in general takes us out of tangent space. In this situation we can define a *covariant derivative* of tangent vector fields by orthogonal projection $D_i V = \mathrm{tpr}(\partial_i V)$ of $\partial_i V$ to the tangent space at the appropriate point (the linear operator 'tpr' denotes this projection). We will see below that this leads to the unique metric-compatible torsion-free connection on the Riemannian manifold (M, g).

In general we would like to have a covariant notion of a derivative of a vector or tensor in some direction ξ, like, for example, a covariant derivative along the tangent vector to a coordinate line $\xi = \partial_i$. The problem is that the partial derivatives of vector components $\partial_i v^m$ compare components of vectors at different points. The corresponding tangent spaces are, however, different spaces that are not related in a coordinate independent way. We thus need to introduce a relation between tangent spaces at neighboring points.[1] (The notion of *parallel transport* is closely related to this: A vector field along a curve will be called parallel iff its covariant derivative in the direction of the tangent vector to the curve vanishes at each point). Such a relation is provided by an **affine connection**

$$D : TM \times TM \to TM, \qquad (\xi, v) \to D_\xi v \tag{3.1}$$

which by definition is bilinear and satisfies

$$D_{f\xi} v = f D_\xi v, \qquad D_\xi (fv) = \xi(f)v + f D_\xi v. \tag{3.2}$$

It can be thought of as a GL_n-valued 1-form, since we have a linear map $TM \to TM$ for each ξ. (The first of these equation states that $D_.()$ is a 1-form and the second that D_ξ acts as a derivation on products; by demanding that D_ξ is a derivation on arbitrary tensor products we can extend its definition to the tensor algebra.)

To understand the content of this defintion in terms of coordinates we evaluate the **covariant derivative** $D_i v^n := (D_{\partial_i} v)^n$ using the **connection coefficients** ${\Gamma_{ij}}^k$ that are defined by ${\Gamma_{ij}}^k \partial_k := D_{\partial_i} \partial_j$. Evaluating the components of $D_i(v) = D_i(v^n \partial_n)$ we obtain

$$D_i v^n = \partial_i v^n + {\Gamma_{ij}}^n v^j, \qquad D_i = \partial_i + {\Gamma_{ij}}^n {\Delta_n}^j. \tag{3.3}$$

The second formula, which involves the GL_n symbol, extends the definition of the covariant derivative to a derivation on the tensor algebra.

Exercise: Show that the connection coefficients transform as

$${\Gamma_{ij}}^k(y) = \left(\frac{\partial^2 x^l}{\partial y^i \partial y^j} + \frac{\partial x^m}{\partial y^i} \frac{\partial x^n}{\partial y^j} {\Gamma_{mn}}^l(x) \right) \frac{\partial y^k}{\partial x^l} \tag{3.4}$$

under coordinate transformations (use, for example, the definition ${\Gamma_{ij}}^k(y) \partial_{y^k} = D_{\partial_{y^i}}(\partial_{y^j})$, the equations (3.2) and $\partial_{y^l} = \frac{\partial x^n}{\partial y^l} \partial_{x^n}$). Inserting $x = y + \xi + O(\xi^2)$ this implies the infinitesimal transformation law

$$\delta_\xi {\Gamma_{ij}}^n = \partial_i \partial_j \xi^n + \mathcal{L}_\xi {\Gamma_{ij}}^n, \tag{3.5}$$

where $\mathcal{L}_\xi {\Gamma_{ij}}^n$ denotes the terms that would arise if ${\Gamma_{ij}}^n$ were a tensor.[2]

[1] The Lie derivative $\mathcal{L}_\xi$ of a vector depends on $\partial_i \xi$, so it is not a (coordinate independent) directional derivative.

[2] Note that ${\Gamma_{ij}}^k dx_j = D_i dx^k$ can be interpreted as a 'tensor' is some sense if we regard it as the difference to the flat connection on a chart $\mathcal{U}_x$; this is, however, not coordinate independent.

3.2 Curvature and torsion

The transformation law (3.4) implies that the anti-symmetric parts of the connection coefficients

$$T_{ij}{}^k := \Gamma_{ij}{}^k - \Gamma_{ji}{}^k \tag{3.6}$$

are the components of a tensor, which is called the **torsion tensor**; by an appropiate choice of local coodinates we may, on the other hand, make the symmetric part $\Gamma_{(ij)}{}^k$ vanish at any given point. Furthermore, the difference of two connections and variations of a connection are tensors.

Since the commutator $[D_m, D_n]$ is again a derivation it must be proportional to D_l and $\Delta_i{}^k$. We find the following algebra of derivations:

$$[D_i, D_j] = -T_{ij}{}^l D_l + R_{ijk}{}^l \Delta_l{}^k, \tag{3.7}$$

$[\Delta_i{}^k, D_j] = -\delta_j^k D_i$ and $[\Delta_i{}^k, \Delta_j{}^l] = \delta_i^l \Delta_j{}^k - \delta_j^k \Delta_i{}^l$, with the **curvature tensor**

$$R_{ijk}{}^l = \partial_i \Gamma_{jk}{}^l - \partial_j \Gamma_{ik}{}^l + \Gamma_{in}{}^l \Gamma_{jk}{}^n - \Gamma_{jn}{}^l \Gamma_{ik}{}^n. \tag{3.8}$$

Exercise: Check this results for the curvature $R_{ijk}{}^l$ be direct evaluation of $[D_i, D_j]$ or by application of (3.7) to a vector field. (Curvature and torsion are coefficients of independent covariant derivations in a tensorial equation and hence must themselves transform as tensors.)

Exercise: Show that the Jacobi identity $\sum_{ijk}[D_i, [D_j, D_k]]$ implies the **Bianchi identities**

$$1^{\text{st}}\ \text{BI}: \qquad \sum_{ijk} \left(R_{ijk}{}^l - D_i T_{jk}{}^l + T_{jk}{}^n T_{in}{}^l\right) = 0, \tag{3.9}$$

$$2^{\text{nd}}\ \text{BI}: \qquad \sum_{ijk} \left(D_i R_{jkl}{}^m - T_{jk}{}^n R_{inl}{}^m\right) = 0, \tag{3.10}$$

for curvature and torsion, where the *cyclic sum* $\sum_{ijk}$ denotes the sum over cyclic permutations of the ordered set (i, j, k).

Interpreting the torsion as vector valued two form $T = \frac{1}{2} dx^i dx^j T_{ij}{}^l \partial_l$ and the curvature as a GL_n-valued two form $R = \frac{1}{2} dx^i dx^j R_{ijk}{}^l \Delta_l{}^k$ it is straightforward to check the coordinate independent formulas

$$T(X,Y) = D_X Y - D_Y X - [X,Y], \qquad R(X,Y)Z = D_X D_Y Z - D_Y D_X Z - D_{[X,Y]} Z \tag{3.11}$$

for arbitrary vector fields X, Y, Z, which can be used as a coordinate independent definition of the tensors T and R. The geometrical meaning of curvature and torsion can be clarified by inserting $X = \partial_i$ and $Y = \partial_j$: Then the Lie bracket $[X, Y]$ does not contribute and nonvanishing torsion implies that 'parallelograms do not close'. Nonvanishing curvature implies that a vector changes when transported along an infinitesimal closed loop (like, for example, along a parallelogram of coordinate lines). The subgroup of GL_n that is generated by parallel transport of vectors along arbitrary closed loops on a manifold is called **holonomy group**.

A **Riemannian manifold** is a manifold with a **metric** $g = dx^m \otimes dx^n g_{mn}$, which by definition is a positive definite[3] symmetric bilinear form on tangent space

$$g(v,w) = v^m g_{mn} w^n, \qquad g_{mn} = g_{nm}, \qquad g := \det g_{mn} \neq 0. \tag{3.12}$$

A metric defines lengths and angles among tangent vectors at a given point and it is natural to demand that parallel transport does not change these quantities. This is equivalent to covariant constance $D_m g_{ij} = 0$ of the metric tensor. This property is called **metric compatibility**

$$D_m g_{ij} = \partial_m g_{ij} - \Gamma_{mij} - \Gamma_{mji} = 0, \qquad \text{with} \quad \Gamma_{ijk} := g_{kl}\Gamma_{ij}{}^l. \tag{3.13}$$

Exercise: Show that metric compatibility implies that the connection coefficients are

$$\Gamma_{ijn} = \hat\Gamma_{ijn} + \tfrac{1}{2}(T_{ijn} + T_{nij} - T_{jni}), \qquad \hat\Gamma_{ijn} := \tfrac{1}{2}(\partial_i g_{jn} + \partial_j g_{in} - \partial_n g_{ij}). \tag{3.14}$$

$\hat\Gamma_{ijn} = g_{nl}\hat\Gamma_{ij}{}^l$ is called **Christoffel symbol** or **Levi-Civita connection**. Thus the tensors metric and torsion fix a unique metric-compatible connection.
The torsion dependent contribution $K_{ijn} = K_{i[jn]} = \frac{1}{2}(T_{ijn} + T_{nij} - T_{jni})$ is called **contorsion**. Note that $\Gamma_{in}{}^n = \hat\Gamma_{in}{}^n$ but $\Gamma_{in}{}^i = \hat\Gamma_{in}{}^i + T_{in}{}^i$ and $g^{ij}\Gamma_{ij}{}^n = g^{ij}\hat\Gamma_{ij}{}^n + g^{nl}T_{li}{}^i$.

The metric provides a natural n form, the **volume form** $\sqrt{g}\, d^n x$ so that any scalar ϕ can be used to build an n form $\sqrt{g}\, d^n x \phi$ whose integral over the manifold is coordinate independent. It also can be used to define curvature tensors with fewer indices: A single contraction gives the **Ricci tensor** $\mathcal{R}_{ik} := R_{ijk}{}^j$ and a second contraction gives the **curvature scalar** $\mathcal{R} = g^{ik}R_{ijk}{}^j$, which is used to build the Einstein Hilbert action in general relativity

$$S_{EH} = \frac{1}{16\pi G_N}\int d^n x \sqrt{-g}\,\mathcal{R} \tag{3.15}$$

The following are some variational formulas that are useful for the derivation of the field equations:

$$\delta g^{ij} = -g^{im}g^{jn}\delta g_{mn}, \qquad \delta g = g\, g^{mn}\delta g_{mn}, \tag{3.16}$$

$$\delta\mathcal{R}_{mn} = D_m\delta\Gamma_{jn}{}^j - D_j\delta\Gamma_{mn}{}^j + T_{mj}{}^l\delta\Gamma_{ln}{}^j \qquad \textbf{Palatini identity} \tag{3.17}$$

$$\frac{\delta}{\delta g^{mn}}\left(\sqrt{g}e^{\phi}\hat{\mathcal{R}}\right) = \sqrt{g}\left(e^{\phi}\hat G_{mn} + \hat D_m \hat D_n e^{\phi} - g_{mn}g^{kl}\hat D_k \hat D_l e^{\phi}\right) \tag{3.18}$$

where $G_{mn} = \mathcal{R}_{mn} - \frac{1}{2}g_{mn}\mathcal{R}$ is the **Einstein tensor** and ϕ an arbitrary scalar field. The Palatini identity is independent of the metric and holds for arbitrary affine connections.[4] The formula for

[3] If g is non-degenerate but not positive definite then the manifold is called pseudo-Riemannian; the signature of g cannot change in a connected component of the manifold.

[4] It can be used to simplify the computation of the Einstein equations by treating $G_{ij}{}^l$ as an independent field whose usual dependence on the metric follows from its variational equation.

the veriation of the determinant of the metric is easily obtained by using $\ln\det X_{ij} = \operatorname{tr}\ln X_{ij}$, so that $\delta\ln g = \operatorname{tr}(g_{..})^{-1}\delta g_{..} = g^{mn}\delta g_{mn}$ with $g^{mn} := (g_{mn})^{-1}$.

For derivatives of the determinant $g = \det g_{mn}$ of the metric we find

$$\partial_p \ln\sqrt{g} = \hat\Gamma_{pm}{}^m = \Gamma_{pm}{}^m, \quad \partial_p(\sqrt{g}v^p) = \sqrt{g}(D_p + T_{pl}{}^l)v^p, \quad \mathcal{L}_\xi\sqrt{g} = \partial_p(\xi^p\sqrt{g}). \tag{3.19}$$

Therefore $\sqrt{g}\,(D_n X_{i\ldots j}Y^{ni\ldots j} + X_{i\ldots j}D_nY^{ni\ldots j} + T_{nl}{}^l X_{i\ldots j}Y^{ni\ldots j}) = \partial_n(\sqrt{g}\,X_{i\ldots j}Y^{ni\ldots j})$ is a total derivative, which implies a covariant partial integration rule. These formulas also confirm that $\sqrt{g}\,\phi$ gives coordinate independ integrals: Under an infinitesimal coordinate transformation a *scalar density* $\sqrt{g}\,\phi$ transforms into the total derivative $\mathcal{L}_\xi(\sqrt{g}\,\phi) = \partial_m(\xi^m\sqrt{g}\,\phi)$, whose integral gives a surface term that just compensates the shift in the integration domain. (For a tensor $T_{\cdots}^{\cdots}$ the product $\sqrt{g}\,T_{\cdots}^{\cdots}$ is called a **tensor density**.)

Exercise: Show that the 'orthogonal projection to tangent space' definition of the covariant derivative that can be used if a Riemannian manifold is realized by an embedding into Euclidean space is equal to the Levi-Civita connection. Hint: Use the definition $\Gamma_{mn}{}^l\partial_l X^\mu := D_m(\partial_n X^\mu) := \operatorname{tpr}(\partial_m(\partial_n X^\mu))$ and relate Γ_{mnl} to the partial derivatives of the induced metric $g_{mn} = \partial_m X^\mu \partial_n X^\nu \delta_{\mu\nu}$. (Note that $V_\mu \operatorname{tpr}(W^\mu) = V_\mu W^\mu$ if V^μ is tangential.)

The length of a curve $x(t)$ with curve parameter t on a Riemannian manifold is given by the integral $S = \int ds = \int dt L(t)$ with $L = \sqrt{\dot x^m \dot x^n g_{mn}(x)}$. A curve of extremal length is called a **geodesic**. Variation of $x(t)$ implies

$$\frac{\delta S}{\delta x^p} = \frac{1}{2L}\dot x^m \dot x^n \partial_p g_{mn} - \frac{d}{dt}\left(\frac{2\dot x^n g_{np}}{2L}\right) = g_{pn}\dot x^n \frac{\dot L}{L^2} - \frac{g_{pn}}{L}\left(\ddot x^n + \dot x^i \dot x^j \Gamma_{ij}{}^n\right) = 0 \tag{3.20}$$

This equation does not fix $x(t)$ as a function of t because the 'action' S is reparametrization invariant. We may thus choose an **affine parametrization** of the curve, with the curve parameter proportional to the length (i.e. we impose $\dot L = 0$), to simplify the equation. Then only the last term remains and we obtain the geodesic equation

$$\ddot x^n + \dot x^i \dot x^j \hat\Gamma_{ij}{}^n. \tag{3.21}$$

This equation can be obtained directly using the action $S = \int dt L^2$. (In general relativity the action of a structureless free particle is proportional to the proper time; such particles therefore move on geodesics.)

There is an alternative definition of a geodesic as a curve whose tangent vectors are parallel and – with affine parametrization – of constant length along the curve (the curve is *autoparallel*). This means that the covariant derivative of $\dot x$ along $\dot x^m\partial_m$ vanishes, i.e. $D_{\dot x}\dot x^m = \ddot x^n + \dot x^i \dot x^j \Gamma_{ij}{}^n$. The two definitions agree for the Levi-Civita connection $\Gamma = \hat\Gamma$.

Since R and $\hat{R}$ both are tensors the difference has to be a tensor as well. It is, indeed, straightforward to check that

$$R_{ijk}{}^l = \hat{R}_{ijk}{}^l + \hat{D}_i K_{jk}{}^l - \hat{D}_j K_{ik}{}^l + K_{in}{}^l K_{jk}{}^n - K_{jn}{}^l K_{ik}{}^n, \tag{3.22}$$

where $K_{ijn} = \Gamma_{ijn} - \hat{\Gamma}_{ijn} = \frac{1}{2}(T_{ijn} + T_{nij} - T_{jni})$ is the contorsion. There is thus no loss in generality if we use the Levi-Civita connection and consider the torsion as an independent tensor field (the same is true for non-metricity components of the connection, as soon as a metric tensor if available).

The curvature tensor $\hat{R}$ of the Levi-Civita connection has additional symmetries: Since

$$\hat{R}_{ijkl} = \tfrac{1}{2}(\partial_i\partial_k g_{jl} - \partial_i\partial_l g_{jk} - \partial_j\partial_k g_{il} + \partial_j\partial_l g_{ik}) + \hat{\Gamma}_{ik}{}^n\hat{\Gamma}_{jln} - \hat{\Gamma}_{jk}{}^n\hat{\Gamma}_{iln} \tag{3.23}$$

symmetry of $\hat{\Gamma}_{ij}{}^k$ implies $\hat{R}_{ijkl} = \hat{R}_{klij} = -\hat{R}_{jikl}$; the first Bianchi identity reads $\sum_{ijk} \hat{R}_{ijk}{}^l = 0$.

The following formulas for a **Weyl rescaling** of the metric are valid for the Levi-Civita connection: Let $g^*_{ij} := e^{-2\sigma} g_{ij}$ and define $\sigma_l := \partial_l\sigma$, $\sigma_{mn} := D_m D_n\sigma + \sigma_m\sigma_n$. Then

$$e^{2\sigma} R^*_{ijkl} = R_{ijkl} + g_{il}\sigma_{jk} - g_{ik}\sigma_{jl} - g_{jl}\sigma_{ik} + g_{jk}\sigma_{il} + \sigma^n\sigma_n(g_{ik}g_{jl} - g_{jk}g_{il}) \tag{3.24}$$

$$R^*_{ij} = R_{ij} + (n-2)(g_{ij}\sigma^n\sigma_n - \sigma_{ij}) - g_{ij}\Delta\sigma, \quad e^{-2\sigma}R^* = R + (n-1)\left((n-2)\sigma^n\sigma_n - 2\Delta\sigma\right), \tag{3.25}$$

$$\Delta f := D^n D_n f = (g^{mn}\partial_m\partial_n - g^{ij}\Gamma_{ij}{}^n\partial_n) f, \tag{3.26}$$

$$\overline{\Gamma}^l := g^{mn}\Gamma_{mn}{}^l - \tfrac{2}{n+1} g^{ml}\Gamma_{mn}{}^n \;\Rightarrow\; \overline{\Gamma}^*_l = \overline{\Gamma}_l + \tfrac{(n+2)(n-1)}{n+1}\sigma_l \tag{3.27}$$

$$e^{-2\sigma}\Delta^* = \Delta - (n-2)\sigma^n\partial_n, \quad (\Delta^* + \tfrac{1}{4}\tfrac{n-2}{n-1}R^*)\phi^* = e^{\frac{n+2}{2}\sigma}(\Delta + \tfrac{1}{4}\tfrac{n-2}{n-1}R)\phi \quad \text{with} \quad \phi^* = e^{\frac{n-2}{2}\sigma}\phi \tag{3.28}$$

The conformal (Weyl) curvature tensor $C_{ijk}{}^l = C^*_{ijk}{}^l$ is given by

$$C_{ijkl} := R_{ijkl} + \frac{g_{il}R_{jk} - g_{ik}R_{jl} - g_{jl}R_{ik} - g_{jk}R_{il}}{n-2} + \frac{R(g_{ik}g_{jl} - g_{il}g_{jk})}{(n-2)(n-1)}, \tag{3.29}$$

$$C_{ijkl}C^{ijkl} = R_{ijkl}R^{ijkl} - \frac{4R_{ik}R^{ik}}{n-2} + \frac{2R^2}{(n-1)(n-2)}, \qquad \sqrt{g}C^2 = \left(\sqrt{g}C^2\right)^* \Leftrightarrow n=4 \tag{3.30}$$

3.3 The Killing equation and the conformal group

A Riemannian manifold has a (continuous) symmetry if there is a family of coordinate transformations that leaves a fixed metric invariant. The vector field ξ that corresponds to an infinitesimal symmetry thus has to satisfy the **Killing equation**

$$\mathcal{L}_\xi g_{mn} = D_m\xi_n + D_n\xi_m = 0 \tag{3.31}$$

A **conformal transformation** is a coordinate transformation $x \to x'(x)$ that amounts to Weyl rescaling of the metric, i.e $g'_{mn}(x') = e^{-2\sigma(x)} g_{mn}(x)$. The existence of an infinitesimal conformal transformation thus requires the existence of a solution to the *conformal Killing equation*

$$\mathcal{L}_\xi g_{mn} = D_m\xi_n + D_n\xi_m = -2\sigma(x) g_{mn} \tag{3.32}$$

($\sigma(x) = -\frac{1}{n} D_l\xi^l$ follows from taking the trace). Such solutions exist, for example, in flat space.

The conformal group $SO(p+1, q+1)$ of a flat space with signature (p, q) can be obtained by solving the equation

$$h_{mn} := \partial_m\xi_n + \partial_n\xi_m + 2\sigma(x)\eta_{mn} = 0, \tag{3.33}$$

where $\eta_{mn} = \mathrm{diag}(1, \ldots, 1, -1, \ldots, -1)$ with $q = n - p$ negative entries. Taking the trace (i.e. contracting with the inverse metric) and the double divergence (i.e. contracting with $\partial_m\partial_n$) we obtain

$$\eta^{mn} h_{mn} = 2(\partial\xi + n\sigma) = 0 \qquad \Rightarrow \qquad \partial^m\partial^n h_{mn} = \left(2 - \tfrac{2}{n}\right) \Box\partial\xi = 0. \tag{3.34}$$

For $n \neq 1$ this implies $\Box\Lambda = \Box\partial\xi = 0$. (In one dimension there is, of course, no restriction on ξ.) Now we compute the symmetrized derivative of the divergence of h_{mn},

$$\partial_l\partial^m h_{mn} + \partial_n\partial^m h_{ml} = \Box(\partial_l\xi_n + \partial_n\xi_l) + 2\partial_l\partial_n\partial\xi + 4\partial_l\partial_n\Lambda = 2\left(1 - \tfrac{2}{n}\right) \partial_l\partial_n\partial\xi = 0, \tag{3.35}$$

where we used $\Box(\partial_l\xi_n + \partial_n\xi_l) = -2\eta_{ln}\Box\Lambda = 0$. In more than two dimensions this implies that all second derivatives of Λ vanish, i.e.

$$n > 2 \qquad \Rightarrow \qquad \Lambda = -\tfrac{1}{n}\partial\xi = 2\, bx - \lambda \tag{3.36}$$

for some constants λ and b_m. In order to solve for ξ we still need the antisymmetric part of $\partial_m\xi_n$, whose derivative is

$$\partial_l(\partial_m\xi_n - \partial_n\xi_m) = \partial_m\partial_l\xi_n - \partial_n\partial_l\xi_m = 2(\eta_{lm}\partial_n\Lambda - \eta_{ln}\partial_m\Lambda) = 4(\eta_{lm}b_n - \eta_{ln}b_m). \tag{3.37}$$

Integrating this equation we find

$$\tfrac{1}{2}(\partial_m\xi_n - \partial_n\xi_m) = \omega_{mn} + 2x_m b_n - 2x_n b_m \tag{3.38}$$

with an antisymmetric integration constant $\omega_{mn} = -\omega_{nm}$. Putting the pieces together

$$\partial_m\xi_n = \omega_{mn} + 2x_m b_n - 2x_n b_m + \eta_{mn}(\lambda - 2bx) \tag{3.39}$$

and thus

$$\xi^n = a^n + x^m\omega_m{}^n + \lambda\, x^n + x^2 b^n - 2bx\, x^n. \tag{3.40}$$

a, ω, λ and b generate translations, Lorentz transformations, dilatations and 'special conformal transformations', respectively.

Computing the Lie brackets of these vector fields it can be shown that the conformal group is isomorphic to $SO(p+1, q+1)$ for a space with signature (p, q). The finite form of the special conformal transformations is $x^n \to y^n = (x^n + x^2\, b^n)/(1 + 2bx + b^2x^2)$. They form a subgroup, as can be seen by writing the transformation as a combination of two inversions and a translation $\vec{y}/y^2 = \vec{x}/x^2 + \vec{b}$ (note that $x^2/y^2 = 1 + 2bx + b^2x^2$; the inversion $\vec{x} \to \vec{x}/x^2$ itself is also a conformal map, but it has negative functional determinant – the radial direction is reversed – and hence is not continuously connected to the identity). The functional determinant is $|\frac{\partial y}{\partial x}| = (\frac{y^2}{x^2})^n = (1 + 2bx + b^2x^2)^{-n}$ and $\eta^{mn} \frac{\partial y^i}{\partial x^m} \frac{\partial y^j}{\partial x^n} = \eta^{ij}/(1 + 2bx + b^2x^2)^2$.

3.4 Vielbein and Lorentz connection

The structure group of Riemannian geometry can always be chosen to be $SO(n)$, which also has spinor representations. So far, however, we have only considered tensors. In order to describe spinors we introduce an orthonormal basis $e^a = dx^m e_m{}^a$ for the cotangent bundle and the dual basis $E_a = E_a{}^m \partial_m$ with $g(E_a, E_b) = \eta_{ab}$ for the tangent bundle, so that

$$g_{mn} E_a{}^m E_b{}^n = \eta_{ab}, \qquad e_m{}^a E_a{}^n = \delta_m^n, \qquad g_{mn} = \eta_{ab} e_m{}^a e_n{}^b. \tag{3.41}$$

For a Riemannian metric we have $\eta_{ab} = \delta_{ab}$; in the pseudo-Riemannian case the diagonal matrix η_{ab} has entries ± 1. In physics $e_m{}^a$ is called *vielbein*,[5] and it allows, for example, to write down a Dirac operator $\partial\!\!\!/ = \gamma^a E_a{}^m \partial_m$ and an action $\int e\, \overline{\psi}(i\partial\!\!\!/ - m)\psi$ for spinor fields in curved space, with the usual γ-matrices satisfying $\{\gamma^a, \gamma^b\} = 2\eta^{ab}$ and with $e = \det(e_m{}^a) = \sqrt{g}\,)$. We will denote *Lorentz indices* by letters from the beginning of the alphabet and *world indices* by $k, l, m, \ldots$. In addition, we introduce the *Lorentz connection* (or *spin connection*) $\omega_a{}^b = dx^m \omega_{ma}{}^b$ for tensors with Lorentz indices, which has to be antisymmetric in the last two indices $\omega_{ab} + \omega_{ba} = 0$ in order to preseve orthonormality under parallel transport (this replaces the metricity condition of an affine connection).

If we want to use holonomous and orthogonal bases (i.e. tensors with both types of indices) at the same time we may do so by defining the covariant derivative as

$$D = d + \Gamma_l{}^m \Delta_m{}^l + \tfrac{1}{2}\, \omega_{ab}\, l^{ab}, \qquad l_{ab} v_c = \eta_{ac} v_b - \eta_{bc} v_a. \tag{3.42}$$

with $D = dx^n D_n$, $\Gamma_l{}^m = dx^i \Gamma_{il}{}^m$ and $\omega_{ab} = dx^i \omega_{iab}$. [6] The constant flat metric η_{ab} is invariant under l_{ab} so that upper and lower Lorentz indices transform with the same sign.

$$[l_{ab}, l_{cd}] = \eta_{ac} l_{bd} - \eta_{bc} l_{ad} - \eta_{ad} l_{bc} - \eta_{bd} l_{ac} \tag{3.43}$$

is the algebra of the Lorentz transformations.

Cartan invented a very efficient calculus by introducing differential forms and orthonormal bases e^a of contangent space: On a vector field v_a the (spin) connection term of the covariant derivative acts by matrix multiplication $Dv_a = dv_a + \omega_a{}^b v_b$. Curvature and torsion can thus be defined by Cartan's structure equations (CSEq)

$$R_a{}^c = d\omega_a{}^c + \omega_a{}^b \omega_b{}^c, \qquad T^a = de^a + \omega^a{}_b e^b, \tag{3.44}$$

[5] A more fancy name for $e^a = dx^m e_m{}^a$ is *soldering form*: The vielbein provides a soldering of the (principal bundle of the) cotangent bundle with an orthonormal frame bunde.

[6] It is, however, usually more efficient to convert all indices – except for anti-symmetrized form indices – to the orthonormal basis. Note that the connection coefficients drop out of $D_{[i_0}\omega_{i_1\ldots i_p]}$ up to torsion terms (which are tensors).

or, in an even more compact symbolic form,[7] $R = D^2$ and $T = De$. The Bianchi identities

$$dT + \omega \wedge T = R \wedge e, \qquad dR + \omega \wedge R - R \wedge \omega = 0. \tag{3.45}$$

are now a trivial consequence of $d^2 = 0$. The total antisymmetrization that is implicit in these 3-forms replaces the cyclic sum over the respective indices.

Since parallel transport should not depend on which basis we use for tangent space the vielbein should be covariantly constant, i.e.

$$D_m e_n{}^a = \partial_m e_n{}^a - \Gamma_{mn}{}^a - \omega_{mn}{}^a = 0, \tag{3.46}$$

which provides a relation between Γ and the spin connection (here we use the vielbein and its inverse to convert the second index of ω and the third index of Γ to the appropriate basis). Contraction of this equation with $dx^m dx^n$ leads to $de^a - dx^m dx^n \Gamma_{mn}{}^a - dx^m dx^n \omega_{mb}{}^a e_n{}^b = de^a + \omega^a{}_b e^b - dx^m dx^n \Gamma_{mn}{}^a = 0$, which shows that the definition of the torsion via the CSEq agrees with our previous definition. The equivalence of the definitions of the curvature can be seen by evaluating the matrix valued curvature form $R = D^2$ in a holonomous basis:

$$R = D^2 = dx^i D_i(dx^j D_j) = dx^i \Gamma_{il}{}^j dx^l D_j + dx^i dx^j D_i D_j = \tfrac{1}{2} dx^i dx^j (T_{ij}{}^l D_l + [D_i, D_j]) \tag{3.47}$$

Here we used $D_i dx^j = \Gamma_{il}{}^j dx^l$, which seems to show that the connection coefficients are tensors. This paradox can be understood by regarding Γ as the difference between the flat connection in the coordinate patch and the connection Γ, and we already know that differences between connection coefficients transform as tensors.

A formula for the spin connection as a function of vielbein and torsion can be obtained by using $\omega_{ma}{}^b = E_a{}^n \partial_m e_n{}^b - \Gamma_{ma}{}^b$, $g_{mn} = e_m{}^a e_n{}^b \eta_{ab}$ and our formula (3.14) for $\Gamma(g, T)$. The same result can also be obtained directly from the structure equation $T^a = de^a + \omega^a{}_b e^b$ if we use the formula $2\omega_{[mn]r} = \sum_{mnr} \omega_{mnr} - \omega_{rmn}$ that allows to compute a tensor that is antisymmetric in its last 2 indices from its antisymmetrization in the first two indices:[8]

$$-\omega_{mnl} = e_{ma} \partial_{[n} e_{l]}{}^a + e_{la} \partial_{[n} e_{m]}{}^a + e_{na} \partial_{[m} e_{l]}{}^a + \tfrac{1}{2}(T_{mnl} + T_{lmn} - T_{nlm}) \tag{3.48}$$

For fixed vielbein the components of spin connection and torsion are thus related by invertible linear equations, so that it is equivalent to use $\{e_m{}^a, \omega_{ma}{}^b\}$ or $\{e_m{}^a, T_{mn}{}^a\}$ as independent of fields. Usually it is more convenient to work with the former set.

Since the Lie derivative of a tensor field is again a tensor field it should be possible to rewrite it in terms of covariant derivatives. If Lorentz indices are involved this is, however, only

[7] Derivatives of the vector field components drop out in $D^2 v = d^2 v + d(\omega v) + \omega dv + \omega^2 v = (d\omega + \omega^2) v$.

[8] The cyclic sum gives the total antisymmetrization of such a tensor, so that both, $\omega_{[mn]r}$ and $\sum_{mnr} \omega_{mnr}$, are functions of vielbein and torsion. We thus obtain $\omega_{mab} = E_a{}^n E_b{}^r (\omega_{[mn]r} - \omega_{[nr]m} + \omega_{[rm]n})$ with $\omega_{[mn]r} = \eta_{ab} e_r{}^a \partial_{[m} e_{n]}{}^b - \frac{1}{2} T_{mnr}$.

possible if we combine local coordinate and Lorentz transformations to a total transformation

$$s := \mathcal{L}_\xi + \tfrac{1}{2}\Lambda_{ab}l^{ab} = \xi^l D_l - (D_i\xi^k + \xi^l T_{li}{}^k)\Delta_k{}^i + \tfrac{1}{2}\hat{\Lambda}_{ab}l^{ab}, \qquad \hat{\Lambda}_{ab} := \Lambda_{ab} - \xi^l\omega_{lab}. \tag{3.49}$$

The same ξ-dependent redefinition of the Lorentz transformation has to be used if we want to write the variation of the spin connection under such a transformation in a manifestly covariant form:

$$s\Gamma_{nl}{}^m = D_nD_l\xi^m + D_n(\xi^k T_{kl}{}^m) + \xi^k R_{knl}{}^m, \qquad s\omega_{na}{}^b = -D_n\hat{\Lambda}_a{}^b - \xi^l R_{lna}{}^b. \tag{3.50}$$

To derive these formulas we can use $[s, D_n]v^j = (s\Gamma_{nl}{}^m)\Delta_m{}^l v^j$ and $[s, D_n]v^c = \frac{1}{2}(s\omega_{nab})l^{ab}v^c$. In the BRST formalism commuting transformation parameters are turned into anti-commuting *ghost fields* ξ and Λ; then $\hat{\Lambda}$ are called *covariant* Lorentz ghosts; for more details on transformation properties of ghosts see [$\mathtt{br}_1\mathtt{96}$].

3.5 Hodge duality and inner products

Given a metric with sign of the determinant $\mathrm{sign}(g) = (-)^s$ the natural **volume form** is $\varepsilon = \sqrt{|g|}dx^1 \ldots dx^n$ and we can define the **inner product** for p forms $(\alpha|\beta) := \frac{1}{p!}\alpha_{i_1\ldots i_p}\beta^{i_1\ldots i_p}$, where

$$(dx^{i_1}\ldots dx^{i_p}|dx^{j_1}\ldots dx^{j_p}) = \det g^{i_a j_b} \tag{3.51}$$

The **Hodge star** operation $* : \Lambda^p \to \Lambda^{n-p}$ is defined by $(*\omega)_{i_{p+1}\ldots i_n} = 1/p!\, \varepsilon_{i_1\ldots i_n}\omega^{i_1\ldots i_p}$ with

$$\varepsilon(\alpha|\beta) = \alpha\wedge *\beta, \quad *^2 = (-)^{p(n-p)+s}, \quad (*\alpha|*\beta) = (-)^s(\alpha|\beta), \tag{3.52}$$

as well as

$$*1 = \varepsilon, \qquad \alpha\wedge *\beta = \beta\wedge *\alpha, \qquad \alpha\wedge *\alpha = 0 \Leftrightarrow \alpha = 0. \tag{3.53}$$

The **codifferential** δ, with

$$\delta\omega := (-)^p *^{-1} d * \omega = (-)^{pD+D+s+1} * d * \omega, \quad *\,\delta = (-)^p d*, \quad *\,d = (-)^{p+1}\delta *, \tag{3.54}$$

is, up to surface terms, the adjoint of the differential d w.r.t. the **scalar product**

$$[\alpha|\beta] := \int \alpha\wedge *\beta = [\beta|\alpha], \quad [d\alpha|\beta] = [\alpha|\delta\beta] + \int d(\alpha\wedge *\beta). \tag{3.55}$$

It allows us to define the **Laplace–Beltrami** operator $\Delta := d\delta + \delta d$ on p-forms in curved space, for which we find

$$(\Delta\omega)_{i_1\ldots i_p} = -g^{mn}D_mD_n\omega_{i_1\ldots i_p} - \sum_{1\le a\le p}\omega_{i_1\ldots j\ldots i_p}R_{i_a}{}^j + \sum_{0\le a<b\le p}\omega_{i_1\ldots j\ldots k\ldots i_p}R_{i_a i_b}{}^{jk}. \tag{3.56}$$

(In calculations it is sometimes useful to have, instead of an equivalent factor $1/p!$, angles $\langle\ldots\rangle$ around indices that enforce their ordering. Then $(\delta\omega)_{i_2\ldots i_p} = -\delta^{\langle j_1\ldots j_p\rangle}_{i_1\ldots i_p} D^{i_1}\omega_{\langle j_1\ldots j_p\rangle}$ with the symbol $\delta^{j_1\ldots j_p}_{i_1\ldots i_p} := \det \delta^{j_b}_{i_a}$ for $1 \le a, b \le p$.)

On a compact Riemannian manifold the Laplacian is a positive operator because

$$[\omega|\Delta\omega] = [d\omega|d\omega] + [\delta\omega|\delta\omega] \geq 0. \tag{3.57}$$

A p-form ω is called *harmonic* if $\Delta\omega = 0$. On a compact orientable Riemannian manifold ω is thus harmonic iff it is closed and co-closed [eg80,NA90]. The **Hodge decomposition** theorem states that there is a unique decomposition of a p-form into an exact, a co-exact and a harmonic piece:

$$\omega_p = d\omega_{p-1} + \delta\omega_{p+1} + \omega_{harm.} \tag{3.58}$$

Each cohomology class therefore has a unique harmonic representative, which implies the Hodge duality $b^r = b^{n-r}$ of Betti numbers. (The linear space of harmonic forms has the same dimension as the de Rham cohomology group, but no ring structure because δ is not an anti-derivation on the exterior algebra and the wedge product of harmonic forms need not be harmonic.)

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