**SKRIPTUM**

# Einführung in die Superstring–Theorie

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# Introduction

As it became clear that general relativity and Maxwell theory are both intimately tied to the concept of local symmetries, a unified description of the forces of nature became conceivable. In general relativity (GR), a local choice of coordinates has to be made and the action is a local functional of the metric and of the matter fields that is independent of this choice. Electrodynamics (ED) can be described very efficiently by a vector potential $A^\mu = (\phi, \vec{A})$, with the field strengths $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ being invariant under gauge transformations $\delta A_\mu = \partial_\mu \Lambda(x^\mu)$ [H. Weyl, 1918]. While classical physics can be formulated in terms of the field strengths, a local description of the coupling to quantum mechanical wave functions requires the gauge potentials, as is illustated by the Aharonov–Bohm effect [ah59].

A promising framework for unification was suggested by Kaluza and Klein [ka21]: They proposed that space-time is 5-dimensional, but with only 4 approximately flat directions and with one direction curled up on a small circle. Then the off-diagonal entry of the metric $A_\mu := g_{4\mu}$ transforms as a vector from the 4-dimensional point of view and serves as the gauge potential. It is still unknown, however, how the dynamics of the gravitational field generates the vacuum expectation value (VEV) for the metric. In the 20s it was not even possible to pose this question within any proper framework, which certainly has to incorporate a quantum mechanical treatment of the gravitational interactions.

After a long development of quantum field theory, techniques for a perturbative analysis of quantum gravity became available in the 60s [fe63, De65]. At about the same time the standard model of strong and electroweak particle interations, an $SU_3 \times SU_2 \times U_1$ gauge theory that is spontaneously broken to $SU_3 \times U_1$ below $100GeV$, was constructed. This led to the discovery of asymptotic freedom [co73] of QCD and – a decade later – to the detection of the $W$ and $Z$ bosons, which mediate weak interactions. Attempts at a group theoretical 'grand unification' of the standard model by a gauge theory with $SU_5$ [ge74] or even larger gauge groups [sl81] produced surprisingly good predictions for the ratio of the $W$ and $Z$ masses and, at the same time, lead to a unification scale of $10^{14} - 10^{16} GeV$, far above the weak (Fermi) scale of $300 GeV$. Since quantum gravity is bound to become important at the Planck mass[1] $M_{Pl} = \sqrt{\hbar c/G_N} = 1.22 \times 10^{19} GeV/c^2$, this may be regarded as an indication that gravity should no longer be ignored in particle physics.

It turned out that a perturbative quantization of gravity is spoiled by non-renormalizability, i.e. an infinite number of divergent quantum corrections that cannot be controlled by symmetries. An important example of such a correction is the one that modifies the cosmological constant $\Lambda$ (the energy density of the vacuum becomes observable in gravitational interactions). The experimental bound for the physical value is best characterized by the tiny dimensionless ratio $|\Lambda|/M_{Pl}^2 < 10^{-121}$ [data].[2] It is clear that such a tiny quantity should be explained by a symmetry, the best candidate for which is supersymmetry (SUSY) [WE83]: Note that the energy of a harmonic oscillator is $E = \frac{1}{2}\omega(a^\dagger a \pm a a^\dagger) = \omega(a^\dagger a \pm \frac{1}{2})$ for excitation modes that are quantized according to bosonic/fermionic statistics $a^\dagger a \mp a a^\dagger = 1$. The zero point energies are, therefore, of equal size and opposite sign. The energy operators for second quantized free fields consist of an infinite sum of such oscillator terms. In order to have a cancellation of zero point energies we should thus have an equal number of bosonic and fermionic degrees of freedom and a symmetry that controlls the cancellation when interactions are turned on. Due to the spin statistics theorem [ST64] a *physical* symmetry that transforms commuting into anti-commuting fields should be in a spin 1/2 representation of the Lorentz group and should be implemented by an anticommuting operator in Hilbert space.

Because the anti-commutator of two SUSY transformations generates translations, the local (or gauged) version or supersymmetry automatically contains gravity and is hence called supergravity (SUGRA). In the late 70s and early 80s it was hoped that SUGRA might cure the divergences of quantum gravity. This also lead to a revival of the ideas of Kaluza and Klein, but now with a higher dimensional compactification space in order to be able to incorporate the whole standard model of particle interactions into a (super)geometrical picture. It was shown that the standard model can only be obtained from at least 11 dimensions, which, at the same time, is the maximal dimension allowed for supergravity (a Weyl fermion has $2^{d/2-1}$ components in $d$ (even) dimensions while a massless vector field has $d-2$ transversal degrees of freedom; this leads to a mismatch for $d > 11$). But it is hard to get chiral fermions by starting in an odd number of dimensions [ba87]. The alternative of adding gauge symmetries to a 10-dimensional theory by hand goes against the original spirit of the ideas of Kaluza and Klein. Even worse, it turned out that SUGRA could not solve the problem with divergences.

1 In $cgs$ units $M_{Pl} = 2.2 \times 10^{-5} g$; the corresponding length scale is $\sqrt{\hbar G_N/c^3} = 1.6 \times 10^{-33} cm$.

2 The experimental bound for the photon mass is $m_\gamma < 3 \times 10^{-33} MeV$, so that electromagnetic gauge invariance appears to be an exact symmetry of nature.

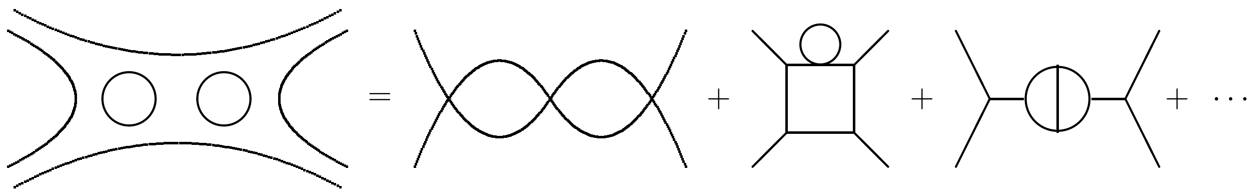

**Fig. 1:** $g = 2$ world sheet and some corresponding 2-loop Feynman graphs.

## 1.1 String unification

String unification apparently works in a rather different way: Here the fundamental object is a thread or a loop in space-time which, during time evolution, sweeps out a surface that is called **world sheet** (in analogy to the world line of a point particle). The dynamics is described by an action that is proportional to the area of that surface, and hence in purely geometrical terms. Particles are oscillation modes of the string, and interactions occur by joining and splitting of string configurations, as is shown in Fig. 1. This has two important consequences:

- There is no interaction point (the apparent splitting point changes under Lorentz transformations), which avoids the UV divergences of second quantized point particle theory.
- There is a unique geometrical interaction, which unifies an a priory infinite number of independent couplings among different fields.

Alltogether, string theory leads to a unification of interactions *and* to a unification of particles.

From a more modern point of view we may think of the world sheet as an independent two-dimensional space with local coordinates $\sigma^m$. The string coordinate functions $X^\mu(\sigma)$ are quantum fields on that space and describe its embedding into a **target space**, which may itself be a topologically non-trivial manifold with local coordinates $X^\mu$. The geometrical description of the action ensures that is can be constructed in a coordinate invariant way as a sum over terms that are defined via local coordinates. For historical reasons such a quantum field theory is called a $\boldsymbol{\sigma}$ **model**. Unfortunately, there are two big problems with this approach:

- Scattering amplitudes are only defined as an infinite sum over different world sheet topologies, i.e. we do not have a non-perturbative definition of string theory. The sum over topologies may be badly divergent.
- To define a string theory we need to choose some fixed **background** target space geometry (or, in a more abstract description, a conformal field theory). Although different choices of this background should lead to equivalent physics, this is not manifest in the construction.

It may or may not be the case that some more elaborate version of string field theory [zw93], which is sometimes referred to by the name 'third quantization', will eventually solve this problem. At present it is not at all clear what string theory really is.

In any case, we do have a well-defined prescription for constructing a finite perturbative expansion of scattering amplitudes for the particles that effectively describe the physics of a string model at large distances (i.e. distances larger than $10^{-33}cm$). This is done in terms of 2-dimensional conformally invariant quantum field theories and a lot has been learned about how the properties of these world sheet models are related to the resulting space-time physics that we can proble in accelerator experiments or astrophysical observations. An incomplete dictionary is compiled at this point for later reference:

| **Space-time** | $\longleftrightarrow$ | **world sheet** |
|---|---|---|
| Einstein equations | $\longleftrightarrow$ | conformal invariance |
| gauge invariance | $\longleftrightarrow$ | current algebra (Kac–Moody) |
| anomaly cancellation | $\longleftrightarrow$ | modular invariance |
| $N=1$ supergravity | $\longleftrightarrow$ | $N=2$ supersymmetry |
| space-time geometry | $\searrow$ | conformal field theory |
| particle spectrum | $\swarrow$ | |

In the last entry of this table it is indicated that a given background space-time geometry leads to a well-defined conformal field theory on the world sheet. Note, however, that this arrow cannot be reversed: In case of strong curvature quantum corrections can be large so that classically different background geometries can lead to quantum mechanically equivalent string theories (and therefore to the same space time physics).

It was already indicated above that we actually need a supersymmetric version of string theory. This has two reasons: The bosonic string has a tachyonic excitation in its spectrum, which indicates that it is unstable and which leads to IR divergences in perturbation theory. Furthermore, we want to describe spin 1/2 particles like electrons or nucleons, and these are missing in the excitation spectrum of the bosonic theory. This leads to the superstring whose conformal anomaly vanishes in 10 dimensions, which is also consistent with an effective low energy supergravity theory.

Actually, there seem to be 5 consistent supersymmetric string theories in 10 dimensions [GR87]: For a flat target space the coordinate fields satisfy the 2D wave equation $\Box X^\mu = (\partial_1+\partial_0)(\partial_1-\partial_0)X^\mu = 0$, whose general solution is a superposition of left-moving and right-moving excitations. In case of open superstrings, called type I, boundary conditions lead to a reflection of these modes at the string ends (the type I theory also contains closed string states in its spectrum since they can be formed by interactions, and its consistency requires to consider unoriented world sheets and Chan–Paton factors for the gauge group $SO_{32}$ [GR87]). For oriented closed strings we have to make a choice in the relative chirality of the left and right moving supersymmetries. This leads to the type IIA and type IIB theories with $N=2$

space-time SUSY, the latter of which is chiral. Moreover, in the closed string case we may even chose to combine a left-moving bosonic string with a right-moving fermionic string. The $D_{bos.} - D_{ferm.} = 26 - 10 = 16$ single left-moving bosons cannot be interpreted as space-time coordinates but rather show up as gauge degrees of freedom. This asymmetric construction is strongly constrained by potential quantum violation of symmetries (space-time anomalies coming from a violation of WS modular invariance), so that only two consistent choices exist: The heterotic strings with gauge groups $E_8 \times E_8$ and $SO_{32}$, respectively.

An important phenomenon in string theory (and many of its building blocks and effective theories) is duality, which means that different classical theories can lead to the identical quantum mechanical models. The oldest example of this type – except for bosonization – is the $R \longleftrightarrow 1/R$ duality of strings compactified on a circle with radius $R$. This duality exchanges winding modes and oscillation modes and is a stringy phenomenon that has no analogue in Kalaza–Klein compactification. Mirror symmetry is a generalization of this duality to certain 3-dimensional curved complex manifolds that can be used to construct more realistic models. In that case quantum mechanically equivalent backgrounds differ not only in size but also in shape and even topology, which leads to exciting implications for both, mathematics and physics [as94, mo95].

Quite recently, this duality business has even been extended to dualities among the above 5 different string theories, or rather their lower dimensional relatives which are continuously connected to 10-dimensional theories by letting some compactification radii go to infinity [as95, fe95, ka95, va95, wi95]. While most of these string-string dualities are still hypothetical, they already survived a number on non-trivial tests [ka$_1$95] and they may well teach us some important lessons towards understanding what string theory really is. There are attempts to understand these dualities in terms of hypothetical 11- or 12-dimensional theories, called $M$ and $F$ theory, respectively [wi$_1$96, va96, be$_3$96, ma96].

The present lecture notes on strings are largely based on the books by Green, Schwarz and Witten [GR87] and by Lüst and Theisen [LU89]. There are many other good sources, like the book by Kaku [KA88] and the lecture notes by Kiritsis [Ki97], which can be obtained via internet. In particular I recommend the excellent books by Polchinski [PO98]. Most books on string theory use the sign convention $\eta_{\mu\nu} = \mathrm{diag}(-,+,+,+)$ for the Minkowski space metric, so that mass and momentum are related by $m^2 = -p^2$ (we use natural units $\hbar = c = 1$). This convention facilitates to keep equations consistent while performing the Wick rotation to Euclidean space. We will, however, use the convention $\eta_{\mu\nu} = \mathrm{diag}(+,-,-,-)$, which is mostly preferred in QFT textbooks, so that $t = x^0 = x_0$, which is somewhat nicer in the Hamilton formalism.

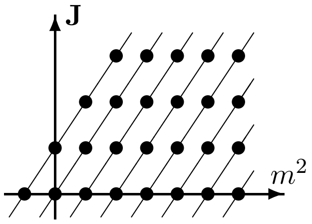


**Fig. 2:** Regge trajectories

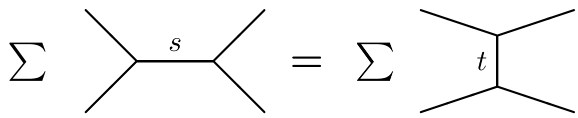


**Fig. 3:** Duality

## 1.2 History of string theory

String theory was discovered in the late 60s as a model for hadron resonances, large numbers of which were found with a spin–mass relation described by Regge trajectories $J = \alpha_0 + m^2\alpha'$, as shown in Fig. 2. Renormalizable QFTs, however, were and are known only for spin $J \in \{0, \frac{1}{2}, 1\}$: Scalars interacting by exchange of a spin $J$ particle, for example, have an amplitude $A_J(s,t) \sim \frac{s^J}{t-m^2}$ where $s = (p_1 + p_2)^2$, $t = (p_1 + p_3)^2$ and $u = (p_1 + p_4)^2$ are the Mandelstam variables[3] for scattering of two particles with momenta $p_1$ and $p_2$ to particles with outgoing momenta $-p_3$ and $-p_4$, because there are $J$ derivatives in the interaction term[4] [GR87]. This generated doubts that hadron resonances were really fundamental particles.

At that time analytical properties of the S-matrix, like the relation between $s$ and $t$ channel amplitudes, were studied extensively, and the idea of duality was born [do68]. It states that $s$ and $t$ channel contributions should be equal, instead of being added as in QFT (see Fig. 3). This hypothesis had only marginal experimental support, but Veneziano [ve68] guessed an amplitude with the desired property, namely $A(s,t) + A(t,u) + A(u,s)$ with

$$A(s,t) = \frac{\Gamma(-\alpha(s))\Gamma(-\alpha(t))}{\Gamma(-\alpha(s)-\alpha(t))} = \int_0^1 dz\, z^{-1-\alpha(s)}(1-z)^{-1-\alpha(t)}, \qquad \alpha(s) = \alpha_0 + \alpha' s. \tag{1.1}$$

It has exponentially soft UV behaviour, whereas for QFTs cross sections only decrease like inverse powers, and it has infinitely many poles, i.e. describes infinitely many particles.

It turned out that this dynamics can be described by a string picture, with the observed particles being the excitation modes of the string. The Nambu-Goto action for the string is proportional to the area of the world sheet, just as the action for a relativistic point particle is proportional to its proper time $S[x] = -m\int ds = -m\int d\tau\,\sqrt{\dot{x}^2}$, where $\tau$ parametrizes the world line $x^\mu(\tau)$. From this geometrical picture of string interactions (see Fig. 1) duality is now apparent. Furthermore, the UV behaviour of string amplitudes is exponentially soft because there are no localizable interaction points on a smooth surface: The symmetries of the string organize the contributions of infinitely many massive particles of high spin in such a way that

[3] $s/t/u$ is the total energy squared in the rest frame of the $s/t/u$ chanel, and $s + t + u = \sum m_i^2$.

[4] Loop amplitudes $\sim \int dp^n A^2(p)/p^4$ are UV finite for $J < 1$ and have a potentially renormalizable logarithmic divergence for $J = 1$ in $n = 4$ dimensions.

the sum of an infinite number of terms with polynomial growth is exponentially small, like in the Taylor expansion of $\exp(-x)$.

In the early 70s QCD turned out to do better in describing hadron interactions[5] (asymptotic freedom in 1973, etc.). But Scherk and Schwarz showed that strings provide a promising theory for quantum gravity [sc74]: There always is a massless spin 2 excitation – the graviton – and there are no UV divergences, because there are no point-like interactions. The bad news, however, was that, in light-cone quantization [go73], Lorentz invariance is broken in $D \neq 26$, and that the intercept $\alpha_0$ turned out to be positive so that the squared mass of the ground state is negative (tachyonic) and the theory is, at best, formulated in an unstable 'vacuum'.

This inconsistency was eventually cured by fermions, which had already been introduced into dual models by Ramond [ra71] and by Neveu and Schwarz [ne71] in order to describe fermionic hadron resonances. A generalization of the Nambu action to the 'spinning string' [br76, de76] was possible, however, only after some development of supersymmetry. Due to the additional fermionic degrees of freedom on the world sheet the critical dimension of the spinning string reduces to $D = 10$. But this model still is plagued by inconsistencies related to a tachyon, which eventually was thrown out by the GSO projection [gl76]. The resulting spectrum of states then turned out to be space-time supersymmetric, i.e. contains an equal number of bosonic and fermionic degrees of freedom, which are related by an anticommuting symmetry.[6] There is an alternative formulation, called the Green–Schwarz superstring [GR87], which is manifestly space-time supersymmetric. We will, however, mainly consider the RNS model with the *manifest* supersymmetry living on the world sheet.

After almost 10 years of underground development of string theory and many fruitless efforts to find a viable model for SUGRA Kaluza–Klein unification it was time for the string revolution, which came in 1984 with the discovery of the Green–Schwarz mechanism [gr84]: In the 'zero slope' limit the superstring leads to a chiral 10-dimensional supergravity theory, and anomaly cancellation fixes the gauge group almost uniquely. The Kaluza–Klein scenario thus eventually obtained a solid basis, but this time including an (almost) unique additional gauge group $E_8 \times E_8$ (or $SO_{32}$). As it turned out, however, the vacuum structure is not so unique after compactification or when string theories are constructed directly in 4 dimensions. There remain many open problems concerning the quantum mechanics that (hopefully) selects a ground state resembling the observable universe (which includes a small cosmological term *after* SUSY breaking). Moreover, it is still not at all clear what string theory really is.

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[5] 'Color strings' are, however, still in use for describing quark interaction at long distances; 'cosmic strings' could form from topological defects in spontaneous symmetry breaking. In both cases, the Nambu–Goto action is only an approximation. We will only be interested in 'fundamental' strings.

[6] It turned out that the GSO projection is not only possible but is mandatory in order to avoid global anomalies at higher genus (this requirement is called 'modular invariance'). Thus supersymmetry, and in particular the presence of fermions, presumably is an unavoidable consequence of string unification.

# Chapter 2

# The bosonic string

The Nambu–Goto action for the bosonic string is given by the area of the world sheet that is embedded in some $D$-dimensional space. If the target space is itself a general manifold with a metric $G_{\mu\nu}(X)$ depending on local coordinates $X^\mu$ then the resulting theory is called a (non-linear) $\sigma$-model. So we start with the action[1]

$$S_N[X] = -T \int d^2\sigma \sqrt{-\det G^*} \qquad \text{with} \;\; G^*_{mn} := (X^*G)_{mn} = \frac{\partial X^\mu}{\partial\sigma^m}\frac{\partial X^\nu}{\partial\sigma^n} G_{\mu\nu}(X), \tag{2.1}$$

where $\sigma^0$ and $\sigma^1$ are local coordinates of the world sheet and the embedding is described by $D$ coordinate functions $X^\mu(\sigma)$. The *induced metric* $G^*_{mn}(\sigma)$ on the world sheet is the *pull back* $X^*G$ of the target space metric $G_{\mu\nu}(X)$ to the parameter space of the embedded surface. The *string tension* $T$ is a constant with the dimension of an inverse length squared and the sign of the action is chosen such that the kinetic energy will be positive for the space-like coordinates $X^1, \ldots, X^{D-1}$ of the target space (see below).

## 2.1 The Polyakov action

From the $\sigma$-model point of view another natural action for the string is

$$S_P[X, g] = -\frac{T}{2} \int d^2\sigma \sqrt{-g}\; g^{mn} \frac{\partial X^\mu}{\partial\sigma^m}\frac{\partial X^\nu}{\partial\sigma^n} G_{\mu\nu}. \tag{2.2}$$

Although this action was already used in refs. [de76, br76] as the bosonic part of a supersymmetric action for the spinning string, it usually goes under the name *Polyakov*, who emphasized the role of the 2-dimensional geometry on the world sheet and showed how to quantize the string

[1] The area element spanned by the vectors $\partial_1 X$ and $\partial_2 X$, which are tangential to target space, is given by $|\partial_1 X|\,|\partial_2 X| \sin(\partial_1 X, \partial_2 X) \;=\; |\partial_1 X|\,|\partial_2 X| \sqrt{1 - \left(\frac{\partial_1 X \cdot \partial_2 X}{|\partial_1 X|\,|\partial_2 X|}\right)^2} \;=\; \sqrt{\det\left(\frac{\partial X}{\partial\sigma^m} \cdot \frac{\partial X}{\partial\sigma^n}\right)}$, where $\sin(\partial_1 X, \partial_2 X)$ denotes the sinus of the angle between the tangent vectors.

in arbitrary dimensions [po81]. The two actions (2.1) and (2.2) are classically equivalent, as they lead to the same equations of motion for the string world sheet. To see this we calculate the variation of $S_P$ with respect to the metric, which by definition[2] is proportional to the energy–momentum tensor $T_{mn}$. Using $\delta(\ln\det M)=\delta(\mathrm{tr}\ln M)=\mathrm{tr}(M^{-1}\delta M)$ we obtain

$$T_{mn}:=\frac{2}{T}\frac{1}{\sqrt{-g}}\frac{\delta S_P}{\delta g^{mn}}=\frac{1}{2}g_{mn}g^{kl}G^*_{kl}-G^*_{mn}=0. \qquad (2.3)$$

The equation of motion $T_{mn}=0$ implies that the world sheet metric must be proportional to the induced metric, i.e. $g_{mn}=\rho G^*_{mn}$, where the factor $\rho=2/(g^{kl}G^*_{kl})$ drops out of all equations and remains arbitrary. Since the Polyakov action does not depend on derivatives of the metric, $\frac{\delta S_P}{\delta g^{mn}}=0$ is algebraic in $g_{mn}$ and we may insert it back into the action without changing the equations of motion for the 'matter fields' $X^\mu$. Taking determinants, we thus observe classical equivalence. For quantization, however, $S_P$ is more convenient because the world sheet scalars $X^\mu$ now have their usual kinetic terms rather than appearing in the square root of a determinant.

Now we compute the total variation of the action to obtain the equations of motion[3] for minimal area surfaces (since $G_{\mu\nu}$ depends on $X^\alpha(\sigma^m)$ we have $\partial_m G_{\mu\nu}=\partial_m X^\alpha\partial_a G_{\mu\nu}$):

$$-\tfrac{2}{T}\,\delta S_P=\int d^2\sigma\,\sqrt{-g}\,\Big(\delta X^\alpha\partial_\alpha G_{\mu\nu}D_nX^\mu D^nX^\nu+2D_n(\delta X^\alpha)D^nX^\rho\,G_{\alpha\rho}-\delta g^{mn}T_{mn}\Big) \qquad (2.4)$$

$$=\int d^2\sigma\,\sqrt{-g}\,\Big(\delta X^\alpha\Big(\partial_\alpha G_{\mu\nu}D_nX^\mu D^nX^\nu-2D^2X^\rho G_{\alpha\rho}-2D^nX^\rho D_nX^\nu\partial_\nu G_{\alpha\rho}\Big)-\delta g^{mn}T_{mn}\Big)$$

Here we ignored surface terms which have to be taken into account for open strings (see below). The last term $\partial_\nu G_{\alpha\rho}$ is symmetrized in $\nu$ and $\rho$, hence all derivatives of the target space metric combine to give the Christoffel symbol $\hat\Gamma_{\mu\nu\alpha}=\frac{1}{2}(\partial_\mu G_{\alpha\nu}+\partial_\nu G_{\alpha\mu}-\partial_\alpha G_{\mu\nu})$ of the target space metric. Contracting $\delta S_P/\delta X^\alpha$ with $G^{\lambda\alpha}$ we thus arrive at the equations of motion

$$\Delta X^\lambda+g^{mn}\partial_m X^\mu\partial_n X^\nu\hat\Gamma_{\mu\nu}{}^\lambda=0, \qquad (2.5)$$

$\Delta:=D^2=g^{mn}D_mD_n=g^{mn}(\partial_m\partial_n-\hat\Gamma_{mn}{}^l\partial_l)$ is the Laplace–Beltrami operator for scalars on the world sheet. (In a non-covariant evaluation of the variational derivative the Christoffel symbol $\hat\Gamma_{mn}{}^l$ of the world sheet metric comes from the term $\partial_m(\sqrt{g}\,g^{mn})=-\sqrt{g}\,g^{kl}\hat\Gamma_{kl}{}^n$.) Note that we recover the geodesic equation if we let the string collapse to a point, so that all derivatives with respect to the space coordinate $\sigma^1$ are zero, and use an affine parametrization $g_{00}=1$ of the resulting world line.

[2] In string theory it is common to deviate from the usual normalization $T_{mn}=\frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{mn}}$, which is consistent with the Noether formula $\hat T_l{}^m=\partial_l\phi^i\frac{\partial\mathcal{L}}{\partial\partial_m\phi^i}-\delta_l^m\mathcal{L}$ for the canonical energy–momentum tensor in Minkowski space. $\hat T^{nm}=\eta^{nl}\hat T_l{}^m$ is in general neither symmetric nor gauge invariant and differs from the flat space limit of $T^{mn}$ by the Belinfante improvement term, which is a divergence of an antisymmetric tensor, plus terms that are proportional to the equations of motion.

[3] The variational derivatives of an action $S=\int\mathcal{L}(\phi,\partial_m\phi)$ are defined by $\frac{\delta S}{\delta\phi^i}:=\frac{\partial\mathcal{L}}{\partial\phi^i}-\partial_m\frac{\partial\mathcal{L}}{\partial\partial_m\phi^i}$. In curved space it is, however, often more efficient to compute the variation directly with covariant partial integration in scalar densities using $\int\partial_{m_1}(\sqrt{g}\,A_{m_2\ldots m_I}B^{m_1\ldots m_I})=\int\sqrt{g}\,D_{m_1}(A_{m_2\ldots m_I}B^{m_1\ldots m_I})=\int\sqrt{g}\,(D_{m_1}A_{m_2\ldots m_I}B^{m_1\ldots m_I}+A_{m_2\ldots m_I}D_{m_1}B^{m_1\ldots m_I})$.

## 2.2 Local symmetries and gauge fixing

Before we try to solve the equations of motion we should have a look at the symmetries of the Polyakov action $\mathcal{L}_P$. By construction the Nambu–Goto action is coordinate invariant in the target space as well as on the world sheet. This carries over to $\mathcal{L}_P$, but for that action we have, in addition, the Weyl invariance $g_{mn}(\sigma) \to e^{2\Lambda(\sigma)}\, g_{mn}(\sigma)$ on the world sheet. Together with two coordinate functions $\tilde\sigma^m(\sigma)$ this gives a total of 3 functions of $\sigma^m$ that we are free to choose.

The number of gauge degrees of freedom thus coincides with the degrees of freedom in the world sheet metric. This suggests that we should be able to use a flat metric $g_{mn} = \eta_{mn}$ on the world sheet, which indeed is true locally. To see this note that in two dimensions for any two linearly independent vector fields there exists a coordinate system whose coordinate lines coincide with the integral curves of the vector fields. Having a metric with Lorentzian signature, there are two natural vector fields defined by the two independent null vectors at each point. In a corresponding coordinate system with coordinates $\sigma^+$ and $\sigma^-$ the metric has only off-diagonal entries. With $\partial_\pm := \partial/\partial\sigma^\pm$ we thus have

$$g_{+-} = g(\partial_+, \partial_-) = \tfrac{1}{2}e^{\varphi}, \qquad g_{++} = g_{--} = 0. \tag{2.6}$$

$\sigma^\pm$ are called light-cone coordinates. They are unique up to reparametrizations $\sigma^\pm \to f_\pm(\sigma^\pm)$ and we choose them in such a way that $\tau = \sigma^0 := (\sigma^+ + \sigma^-)/2$ is time-like and increasing with the target-space time $X^0$, whereas $\sigma = \sigma^1 := (\sigma^+ - \sigma^-)/2$ is space-like. $g_{+-} > 0$ is required by $g_{00} = g_{++} + 2g_{+-} + g_{--} > 0$ and $g_{11} = g_{++} - 2g_{+-} + g_{--} < 0$. These equations, as well as $g_{01} = g_{++} - g_{--}$, follow from $\partial_\tau = \partial_+ + \partial_-$ and $\partial_\sigma = \partial_+ - \partial_-$. We also find

$$\sigma^\pm = \tau \pm \sigma, \qquad g^{+-} = 2e^{-\varphi} = 1/g_{+-}, \qquad g_{+-} = \tfrac{1}{4}(g_{00} - g_{11}), \tag{2.7}$$

$$\partial_\pm = \tfrac{1}{2}(\partial_\tau \pm \partial_\sigma), \qquad g_{mn} = e^{\varphi}\eta_{mn}, \qquad g_{\pm\pm} = \tfrac{1}{4}(g_{00} \pm 2g_{01} + g_{11}). \tag{2.8}$$

Now we can perform a Weyl rescaling to get a flat world sheet metric $g_{mn} = \eta_{mn}$.

In light-cone coordinates we obtain very simple expressions for the Christoffel symbol, whose only non-vanishing components are

$$\hat\Gamma_{++}{}^{+} = \partial_+ \ln\sqrt{-g} = \partial_+\varphi, \qquad \hat\Gamma_{--}{}^{-} = \partial_- \ln\sqrt{-g} = \partial_-\varphi, \tag{2.9}$$

since $\hat\Gamma_{m++} = \hat\Gamma_{m--} = 0$. For the energy–momentum tensor (2.3) we find

$$T_{++} = -\partial_+X\cdot\partial_+X, \qquad T_{+-} = 0, \qquad T_{--} = -\partial_-X\cdot\partial_-X. \tag{2.10}$$

There also is a simple geometrical interpretation of the minimal area equation: Observing that $\partial_+X^\mu$ and $\partial_-X^\mu$ are the light-like tangent vectors defined by the coordinate lines $\sigma_+$ and $\sigma_-$ it is easy to see that (2.5) is nothing but $D_{\partial_-X}\partial_+X = 0$, i.e. the covariant derivative (with respect

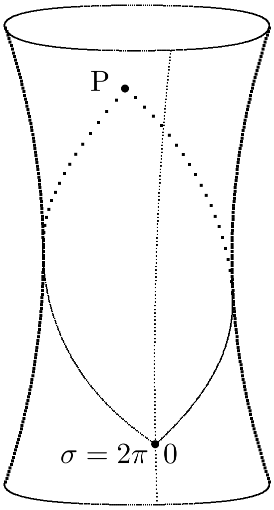


**Fig. 4:** Closed string with $0 < \sigma < 2\pi$

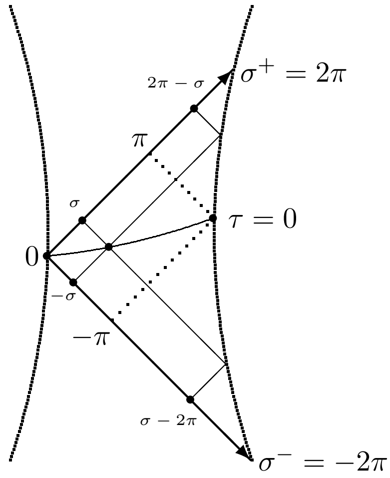


**Fig. 5:** Open string with $0 < \sigma < \pi$

to the Levi–Civita connection in target space) of $\partial_+ X$ along $\partial_- X$ has to vanish. For a flat target space $G_{\mu\nu} = \eta_{\mu\nu}$ this reduces to the wave equation $\Box X^\mu = \partial_+ \partial_- X^\mu = 0$, whose general solution is a superposition of a left-moving mode $X_L^\mu(\sigma^+)$ and a right-moving mode $X_R^\mu(\sigma^-)$.

We now turn to global properties of our choice of parametrization. The basic assumption for our local construction was that the metric has Minkowski signature. This cannot be true globally for interacting strings, as can be seen for a 'pant' representing a smooth joining of two closed strings: For such a world sheet there is always some region where the induced metric is Euclidean. It is therefore convenient to restrict our attention to free strings with 'generic' world sheets and to postpone the study of the interacting case and the rigorous treatment of global questions till after a Wick rotation to Euclidean space. In particular, we exclude world sheets with closed time-like curves and degenerations of the light cone.

The choice of light cone coodinates still allows for reparametrizations of the coordinate lines $\sigma^\pm \to f^\pm(\sigma^\pm)$. This freedom can now be used to choose a parametrization that is $2\pi$-periodic in $\sigma$ for closed strings:

$$X^\mu(\tau, \sigma) = X^\mu(\tau, \sigma + 2\pi) \tag{2.11}$$

(the case of open strings will be discussed in section 2.3). In order to see that (2.11) is consistent with the conformal gauge $g_{mn} \propto \eta_{mn}$ we choose an arbitrary point on the closed string surface as the coordinate origin $\sigma = \tau = 0$ (see Fig. 4). Then we go along the two light-like curves in positive time direction till we arrive at the first intersection point $P$ and choose some parametrization of these two pieces of coordinate lines in such a way that the coordinate labels are smooth and monotonic and reach $2\pi$ at $P$. We can now assign coordinates $\sigma^\pm$ modulo $2\pi$ to any point on the surface by going along the two light cones till we meet one of the sections of coordinate lines where the coordinates have been chosen. In order to fix the coordinates completely we cut the surface along a time-like curve through the origin and demand that the coordinate functions are continuous on the resulting strip. In order to see what happens at the cut we go from the origin to the intersection point along the $\sigma^+$ coordinate line in positive direction till we reach the point $P$ with $\sigma^+ = 2\pi$ and $\sigma^- = 0$. Then we continue along the other coordinate line till we return to the origin, with $\sigma^+$ constant and $\sigma^-$ decreasing by $2\pi$ till we arrive at the origin. We thus observe that the coordinate $\sigma$ jumps by $2\pi$ and that $\tau$ is continuous if we cross the cut at the origin. Because of continuity of the coordinates away from the cut the same has to happen everywhere along the cut.

If we restrict ourselves to parametrizations satisfying (2.11) then we are only free to choose a parametrization of the coordinate lines $\sigma^\pm$ in the intervals $0 < \sigma^\pm < 2\pi$. Taking into account the additional freedom of choosing the origin and imposing smoothness of the coordinate transformation at the origin we end up with a residual reparametrization freedom $\sigma^\pm \to \sigma^\pm - \xi^\pm$ with smooth $2\pi$-periodic functions $\xi^+(\sigma^+)$ and $\xi^-(\sigma^-)$, where we also require $|\partial_\pm \xi^\pm| < 1$ in order that the new coordinates be monotonic.

The restriction that the parametrization of the surface should be $2\pi$-periodic in $\sigma$ with the metric $g_{mn}$ being proportional to $\eta_{mn}$ is called *conformal gauge.* A diffeomorphism $\tilde\sigma^m = \sigma^m - \xi^m(\sigma)$ that changes the metric only by a Weyl rescaling $\tilde g_{mn}(\tilde\sigma) = e^{2\Lambda(\sigma)} g_{mn}(\sigma)$ is called a *conformal transformation* (in other words, this is a reparametrization that preserves angles between coordinate lines; of course we may also think about such a transformation in an active way as giving us a new surface parametrized by the old coordinates). Such transformations respect the conformal gauge. Considering infinitesimal transformations (i.e. keeping only terms that are linear in the small quantities $\xi^m$ and $\Lambda$) this leads to the *conformal Killing equation*

$$\delta g_{mn} = \mathcal{L}_\xi g_{mn} - 2\Lambda g_{mn} = D_m \xi_n + D_n \xi_m - 2\Lambda g_{mn} = 0. \tag{2.12}$$

This means that variation of the metric under infinitesimal coordinate transformations, which is given by the Lie derivative $\mathcal{L}_\xi g_{mn} = D_m\xi_n + D_n\xi_m$ with respect to the vector field $\xi^m$, can be compensated by a Weyl transformation. Taking the trace we find $D_n\xi^n = \Lambda g^{mn} g_{mn} = \Lambda\, d$, so that the Weyl factor becomes proportional to the covariant divergence of $\xi$. In $d = 2$ dimensions this yields $D_m\xi_n + D_n\xi_m - g_{mn} D_l \xi^l = 0$. Using light cone coordinates this equation is an identity for $(m,n) = (\pm,\mp)$ and we recover the conditions $D_\pm \xi_\pm = g_{+-} D_\pm \xi^\mp = 0 \;\Leftrightarrow\; \partial_\pm \xi^\mp = 0$.

## 2.3 Open strings

For open strings the Euler-Lagrange equations of motion still have to be supplemented by boundary conditions that come from the surface terms of a general variation of the action. But first we construct, in analogy to our discussion of closed strings, a conformally flat coordinate system whose space coordinate $\sigma$ ranges from 0 to $\pi$ for all $\tau$. To this end we choose a point at the left boundary[4] as the origin and choose coordinate labels $0 < \sigma^+ < 2\pi$ along the future light cone, as shown in Fig. 5. Imposing the condition that the left and the right boundary of the string is parametrized by $\sigma = 0$ and $\sigma = \pi$, respectively, we can assign coordinates to all points on the string surface by following the light rays till we intersect the original piece of $\sigma^+$-coordinate line. The only difference to the case of closed strings is that this time the coordinate lines are 'reflected' at the boundary. On the $\sigma^-$ coordinate line through the origin, for example, we find coordinate labels between 0 and $-2\pi$ ($\sigma^\pm$ is constant along the $\sigma^\mp$ coordinate lines).

From this construction it follows that the functions $f^+(\sigma^+)$ and $f^-(\sigma^-)$ that correspond to the residual gauge invariance in conformal gauge must be identical $f^+ \equiv f^-$ and $2\pi$ periodic to be consistent with a $\sigma$ coordinate that runs from 0 to $\pi$. The freedom of parametrizing the $\sigma^+$ coordinate line (and thereby also the $\sigma^-$ line) can also be interpreted in a different way: As a consequence of the choice of $\sigma^\pm$ labels the line $\tau = (\sigma^+ + \sigma^-)/2 = 0$ and the $\sigma$ coordinate labels on that line are fixed. In turn, we can first choose the line of vanishing time $\tau = 0$ and assign $\sigma$ coordinate labels between 0 and $\pi$ on that line. Then the $\sigma^\pm$ coordinate labels can be constructed as shown in Fig. 5. Hence the choice of the line of equal time $\tau = \tau_0$ corresponds to the even part of $2\pi$-periodic infinitesimal reparametrizations $\delta\sigma^\pm = f(\sigma^\pm)$, and the freedom of assigning the $\sigma$ coordinate labels corresponds to $f_{odd}$ in the unique decomposition $f = f_{even} + f_{odd}$.

Now we turn to the derivation of the boundary conditions at the ends of the string. We require that the action should be stationary if the variation vanishes at the initial and final times, but is arbitrary at the string ends. To avoid terms coming from a variation of the integration domain we assume a parametrization with $0 < \sigma < \pi$. We thus pick up a surface term

$$\int d^2\sigma \partial_m \left( \delta X^\mu \frac{\partial S}{\partial \partial_m X^\mu} \right) = -T \int_{t_0}^{t_1} d\tau \, \left( \delta X^\mu \sqrt{-g}\, G_{\mu\nu}\, g^{1n} \partial_n X^\nu \right) \Big|_{\sigma=0}^{\sigma=\pi}. \tag{2.13}$$

If we require this expression to vanish for arbitrary variations $\delta X^\mu$ we conclude that for $\sigma = 0$ and for $\sigma = \pi$ we must have

$$\sqrt{-g}\,(g^{10}\dot{X} + g^{11}X') = (g_{01}\dot{X} - g_{00}X')/\sqrt{-g} = 0. \tag{2.14}$$

This implies that the induced metric $G^*_{mn}$ becomes singular at the boundary, which can be seen as follows: First assume that $\dot{X}$ and $X'$ are linearly independent, implying that $g_{01}/\sqrt{-g} = g_{00}/\sqrt{-g} = 0$. The equations of motion imply that $g_{mn}$ is proportional to the induced metric $G^*_{mn} = \rho g_{mn}$ everywhere on the world sheet, with $\rho$ dropping out in the ratio $g_{mn}/\sqrt{-g}$. This cannot happen if $G^*_{mn}$ has a non-singular limit at the boundary. If, on the other hand, $\dot{X}$ and

[4] 'left' refers to decreasing space-like coordinate $\sigma^1$ for an oriented parametrization with $X^0$ increasing with a time-like coordinate $\sigma^0$.

$X'$ become proportional at the boundary, then the matrix $G^*_{mn} = \partial_m X \cdot \partial_n X$ also is singular. The discussion of boundary conditions is therefore very delicate in general and we better first choose a convenient gauge.

In conformal gauge the induced metric $G^*_{mn}$, with entries $\dot{X}^2$, $\dot{X} \cdot X'$ and $(X')^2$, becomes proportional to the flat metric, i.e. $G^*_{mn} = \rho\, \eta_{mn}$, and $\rho$ has to vanish at the boundary. Inserting this back into eq. (2.14) we find the boundary conditions $X' = 0$ and $\dot{X}^2 = 0$. The second condition has the geometrical interpretation that the string ends move with the speed of light, and therefore is independent of the gauge. The $D$ conditions $\partial_\sigma X^\mu = 0$, on the other hand, are valid only in the conformal gauge, as can be seen by choosing a gauge for which the coordinate lines are not orthogonal.[5] Sticking to the conformal gauge, we have *von Neumann* boundary conditions. We therefore can continue the coordinate functions $X^\mu(\tau, \sigma)$ beyond $0 < \sigma < \pi$ to get even and $2\pi$-periodic functions of $\sigma$. Hence all open string solutions can be obtained in the conformal gauge as special cases of closed string solutions.

Von Neumann boundary conditions (in conformal gauge) are the only Lorentz-invariant possibility to make surface terms vanish. If we relax that condition, however, it is also possible to make (2.14) vanish by choosing Dirichlet boundary conditions $X^\mu = const.$ for $p$ of the space-like string coordinates (in a flat target space). The string ends are then constrained to move on a $p$-dimensional submanifold, a so-called $D$-brane (a $p$-brane is an extended object of space-time dimension $p + 1$, so that a 2-brane is a membrane and a 1-brane is a string; here, however, the '$D$' stands for 'Dirichlet', indicating that open strings have to end on that brane without specifying its dimension). The consistency and importance of these boundary conditions was discovered in the context of $T$-duality [da89, ho89] (see below) and the presense and dynamics of the associated (solitonic) extended objects, i.e. $p$-branes acting as $D$-branes, plays an important role in recent results on non-perturbative string dualities [po$_1$95, po$_1$96].

## 2.4 Target space symmetries and conservation laws

By construction, the Polyakov action is invariant under arbitrary coordinate transformations of the world sheet and of the target space. The local world sheet invariances imply gauge symmetries of the action, as we discussed above. Target space coordinate invariance, on the other hand, in general does not imply any symmetry of the $\sigma$ model, since $\mathcal{L}_P$ is only invariant if we also transform $G_{\mu\nu}$. Since the functions $G_{\mu\nu}(X)$ can be interpreted as (an infinite number of) coupling constants, target space coordinate transformations relate *different* $\sigma$ models by

[5] Consider, for example, the solution $X^\mu = (\tau, \cos\sigma \cos\tau, \cos\sigma \sin\tau, 0, \ldots)$ to the equations of motion in conformal-gauge, which satisfies all boundary conditions. Changing the parametrization by $\tau = \overline{\tau} + a\sigma$ we find $\dot{X}^2 = \sin^2\sigma$, $X'^2 = (a^2 - 1)\sin^2\sigma$, and $\dot{X}\, X' = a \sin^2\sigma$. Then $X' = (a, -a\cos\sigma \sin\overline{\tau} - \sin\sigma\cos\overline{\tau}, a\cos\sigma\cos\overline{\tau} - \sin\sigma\sin\overline{\tau}, \ldots)$, which shows that $X'$ does not have to vanish at the boundary in a general gauge.

a reparametrization of the dynamical fields $X^\mu$. We do, however, have a symmetry of the $\sigma$ model if the new functions $G'_{\mu\nu}(X')$ turn out to be identical to the old metric $G_{\mu\nu}(X)$. Then the target manifold has a (geometrical) symmetry, which corresponds to a global symmetry of the $\sigma$ model, because the transformation $X\to X'$ of the dynamical fields is independent of $\sigma$.

Continuous target space symmetries are equivalent to the global existence of Killing vector fields $\Xi^\mu(X)$ with $\mathcal{L}_\Xi G_{\mu\nu}=D_\mu\Xi_\nu+D_\nu\Xi_\mu=0$. According to the Noether theorem[6] they imply the existence of conserved quantities. In the case of a flat target space, for example, we have $G_{\mu\nu}=\eta_{\mu\nu}$ and the general solution to the Killing equation is

$$\Xi^\mu=A^\mu+X^\nu\Omega_\nu{}^\mu \tag{2.15}$$

with $\Omega_{\mu\nu}$ antisymmetric. Invariance under the $D$ independent translations $\delta_\mu X^\rho=-\delta_\mu^\rho$ implies conservation of the target space energy–momentum currents $P_\mu^m$ with the corresponding conserved charges $P_\mu=\int d\sigma P_\mu^0$ (for convenience we use the conformal gauge):

$$P_\mu^m=\delta_\mu^\rho\frac{\partial\mathcal{L}}{\partial\partial_mX^\rho}=-T\eta^{mn}\partial_nX_\mu,\qquad P_\mu=-T\int d\sigma\,\dot X_\mu. \tag{2.16}$$

Note that the object $P_\mu^m$ is different from the canonical (flat) *world sheet* energy–momentum tensor $\hat T_l{}^m=K_l{}^m-\delta_lX^\mu\frac{\partial\mathcal{L}}{\partial\partial_mX^\mu}=-T(\partial_lX^\mu\partial^mX_\mu-\frac12\delta_l^m\delta_k^j\partial_jX^\mu\partial^kX_\mu)$, where $\delta_l\mathcal{L}=\partial_mK_l{}^m=-\partial_l\mathcal{L}$ is the infinitesimal change of the Lagrangian under translations $\delta_l\phi^i=-\partial_l\phi^i$, so that $K_l{}^m=-\delta_l{}^m\mathcal{L}$. In particular, $H=\int d\sigma\,\hat T_0{}^0=\int d\sigma(\dot X^\mu\Pi_\mu-\mathcal{L})$, with the canonical momenta $\Pi_\mu=\partial\mathcal{L}/\partial\dot X^\mu=-T\dot X_\mu$, is the Hamiltonian of our 2-dimensional field theory, which generates time translations (up to a factor $T$, which is due to our convention in eq. (2.3), $\hat T_l{}^m$ is the flat limit of $T_l{}^m$).

Infinitesimal Lorentz transformations $\delta_{\mu\nu}X^\rho=\delta_\mu^\rho X_\nu-\delta_\nu^\rho X_\mu$ yield the angular momenta

$$J_{\mu\nu}^m=(X_\mu\delta_\nu^\rho-X_\nu\delta_\mu^\rho)\frac{\partial\mathcal{L}}{\partial\partial_mX^\rho}=-T\left(X_\mu\partial^mX_\nu-X_\nu\partial^mX_\mu\right),\quad J_{\mu\nu}=-T\int d\sigma\left(X_\mu\dot X_\nu-X_\nu\dot X_\mu\right). \tag{2.17}$$

The Poisson brackets of these charges represent the Poincaré algebra (we include a factor $i$ since $i\{A,B\}_{PB}$ will be replaced by the commutator $[A,B]$ upon quantization):

$$\{P_\alpha,P_\beta\}_{PB}=0,\qquad i\{J_{\mu\nu},P_\alpha\}_{PB}=i\eta_{\mu\alpha}P_\nu-i\eta_{\nu\alpha}P_\mu, \tag{2.18}$$

$$i\{J_{\mu\nu},J_{\alpha\beta}\}_{PB}=i\eta_{\mu\alpha}J_{\nu\beta}-i\eta_{\nu\alpha}J_{\mu\beta}-i\eta_{\mu\beta}J_{\nu\alpha}+i\eta_{\nu\beta}J_{\mu\alpha}. \tag{2.19}$$

The brackets among coordinates and momenta are $\{\Pi_\mu(\tau,\sigma),X^\nu(\tau,\sigma')\}_{PB}=-\delta(\sigma-\sigma')\delta_\mu{}^\nu$.

[6] The first Noether theorem states that continuous symmetries are in one-to-one correspondence with conserved charges: In a field theory with an action $S=\int d^4x\mathcal{L}(\phi^i,\partial\phi^i)$ that is invariant under the infinitesimal transformations $\delta_I\phi^i=R_I{}^i(\phi,\partial\phi)$, i.e. with $\mathcal{L}$ transforming into total derivatives $\delta_I\mathcal{L}=\partial_mK_I{}^m$, the explicit formula for the corresponding Noether currents is $J_I{}^m:=K_I{}^m-\delta_I\phi\frac{\partial\mathcal{L}}{\partial\partial_m\phi}$. Since $\delta_I\mathcal{L}=\partial K_I=\delta_I\phi^i\left(\frac{\partial\mathcal{L}}{\partial\phi^i}-\partial_m\frac{\partial\mathcal{L}}{\partial(\partial_m\phi^i)}\right)+\partial_m\left(\delta_I\phi^i\frac{\partial\mathcal{L}}{\partial(\partial_m\phi^i)}\right)$ the equations of motion imply that the divergence $\partial_mJ_I{}^m$ vanishes, i.e. the currents $J_I$ are *conserved*, so that the charges $Q_I=\int d^3x\ J_I{}^0$ are time independent on shell up to surface terms $\dot Q_I=\int d^3x\ \vec\partial\vec J_I$.

## 2.5 Classical solutions and light cone gauge

Now we want to solve the equations of motion, so we restrict our discussion to the case of a flat target space and use the conformal gauge with the appropriate boundary conditions. Then the coordinate functions fulfill $\partial_+\partial_- X^\mu = 0$ with $\sigma^\pm = \tau \pm \sigma$, hence $X^\mu(\tau,\sigma) = X^\mu_L(\sigma^+) + X^\mu_R(\sigma^-)$ with

$$\partial_+ X^\mu = \partial_+ X^\mu_L = \sum_{n=-\infty}^{\infty} \frac{\alpha^\mu_n}{\sqrt{4\pi T}} e^{-in\sigma^+}, \qquad \partial_- X^\mu = \partial_- X^\mu_R = \sum_{n=-\infty}^{\infty} \frac{\tilde\alpha^\mu_n}{\sqrt{4\pi T}} e^{-in\sigma^-}. \tag{2.20}$$

Integrating these equations we obtain an integration constant, which we choose to be equal $x^\mu_L = x^\mu_R = x^\mu$ for $X_L$ and $X_R$. The boundary conditions imply that the zero modes $\alpha^\mu_0$ and $\tilde\alpha^\mu_0$ must be equal, because there can be no linear $\sigma$ dependence for a periodic function, and we set $\alpha^\mu_0 = \tilde\alpha^\mu_0 = p^\mu/\sqrt{4\pi T}$. Hence

$$X^\mu_L(\tau+\sigma) = \frac{1}{2}x^\mu_L + \frac{1}{4\pi T} p^\mu\,\sigma^+ + \frac{i}{\sqrt{4\pi T}} \sum_{n\neq 0} \frac{1}{n}\alpha^\mu_n e^{-in\sigma^+}, \tag{2.21}$$

$$X^\mu_R(\tau-\sigma) = \frac{1}{2}x^\mu_R + \frac{1}{4\pi T} p^\mu\,\sigma^- + \frac{i}{\sqrt{4\pi T}} \sum_{n\neq 0} \frac{1}{n}\tilde\alpha^\mu_n e^{-in\sigma^-} \tag{2.22}$$

is the most general solution.[7] Reality of the coordinate functions $X^\mu$ implies $\alpha^*_n = \alpha_{-n}$. From the closed string solutions we obtain all solutions for open strings by restricting to even functions. This means that for open strings the left-moving and the right-moving modes must be equal $\tilde\alpha_n = \alpha_n$ and we have much less freedom.

We must be careful to remember that so far we only fulfilled the equations of motion $\delta S/\delta X^\mu = 0$. We still have to set the energy–momentum tensor to zero, i.e. we must impose $T_{++} = -(\partial_+ X)^2 = 0$ and $T_{--} = -(\partial_- X)^2 = 0$: For the left-moving part this means that

$$T_{++} = \frac{1}{2\pi T} \sum_{n=-\infty}^{\infty} L_n e^{-in\sigma^+} = 0, \qquad T_{--} = \frac{1}{2\pi T} \sum_{n=-\infty}^{\infty} \tilde L_n e^{-in\sigma^-} = 0. \tag{2.23}$$

The Virasoro generators $L_n := T\int_0^{2\pi} d\sigma^+ T_{++} e^{in\sigma^+}$ and $\tilde L_n := T\int_0^{2\pi} d\sigma^- T_{--} e^{in\sigma^-}$ are given by

$$L_n = -\frac{1}{2}\sum_{m=-\infty}^{\infty} \eta_{\mu\nu}\alpha^\mu_m \alpha^\nu_{n-m}, \qquad \tilde L_n = -\frac{1}{2}\sum_{m=-\infty}^{\infty} \eta_{\mu\nu}\tilde\alpha^\mu_m \tilde\alpha^\nu_{n-m}. \tag{2.24}$$

[7] The situation is different for compactified dimensions: If $X^\mu$ lives on a circle of radius $R$ then we must allow $X^\mu(\sigma+2\pi) = 2R\pi n + X^\mu(\sigma)$ since the string may wind $n$ times around the loop. Then $\frac{1}{2T}(p^\mu_L - p^\mu_R) = n\,2\pi R$ for some $n \in \mathbb{Z}$. The total momentum $P^\mu = p^\mu_L + p^\mu_R$, on the other hand, will be quantized in units of $1/R$ in the quantum theory because $\exp(2\pi i\,RP^\mu)$ generates a translation by $2\pi R$ and thus has to be the identity operator. This is our first indication of the large/small radius duality $R \to 1/(4\pi T\,R)$. Upon quantization $p_L - p_R$ becomes the winding number operator. Choosing arbitrary integration constants $x^\mu_L$ and $x^\mu_R$, the *collective coordinates* are $x^\mu = \frac{1}{2}(x^\mu_L + x^\mu_R)$ and we may use the combination $x^\mu_L - x^\mu_R$, which does not contribute to $X^\mu$, as the conjugate variable for the winding number. In this way we decompose the operator algebra into a left-moving and a right-moving part. For uncompactified dimensions, on the other hand, left-movers and right-movers are always coupled through the momentum zero modes $p^\mu_L = p^\mu_R$.

The constraints $L_0 = 0 = \tilde{L}_0$, which generate global translations on the world sheet (see below), play a special role. Inserting the definition $p^\mu = \sqrt{4\pi T}\alpha_0^\mu$ for the zero mode, they read

$$p^2 = -8\pi T \sum_{n>0} \alpha_n^* \cdot \alpha_n = -8\pi T \sum_{n>0} \tilde{\alpha}_n^* \cdot \tilde{\alpha}_n. \tag{2.25}$$

Vanishing of $H = (L_0 + \tilde{L}_0)/2$, the 2-dimensional Hamiltonian, tells us the mass $m^2 = P^2$ of a string in terms of the oscillators $\alpha_n$ with $n \neq 0$ (recall that $P^2 = p^2/4$ in case of open strings). This constraint is called the mass shell condition; the generator $L_0 - \tilde{L}_0$ of translations in the space direction equates the masses of left and right movers.

The Virasoro constraints $L_n = 0$ are infinitely many quadratic equations and hard to solve in general. It is therefore time to remember that we still have some gauge freedom left, which we may use to simplify these equations. Note that the periodic reparametrizations of the light-cone coordinates, which are still allowed, lead to the freedom $\tau \to \frac{1}{2}(f^+(\sigma^+) + f^-(\sigma^-))$, which just corresponds to a solution of the wave equation for the coordinate functions. We may therefore choose $\tau$ proportional to $c_\mu X^\mu$ for some fixed time-like or light-like vector $c_\mu$ (a space-like $c_\mu$ would lead to a space-like time direction on the world sheet). Because of the identity

$$V^\pm = V^0 \pm V^{D-1} \qquad \Rightarrow \qquad V_\mu W^\mu = \frac{1}{2}(V^+W^- + V^-W^+) - \sum_{i=1}^{D-2} V^i W^i \tag{2.26}$$

a light-like choice $c_\mu = (1, 0\ldots, 0, 1)$ is particularly useful. In the resulting **light cone gauge** we impose $X^+ = 2\pi T\, p^+\tau$, which implies that all oscillator coefficients $\alpha_n^+ = \tilde{\alpha}_n^+ = 0$ vanish for $n \neq 0$. Now the Virasoro constraints obtain linear terms and can be solved explicitly for

$$\alpha_n^- = \frac{\sqrt{4\pi T}}{p^+} \sum_{m=-\infty}^{\infty} \sum_{i=1}^{D-2} \alpha_m^i \alpha_{n-m}^i, \qquad \tilde{\alpha}_n^- = \frac{\sqrt{4\pi T}}{p^+} \sum_{m=-\infty}^{\infty} \sum_{i=1}^{D-2} \tilde{\alpha}_m^i \tilde{\alpha}_{n-m}^i \tag{2.27}$$

(recall that $p^+ = \alpha_0^+\sqrt{4\pi T}$). This gauge, however, abandons manifest Lorentz invariance in target space and it turns out that the quantum theory violates the Lorentz algebra (2.19) if $D \neq 26$ [go73]; historically this was the first derivation of the critical dimension of the bosonic string. Note that the light cone gauge assumes $p^+ \neq 0$, which can always be achieved by a Lorentz transformation unless $p^\mu \equiv 0$. This is o.k. for a single free string, but in case of interactions intermediate strings may have arbitrary momenta and we should expect sublte technical problems in perturbation theory [gr88].

In order to find a simple solution to the equations of motion (including the Virasoro constraints) we assume that only one frequency is excited (i.e. only one oscillator $\alpha_n$ and its complex conjugate $\alpha_{-n}$, as well as the zero mode $p^\mu$, are non-zero). Then the only relevant left-moving constraints are

$$-L_0 = \tfrac{1}{2}|\alpha_0|^2 + \alpha_{-n} \cdot \alpha_n = 0, \qquad -L_n = \alpha_0 \cdot \alpha_n = 0, \qquad -L_{2n} = \tfrac{1}{2}\alpha_n \cdot \alpha_n = 0. \tag{2.28}$$

The first constraint is the mass shell condition, $L_n = 0$ implies transversality with respect to the momentum $p^\mu$, and the third condition implies a light-like polarization vector $\alpha_n$; by complex conjugation $L_{-n} = 0$ and $L_{-2n} = 0$ are redundant. To simplify things further we keep the string oscillation in the $X^1 - X^2$ plane, and we go to the rest frame and set $x^\mu = \vec{p} = 0$. We now use the light cone gauge[8] and choose $\alpha_n^\mu = \rho\frac{n\sqrt{\pi T}}{2}(0,1,i,0\ldots)$ and $\tilde{\alpha}_n^\mu = \rho\frac{n\sqrt{\pi T}}{2}(0,1,\pm i,0\ldots)$. This is no further restriction since overall phases of $\alpha_n$ and $\tilde{\alpha}_n$ corresponds to shifts in $\sigma^\pm$ and the sign of $\alpha_n^2$ corresponds to a choice of the $X^2$–direction. Then

$$X^\mu_{(\pm)} = \tfrac{\rho}{2}\Big(\tfrac{p^0\tau}{2\pi T},\, \mathrm{Re}(ie^{-in\sigma^+} + ie^{-in\sigma^-}),\, \mathrm{Re}(-e^{-in\sigma^+} \mp e^{-in\sigma^-}), 0\,, \ldots\Big) \tag{2.29}$$

$$= \tfrac{\rho}{2}\Big(\tfrac{p^0\tau}{2\pi T},\; \sin n\sigma^+ + \sin n\sigma^-,\; -\cos n\sigma^+ \mp \cos n\sigma^-, 0\,, \ldots\Big). \tag{2.30}$$

Using the formulas $\sin\alpha + \sin\beta = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$, $\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$, and $\cos\alpha - \cos\beta = -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$, we obtain

$$X^\mu_{(+)} = \rho\Big(\tfrac{p^0\tau}{4\pi T}, \sin\, n\tau\, \cos\, n\sigma, -\cos\, n\tau\, \cos\, n\sigma,\, 0, \ldots\Big), \tag{2.31}$$

$$X^\mu_{(-)} = \rho\Big(\tfrac{p^0\tau}{4\pi T},\; \sin\, n\tau\, \cos\, n\sigma,\; \sin\, n\tau\, \sin\, n\sigma,\, 0, \ldots\Big). \tag{2.32}$$

$X^\mu_{(+)}$ is a solution of total length $4n\rho$ and $2n\rho$ for closed and open strings, respectively. It corresponds to a rotating (multiply covered) rod of length $2\rho$. According to (2.25) we have $p^0 = 2\rho n\,\pi T$, so that the ends indeed move with the speed of light (the tangential vector $\dot{X}^\mu_{(+)}$ is light-like at the boundary). The solution $X^\mu_{(-)}$ only exists for closed strings and corresponds to a periodically collapsing (multiply covered) circle of maximal radius $\rho$, i.e. maximal length $2\rho n\pi$. At the maximal radius there is no kinetic energy and we can check that $(mass)/(length) = T$ is the string tension. For open strings we always have kinetic energy and the factor of 2 in the string length matches the relative factor of 2 in the ratio $\sqrt{-p^2}/m$ for the two types of strings.

Evaluation of (2.17) shows that the angular momentum tensor decomposes into an orbit contribution $x_\mu P_\nu - x_\nu P_\mu$ and the left- and right-moving spin contributions $\Sigma$ and $\tilde{\Sigma}$,

$$J_{\mu\nu} = x_\mu P_\nu - x_\nu P_\mu + \Sigma_{\mu\nu} + \tilde{\Sigma}_{\mu\nu}, \qquad \Sigma^{\mu\nu} = -i\sum_{n>0}\frac{1}{n}\left(\alpha^\mu_{-n}\alpha^\nu_n - \alpha^\nu_{-n}\alpha^\mu_n\right), \tag{2.33}$$

where $\tilde{\Sigma}$ should be omitted in case of open strings (in that case the $\sigma$ integral only extends from 0 to $\pi$ and the spin contribution has to be divided by 2). Inserting the above solutions we find

$$p^2 = (2\rho n\pi T)^2, \qquad\qquad \Sigma^{12} = \pm\tilde{\Sigma}^{12} = \rho^2 n\pi T. \tag{2.34}$$

The classical spin is given by $J = \Sigma^{12}$ and $J = \Sigma^{12}+\tilde{\Sigma}^{12}$ for open and closed strings, respectively. The length scale $\rho$ drops out in the ratio $J/m^2$, whose maximal value is obtained for the lowest

[8] There are spurious solutions to (2.28) that are missed by the light cone gauge condition: Consider, for example, $p^\mu = 0$ and $\alpha^\mu_{\pm n} = \tilde{\alpha}^\mu_{\pm n} = (1,1,0,\ldots,0)$.

frequency $n = 1$. This shows that the slope $\alpha'$ of the leading Regge trajectory is

$$\alpha'_{closed} = \frac{1}{4\pi T}, \qquad \alpha'_{open} = \frac{1}{2\pi T}, \qquad \frac{J}{m^2} \le \alpha'. \tag{2.35}$$

According to the literature it can be shown that all classical solutions obey this inequality. In the quantum theory it will be corrected by a constant shift $\alpha_0$.

## 2.6 Poisson brackets and Virasoro algebra

As a first step towards quantization we now compute the Poisson brackets among the oscillators $\alpha$ and $\tilde{\alpha}$, which follow from the canonical brackets $\{\Pi^\mu(\tau,\sigma), X^\nu(\tau,\sigma')\}_{PB} = -\delta(\sigma-\sigma')\eta^{\mu\nu}$ with $\Pi^\mu = -T\dot{X}^\mu$, $\{X^\mu(\tau,\sigma), X^\nu(\tau,\sigma')\}_{PB} = 0$ and $\{\Pi^\mu(\tau,\sigma), \Pi^\nu(\tau,\sigma')\}_{PB} = 0$ by a Fourier analysis. $\delta(\sigma-\sigma')$ is understood to be $2\pi$–periodic. Inserting (2.21) and (2.22) we find

$$\begin{aligned} \frac{\delta(\sigma-\sigma')\eta^{\mu\nu}}{T} \quad &= \quad \{\frac{p^\mu}{2\pi T} + \sum_{m\neq 0} \frac{\alpha_m^\mu e^{-im(\tau+\sigma)} + \tilde{\alpha}_m^\mu e^{-im(\tau-\sigma)}}{\sqrt{4\pi T}}, \\ &\qquad x^\nu + \frac{p^\nu \tau}{2\pi T} + \sum_{n\neq 0} \frac{i}{n} \frac{\alpha_n^\nu e^{-in(\tau+\sigma')} + \tilde{\alpha}_n^\nu e^{-in(\tau-\sigma')}}{\sqrt{4\pi T}}\}_{PB}. \end{aligned} \tag{2.36}$$

Since the variables $p^\mu$, $x^\mu$, $\alpha_m^\mu$ and $\tilde{\alpha}_m^\mu$ parametrize the general solution to the equations of motions, general results of the canonical formalism tell us that we have to fulfill these relations at a fixed time, say $\tau = 0$. This fixes all brackets among the coefficients in the Fourier representation of $X^\mu(\tau,\sigma)$ and guarantees the canonical brackets for all times.

We first consider the closed string and pick out the brackets among the individual coefficients by evaluating the double integrals $\iint d\sigma\, d\sigma' e^{i(k\sigma + k'\sigma')}$. For $k = k' = 0$ we obtain

$$\{p^\mu, x^\nu\}_{PB} = \eta^{\mu\nu} \tag{2.37}$$

and $\{x^\mu, x^\nu\}_{PB} = \{p^\mu, p^\nu\}_{PB} = 0$. For $k = 0 \neq k'$ we obtain from the brackets $\{\dot{X}, \dot{X}\}_{PB}$ and $\{\dot{X}, X\}_{PB}$ at $\tau = 0$ that

$$\{p^\mu, \alpha_{k'}^\nu + \tilde{\alpha}_{-k'}^\nu\}_{PB} = 0 = \{p^\mu, \frac{1}{k'}\alpha_{k'}^\nu + \frac{1}{-k'}\tilde{\alpha}_{-k'}^\nu\}_{PB}, \tag{2.38}$$

hence $\{p^\mu, \alpha_n^\nu\}_{PB} = \{p^\mu, \tilde{\alpha}_n^\nu\}_{PB} = 0$. Similarly, for $k' = 0 \neq k$ the brackets $\{X, X\}_{PB}$ and $\{\dot{X}, X\}_{PB}$ imply that $x^\mu$ has vanishing brackets with all oscillators $\alpha$ and $\tilde{\alpha}$. Eventually, for $k$ and $k'$ non-zero we find that all brackets among $\alpha$ and $\tilde{\alpha}$ vanish and we conclude that

$$i\{\alpha_m^\mu, \alpha_n^\nu\}_{PB} = i\{\tilde{\alpha}_m^\mu, \tilde{\alpha}_n^\nu\}_{PB} = n\,\delta_{m+n}\,\eta^{\mu\nu}, \qquad i\{x^\mu, p^\nu\}_{PB} = -i\eta^{\mu\nu} \tag{2.39}$$

are the non-vanishing brackets, where $\delta_l$ is an abbreviation for $\delta_{l,0}$.

In case of open strings we must set $\tilde{\alpha}_n^\mu = \alpha_n^\mu$, which makes $X^\mu$ even and $2\pi$ periodic. It is convenient to integrate again over the interval $0 < \sigma, \sigma' < 2\pi$ in order that the exponentials provide a complete set of orthogonal vectors. But then we have to take into account a second contribution from the $\delta$ function, i.e. we must let $\delta(\sigma-\sigma') \to \delta(\sigma-\sigma')+\delta(\sigma+\sigma')$ in eq. (2.36). The double integral for the Fourier coefficients then gives $\frac{2\pi}{T}\eta^{\mu\nu}(\delta_{k-k'}+\delta_{k+k'})$ on the l.h.s. of that equation. We can now repeat the same calculation as above, with the only difference that the second $\delta$ function $\delta_{k+k'}$ now doubles the result for the bracket $\{x^\mu, p^\nu\}_{PB}$,

$$i\{x^\mu, p^\nu\}_{PB} = -2i\eta^{\mu\nu} \qquad \text{(open string).} \tag{2.40}$$

This can be understood easily because, with the ansatz (2.21), the total momentum $P_\mu$ is $p_\mu$ for closed strings and $p_\mu/2$ for open strings, so that $\{x_\mu, P_\nu\}_{PB} = -\eta_{\mu\nu}$ in both cases, as we should expect. $\delta_{k+k'}$ also allows for a non-vanishing bracket $\{\alpha_n, \tilde{\alpha}_{-n}\}_{PB}$, which is necessary because of the identification of $\tilde{\alpha}$ and $\alpha$. Otherwise the Poisson brackets are the same for open and closed strings.

Recall that the Virasoro generators $L_n := T\int_0^{2\pi} d\sigma^+ T_{++}e^{in\sigma^+}$ and $\tilde{L}_n := T\int_0^{2\pi} d\sigma^- T_{--}e^{in\sigma^-}$, which are the Fourier modes of the energy momentum tensor, are given by

$$L_n = -\frac{1}{2}\sum_{m=-\infty}^{\infty}\eta_{\mu\nu}\alpha_m^\mu\alpha_{n-m}^\nu, \qquad \tilde{L}_n = -\frac{1}{2}\sum_{m=-\infty}^{\infty}\eta_{\mu\nu}\tilde{\alpha}_m^\mu\tilde{\alpha}_{n-m}^\nu. \tag{2.41}$$

They satisfy an infinite-dimensional Lie algebra, the **Virasoro algebra**

$$i\{L_m, L_n\}_{PB} = (m-n)L_{m+n}, \qquad i\{\tilde{L}_m, \tilde{L}_n\}_{PB} = (m-n)\tilde{L}_{m+n}, \tag{2.42}$$

as is easily verified using that $i\{L_n, \alpha_l^\rho\}_{PB} = -l\alpha_{n+l}^\rho$.

A more direct way to understand this algebra is to observe that any *conformal* Killing vector field $\xi$ defines a conserved current $J_\xi^m = \sqrt{-g}\,\xi^l T_l{}^m$ if the energy–momentum tensor is traceless and conserved (which can be shown to be consequences of Weyl invariance and the equations of motion of the matter fields, respectively). For left-moving conformal reparametrizations with $\xi = \xi^+(\sigma^+)\partial_+$ the corresponding conserved quantity is $L_\xi := \int d\sigma\,\xi^+ T_+{}^0 = \int d\sigma^+\xi^+(T_+{}^+ + T_+{}^-)/2 = \int d\sigma^+\xi^+T_{++}$ (recall that $\eta^{+-} = 2$; $d\sigma$ can be replaced by $d\sigma^+$ because the integrand only depends on $\sigma^+ = \sigma+\tau$). If we choose the basis $\xi_n = e^{in\sigma^+}\partial_+$ for periodic infinitesimal reparametrizations of $\sigma^+$, we find the Lie brackets $[\xi_m, \xi_n] = [e^{im\sigma^+}\partial_+, e^{in\sigma^+}\partial_+] = i(n-m)e^{i(m+n)\sigma^+}\partial_+$. Since $L_\xi$ generates the Lie derivative $\{L_\xi, X\}_{PB} = -\xi^m\partial_m X$, and because of the Jacobi identity, the Poisson algebra of the charges $L_n = L_{\xi_n}$ must have the same structure constants. Note that the Virasoro constraints $L_n$ and $\tilde{L}_n$ are conserved quantities, i.e. it is sufficient to impose $T_{mn} = 0$ at some initial time. Obviously, the conformal algebra in two dimensions is the direct product of two identical, infinite dimensional Lie algebras.

# Chapter 3

# Quantization of bosonic strings

Before actually performing the quantization in section 2.1 let us discuss some general aspects of canonical quantization of gauge invariant systems. It is clear by now that the 'matter fields' $X^\mu$ are the dynamical fields of the bosonic string, and that, at least locally, the metric only consists of gauge degrees of freedom. In the Hamiltonian formalism this is indicated by the fact that the conjugate momenta $(\pi_g)^{mn} = \partial L/\partial \dot g_{mn}$ vanish identically. Therefore we cannot naively impose the Poisson brackets, at least not as a 'strong' identity: In the process of quantization it is certainly not consistent to impose a commutator $[g_{mn}, (\pi_g)^{kl}] = i(\delta_m^k\delta_n^l + \delta_n^k\delta_m^l)/2$ if $(\pi_g)^{kl} \equiv 0$.

Dirac and Bergmann [DI64] developed a method for obtaining a Hamiltonian description if the Legendre transformation is singular (as it happens in our case): The defining equations for the momenta cannot be solved for the time derivatives of the coordinates *iff* there are relations among the coordinates and momenta, the *primary constraints* $\Phi_{\hat I}(p,q) = 0$. If this happens, then the Hamiltonian is only defined up to terms proportional to the constraints. These must be fulfilled at all times, so we must have $\{\Phi_{\hat I}, H\}_{PB} = 0$ (here we ignore the fact that the coordinates and momenta are constrained and compute the naive PB). If there is no choice for the Hamiltonian that makes the l.h.s. of this equation proportional to the $\Phi_{\hat I}$, then we get additional constraints, which we call *secondary.* (In our case the secondary constraint $\{(\pi_g)^{kl}, H\}_{PB} = 0$ is equivalent to the vanishing of the energy momentum tensor.)

After all constraints $\Phi_I = 0$, primary and secondary, are known, we have to calculate their Poisson algebra: If the antisymmetric matrix $c_{IJ} = \{\Phi_I, \Phi_J\}_{PB}$ vanishes on the constraint surface $\Phi_I = 0$, i.e. if it is a linear combination of constraints $\{\Phi_I, \Phi_J\}_{PB} = f_{IJ}{}^K\Phi_K$, then the $\Phi_I$ are called *first class.* This type of constraints indicates gauge symmetries (the Virasoro algebra is an example). Indeed, in case of gauge symmetries the equations of motion cannot be of the form $\dot f = \{f, H\}_{PB}$ with a *unique* Hamiltionian, because the time evolution is not fixed by the Euler–Lagrange equations of motion.[1] For quantization it is necessary to get rid of the

[1] The full gauge freedom is recovered in the form of an arbitrary linear combination of the constraints $\sum \lambda^I\Phi_I$

first class constraints by imposing additional gauge fixing constraints (this leads to the 'reduced phase space'). One problem of the Dirac procedure is that the set of available gauge fixings is too restricted. In the case of QED, for example, we cannot impose a Lorentz covariant gauge, simply because the time derivative of $A_0$ is not available. This shortcoming has been cured by the BFV formalism [fr75, HE92]: The phase space is extended with dynamical Lagrange multipliers and ghosts, making available, among other benefits, covariant gauge fixings. We will, however, arrive at the same result with the following short cut: We fix the gauge before we perform the Legendre transformation, and thus never get first class constraints. In order to compensate the resulting propagation of unphysical degrees of freedom we introduce ghosts, and the cancellation of all unphysical contributions to physical quantities is controlled by BRST invariance (see below).

If $c_{IJ} = \{\Phi_I, \Phi_J\}_{PB}$ has maximal rank on the constraint surface $\Phi_I = 0$ then the $\Phi_I$ are called *second class.* This type of constraints are caused by having too many degrees of freedom: Usually the Lagrangian is quadratic in the time derivatives so that the equations of motion are second order, i.e. two functions have to be specified on a Cauchy surface (the fields and their time derivatives). The Hamiltonian equations of motion, on the other hand, are first order, but now we have twice as many 'off shell' degrees of freedom (for each coordinate we introduce a momentum). This counting is spoiled if the Lagrangian is only linear in the time derivatives. Accordingly, $p_i = \partial L/\partial\dot{q}^i$ is a function of the coordinates only, and, instead of defining $\dot{q}^i$ in terms of phase space variables, this equation is a second class constraint (as $c_{IJ}$ is antisymmetric and invertible such constraints must occur in pairs).

So what happens is that we introduce too many phase space variables and that the redundant momenta can be eliminated by the constraint equations. Accordingly, the PBs have to be replaced by the Dirac brackets, which only take into account the true degrees of freedom: $\{f, g\}_{DB} = \{f, g\}_{PB} - \{f, \Phi_I\}_{PB}\, c^{IJ}\, \{\Phi_J, g\}_{PB}$ with $c_{IJ} = \{\Phi_I, \Phi_J\}_{PB}$ and $c_{IJ}c^{JK} = \delta_I^K$, so that $\{f, \Phi_I\}_{DB} = 0$ for all functions $f$ on phase space and for all constraints $\Phi_I$. But we will again use a short cut. Consider the inverse Legendre transformation and the resulting variational equations,

$$L = \dot{q}^i p_i - H(p, q), \qquad \frac{\delta L}{\delta q^i} = -\dot{p}_i - \frac{\partial H}{\partial q^i} = 0 \qquad \frac{\delta L}{\delta p_i} = (-)^i \dot{q}^i - \frac{\partial H}{\partial p_i} = 0. \tag{3.1}$$

If we do not eliminate the momenta from their variational equation we always find the situation with second class constraints (this is sometimes called *first order formalism*). For a Lagrangian of this form we can therefore directly read off the conjugate pairs of phase space variables and introduce the corresponding brackets, instead of introducing momenta for $q^i$ and for $p_i$ and eliminating them in a second step with the Dirac procedure.

that can be added to the Hamiltonian. This is know as Dirac's conjecture and has been proven under certain regularity assumptions. The coefficients $\lambda^I$ are called Lagrange multiplier (fields).

The final ingredient that we need for quantization is the BRST formalism [be76]. The introduction of 'ghost fields' that have the same quantum numbers – including spin – as the gauge degrees of freedom, but the opposite statistics, was suggested already in 1963 by Feynman in the context of quantum gravity [fe63]. This was motivated by the observation that the gauge degrees of freedom propagate after gauge fixing and that their contribution to loop diagrams is not transversal, like in QED, and therefore spoils unitarity and gauge independence. In 1967 Faddeev and Popov used ghosts to bring the field dependent functional determinant that arises from gauge fixing in non-abelian gauge theories into the exponent of the path integral [IT80].

Later it was observed by Becchi, Rouet and Stora [be76] that the resulting action has a global fermionic nilpotent symmetry $s^2 = 0$ with $s\phi^i = c^I\delta_I\phi^i$, i.e. for a $\delta_I$-invariant action the BRST transformation of a matter field $\phi^i$ is equal to its gauge transformation with the gauge parameters replaced by ghost fields. In order to have a well-defined ghost number that is consistent with the dynamics of the ghosts, we also need to introduce anti-ghost fields $\bar{c}^I$ with ghost number $-1$, whose BRST transform $b^I := s\bar{c}^I$ is usually called *lagrange multiplier field.* For a general gauge theory with an irreducible closed gauge algebra $[\delta_I, \delta_J] = \mathcal{F}_{IJ}{}^K\delta_K$ it can be shown that $s^2\phi^i = 0$ implies that the BRST transformation of the ghosts is

$$sc^K = \tfrac{(-)^I}{2}\, c^I c^J \mathcal{F}_{JI}{}^K. \tag{3.2}$$

Nilpotency of $s$, i.e. the equation $s^2c^I = 0$, is then equivalent to the Jacobi identity

$$\sum\!\!\!\!\!\!\bigcirc_{IJK} (-)^{IK} \left(\delta_I\mathcal{F}_{JK}{}^L + \mathcal{F}_{IJ}{}^M\mathcal{F}_{MK}{}^L\right) = 0, \tag{3.3}$$

which follows from $\sum\!\!\!\!\!\!\bigcirc_{IJK}(-)^{IK}[[\delta_I, \delta_J], \delta_K] = 0$. (We have an *open* gauge algebra if the graded commutator $[\delta_I, \delta_J]$ is proportional to $\delta_K$ only on shell, i.e. up to the equations of motion. In that case on needs the BV antibracket formalism [ba81]. Irreducibility of the gauge algebra means that the gauge transformations $\delta_I$ are linearly independent.) The BRST algebra thus encodes the structure of the symmetry algebra in a very efficient way.

The role of the BRST symmetry in canonical quantization was eventually clarified by Kugo and Ojima [ku79]: Initially it was assumed that $\bar{c}$ is the complex conjugate of $c$, but then gauge fixing does not give a real Hamiltonian and thus formally spoils unitarity. Rather, $c$ and $i\bar{c}$ are independent real fields.[2] In the quantum theory the conserved charge $Q_{BRST}$ that corresponds to the BRST symmetry commutes with the Hamiltonian (up to anomalies), so it can be used to define a 'physical' subspace of the Fock space with the physical states defined by the condition $Q|phys\rangle = 0$, which is consistent with time evolution.

If there is no anomaly in the commutation relation $\{Q, Q\} = 2Q^2 = 0$ then all states fall into doublet and singlet representations $(|\psi\rangle, |Q\psi\rangle)$ and $|\psi_{phys}\rangle$ of $Q$. For the doublets the dual

[2] With this assignment gauge transforms $s\phi^i$ of real fields, lagrange multiplier fields $s\bar{c}$, and gauge fixing terms $s\psi$ with imaginary anticommuting $\psi$ are real. Note that $(XY)^* = (-)^{XY}X^*Y^*$ and $\mathcal{O}^*\phi = (-)^{\mathcal{O}\phi}(\mathcal{O}\phi^*)^*$.

states must also form a doublet since the BRST charge $Q$, which generates a real symmetry transformation, should be hermitian. Furthermore, BRST-trivial states $Q|\psi\rangle$ have vanishing scalar product with all physical states, which therefore correspond to cohomology classes of $Q$-invariant states modulo $Q$-exact states. This is the 'quartet mechanism' by which doublet states cannot contribute to negative norm states in the physical Hilbert space. What remains to be checked for a given theory is that the 'physical Hilbert space' that we end up in this way does not contain any negative norm states.

Expectation values of physical operators, i.e. observables, should not depend on the representative we choose for a physical state. This is guaranteed if $\mathcal{O}$ (anti)commutes with $Q$, i.e. $[Q, \mathcal{O}_{phys}] = 0$. In turn, physical expectation values of $Q$-exact operators $\mathcal{O} = [Q, \mathcal{O}']$ vanish, so that physical obervables also correspond to cohomology classes. In particular, the sum of gauge dependent and ghost dependent terms of an $s$-invariant classical action with vanishing ghost number can be shown to be $s$-exact: $\mathcal{L}(\phi, c, \bar{c}, b) = \mathcal{L}_{inv.}(\phi) + s\Psi(\phi, c, \bar{c}, b)$ ($\psi$ is often called *gauge fermion*). This suggests that physical quantities should be independent of the choice of the gauge fixing term $s\psi$, which is known as the Fradkin–Vilkovisky theorem [HE92].

## 3.1 BRST quantization

Now we are ready to apply the above machinery to the case of the bosonic string. The Polyakov action is invariant under the nilpotent transformation

$$sX^\mu = c^l\partial_l X^\mu, \qquad sg_{mn} = D_m c_n + D_n c_m - 2g_{mn}\lambda, \qquad sc^m = c^l\partial_l c^m, \qquad s\lambda = c^l\partial_l\lambda, \tag{3.4}$$

where $c^m$ are the diffeomorphism ghosts and $\lambda$ is the Weyl ghost. We want to fix the metric to a background value $\hat{g}_{mn}$, which we initially keep arbitrary. We will see that the equations of motion for the antighost field $b_{mn}$ imply that $b_{++}$ is a function of $\sigma^+$, so that this field naturally has lower indices. It is thus convenient to fix the inverse metric $g^{mn}(\sigma) = \hat{g}^{mn}(\sigma)$ and we add the gauge fixing and ghost term $\int d^2\sigma\mathcal{L}^{(c)}$ with

$$\tfrac{2}{T}\mathcal{L}^{(c)} = s(\sqrt{-g}\, b_{mn}(g^{mn} - \hat{g}^{mn})) = \tilde{L}_{mn}(g^{mn} - \hat{g}^{mn}) + 2\sqrt{-g}\, b_{mn}(g^{ml}D_l c^n - g^{mn}\lambda), \tag{3.5}$$

and $\tilde{L}_{mn} = \sqrt{-g}\, L_{mn} = s(\sqrt{-g}\, b_{mn})$ to the Polyakov action. Note that the quantum numbers of the anti-ghost field come from the gauge fixing term, whereas those of the ghosts are inherited from the gauge transformation (only the numbers of degrees of freedom must coincide for ghosts and anti-ghosts, but quantum numbers like the spin can be different). The factor $\sqrt{-g}$ is inserted to make $b_{mn}$ a symmetric tensor rather than a tensor density.

Variation with respect to $g^{mn}$, $L_{mn}$, $\lambda$ and $b_{mn}$ implies the equations of motion

$$L_{mn} + T^{(X)}_{mn} + T^{(c)}_{mn} = 0, \qquad g^{mn} = \hat{g}^{mn}, \qquad b_{mn}g^{mn} = 0, \qquad 2\lambda = D_n c^n, \tag{3.6}$$

which are algebraic for the fields $L_{mn}$, $g^{mn}$, $\lambda$ and for the trace of the anti-ghost.

$$
\begin{aligned}
T^{(c)}_{mn} &= (b_{mj}D_nc^j + b_{nj}D_mc^j + D_j(b_{mn}c^j) - g_{mn}g^{ij}b_{jk}D_ic^k) + (g_{mn}g^{ij}b_{ij} - 2b_{mn})\lambda && (3.7)\\
&= b_{mj}D_nc^j + b_{nj}D_mc^j + D_jb_{mn}c^j - g_{mn}g^{ij}b_{jk}D_ic^k && (3.8)
\end{aligned}
$$

is the ghost contribution to the energy–momentum tensor[3] and $T^{(X)}_{mn}$ is the 'matter' contribution coming from the Polyakov action. $g^{mn}b_{mn} = 0$ implies that $T^{(c)}_{mn}$ is traceless. Furthermore, the total energy–momentum $T_{mn} = T^{(X)}_{mn} + T^{(c)}_{mn}$, corresponding to the action

$$
\mathcal{L} = \mathcal{L}_P + T\sqrt{-g}\, b_{mn}(g^{ml}D_lc^n - \tfrac{1}{2}g^{mn}D_lc^l), \tag{3.9}
$$

is proportional to the BRST variation of the traceless anti-ghost $b_{mn}$. Note that we can eliminate a set of fields whose *own* equations of motion are algebraic by inserting their values back into the action.

In light-cone coordinates we find $T^{(c)}_{+-} = 0$ and $T^{(c)}_{++} = 2b_{++}D_+c^+ + D_+b_{++}c^+$, where we used the equation of motion $\delta S/\delta c^n = g^{ml}D_lb_{mn} = 0$, implying $D_-b_{++}c^- = 0$. The Christoffel symbols drop out of this expression so that

$$
T^{(c)}_{++} = 2b_{++}\partial_+c^+ + \partial_+b_{++}c^+, \qquad T^{(c)}_{--} = 2b_{--}\partial_-c^- + \partial_-b_{--}c^-, \tag{3.10}
$$

Since $\hat\Gamma_{++}{}^{+}$ and $\hat\Gamma_{--}{}^{-}$ are the only non-vanishing components of the Christoffel symbol and $\sqrt{-g}\, g^{+-} = 1$, the complete Lagrangian in light-cone coordinates is

$$
\mathcal{L} = T\partial_+X^\mu\partial_-X^\nu G_{\mu\nu} + T(b_{++}\partial_-c^+ + b_{--}\partial_+c^-) \tag{3.11}
$$

and the equations of motion imply that $b_{++}$ and $c^+$ only depend on $\sigma^+$.

From (3.9) it follows that the imaginary field $-Tb_{++}$ is the conjugate momentum to $c^+$,

$$
i\{b_{++}(\tau+\sigma), c^+(\tau+\sigma')\}_{PB} = \tfrac{i}{T}\delta(\sigma-\sigma') = i\{b_{--}(\tau-\sigma), c^-(\tau-\sigma')\}_{PB}. \tag{3.12}
$$

For the Fourier modes $b_n = (b_{-n})^\dagger = -iT\int d\sigma\, b_{++}e^{in\sigma^+}$ and $c_n = (c_{-n})^\dagger = \frac{1}{2\pi}\int d\sigma\, c^+e^{in\sigma^+}$, and their right-moving relatives with

$$
b_{--} = \frac{i}{2\pi T}\sum_{n=-\infty}^{\infty}\tilde b_n e^{-in\sigma^-}, \qquad c^- = \sum_{n=-\infty}^{\infty}\tilde c_n e^{-in\sigma^-}, \tag{3.13}
$$

this implies the Poisson brackets

$$
i\{b_n, c_m\}_{PB} = \delta_{m+n} = i\{\tilde b_n, \tilde c_m\}_{PB} \tag{3.14}
$$

[3] The most tedious part of the computation of $T^{(c)}_{mn} = \frac{1}{T}\frac{2}{\sqrt{-g}}\frac{\delta\mathcal{L}^{(c)}}{\delta g^{mn}}$ is the variation of the Christoffel symbol contained in the covariant derivative: $\delta(D_lc^n) = \delta\hat\Gamma_{lm}{}^nc^m = \frac{1}{2}c^m\delta(g^{nk}(\partial_lg_{mk} + \partial_mg_{lk} - \partial_kg_{ml}))$. Since both sides of this equation are covariant, all terms linear in Christoffel symbols or partial derivatives of the metric must cancel and we immediately obtain $\delta(D_lc^n) = \frac{1}{2}g^{nk}c^m(D_l\delta g_{mk} + D_m\delta g_{lk} - D_k\delta g_{ml})$. Covariant partial integration of the variation of the action then leads to (3.7).

which have to be replaced by anticommutators for the quantized oscillator modes.

In light-cone coordinates the BRST current $J_S^m = -sX^\mu \frac{\partial \mathcal{L}}{\partial\partial_m X^\mu} - sc^l \frac{\partial \mathcal{L}}{\partial D_m c^l} + c^m \mathcal{L}_P$ reads

$$\begin{aligned} J_S^- &= -2Tc^l\partial_l X^\mu\ \partial_+ X^\nu G_{\mu\nu} - 2Tc^n\partial_n c^+\ b_{++} + 2Tc^-\ \partial_+ X^\mu \partial_- X^\nu G_{\mu\nu} &(3.15)\\ &= -2T(c^+\partial_+ X^\mu \partial_+ X^\nu G_{\mu\nu} + b_{++}c^+\partial_+ c^+). &(3.16)\end{aligned}$$

This suggests to define left- and right-moving BRST charges $Q_\pm$ with $Q = -\int d\sigma\, J_S^0 = -\frac{1}{2}(\int d\sigma^+ J_S^- + \int d\sigma^- J_S^+) = Q_+ + Q_-$,

$$Q_+ = T\int d\sigma^+\ c^+(\partial_+ X^\mu \partial_+ X^\nu G_{\mu\nu} + b_{++}\partial_+ c^+) = T\int d\sigma^+\ c^+(T_{++}^{(X)} + \frac{1}{2}T_{++}^{(c)}) \qquad (3.17)$$

and its right-moving partner $Q_-$; note that $(c^+)^2 = 0$ and $c^+D_+c^+ = c^+\partial_+c^+$.

For a flat target space we can insert the solutions to the equations of motion. For the Fourier modes $L_n$ of $T_{++} = \frac{1}{2\pi T}\sum L_n e^{-in\sigma^+}$ we obtain

$$L_n^{(X)} = -\tfrac{1}{2}\sum_{m=-\infty}^{\infty} :\alpha_{n-m}\cdot\alpha_m:, \qquad L_n^{(c)} = \sum_{m=-\infty}^{\infty}(n+m) : b_{n-m}c_m : \qquad (3.18)$$

and for the BRST charge

$$\begin{aligned} Q_+ &= \sum_{n=-\infty}^{\infty} :(L_n^{(X)} + \frac{1}{2}L_n^{(c)})c_{-n}: - ac_0 &(3.19)\\ &= \sum_{n=-\infty}^{\infty} L_n^{(X)}\ c_{-n} - \frac{1}{2}\sum_{n,m=-\infty}^{\infty}(m-n) : c_{-m}c_{-n}b_{m+n} : - ac_0, &(3.20)\end{aligned}$$

where we introduced normal ordering symbol :: that puts creation operators (negative index) to the left and a coefficient $a$ parametrizing the ordering ambiguity in $Q_+$. We find

$$[L_n, b_l] = (n-l)b_{n+l}, \qquad [L_n, c_l] = -(2n+l)c_{n+l} \qquad (3.21)$$

with $\{Q_+, b_n\} = L_n := L_n^{(X)} + L_n^{(c)} - a\delta_n$, and $\{Q_+, c_l\} = \sum_n\ (n + l/2)c_{l+n}c_{-n}$.

Next we turn to the construction of a Fock space representation of our operator algebra. Recall the commutation relations

$$[\alpha_m^\mu, \alpha_n^\nu] = -m\delta_{m+n}\eta^{\mu\nu}, \qquad [x^\nu, P^\mu] = -i\eta^{\mu\nu}, \qquad \{b_m, c_n\} = \delta_{m+n}. \qquad (3.22)$$

with $p^\mu = \sqrt{4\pi T}\alpha_0^\mu$. We define a vacuum state that is annihilated by all oscillators with positive mode number, i.e. $\alpha_n^\mu|0\rangle = b_n^\mu|0\rangle = c_n^\mu|0\rangle = 0$ for $n > 0$. The difficult part is the treatment of the zero modes. All states can be constructed as sums of tensor products of a coordinate factor and a ghost factor. For the bosonic part we can, for example, diagonalize the momentum and define eigenstates $P_\mu|k\rangle = k_\mu|k\rangle$, so that $|k\rangle = \ :e^{ikX}: |0\rangle$. In the ghost sector the zero mode algebra is $b_0^2 = c_0^2 = 0$ and $\{b_0, c_0\} = 1$. We cannot diagonalize a nilpotent operator, so we need to introduce a 2-dimensional representation space with

$$\begin{array}{lll} b_0|\uparrow\rangle = |\downarrow\rangle & c_0|\uparrow\rangle = 0 & \langle\uparrow|\downarrow\rangle = \langle\downarrow|\uparrow\rangle = 1 \\ b_0|\downarrow\rangle = 0 & c_0|\downarrow\rangle = |\uparrow\rangle & \langle\uparrow|\uparrow\rangle = \langle\downarrow|\downarrow\rangle = 0 \end{array} \tag{3.23}$$

and $\langle\downarrow|c_0 = \langle\uparrow|$, $\langle\uparrow|b_0 = \langle\downarrow|$, $\langle\uparrow|b_0|\uparrow\rangle = 1 = \langle\downarrow|c_0|\downarrow\rangle$.

## 3.2 Conformal anomaly and critical dimension

For the consistency of the BRST quantization program we have to check that $Q^2 = 0$. This will fix the constant $a$ in eq. (3.20) and also gives us the critical dimension. First we observe that $Q_+^2 = 0$ implies that the Virasoro algebra has no anomalous contribution (the anticommutator of $Q_+$ and $Q_-$ vanishes trivially, so we only need to consider left movers). Indeed, $0 = [\{Q_+, Q_+\}, b_n] = [Q_+, \{Q_+, b_n\}] - [\{Q_+, b_n\}, Q_+] = 2[Q_+, L_n]$, hence

$$[L_m, L_n] = [L_m, \{Q_+, b_n\}] = \{Q_+, [L_m, b_n]\} = (m-n)\{Q_+, b_{m+n}\} = (m-n)L_{m+n}. \tag{3.24}$$

The converse is also true since one can show that

$$Q_+^2 = \frac{1}{2}\sum c_{-m}c_{-n}([L_m, L_n] - (m-n)L_{m+n}). \tag{3.25}$$

This calculation, however, is very tedious, so we postpone it till we have more efficient tools for computing commutators of normal ordered expressions when we come to operator products and contour integrals in the complex plane.

In any case, absence of anomalies in the Virasoro algebra is necessary for $Q^2 = 0$. Since $L_0$ is the only mode for which there is an ordering ambiguity it is easy to see that

$$[L_m, L_n] = (m-n)L_{m+n} + A_m\delta_{m+n}. \tag{3.26}$$

Obviously, $A_{-m} = -A_m$ and $A_0 = 0$. From the Jacobi identity $\sum_{lmn}[L_l, [L_m, L_n]] = 0$ if follows for $l + m + n = 0$ that

$$(m-n)A_l + (n-l)A_m + (l-m)A_n = 0. \tag{3.27}$$

For $l = 1$ we get $(n-1)A_{n+1} = (n+2)A_n - (2n+1)A_1$ which determines all $A_m$ in terms of $A_1$ and $A_2$. Since $A_m = m$ and $A_m = m^3$ solve this equation, we find

$$A_m = \tfrac{A_2 - 2A_1}{6}\, m^3 - \tfrac{A_2 - 8A_1}{6}\, m. \tag{3.28}$$

The final step in the calculation of the anomaly is to fix the two remaining constants by evaluating expectation values $A_m = \langle\uparrow|\, L_mL_{-m} - 2mL_0\, |\downarrow\rangle$ of (3.26) where $m > 0$.

$$A_1 \;=\; \langle\uparrow|\,(\alpha_0\cdot\alpha_1)(\alpha_0\cdot\alpha_{-1}) - 2(-\tfrac{\alpha_0^2}{2} - a) + (b_1c_0 + 2b_0c_1)(-b_{-1}c_0 - 2b_0c_{-1})\,|\downarrow\rangle$$

$$= \langle\uparrow|\, 2a-(2b_0c_1)(b_{-1}c_0)\,|\downarrow\rangle = 2a-2, \tag{3.29}$$

$$A_2 = \langle\uparrow|\,(\tfrac{\alpha_1^2}{2}+\alpha_0\cdot\alpha_2)(\tfrac{\alpha_{-1}^2}{2}+\alpha_0\cdot\alpha_{-2})+4(\tfrac{\alpha_0^2}{2}+a)\,|\downarrow\rangle$$
$$-\langle\uparrow|\,(2b_2c_0+3b_1c_1+4b_0c_2)(2b_{-2}c_0+3b_{-1}c_{-1}+4b_0c_{-2})\,|\downarrow\rangle$$
$$= \langle\uparrow|\,\tfrac14\alpha_1^2\alpha_{-1}^2+4a-(3\cdot3+4\cdot2)\,|\downarrow\rangle = D/2+4a-17. \tag{3.30}$$

Putting the pieces together we find

$$A_m = \frac{m^3}{12}(D-26)-\frac{m}{12}(D-2-24a), \tag{3.31}$$

so that $Q^2=0$ implies $a=1$ and $D=26$. Note that the term linear in $m$ depends on $a$. This can be used to eliminate $A_1$ even if $Q^2\neq0$, i.e. to bring the anomaly into the form

$$A_m = \frac{m^3-m}{12}c, \tag{3.32}$$

where $c$ is called central charge (in our case $a=1$ is the appropriate value for any $D$). Then the $SL(2)$ subalgebra of the Virasoro algebra that is generated by $L_0$ and $L_{\pm1}$ is free of anomalies[4] (the non-vanishing commutators are $[L_{\pm1},L_0]=\pm L_{\pm1}$ and $[L_1,L_{-1}]=2L_0$).

In addition to the BRST charge there is another (classically) conserved quantity: We assign ghost number $\pm1$ to ghosts $c^m,\lambda$ and antighosts $b_{mn}$, respectively, and observe that our BRST-invariant classical action has ghost number 0. It is thus invariant under the infinitesimal transformation $\delta c^n=c^n$, $\delta\lambda=\lambda$ and $\delta b_{mn}=-b_{mn}$. This leads to the conserved Noether current $J_{gh}^m=-\delta c^n\frac{\partial\mathcal{L}}{\partial\partial_m c^n}=T\sqrt{-g}\,g^{ml}b_{ln}c^n$, which again simplifies nicely in the light cone gauge: $J_{gh}^{+}=2Tb_{--}c^-$. There is thus a left-moving and a right-moving contribution to the ghost number, $\mathcal{N}=-i\int d\sigma J_{gh}^0=-\frac{i}{2}\int d\sigma(J_{gh}^{+}+J_{gh}^{-})=\mathcal{N}_++\mathcal{N}_-$,

$$\mathcal{N}_+=\int\tfrac{d\sigma^+}{2i}J_{gh}^- = \sum_{n=-\infty}^{\infty}:c_nb_{-n}: \;+\text{const.} = \tfrac12(c_0b_0-b_0c_0)+\sum_{n>0}(c_{-n}b_n-b_{-n}c_n)+\tfrac32, \tag{3.33}$$

where we include a factor of $i$ in the definition of the charge to make the eigenvalues real. The reason for our asymmetric choice of the constant coming from the operator ordering ambiguity will become clear below. It leads to $\mathcal{N}_+|\uparrow\rangle=2|\uparrow\rangle$ and $\mathcal{N}_+|\downarrow\rangle=|\downarrow\rangle$. We will see later that $J_{gh}$ is not conserved in the quantum theory. An anomaly of a global symmetry, however, does not spoil the consistency of a theory; the anomalous violation of ghost number conservation is, in fact, related to the topology of the world sheet and will play an important role in interactions.

[4] This is a special case of the following result: An anomalous term like the one in eq. (3.26) is called a central extension if it is consistent with the Jacobi identity. It is easy to show that semi-simple (finite-dimensional) Lie algebras only admit trivial central extensions, i.e. the 'central' terms in the algebra can be eliminated by adding constants to the generators.

## 3.3 Physical states

The physical subspace of our Fock space is defined by the cohomology of $Q$. We first consider states of the form $P(\alpha)|k\rangle \otimes |\uparrow\rangle$ or $P(\alpha)|k\rangle \otimes |\downarrow\rangle$, where $P(\alpha)$ is a polynomial in the physical creation operators $\alpha^\mu_{-m}$. For such states

$$Q_+\left(P(\alpha)|k\rangle \otimes |\downarrow\rangle\right) = \sum_{n\geq 0}\left(L_n^{(X)} - \delta_{n,0}\right) P(\alpha)|k\rangle \otimes c_{-n}|\downarrow\rangle, \tag{3.34}$$

$$Q_+\left(P(\alpha)|k\rangle \otimes |\uparrow\rangle\right) = \sum_{n>0} L_n^{(X)} P(\alpha)|k\rangle \otimes c_{-n}|\uparrow\rangle. \tag{3.35}$$

This looks similar to Gupta–Bleuler in QED, where the annihilation part of the gauge condition is imposed as a constraint on physical states. In the present context we need to make sure that physical expectation values of $T_{++}$ vanish. Since the states built on $|\downarrow\rangle$ are dual to the states built on $|\uparrow\rangle$ the above formulas imply that all expectation values of $L_n$ between physical states that do not contain ghost or antighost creation operators vanish (it is, of course, true in general, since $L_n = \{Q, b_n\}$). Hence our formalism reduces to the 'old covariant approach' [GR87] in the ghost-free sector.

Since the mass shell operator $L_0 = \{Q, b_0\}$, the momentum operator $P^\mu$ and $Q$ all commute with one another, we can compute the cohomology for $Q$ for fixed eigenspaces with eigenvalues. If $(L_0 - \lambda)|\Phi\rangle = 0$ with $\lambda \neq 0$ for some $Q$-invariant state $|\Phi\rangle$ then $|\Phi\rangle = Q(\frac{1}{\lambda} b_0|\Phi\rangle)$ is $Q$-exact, so that non-trivial physical states must be on-shell states. In this way we recover the mass shell condition $L_0 = 0$ also for the states of the form (3.35) that are built on $|\uparrow\rangle$. Moreover, it can be shown that representatives of all physical states can be chosen to be of the form (3.34) or of the form (3.35) [th89]; there is a one-to-one correspondence of these states, which can be obtained from one another by application of $b_0$ or $c_0$. So we have a two-fold degeneracy, which follows from the existence of ghost zero modes and from the 'quartet mechanism' [ku79], i.e. the fact that dual states of BRST singlets and doublets form singlets and doublets, respectively.

This can be used to give a simple proof of the fact that the ghosts drop out of the cohomology, except for their zero modes: Since $\{Q, b_n^\dagger\} = L_n^\dagger$, standard arguments of homological algebra show that all $b_n^\dagger$ with $n > 0$ drop out of the cohomology if the $L_n^\dagger$ can be used as part of a basis for the algebra of creation operators, which can be shown to be true if $P^\mu \neq 0$. For $P^\mu = 0$ there are only $2D+4$ on-shell states, which are the singlets $b_{-1}|\downarrow\rangle$, $\alpha^\mu_{-1}|\downarrow\rangle$ and their dual states $c_{-1}|\uparrow\rangle$, $\alpha^\mu_{-1}|\uparrow\rangle$, and the self-dual doublet $b_{-1}|\uparrow\rangle$ and $Q(b_{-1}|\uparrow\rangle) = -2c_{-1}|\downarrow\rangle$. The '$SL(2,\mathbb{C})$ invariant vacuum' $|0\rangle = b_{-1}|\downarrow\rangle$ and its dual are the only Lorentz-invariant physical states. It will play an important role in conformal field theory.

Since the physical states that are built on $|\downarrow\rangle$ are automatically on-shell, whereas those on $|\uparrow\rangle$ are off-shell null states or on-shell limits of null states, the states $P(\alpha)|\downarrow\rangle$ seem to be somewhat preferable. So we may choose the 'Siegel gauge' $b_0|\Phi\rangle = 0$ in addition to the physical state condition $Q|\Phi\rangle = 0$, a relaxed form of which plays an important role in closed string field theory [zw93, be94, ne89, di$_b$91].

It has been shown a long time ago that in $D = 26$ dimensions all physical states can be generated from tachyonic vacua $|k\rangle$ with $\frac{1}{4\pi T}k^2 = -2$ by repeated application of the so-called DDF creation operators $(A^i_m)^\dagger = A^i_{-m}$ with $m > 0$ [de72, GR87], which are zero modes of 'transversal vertex operators for massless states' and satisfy

$$[A^i_m, A^j_n] = m\delta_{ij}\delta_{m+n}, \qquad A^i_m = \int \frac{d\tau}{2\pi}\varepsilon^i_\mu \dot{X}^\mu e^{imqX}, \tag{3.36}$$

where $q^2 = 0$, $qk = 1$, $\varepsilon^i k = \varepsilon^i q = 0$ and $\varepsilon^i\varepsilon^j = -\delta^{ij}$. This implies that all physical states have positive norm.

We now consider the tachyon and the massless states in more detail. Since $[L^{(X)}_m, L^{(X)}_1] = (m-1)L^{(X)}_{m+1}$ for $m > 0$ it is sufficient to impose

$$L^{(X)}_0 = -(\frac{1}{2}\alpha_0^2 + \alpha_{-1}\cdot\alpha_1 + \alpha_{-2}\cdot\alpha_2 + \ldots) = 1, \tag{3.37}$$

$$L^{(X)}_1 = -(\alpha_0\cdot\alpha_1 + \alpha_{-1}\cdot\alpha_2 + \alpha_{-2}\cdot\alpha_3 + \ldots) = 0, \tag{3.38}$$

$$L^{(X)}_2 = -(\frac{1}{2}\alpha_1^2 + \alpha_0\cdot\alpha_2 + \alpha_{-1}\cdot\alpha_3 + \alpha_{-2}\cdot\alpha_4 + \ldots) = 0 \tag{3.39}$$

on $P(\alpha)|k\rangle$. For $P(\alpha) = 1$ we obtain $\hat{k}^2 = k^2/(4\pi T) = -2$ with $\alpha^\mu_0|k\rangle = \hat{k}^\mu|k\rangle$, i.e. we find a scalar, tachyonic state in the string spectrum. On the next level $P(\alpha) = t_\mu\alpha^\mu_{-1}$ the mass shell condition is $k^2 = 0$ and $L^{(X)}_1 = 0$ implies transversality $t_\mu k^\mu = 0$ of the polarization vector $t_\mu$. The norm of this state is proportional to $t^2$, i.e. it vanishes for a longitudinal polarization $t_\mu \sim k_\mu$. We expect that such a state is $Q$-exact, and indeed,

$$Q(b_{-1}|\downarrow\rangle) = L^{(X)}_{-1}|\downarrow\rangle + L^{(c)}_{-1}|\downarrow\rangle - b_{-1}Q|\downarrow\rangle \tag{3.40}$$

$$= \hat{k}\cdot\alpha_{-1}|\downarrow\rangle - b_{-1}c_0|\downarrow\rangle - b_{-1}c_0(L_0-1)|\downarrow\rangle \tag{3.41}$$

$$= \hat{k}\cdot\alpha_{-1}|\downarrow\rangle + \tfrac{1}{2}\hat{k}^2 b_{-1}c_0|\downarrow\rangle. \tag{3.42}$$

In the case of open strings this is the whole story: We have a massless vector excitation in the target space, whose polarization must be transversal since $Q\alpha^\mu_{-1}|\downarrow\rangle \sim \hat{k}^\mu c_{-1}|\downarrow\rangle$ should vanish, and there is a null (i.e. zero norm) polarization, which is $Q$-exact. For closed strings we have to include the right-movers and therefore have a polarization tensor $t_{\mu\nu}$ which is transversal in both indices. Now the gauge invariance corresponds to $\delta t_{\mu\nu} = k_\mu v^{(R)}_\nu + k_\nu v^{(L)}_\mu$. The physical interpretation requires the decomposition of the polarization tensor into irreducible representations of the Lorentz group: $t_{\mu\nu}$ has a traceless symmetric part, the graviton, an antisymmetric tensor field $B_{\mu\nu} = t_{\mu\nu} - t_{\nu\mu}$, and a scalar degree of freedom due to the trace, which is called dilaton. The antisymmetric part of the gauge invariance implies that only the field strength $H_{\mu\nu\rho} = \sum \partial_\mu B_{\nu\rho}$ enters physical quantities.

Summarizing our observations, we recover the essential ingredients of QED and of linearized gravity. In addition, we have an infinite tower of gauge symmetries at higher levels which controll the interplay of the infinite set of massive string modes at and above the Planck mass, which, in our picture of string unification, is proportional to $\sqrt{T}$. In order to obtain interaction terms for gravitons and the other target-space fields we need to compute string interactions.

So it seems the 'old covariant approach' to string quantization is sufficient. Ghosts will become important in case of interactions. Before coming to this subject, however, we first continue to the Euclidean domain and set up the machinery of CFT. Since the $SL(2)$ subalgebra of the Virasoro algebra has no anomaly we could, in fact, require $L_n = 0$ for $n \geq -1$. Accordingly, it is useful to work with the $SL(2)$ invariant ghost vacuum $|0\rangle_{gh}$ which is defined by

$$b_n|0\rangle_{gh} = 0 \quad n \geq -1, \qquad\qquad c_n|0\rangle_{gh} = 0 \quad n \geq 2. \tag{3.43}$$

It is related to our previous vacuum by $|\downarrow\rangle = c_1|0\rangle_{gh}$ and $|0\rangle_{gh} = b_{-1}|\downarrow\rangle$. The importance of this vacuum will become clear in the context of CFT on the complex plain.

That $D \leq 26$ is a necessary condition for consistent string quantization can be seen easily by computing the norm of a physical scalar state at the second mass level: We make the ansatz

$$|\phi\rangle = (\alpha_{-1}\cdot\alpha_{-1} + A(\alpha_0\cdot\alpha_{-1})^2 + B\alpha_0\cdot\alpha_{-2})|p\rangle \tag{3.44}$$

Since $L_{n+1} = \frac{1}{n-1}[L_n, L_1]$ it is sufficient to impose $L_0 = L_1 = L_2 = 0$ with $L_0 = L_0^{(X)} - 1$. Straightforward evaluation of the commutators gives $L_0|\phi\rangle = (-\frac{1}{2}\alpha_0^2 + 2 - 1)\alpha_{-1}^2|\phi\rangle$, $L_1|\phi\rangle = 2(1 + A\alpha_0^2 + B)\alpha_0\cdot\alpha_{-1})|p\rangle$ and $L_2|\phi\rangle = (-D - (A - 2B)\alpha_0^2)|p\rangle$, so that we find

$$\frac{\langle\phi|\phi\rangle}{\langle p|p\rangle} = 2D + 4\alpha_0^2 A + 2\alpha_0^4 A^2 - 2\alpha_0^2 B^2 = \tfrac{2}{25}(D-1)(26-D), \qquad B = \tfrac{D-1}{5},\ A = -\tfrac{D+4}{10} \tag{3.45}$$

More generally it can be shown that there are no negative norm states if $a = 1$ and $D = 26$ or if $a \leq 1$ and $D \leq 25$ (see [GR87]). A covariant quantization of string theory below the critical dimension has first been persued successfully by Polyakov [po81]. He found that the conformal anomaly makes the conformal mode $\phi$ of the metric $g_{mn} = e^\phi \eta_{mn}$ dynamical, with an effective Lagrangian (Wess-Zumino term)

$$\tfrac{26-D}{48\pi}\ \left(\tfrac{1}{2}(\partial\phi)^2 + \mu^2 e^\phi\right) \tag{3.46}$$

which is positive if $D < 26$. This action has been known for a long time under the name Liouville action and $\phi$ is thus called Liouville field.

## 3.4 Strings in background fields

To recover the full content of gravity it seems that we have to study graviton scattering order by order in perturbation theory, which would require the calculation of correlations functions with an arbitrary number of graviton vertex operator insertions. There is, however, an alternative approach that directly gives us the Einstein equations in curved space [fr85, ca$_1$85]. Recall that we can consider the string with an arbitrary target space matric $G_{\mu\nu}(X)$. We computed the equations of motion of the coordinate fields in such a background. It is not so obvious, however, how the dynamics of the background metric arises. As it turns out, this dynamics is fixed by the absence of conformal anomalies.

Here we should be more general: In principle, all massless fields in our theory can form condensates. Hence there should be a consistent movement of strings in curved target spaces with additional backgrounds that correspond to the dilaton and antisymmetric tensor fields. Indeed, if we write down the most general renormalizable action for the coordinate fields we find $S = S_P + S_B + S_\phi + S_\tau$ with

$$\mathcal{L} = \mathcal{L}_P - \tfrac{T}{2}\varepsilon^{mn}\partial_m X^\mu \partial_n X^\nu B_{\mu\nu}(X) + \tfrac{1}{4\pi}\sqrt{-g}\,\phi(X)R^{(2)} + \sqrt{-g}\,\tau(X), \tag{3.47}$$

where $R^{(2)}$ is the curvature scalar on the world sheet. Equations of motion for the coordinate fields read

$$\frac{G^{\alpha\rho}}{\sqrt{-g}\,T}\frac{\delta S}{\delta X^\rho} = \Delta X^\alpha + \partial_m X^\mu \partial_n X^\nu \Big(g^{mn}\hat\Gamma_{\mu\nu}{}^\alpha - \tfrac12 \tfrac{\varepsilon^{mn}}{\sqrt{-g}} H_{\mu\nu}{}^\alpha\Big) + O(1/T) \tag{3.48}$$

with the totally antisymmetric 'torsion' $H^{\mu\nu\lambda} = \sum_{\mu\nu\lambda}\partial_\mu B_{\nu\lambda}$, i.e.$H = dB$ with $B = \frac12 dx^\mu dx^\nu B_{\mu\nu}$ and $H = \frac{1}{3!}dx^\mu dx^\nu dx^\lambda H_{\mu\nu\lambda}$. The contributions of the last 2 terms are suppressed by powers of $1/T$. $S_\phi$ is conformally invariant only if $\phi$ is constant and should be considered as contributing only at the quantum level. In 2 dimensions $\sqrt{-g}\,R$ is a total derivative whose integral is proportional to the Euler characteristic of the manifold. Therefore the vacuum expectation value of a constant dilaton field $\phi$ determines the strength of the string coupling. ($S_\tau$ is needed as a counterterm to cancel divergences and plays no further role.)

It can be shown that the resulting quantum theory is conformally invariant to leading order in $1/T$ iff the following '$\beta$-functionals' for the coupling functions $G$, $B$ and $\phi$ vanish [fr85,ca$_1$85],

$$0 = R_{\mu\nu} - \tfrac14 H_{\mu\lambda}{}^\rho H_{\nu\rho}{}^\lambda - 2D_\mu D_\nu\phi \tag{3.49}$$

$$0 = D_\lambda H_{\mu\nu}{}^\lambda - 2H_{\mu\nu}{}^\lambda D_\lambda\phi \tag{3.50}$$

$$0 = 4D_\mu\phi D^\mu\phi - 4D_\mu D^\mu\phi + R + \tfrac{1}{12}H_{\mu\nu\lambda}H^{\mu\nu\lambda} \tag{3.51}$$

which can be interpreted as equations of motion for the metric, the (field strength of) the antisymmetric tensor field, and the dilaton, respectively. Actually these equations are the Euler Lagrange equations for the (effective) action

$$S_{26} = \int d^{26}X\sqrt{-G}e^{-2\phi}(R - 4D_\mu\phi D^\mu\phi + \frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho}). \tag{3.52}$$

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