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Geometrische Methoden der Theoretischen Physik

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Thinking about what may be the 'most general theory' that one can write down we always have to start from some reasonable set of assumptions. In flat space we usually describe the dynamics of local fields (or first-quantized point particles) by a Lorentz invariant and renormalizable local action. In curved space Lorentz invariance will refer to tangent space indices and is supplemented by the requirement of general coordinate invariance; in 4 dimensions renormalizability can only be imposed on the matter part of the action since, with the Einstein-Hilbert action $S = \frac{1}{16\pi G_N}\int\sqrt{-g}\,\mathcal{R}$, the gravitational coupling constant $\kappa = \sqrt{16\pi G_N}$ has negative mass dimension.[1]

It is a well-known fact in group theory that, in addition to vector and tensor representations, the Lorentz algebra has an irreducible spinor representation s and its conjugate $c = s^c$; all representations are contained in (Kronecker) products of tensors of $SO(1, D-1)$ – i.e. tensor products of the vector representation – with possibly one additional factor of s or c. Therefore we construct a field theory out of tensor fields $t_{a_1\ldots a_n}{}^{b_1\ldots b_m}$ and spinor fields $\psi^{\underline{\alpha}}{}_{a_1\ldots a_n}{}^{b_1\ldots b_m}$, where $\underline{\alpha}$ can be α for the spinor representation or $\dot\alpha$ for its conjugate.[2] General tensors of this form correspond to reducible representations. One can project to irreducible representations either by setting contractions with invariant tensors to 0 or by making them redundant via gauge symmetries. In 4 dimensions renormalizable Lorentz-invariant interactions exist only for scalar, spinor, and vector fields; consistent couplings to gravitons g_{mn} and gravitinos $\gamma_m{}^{\underline{\alpha}}$ require local coordinate invariance and local supersymmetry, respectively.

The reason for the split into tensor and spinor representations is that $SO(1, D-1)$, and more generally $SO(p, q)$, is not simply connected but has a double covering group, which is called $Spin(p, q)$. The spinor representations are the double valued 'representations' of $SO(p, q)$. We will discuss the properties of spinors of the Lorentz group using representations of the Clifford algebra, which will provide us with a realization of the double covering group $Spin(p, q)$. Then we turn to some elementary aspects of supersymmetry and supergravity. Since $D = 2, 4, 10$ dimensions are all important in string theory, we discuss the situation as far as possible for an arbitrary (even) number of dimensions. Eventually we explicitly construct the so-called (1,1) supergravity in 2 dimensions and its (0,1) restriction, which is relevant for heterotic strings.

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[1] In perturbation theory we use $g_{mn} = \eta_{mn} + \kappa h_{mn}$ with $\kappa = \sqrt{16\pi G_N} = \sqrt{2}M_{Pl}^{(1-D/2)}$ and expand in powers of h, so that we obtain a κ-independent quadratic term (with second derivatives) and corrections proportional to $h^2(\kappa h)^n$; in $D = 4$ dimensions the mass dimension of G_N is $2 - D = -2$. More generally, physical bosonic fields ϕ have 2 derivatives in their kinetic terms, while fermions have a Dirac operator $\not{D} = \gamma^m D_m$, so that the mass dimensions of these fields are $\dim(\phi) = (D-2)/2$ and $\dim(\psi) = (D-1)/2$. Higher derivative kinetic terms for physical fields usually spoil unitarity (or positivity of the energy): Since $((\Box + m_1^2)(\Box + m_2^2))^{-1} = (m_2^2 - m_1^2)^{-1}((\Box + m_1^2)^{-1} - (\Box + m_2^2)^{-1})$, we always have some kinetic term of the wrong sign.

[2] Since the vector representation is contained in $s \otimes c$ we also could use fields with only spinor indices, but in more than 4 dimensions this is not very economic since the dimensions of the spinor representations become too large.

Chapter 1

Spinors

1.1 Clifford algebras, representations and spin

With any vector space with a non-degenerate scalar product there comes a Clifford algebra that is constructed in the following way: We consider a basis in which the metric has the form $\eta_{ab} = \mathrm{diag}(+,\ldots,+,-,\ldots,-)$ with p positive and $q = D - p$ negative entries and objects γ^a that satisfy the relations

$$\{\gamma^a, \gamma^b\} = 2\eta^{ab}\mathbf{1}, \qquad \Gamma^{a_1\ldots a_i} := \gamma^{a_1}\ldots\gamma^{a_i} \tag{1.1}$$

where $\mathbf{1}$ is the identity operator. Then the products $\pm\Gamma^{a_1\ldots a_i}$ with $a_1 < a_2 < \ldots < a_i$ (together with $\pm\mathbf{1}$) form a finite group with $2\sum_{r=0}^{D}\binom{D}{r} = 2^{D+1}$ elements whose formal linear combinations with coefficients in some field $\mathbb{K}$ form an algebra of vector space dimensions 2^D (we will only be interested in $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$; we will also write $\mathcal{C}(p,q)$ instead of $\mathcal{C}_{\mathbb{R}}(p,q)$). This algebra is called Clifford algebra $\mathcal{C}_{\mathbb{K}}(p,q)$: Its generators anticommute, p of them square to $\mathbf{1}$ and the remaining q generators square to $-\mathbf{1}$. Special cases are the Dirac algebra $\mathcal{C}(1,3)$, the Pauli algebra $\mathcal{C}(2,0)$, the complex numbers $\mathcal{C}_{\mathbb{R}}(0,1)$, and the quaternions $\mathbb{H} = \mathcal{C}_{\mathbb{R}}(0,2)$, which are the smallest non-commutative field containing the complex numbers.[1]

$SO(p,q)$ is the group of linear transformations with det $= 1$ that leave a (pseudo) metric tensor η_{ab} invariant. The defining (vector) representation is given by matrices $\Omega_a{}^b$ satisfying $\Omega_a{}^c\Omega_b{}^d\eta_{cd} = \eta_{ab}$. For infinitesimal transformations $\Omega_a{}^b = \delta_a^b + \omega_a{}^b + O(\omega^2)$ this implies that $\omega_{ab} := \omega_a{}^c\eta_{cb}$ is antisymmetric. Using the basis $(l^{ab})_{cd} = \delta_d^a\delta_c^b - \delta_c^a\delta_d^b$ of antisymmetric matrices we can write $\omega_{cd} = \frac{1}{2}\omega_{ab}(l^{ab})_{cd}$ and obtain the structure constants of the Lie algebra $so(p,q)$ as

$$[l_{ab}, l_{cd}] = -\eta_{ac}l_{bd} + \eta_{bc}l_{ad} + \eta_{ad}l_{bc} - \eta_{bd}l_{ac}, \qquad (l_{ab}v)_c = (l_{ab})_c{}^d v_d = -\eta_{ac}v_b + \eta_{bc}v_a \tag{1.2}$$

[1] The symbol $\mathbb{H}$ refers to Hamilton, who discovered the quaternions in 1844 when he tried to find a group structure on the 3-sphere. With $i := \gamma^1$, $j := \gamma^2$ and $k := ij$ we have 3 different 'square roots of -1' that are related by $jk = i$ and $ki = j$. Rotations of vectors (x,y,z) are represented by $q \to uqu^{-1}$ where $q = ix + jy + kz$ is a 'pure quaternion' and u a 'unit quaternion' $u = a_0 + a_1 i + a_2 j + a_3 k$ with $\overline{u}u = \sum a_i^2 = 1$. Quaternionic manifolds appear in extended supersymmetry.

Our next step is to find a spinor representation of this Lie algebra, i.e. a representation such that finite transformations provide a double cover of $SO(p,q)$.

To this end we consider antisymmetrized bilinears $\Sigma_{ab} := \frac{1}{4}[\gamma_a,\gamma_b] = \frac{1}{2}(\gamma_a\gamma_b - \eta_{ab})$, whose commutator with γ_c is $[\Sigma_{ab},\gamma_c] = \frac{1}{2}[\gamma_a\gamma_b,\gamma_c] = \gamma_a\eta_{bc} - \eta_{ac}\gamma_b$ and thus has the same form as a Lorentz transformation of the vector γ_a of Clifford algebra generators. This implies that, for any matrix representation of the Clifford algebra, the Σ_{ab}'s provide a representation of the Lorentz algebra,

$$\Sigma_{ab} := \frac{1}{4}[\gamma_a,\gamma_b], \qquad [\Sigma_{ab},\gamma_c] = -\eta_{ac}\gamma_b + \eta_{bc}\gamma_a, \qquad [\Sigma_{ab},\Sigma_{cd}] = -\eta_{ac}\Sigma_{bd} + \eta_{bc}\Sigma_{ad} + \eta_{ad}\Sigma_{bc} - \eta_{bd}\Sigma_{ac}. \tag{1.3}$$

To see that we actually constructed a double cover of $SO(p,q)$ we perform a rotation by an angle φ in the ij-plane, where we assume that both directions are 'space-like' $(\gamma_i)^2 = (\gamma_j)^2$; this is, of course, not possible for $SO(1,1)$, whose fundamental group is trivial (for $(\gamma_i)^2 = -(\gamma_j)^2$ we would consider boosts and thus get *hyperbolic* sines and cosines). Since $\Sigma^{ij} = \frac{1}{2}\gamma^i\gamma^j$ squares to $(\Sigma^{ij})^2 = -\frac{1}{4}\mathbf{1}$ for $i \neq j$ we find in the spinor representation

$$\omega_{ij} = -\omega_{ji} = \varphi \quad \Rightarrow \quad \exp(\omega) = \exp(\tfrac{1}{2}\omega_{ab}\Sigma^{ab}) = \exp(\varphi\Sigma^{ij}) = \cos\tfrac{\varphi}{2} + (\gamma^i\gamma^j)\sin\tfrac{\varphi}{2}, \tag{1.4}$$

so that only a rotation by 4π leaves a spinor, i.e. a vector in the representation space of the Clifford algebra, invariant. In other words, the spinor representation is a double valued representation of $SO(p,q)$.

In order to understand that the properties of spinors and Clifford algebras are independent of a particular matrix representation we now recall some elementary facts of group theory: A **representation** R of a group G is a map $R: G \to \mathrm{End}(V)$ from G into the group of linear transformations on a vector space V over a field $\mathbb{K}$ that is consistent with the group structures, i.e. $R(g)\circ R(h) = R(gh)$ for all $g,h \in G$. The dimension n of the vector space is called dimension of the representation. A choice of a basis in V provides an identification of $\mathrm{End}(V)$ with the matrix group $GL(n,\mathbb{K})$.

Complex representations, where $\mathbb{K} = \mathbb{C}$, always come in quartets R, R^*, $(R^T)^{-1}$, and $(R^\dagger)^{-1}$, where R^* is the **complex conjugate** representation and $(R^T)^{-1}$ is called **contragradient** or **dual** representation. Note that R^T is not a representation of the group G and that all of these representations live in different vector spaces. An **intertwiner** between two representations R_1 and R_2 is a map $A: V_1 \to V_2$ that is compatible with the representations, i.e. $AR_1(g) = R_2(g)A \;\; \forall g \in G$ (more abstractly, this is a *morphism* of the *category* of representations of G, i.e. a map that is compatible with the 'relevant' algebraic structures). The representations R_1 and R_2 are called **equivalent** if there exists an invertible intertwiner (an isomorphism); for matrix representations this means that there is a matrix $A \in GL(n,\mathbb{K})$ such that $R_2(g) = AR_1(g)A^{-1} \;\; \forall g \in G$.

A representation on a vector space with a hermitian metric is called **unitary** if all linear maps $R(g)$ preserve scalar products or, equivalently, if all representation matrices $R(g)$ are unitary in one (and hence in any) orthonormal basis. For finite groups and for compact groups it can be shown that all finite-dimensional representations are equivalent to unitary representations (an appropriate metric can be constructed by averaging an arbitrary hermitian metric over the group). Furthermore, equivalent unitary representations are unitarily equivalent, i.e. A can be chosen to fulfill $AA^\dagger = 1$.

A representation $R: G \to \mathrm{End}(V)$ is called **irreducible** if $\{0\}$ and V are the only invariant subspaces. It is called **completely reducible** if for all invariant subspaces $V_1 \subset V$ there exists an invariant complement, i.e. a subspace V_2 such that $V = V_1 \oplus V_2$ and all representation matrices $R(g)$ become block diagonal in a basis consisting of elements of V_i. A representation is called **faithful** if R is injective, i.e. if $R(g) = \mathbf{1}$ implies that g is the unit of the group. The **group ring** is the set of formal linear combinations of group elements with coefficients in some ring $\mathbb{K}$ with the natural product operation. If $\mathbb{K}$ is a field, then the group ring is an algebra, since the group ring is a vector space over $\mathbb{K}$. (The Clifford algebra is thus the group algebra of the finite group that is generated by γ_a.)

Schur's Lemma: If A is a homomorphism of a finite-dimensional irreducible representation R with $[A, R(g)] = 0 \ \ \forall g \in G$ then $A = \lambda\mathbf{1}$ is a multiple of unity.

Proof: For finite-dimensional representations A must have some eigenvalue λ. Since A commutes with all representation maps $R(g)$ the kernel of $A - \lambda\mathbf{1}$ is a non-empty invariant subspace and must thus be equal to the whole representation space. □

Corrolary: Intertwiners between irreducible representations are unique up to a factor.

Proof: Assume that $f : V_1 \to V_2$ and $g : V_1 \to V_2$ are intertwiners between two irreducible representations R_1, R_2. Since the kernel of g is an invariant subspace it must be all of V_1 or $\{0\}$. Hence we either have $g = 0 \cdot f$ or g is invertible. In the latter case $A := f \circ g^{-1} : V_2 \to V_2$ is an automorphism with $[A, R_2(g)] = 0 \ \ \forall g \in G$. Hence, $A = \lambda\mathbf{1}$ and $f = \lambda g$. □

Theorem: All unitary representations are irreducible or completely reducible.[2]
(Without proof)

Before turning to the formulation of Lorentz invariant field theories that contain spinors we should first understand the irreducible representations of the Clifford algebra. We define $\gamma_* := \Gamma^{12\ldots D}$ and observe that it satisfies

$$\begin{array}{r}\gamma_* := \gamma^1 \ldots \gamma^D \\ \tilde\gamma := i^{\left[\frac{p-q}{2}\right]}\gamma_*\end{array}, \qquad \gamma_*\gamma^a = (-)^{D-1}\gamma^a\gamma_*, \qquad {\gamma_*}^2 = (-)^{\left[\frac{p-q}{2}\right]}\mathbf{1} = \begin{cases} \mathbf{1} & p-q \equiv 0,1 \mod 4 \\ -\mathbf{1} & p-q \equiv 2,3 \mod 4 \end{cases} \tag{1.5}$$

[2] An example of a reducible representation that is not completely reducible is given by the Galilei transformations $\binom{t}{x} \to \begin{pmatrix} 1 & 0 \\ v & \mathcal{O}\end{pmatrix}\binom{t}{x}$; vectors with $t = 0$ form a non-trivial invariant subspace.

because $q+\binom{D}{2}=2pq+\binom{p-q}{2}\equiv\binom{p-q}{2}\equiv\left[\frac{p-q}{2}\right]\bmod 2$. The first equation shows that in odd dimensions there cannot exist an irreducible faithful representation of the Clifford algebra since γ_* would have to be proportional to the unit element because of Schur's lemma. In even dimensions $\tilde\gamma$ anticommutes with all γ^a's and squares to $\mathbf{1}$. It can therefore be used to define the *helicity* projectors $P_\pm=\frac{1}{2}(\mathbf{1}\pm\tilde\gamma)$, which will be important for the construction of irreducible spin representations since they commute with the generators Σ_{ab} of spin(p,q).

For $D\in 2\mathbb{Z}$ the existence of a faithful irreducible representation of dimension $2^{D/2}$ can be shown by explicit construction: For $D=2$ we can use any two of the three Pauli matrices

$$\sigma_1=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad \sigma_2=\begin{pmatrix}0&-i\\i&0\end{pmatrix},\quad \sigma_3=\begin{pmatrix}1&0\\0&-1\end{pmatrix},\qquad \sigma_i^2=\mathbf{1},\quad \sigma_1\sigma_2\sigma_3=i\mathbf{1} \tag{1.6}$$

to represent $\mathcal{C}(2,0)$ and imaginary multiples to get the other signatures. Representations for larger dimensions can be constructed recursively since

$$\mathcal{C}(p_1,q_1)\otimes\mathcal{C}(p_2,q_2)\cong\begin{cases}\mathcal{C}(p_1+p_2,q_1+q_2) & \text{if } \frac{p_1-q_1}{2} \text{ is even}\\ \mathcal{C}(p_1+q_2,q_1+p_2) & \text{if } \frac{p_1-q_1}{2} \text{ is odd}\end{cases}. \tag{1.7}$$

To see this, an isomorphism can be constructed in the following way: For given representations $\gamma^{(i)}$ of $\mathcal{C}(p_i,q_i)$ we use the matrices $\gamma^{(1)}_{a_1}\otimes\mathbf{1}^{(2)}$ with $a_1=1,\ldots,D_1$ and ${\gamma_*}^{(1)}\otimes\gamma^{(2)}_{a_2}$ with $a_2=1,\ldots,D_2$, which all anticommute, e.g. $\{\gamma^{(1)}_{a_1}\otimes\mathbf{1}^{(2)},{\gamma_*}^{(1)}\otimes\gamma^{(2)}_{a_2}\}=\gamma^{(1)}_{a_1}{\gamma_*}^{(1)}\otimes\gamma^{(2)}_{a_2}+{\gamma_*}^{(1)}\gamma^{(1)}_{a_1}\otimes\gamma^{(2)}_{a_2}=\{\gamma^{(1)}_{a_1},{\gamma_*}^{(1)}\}\otimes\gamma^{(2)}_{a_2}=0$. Moreover, $(\gamma^{(1)}_a\otimes\mathbf{1}^{(2)})^2=\eta^{(1)}_{aa}\mathbf{1}^{(1)}\otimes\mathbf{1}^{(2)}$ and $({\gamma_*}^{(1)}\otimes\gamma^{(2)}_a)^2=(-)^{\frac{p_1-q_1}{2}}\eta^{(2)}_{aa}\mathbf{1}^{(1)}\otimes\mathbf{1}^{(2)}$, which establishes the isomorphism.

Corrolary: $\mathcal{C}(4,0)\cong\mathcal{C}(2,0)\otimes\mathcal{C}(0,2)\cong\mathcal{C}(0,2)\otimes\mathcal{C}(2,0)\cong\mathcal{C}(0,4)$ implies the isomorphism $\mathcal{C}(p,q)\cong\mathcal{C}(4,0)\otimes\mathcal{C}(p-4,q)\cong\mathcal{C}(0,4)\otimes\mathcal{C}(p-4,q)\cong\mathcal{C}(p-4,q+4)$, so that $\mathcal{C}(p,q)$ only depends on D and $p-q\bmod 8$.

For odd dimensions $D=p+q=2n+1$ two irreducible representations can be obtained by using a set of γ–matrices for the case $D=2n$ and in addition the matrices $\pm{\gamma_*}^{(2n)}$ or $\pm i{\gamma_*}^{(2n)}$. These two representations are inequivalent since the product ${\gamma_*}^{(2n+1)}$ of all $D=2n+1$ γ-matrices is proportional to unity, ${\gamma_*}^{(2n+1)}=\pm i^{\left[\frac{p-q}{2}\right]}\mathbf{1}$, but with different signs of the proportionality factor in the two representations. This factor cannot be changed by an equivalence transformation. If $\{\gamma^a,a=1,\ldots,D\}$ is a representation with ${\gamma_*}^{(2n+1)}=+i^{\left[\frac{p-q}{2}\right]}\mathbf{1}$ then $\{-\gamma^a,a=1,\ldots,D\}$ provides a representation with ${\gamma_*}^{(2n+1)}=-i^{\left[\frac{p-q}{2}\right]}\mathbf{1}$. It can be shown that the $2^{D/2}$-dimensional representation for $D\in 2\mathbb{Z}$ and the two $2^{(D-1)/2}$-dimensional representations for $D\notin 2\mathbb{Z}$ are unique up to equivalence.[3] In the odd-dimensional case the direct sum of the two inequivalent representations is isomorphic to $\mathcal{C}(p,q)$ and thus provides the minimal faithful representation.

[3] The proof uses some results about the dimensions of irreducible representations of finite groups: The group ring of a finite group carries the regular representation, which has dimension $|G|$ and whose representation matrices correspond to permutations of the group elements. The number of inequivalent irreducible representations (irreps) of a finite group is equal to the number N_{cc} of conjugacy classes (i.e. classes of group elements h that are related by equivalence transformations $h\to ghg^{-1}$ for some $g\in G$). The regular representation contains each irrep R_λ of dimension d_λ of the group exactly d_λ times, so that $|G|=\sum_{\lambda=1}^{N_{cc}}d_\lambda^2$. In our case $|G|=2^{D+1}$

Note that the tensor product of representations corresponds to 'matrices of matrices' (like e.g. 2×2 matrices whose entries are lower-dimensional γ-matrices). We can obtain, for example, a purely imaginary (Majorana) representation of the Dirac algebra $\mathcal{C}(1,3)\cong\mathcal{C}(1,1)\otimes\mathcal{C}(0,2)$ using $(\sigma_2, i\sigma_1)$ and $(i\sigma_1, i\sigma_3)$ with $\gamma_*^{(1)}=\sigma_2(i\sigma_1)=\sigma_3$, so that, inserting the left factor into the right factor,

$$\gamma^0=\begin{pmatrix}\sigma_2 & 0\\ 0 & \sigma_2\end{pmatrix},\quad \gamma^1=\begin{pmatrix}i\sigma_1 & 0\\ 0 & i\sigma_1\end{pmatrix},\quad \gamma^2=\begin{pmatrix}0 & i\sigma_3\\ i\sigma_3 & 0\end{pmatrix},\quad \gamma^3=\begin{pmatrix}i\sigma_3 & 0\\ 0 & -i\sigma_3\end{pmatrix}. \tag{1.8}$$

We will see later that a representation of $\mathcal{C}(1,3)$ with real matrices cannot exist.

Another non-commutative algebra that comes with any vector space is the exterior algebra $\Lambda=\sum_0^D\Lambda^p$, which also has dimension 2^D. There is, in fact, a natural one-to-one correspondence of k-forms and Clifford algebra elements which is given by $\omega\mapsto\not\omega:=\frac{1}{k!}\Gamma^{a_1\ldots a_k}\omega_{a_1\ldots a_k}$ [co82, CH54]. The Clifford product and the exterior product differ by terms of lower 'form degree'. The Clifford algebra is therefore not a $\mathbb{Z}$-graded algebra. Since the defining anticommutation relation (1.1) has a term of degree 0 on the r.h.s. there is, however, a $\mathbb{Z}_2$ grading left that we can use to define the decomposition $\mathcal{C}(p,q)=\mathcal{C}_+(p,q)\oplus\mathcal{C}_-(p,q)$, where $\mathcal{C}_+(p,q)$ contains all linear combinations of products of an even number of γ matrices and $\mathcal{C}_-(p,q)$ contains all products of an odd number of γ factors.

Obviously $\mathcal{C}_+(p,q)$ is a subalgebra, which is again isomorphic to a Clifford algebra. To see this we use, for some fixed a_0, the products $\gamma^{a_0}\gamma^a$ with $a\neq a_0$ as generators of $\mathcal{C}_+(p,q)$. Since $\{\gamma^{a_0}\gamma^a,\gamma^{a_0}\gamma^b\}=-(\gamma^{a_0})^2\{\gamma^a,\gamma^b\}$ this gives us an isomorphism to $\mathcal{C}(q,p-1)$ or to $\mathcal{C}(p,q-1)$, depending on whether $(\gamma^{a_0})^2$ is positive or negative. We thus find the chain of isomorphisms

$$\mathcal{C}_+(p,q)\cong\mathcal{C}(p,q-1)\cong\mathcal{C}(q,p-1)\cong\mathcal{C}_+(q,p). \tag{1.9}$$

Note that Spin(p,q), the Lie group corresponding to the Lie algebra generated by Σ_{ab}, resides in the even part $\mathcal{C}_+$ of the Clifford algebra. This implies that the dimension dependent properties of the spin representations are just reversed as compared to those of the Clifford algebra: There are two inequivalent spinors in even dimensions, and in odd dimensions there is a unique irreducible representation. In fact, we already found the projectors $P_\pm$ that decompose the Clifford algebra representation for $D\in 2\mathbb{Z}$ into the two inequivalent irreducible spinor representations of $so(p,q)$, which are of dimension $2^{D/2-1}$ and whose elements are called Weyl spinors. Spinors in odd dimensions have $2^{(D-1)/2}$ components. The symmetry of $\mathcal{C}_+(p,q)$ under the exchange of p and q should have been expected since the properties of spin representations of $so(p,q)$ should not depend on which sign convention we choose for the indefinite metric.

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and $N_{cc}=2^D+s$ with $s=1$ ($s=2$) for D even (odd): All products $\Gamma^{a_1\cdots a_p}$ of different γ matrices are combined into conjugate pairs $\pm\Gamma$ except for $p=0$ (and $p=D$ if D is odd, because then $\{\pm 1\}$ *and* $\{\pm\gamma_*\}$ are classes that contain only one element). Since we already have dimension $\sum d_\lambda^2=2^D$ from 1 (2) irreps for D even (odd), all other irreps must be 1-dimensional.

1.2 Dirac adjoint and charge conjugation

From now on we assume, without loss of generality, that our representations of γ-matrices are unitary:

$$(\gamma^a)^\dagger = (\gamma^a)^{-1} = \gamma_a. \tag{1.10}$$

This will be useful for constructing **real** Lorentz tensors that are quadratic in spinor fields, the stuff we need as building blocks of Lagrange densities, and it allows us to use group theoretic results on unitary representations. We also restrict our discussion to the even-dimensional case $D \in 2\mathbb{Z}$, where we have a unique irreducible representation. Analogous results for the odd-dimensional case can then be derived, for example, by using what we know about the γ matrices in dimension $D-1$.

In addition to the contragradient and the complex conjugate representations the structure of the Clifford algebra provides us with a host of other representations for any given one. We may, for example, flip the sign of any number of γ matrices, consider transposed matrices, or even linear combinations $\gamma^a \to \Omega^a{}_b\gamma^b$ with arbitrary $SO(p,q)$ transformations $\Omega^a{}_b$. In the latter case, we loose unitarity of the representation if $\Omega^a{}_b$ contains boosts and does not just mix space-like and time-like directions among themselves. In any case we are guaranteed the existence of an intertwiner (as we only consider even dimensions). The physically most important equivalences [gl76,Re84] are listed in the following table:

$\gamma^{a\dagger} = A\gamma^a A^\dagger$	$A = A^\dagger = (i^{\frac{q}{2}}\gamma_*)^{p-1}\prod_1^p \gamma^a$	$\overline{\psi} := \psi^\dagger A, \quad AA^\dagger = 1$
$-\gamma^{a*} = B^\dagger\gamma^a B$	$B = bB^T = CA^T = \prod_{\gamma^a_{real}}(\gamma_*\gamma^a)$	$\psi^c := B\psi^*, \quad BB^\dagger = 1$
$-\gamma^{aT} = C^\dagger\gamma^a C$	$C = cC^T = BA^* = bcA^\dagger B$	$\psi^c = C\overline{\psi}^T, \quad CC^\dagger = 1$
$\Omega^a{}_b\gamma^b = \Lambda\gamma^a\Lambda^{-1}$	$\Lambda = \mathbf{1} + \frac{1}{8}\omega_{ab}[\gamma^a,\gamma^b] + O(\omega^2)$	$\Lambda^* = B^\dagger\Lambda B, \ \Lambda^\dagger = A\Lambda^{-1}A^\dagger$

Here $\overline{\psi}$ is the Dirac adjoint spinor, which transforms contragradient so that it can be used to write down Lorentz invariant scalar products. The matrix B allows us to impose Lorentz-invariant reality conditions $\psi^c = \psi$ on spinors. The explicit formula that we gave for B assumes that we use a representation with all γ matrices either being real or imaginary (tensor products of Pauli matrices, which we used for our recursive construction of representations, obviously are of this type). The charge conjugate spinor ψ^c is usually defined in terms of the Dirac adjoint spinor and the charge conjugation matrix C. Eventually we recover the existence of a Lorentz transformation $\Lambda(\Omega)$ on spinors. After imposing reality and normalization conditions Λ becomes unique up to a sign. This remaining ambiguity cannot be removed since we are dealing with a double valued representation.

Before we discuss the various equivalences and the entries in the above table in more detail let us first elaborate on what spaces and intertwiners we are talking about. With any vector space V over a field $\mathbb{K}$ there comes its dual space $\tilde{V}$ which is the space of all linear functionals $a : V \to \mathbb{K}$. The elements of $\tilde{V}$ are also called covectors. We define $(a, v) := (v, a) := a(v)$ for all $a \in \tilde{V}, v \in V$. $\tilde{V}^*$ denotes the complex conjugate dual space which is the space of all antilinear functionals on V, $a^*(v) := (a, v)^*$. The complex conjugate vector space V^* is the space of all antilinear functionals on $\tilde{V}$, $v^*(a) := (a, v)^*$ (this is not a perfect notation, but we want to reserve the '$*$' symbol for complex conjugation). If we have a (semi) linear operator $L : V \to W$ intertwining two complex representation spaces, then we have its transposed map $L^T : \tilde{W} \to \tilde{V}$ (defined by $(L^T b, v) := (b, Lv)^{(*)}$), the complex conjugate map $L^* : V^* \to W^*$ (defined by $(L^* v^*, b) := (v^*, L^T b)^{(*)}$) and the hermitian conjugate map $L^\dagger : \tilde{W}^* \to \tilde{V}^*$ (defined by $(L^\dagger b^*, v) := (b^*, Lv)^{(*)}$).

From the left column of the above table we observe how $A : V \to \tilde{V}^*$, $B : V^* \to V$ and $C : \tilde{V} \to V$ intertwine the various representations. Hence $A^T : V^* \to \tilde{V}$, and if we use lower (upper) indices $\underline{\alpha}$ for (dual) spinors and indices $\overline{\alpha}$ for the complex conjugate spaces we have the index pictures $A^{\overline{\alpha}\underline{\beta}}$, $B_{\underline{\alpha}}{}^{\overline{\beta}}$, $C_{\underline{\alpha}\underline{\beta}}$, and $(A^T)^{\underline{\alpha}\overline{\beta}}$ (we underline spinor indices and reserve α for Weyl spinors; see below). Hence C^{-1} plays the role of a bilinear metric in spinor space. A and B are antilinear on spinors ψ, i.e. linear on ψ^*. Implicit in our formulas is that we fix the real subspace of V and a hermitian metric (having the same index picture as A), which allow us to define unitary matrices and transposed and conjugated vectors. With these intertwiners, whose existence in even dimensions is guaranteed by the uniqueness of the irreducible representation, we can now define the Dirac adjoint spinor[4] $\overline{\psi} = \psi^\dagger A \in \tilde{V}$ and the conjugate spinor $\psi^c = B\psi^* = C\overline{\psi}^T = CA^T\psi^*$, which requires a choice of normalizations that is compatible with $B = CA^T$.

Unitarity of our representation of γ matrices implies that γ^a and $\gamma^{a\dagger} = (\gamma^a)^{-1}$ are related by a unitary matrix A. It is easy to see that for p odd (even) the product of all γ matrices with positive (negative) square can be used. Since $A^\dagger$ does the same job as A, Schur's lemma implies that $A^\dagger$ and A are, in fact, proportional. With an appropriate choice of the phase of A, which is not fixed by unitarity, we may choose $A = A^\dagger$, as is done in the formula for A in the table. With the matrix A we can define the **Dirac adjoint** spinor $\overline{\psi} = \psi^\dagger A$, which transforms contragradient to ψ under Lorentz transformations because $\psi \to \Lambda\psi$ implies $\overline{\psi} \to \psi^\dagger\Lambda^\dagger A = \psi^\dagger A\Lambda^{-1}$. This allows us to construct Lorentz-invariant real scalars $\overline{\psi}\psi$, as well as antisymmetric tensors $\overline{\psi}\Gamma^{a_1\cdots a_p}\psi$. The job of A is to compensate the non-unitarity of the spinor representation in the non-compact case (i.e. for boosts). Note that we define hermitian and complex conjugation for anti-commuting fields such that $(XY)^\dagger := Y^\dagger X^\dagger$ and

[4] More precisely we should write $\overline{\psi} := A^T\psi^*$ because $\overline{\psi} := \psi^\dagger A$ only makes sense after a choice of basis, which allows us to write bilinears in terms of matrix multiplications (there is no natural map $\psi \to \psi^\dagger \in \tilde{V}^*$).

$(XY)^* := (-)^{|X||Y|}X^*Y^*$. In the case of Minkowski signature $p = 1$ we have $A = \gamma^0$.

Charge conjugation should change the sign of the coupling of the gauge field to the fermion in the Dirac equation, i.e. we want to transform $(i\partial\!\!\!/ - eA\!\!\!/ - m)\psi = 0$ into the equation $(i\partial\!\!\!/ + eA\!\!\!/ - m)\psi^c = 0$ for the wave function of the charge conjugate particle. Such an equation can easily be obtained by transposition of the Dirac equation $\overline{\psi}(-i\overleftarrow{\partial\!\!\!/} - eA\!\!\!/ - m) = 0$ for the adjoint spinor provided that there exists an intertwiner C that transposes the γ matrices,

$$\big((-\gamma^m)^T(i\partial_m + eA_m) - m\big)\,\overline{\psi}^T = C^{-1}(i\partial\!\!\!/ + eA\!\!\!/ - m)C\overline{\psi}^T = 0, \qquad \psi^c := C\overline{\psi}^T. \tag{1.11}$$

Putting the pieces together we find that $\psi^c = B\psi^*$ with $B := CA^T$. We should indeed expect that ψ^c is proportional to ψ^* since wave functions transform with phases $\psi \to e^{-ie\Lambda}\psi$ under gauge transformations $A \to A + d\Lambda$.

Unitarity of the γ^a implies unitary equivalence, so that C can be fixed up to an irrelevant phase by $CC^\dagger = \mathbf{1}$. Furthermore, the transposed equation $-\gamma^a = C^T\gamma^{aT}(C^T)^{-1}$ of $-\gamma^{aT} = C^{-1}\gamma^a C$ shows that C^T does the same job as C so that these intertwiners must be proportional $C = cC^T$ with $c = \pm 1$. The constant c can be computed if we observe that the matrices $\Gamma^{a_1\ldots a_r}C$, which span the representation space, are all either symmetric or antisymmetric:

$$\gamma C = -C\gamma^T \quad\Rightarrow\quad \Gamma^{a_1\ldots a_r}C = (-)^r C(-)^{\binom{r}{2}}(\Gamma^{a_1\ldots a_r})^T = c(-)^{\binom{r+1}{2}}(\Gamma^{a_1\ldots a_r}C)^T. \tag{1.12}$$

The number of symmetric minus the number of antisymmetric matrices is the dimension of the representation, hence

$$2^{D/2} = c\sum_{r=0}^{D}\binom{D}{r}(-)^{\binom{r+1}{2}} = c\sum_{r=0}^{D}\binom{D}{r}\,\mathrm{Re}((1+i)i^r) \tag{1.13}$$

$$= c\,\mathrm{Re}(1+i)^{D+1} = c\,\mathrm{Re}((1+i)(2i)^{D/2}) = c2^{D/2}(-)^{\binom{D/2+1}{2}}. \tag{1.14}$$

Note that $C' := \gamma_* C$ intertwines γ^a and $+\gamma^{aT}$. Since $C^{-1}(\gamma_* C) = (-)^{\binom{D+1}{2}}{\gamma_*}^T = (-)^{[\frac{D+1}{2}]}{\gamma_*}^T = (-)^{D/2}{\gamma_*}^T$ we obtain

$$C^T = C(-)^{\binom{D/2+1}{2}}, \qquad (\gamma_* C)^T = (\gamma_* C)(-)^{\binom{D/2}{2}} \tag{1.15}$$

For the chiral projector $P_\pm$ we thus find $C^{-1}P_\pm C = P_\pm^T$ if $D \in 4\mathbb{Z}$ and $C^{-1}P_\pm C = P_\mp^T$ if $D \in 4\mathbb{Z}+2$. The metric C thus mixes chiralities in $2 \bmod 4$ dimensions and preserves chirality in $4 \bmod 4$ dimensions: In a Weyl basis, in which $P_\pm$ is diagonal, C must be block-(off)diagonal for even (odd) $D/2$. The same is true for C'.

In the odd-dimensional case $D = 2n+1$ there cannot exist intertwiners between γ^a and both $+\gamma^{aT}$ and $-\gamma^{aT}$ (otherwise γ^{aT} would be equivalent to both γ^a and $-\gamma^a$ in contradiction to the inequivalence of γ^a and $-\gamma^a$). Recall that γ-matrices in $D = 2n+1$ dimensions can be constructed from those in $D-1 = 2n$ dimensions. The first $D-1$ matrices γ^a $(a =$

$1, 2, \ldots, D-1$) are taken to be those in $D-1$ dimensions. The last matrix γ^D is taken to be $\pm\gamma_*{}^{(2n)}$ or $\pm i\gamma_*{}^{(2n)}$, depending on whether $a = D$ is a space-like or a time-like direction. Then the intertwiners C, C' used in $D-1$ dimensions satisfy

$$C^{-1}\gamma^a C = -\gamma^{aT}, \qquad C'^{-1}\gamma^a C' = +\gamma^{aT}, \qquad a = 1, \ldots, D-1, \tag{1.16}$$

$$C^{-1}\gamma^D C = C'^{-1}\gamma^D C' = (-)^{\binom{D-1}{2}}\gamma^{DT} \tag{1.17}$$

Hence, if $(-)^{\binom{D-1}{2}} = -1$ ($(-)^{\binom{D-1}{2}} = +1$) we can use C (C') to intertwine γ^a and $-\gamma^{aT}$ ($+\gamma^{aT}$). Formally, we can define the intertwiner $C'' = \frac{1}{2}\Big(C' + C + (-)^{\binom{D-1}{2}}(C' - C)\Big)$ satisfying

$$C''^{-1}\gamma^a C'' = (-)^{\binom{D-1}{2}}\gamma^{aT}, \qquad a = 1, \ldots, D, \tag{1.18}$$

$$(C'')^T = (-)^{\binom{(D-1)/2+1}{2}}C''. \tag{1.19}$$

Note that the symmetry properties of the *linear* intertwiner C only depend on D; reality properties, like the symmetry of B, will depend on the signature $p - q$.

1.3 Majorana and Weyl spinors

It will be important to know what Lorentz invariant **reality** conditions we may impose on spinors. The **Majorana** condition requires that a spinor is equal to its charge conjugate $\psi = \psi^c := B\psi^*$, which is possible iff $BB^* = \mathbf{1}$, as can be seen as follows: First we note that $\psi = \psi^c$ implies $(\psi^c)^c = B(B\psi^*)^* = \psi$, i.e. BB^* should have an eigenvalue 1. Complex conjugation of $-\gamma^{a*} = B^\dagger\gamma^a B$ yields $\gamma^a = -B^T\gamma^{a*}B^* = B^T B^\dagger\gamma^a BB^* = (BB^*)^\dagger\gamma^a BB^*$, so that Schur's lemma implies $BB^* = b\mathbf{1}$. Using unitarity $B^{-1} = B^\dagger$ we thus find $B = b(B^*)^{-1} = bB^T = b^2 B$ with $b = \pm 1$. The sign factor b can be computed using the symmetry of C: Since A is the product of p (q) γ matrices for p odd (even) we find

$$B = CA^T \;\Rightarrow\; B^T = AC^T = ACc = CA^T b \;\Rightarrow\; b = c(-)^{\frac{s(s+1)}{2}} \quad \text{with } s = \begin{cases} p & p, q \text{ odd} \\ q & p, q \text{ even} \end{cases} \tag{1.20}$$

(recall that $\Gamma^{a_1 \ldots a_r}C = C\Gamma^{a_1 \ldots a_r T}(-)^{\binom{r+1}{2}}$). The sign $b = (-)^\sigma$ thus becomes

$$\sigma \equiv \tfrac{1}{2}\tfrac{p+q}{2}(\tfrac{p+q}{2} + 1) + \tfrac{s^2+s}{2} \equiv \tfrac{1}{2}((\tfrac{q-p}{2})^2 + \tfrac{q-p}{2}) + \tfrac{1}{2}(pq + p + s^2 + s) \tag{1.21}$$

$$\equiv \tfrac{1}{2}\tfrac{q-p}{2}(\tfrac{q-p}{2} - 1) + \tfrac{1}{2}(pq + q + s^2 + s) \equiv [\tfrac{q-p}{4}] \mod 2 \tag{1.22}$$

because for p odd $pq + q + s^2 + s = (p+1)(q+p) \in 4\mathbb{Z}$ and for p even $pq + q + s^2 + s = q(p+1+q+1) \in 4\mathbb{Z}$. Majorana spinors therefore don't exist for $q - p \in \{4, 6\} \mod 8$, i.e. in 6 and 8 even dimensions with Minkowski signature. Somewhat puzzling about this result is that the existence of Majorana spinors depends on the convention that we use for the sign of the metric. There is, however, another spin(p,q)-invariant reality condition that we may impose, namely $\psi = B'\psi^*$ with $B' = \gamma_* B$. As above we find $\gamma_* B(\gamma_*)^* B^* = b'\mathbf{1}$ with $b' = \pm 1$,

$$b = (-)^{[\frac{q-p}{4}]} \quad \Rightarrow \quad b' = \gamma_* B(\gamma_*)^* B^* = (-)^D (\gamma_*)^2 BB^* = (-)^{\frac{p-q}{2}} b = (-)^{[\frac{p-q}{4}]}, \qquad (1.23)$$

so that indeed the role of p and q is exchanged in the modified Majorana condition (Majorana spinors have to be multiplied by γ_* times some phase to conform with such an exchange). Sometimes spinors satisfying the modified Majorana condition are called pseudo Majorana spinors. Note that $B' = \gamma_* B$ intertwines γ^a and $+\gamma^{a*}$. Both reality conditions can be imposed simultaneously iff the chiral projector commutes with B, which happens if $p - q \equiv 0 \bmod 8$. Then we can have Majorana-Weyl spinors, i.e., with an appropriate choice of basis, real Weyl spinors.

Theorem: There is a basis with $B = \mathbf{1}$ and all γ matrices imaginary iff $b = 1$. In this basis the Dirac equation $(i\partial\!\!\!/ + m)\psi$ is real. (There is a basis with real γ matrices iff $b' = 1$.)

Proof: As $UBU^T : V^{*\prime} \to V'$ for $U : V \to V'$ the intertwiner B transforms into (a phase times) UBU^T under a unitary change of basis $\gamma^a \to U\gamma^a U^\dagger$. Decomposing B into real and imaginary part $B = B_1 + iB_2 = B^T$ we find $BB^\dagger = \mathbf{1} = B_1^2 + B_2^2 - i[B_1, B_2]$. Hence B_1 and B_2 are commuting real symmetric matrices, which can be diagonalized simultaneously by a real orthogonal transformation (which preserves unitarity of the representation). Then B becomes a diagonal unitary matrix, which can be transformed into $B = \mathbf{1}$ with $U = 1/\sqrt{B}$. □

For practical calculations it is often convenient to use a **Weyl representation** in which the chiral projectors are diagonal and, consequently, the matrices γ^a (which anti-commute with $\tilde\gamma$) are block off-diagonal (to see this just make an ansatz for the blocks of γ^a). In such a basis we split a Dirac spinor $\psi_{\underline{\alpha}}$ into positive and negative chirality Weyl spinors ψ_α and $\tilde\psi^{\dot\alpha}$; this is called Infeld – van der Waerden notation. We thus have

$$\tilde\gamma = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix} \quad \Rightarrow \quad (\gamma^a)_{\underline{\alpha}}{}^{\underline{\beta}} = \begin{pmatrix} 0 & \sigma^a_{\alpha\dot\beta} \\ (\overline{\sigma}^a)^{\dot\alpha\beta} & 0 \end{pmatrix}, \qquad \psi_{\underline{\alpha}} = \begin{pmatrix} \psi_\alpha \\ \tilde\psi^{\dot\alpha} \end{pmatrix}, \qquad (1.24)$$

$$(\Sigma^{ab})_{\underline{\alpha}}{}^{\underline{\beta}} = \begin{pmatrix} (\sigma^{ab})_\alpha{}^\beta & 0 \\ 0 & (\overline{\sigma}^{ab})^{\dot\alpha}{}_{\dot\beta} \end{pmatrix}, \qquad \begin{aligned} \sigma^{ab} &= \tfrac14(\sigma^a\overline{\sigma}^b - \sigma^b\overline{\sigma}^a) \\ \overline{\sigma}^{ab} &= \tfrac14(\overline{\sigma}^a\sigma^b - \overline{\sigma}^b\sigma^a) \end{aligned} \qquad (1.25)$$

Since $\tilde\gamma$ is real in a Weyl representation γ_* is real iff $\gamma_*^2 = 1$, so that $B\tilde\gamma = (-)^{\frac{p-q}{2}}\tilde\gamma B$. With an ansatz for the block entries of B we thus find the following general form of its matrix elements:

$q-p=D-2$	$0\equiv 8$	2	4	6
$b=BB^*$	1	1	-1	-1
$B=bB^T$	$\begin{pmatrix} B_+^{(s)} & 0 \\ 0 & B_-^{(s)} \end{pmatrix}$	$\begin{pmatrix} 0 & B_{-+}^T \\ B_{-+} & 0 \end{pmatrix}$	$\begin{pmatrix} B_+^{(a)} & 0 \\ 0 & B_-^{(a)} \end{pmatrix}$	$\begin{pmatrix} 0 & -B_{-+}^T \\ B_{-+} & 0 \end{pmatrix}$
Majorana	$\begin{pmatrix} \psi = B_+\psi^* \\ \tilde\psi = B_-\tilde\psi^* \end{pmatrix}$	$\begin{pmatrix} \psi = B_{-+}^T\tilde\psi^* \\ \tilde\psi = B_{-+}\psi^* \end{pmatrix}$	—	—
$b'=B'B'^*$	1	-1	-1	1
$B'=b'B'^T$	$\begin{pmatrix} B_+^{\prime(s)} & 0 \\ 0 & B_-^{\prime(s)} \end{pmatrix}$	$\begin{pmatrix} 0 & -B_{-+}^{\prime T} \\ B'_{-+} & 0 \end{pmatrix}$	$\begin{pmatrix} B_+^{\prime(a)} & 0 \\ 0 & B_-^{\prime(a)} \end{pmatrix}$	$\begin{pmatrix} 0 & B_{-+}^{\prime T} \\ B'_{-+} & 0 \end{pmatrix}$
p-Majorana	$\begin{pmatrix} \psi = B'_+\psi^* \\ \tilde\psi = B'_-\tilde\psi^* \end{pmatrix}$	—	—	$\begin{pmatrix} \psi = B_{-+}^{\prime T}\tilde\psi^* \\ \tilde\psi = B'_{-+}\psi^* \end{pmatrix}$

In odd dimensions $D=2n+1$ there cannot exist intertwiners between γ^a and both $+\gamma^{a*}$ and $-\gamma^{a*}$. If we take the first $D-1$ matrices γ^a $\quad (a=1,2,\ldots,D-1)$ to be those in $D-1$ dimensions and $\gamma^D=\pm(i)\gamma_*{}^{(2n)}$ the intertwiners B,B' used in $D-1$ dimensions satisfy

$$B^{-1}\gamma^a B=-\gamma^{a*},\qquad B'^{-1}\gamma^a B'=+\gamma^{a*},\qquad a=1,\ldots,D-1, \tag{1.26}$$

$$B^{-1}\gamma^D B=B'^{-1}\gamma^D B'=(-)^{\frac{p-q-1}{2}}\gamma^{D*} \tag{1.27}$$

As in the case of charge conjugation, we define $B''=\frac{1}{2}\Big(B'+B+(-)^{\frac{p-q-1}{2}}(B'-B)\Big)$ satisfying

$$B''^{-1}\gamma^a B''=(-)^{\frac{p-q-1}{2}}\gamma^{a*},\qquad a=1,\ldots,D, \tag{1.28}$$

$$B''(B'')^*=b''\mathbf{1}=(-)^{\left(\frac{(p-q-1)/2+1}{2}\right)}\mathbf{1}. \tag{1.29}$$

$q-p=D-2$	1	3	5	7
$b''=B''(B'')^*$	1	-1	-1	1

With Minkowski signature (Pseudo) Majorana spinors thus exist in 1 and 3 mod 8 dimensions.

It can be shown that real Clifford algebras are always isomorphic to (the direct sum of two) 'full matrix algebras' $\mathcal{M}(d,\mathbb{K})$ with $d\times d$ real, complex or quaternionic entries [co82]:

$p-q$ mod 8	$D_{[p=1]}$ mod 8	$\mathcal{C}_\mathbb{R}(p,q)$
1	1	$\mathcal{M}(2^{[D/2]},\mathbb{R})\oplus\mathcal{M}(2^{[D/2]},\mathbb{R})$
2, 0	0, 2	$\mathcal{M}(2^{D/2},\mathbb{R})$
3, 7	7, 3	$\mathcal{M}(2^{[D/2]},\mathbb{C})$
4, 6	6, 4	$\mathcal{M}(2^{D/2-1},\mathbb{H})$
5	5	$\mathcal{M}(2^{[D/2]-1},\mathbb{H})\oplus\mathcal{M}(2^{[D/2]-1},\mathbb{H})$

The matrix algebra is real iff the γ matrices can be chosen to be *real*.

1.4 Towards actions

Eventually we come to the discussion of **Lorentz transformations** $\Omega^a{}_b\gamma^b = \Lambda\gamma^a\Lambda^{-1}$. Since Λ cannot be chosen to be unitary this defines Λ up to a factor $\lambda \in \mathbb{C}^* = \mathbb{C} - \{0\}$. We next show that this ambiguity can be resolved up to a sign if we require A and B to intertwine the Lorentz transformations on the conjugate representation spaces:

$$-\Omega^a{}_b\gamma^{b*} = \Omega^a{}_b B^\dagger\gamma^b B = B^\dagger\Lambda\gamma^a\Lambda^{-1}B = -(\Lambda\gamma^a\Lambda^{-1})^* = (\Lambda^* B^\dagger)\gamma^a(\Lambda^* B^\dagger)^{-1} \tag{1.30}$$

implies that $B^\dagger\Lambda$ and $\Lambda^* B^\dagger$ are proportional, so that we can obtain $B^\dagger\Lambda B = \Lambda^*$ by a choice of the phase of Λ. Similarly,

$$\Omega^a{}_b\gamma^{b\dagger} = \Omega^a{}_b A\gamma^b A^\dagger = A\Lambda\gamma^a\Lambda^{-1}A^\dagger = (\Lambda\gamma^a\Lambda^{-1})^\dagger = (\Lambda^{\dagger-1}A)\gamma^a(A^\dagger\Lambda^\dagger). \tag{1.31}$$

This implies that $\Lambda^{\dagger-1}A$ and $A\Lambda$ are proportional. Now the constant can be fixed to 1 by a choice of a real factor for Λ. But $\Lambda^{\dagger-1}A = A\Lambda$ implies $A^\dagger\Lambda^{-1}A = \Lambda^\dagger = A\Lambda^{-1}A^\dagger$ since A and $A^\dagger$ are proportional, which completes the proof of our proposition.

In particular we have shown now that bilinears $i^{\binom{r+1}{2}}\,\overline{\chi}\Gamma^{a_1\ldots a_r}\psi$ transform as antisymmetric tensors under Lorentz transformations (all of them are real for $\chi = \psi$). Since we are in even dimensions we can insert chiral projectors, which flip chirality $r{+}p$ times if we pull them through the r factors of Γ and through the intertwiner A that defines the Dirac adjoint. In Minkowski space this implies that kinetic terms $\not\partial$ and minimal gauge couplings $\not A$ preserve chirality, while mass terms $\overline{\psi}m\psi$ and anomalous magnetic moments $\overline{\psi}\not F\psi$ flip chirality. In $4 \bmod 4$ dimensions it is, however, still possible to give a mass to a single Weyl spinor by coupling it to its charge conjugate (charge conjugation changes chirality in $D/2 \notin 2\mathbb{Z}+p$ dimensions, where B is off-diagonal in a Weyl representation). Such a mass term is called *Majorana mass* and it violates fermion number by 2 units.

The **Fierz rearrangement** formulas, which are crucial for the existence of supersymmetric actions, follow from completeness of the basis $\{\Gamma^I\} = \{\Gamma^{a_1\ldots a_r}\}$ of the Clifford algebra (for a faithful representation the corresponding matrices thus provide a basis for the space of linear maps of the representation space). We define $\Gamma_I := (\Gamma^I)^{-1} = (\Gamma^I)^3$; $(\Gamma^I)^2 = \varepsilon_{(I)}\mathbf{1}$ with $\varepsilon_{(I)}^2 = 1$.

Lemma: In even dimensions $\operatorname{tr}\Gamma^I = 0$ if $\Gamma^I \neq \mathbf{1}$, $\operatorname{tr}\mathbf{1} = 2^{D/2}$ and $\omega_I = \frac{1}{2^{D/2}}\operatorname{tr}(\not\omega\Gamma_I)$.

Proof: For all $\Gamma^I \neq \mathbf{1}$ there exists a Γ^J that anti-commutes with Γ^I (in odd dimensions also Γ_* has to be excluded). Hence $\operatorname{tr}\Gamma^I = \operatorname{tr}\Gamma^I\Gamma^J\Gamma_J = -\operatorname{tr}\Gamma^J\Gamma^I\Gamma_J = -\operatorname{tr}\Gamma^I$ (in the last step we used cyclicity of the trace). The rest is obvious. □

Theorem: In even dimensions the following (equivalent) rearrangement formulas hold:

$$\sum_I(\Gamma^I\otimes\Gamma_I)_{\underline{\alpha}\underline{\gamma}}^{\beta\delta} := \sum_I(\Gamma^I)_{\underline{\alpha}}{}^{\beta}\,(\Gamma_I)_{\underline{\gamma}}{}^{\delta} = 2^{D/2}\delta_{\underline{\alpha}}^{\delta}\delta_{\underline{\gamma}}^{\beta}, \tag{1.32}$$

$$(\overline{\varphi}\Gamma^A\psi)\,(\overline{\chi}\Gamma^B\eta) = (-)^{\psi\overline{\chi}+\psi\eta+\overline{\chi}\eta}\sum_{IJ}\alpha^{IJ}\,(\overline{\varphi}\Gamma_I\Gamma^B\eta)\,(\overline{\chi}\Gamma_J\Gamma^A\psi), \qquad \alpha^{IJ} = \varepsilon_{(I)}\delta^{IJ}/2^{D/2}. \quad (1.33)$$

This means, in particular, that the partners in 4-Fermi terms of a Lorentz-invariant action can be exchanged. The number of terms that arise in this way, however, grows rapidly with the dimension.

Proof: First we observe that $\sum_I \Gamma^I\Gamma^A\Gamma_I = \rho_A\mathbf{1}$ because it commutes with all elements of the Clifford algebra (which follows from $\sum_I \Gamma^I\Gamma^A\Gamma_I\Gamma^B = \sum_I \Gamma^B(\Gamma_I\Gamma^B)^{-1}\Gamma^A\Gamma_I\Gamma^B = \sum_I \Gamma^B\Gamma^I\Gamma^A\Gamma_I$, where the sum over all Γ^I was replaced by the sum over all $\Gamma_I\Gamma^B$ for fixed Γ^B). Computing the trace we find $\rho_A = 2^D\,\mathrm{tr}\,\Gamma^A/\,\mathrm{tr}(\mathbf{1}) = 2^{D/2}\,\mathrm{tr}\,\Gamma^A$. Contracting (1.32) with $(\Gamma^A)_{\underline{\beta}}{}^{\underline{\gamma}}$ the r.h.s. becomes proportional to $\delta_{\underline{\alpha}}^{\underline{\delta}}$. Similarly we find the proportionality to $\delta_{\underline{\gamma}}^{\underline{\beta}}$, and contraction of all indices gives us the normalization. □

If we decompose the Dirac spinors into their Weyl components completeness of the Γ^I matrices allows us to decompose $s\otimes s$ and $s\otimes c$ into irreducible tensor representations of the Lorentz group. We get the correct position of indices by multiplication with the charge conjugation matrix $(C^{-1}\Gamma^I)^{\underline{\alpha\beta}}$. Thus the tensor product of spinors of equal chirality decomposes into even forms in $D\in 4\mathbb{Z}$ dimensions (where C is block diagonal), and into odd forms in $D\in 4\mathbb{Z}+2$ dimensions. For $c\otimes s$ even and odd forms are exchanged. Comparing the dimension 2^{D-2} of the tensor products of chiral and/or antichiral fermions with the number of independent components of the antisymmetric tensors $\sum_{even}\binom{D}{r} = \sum_{odd}\binom{D}{r} = 2^D/2$ there seems to be a mismatch by a factor of 2. This is easily resolved by observing that γ_* is proportional to unity on Weyl spinors, so that forms of degree r and $D-r$ have to be identified (multiplication by γ_* is analogous to the Hodge $*$ operation on forms, i.e. contraction with the ε tensor, which is invariant under special orthogonal transformations).

For the middle degree $r = D/2$ the $*$ operation is an endomorphism of Λ^r with $*^2 = (-)^{Dr+r+q}$ on r forms, which can thus be decomposed into eigenspace. Inserting $D = p+q$ and $r = D/2$ we obtain $*^2 = (-)^{r-q} = (-)^{\frac{p-q}{2}} = {\gamma_*}^2$. Therefore the eigenvectors of $*$ are (real) selfdual $\omega = *\omega$ or anti-selfdual $\omega = -*\omega$ forms iff $\frac{p-q}{2}\in 2\mathbb{Z}$, which coincides with the signatures where charge conjugation preserves chirality. In particular, $s\otimes s$ contains the scalar in $D\in 4\mathbb{Z}$ (C is block diagonal) and the vector in $D\in 4\mathbb{Z}+2$ (C is off-diagonal). This is in agreement with the result that the center of $D_n = Spin(2n)$ is $\mathbb{Z}_2\times\mathbb{Z}_2$ for even n and $\mathbb{Z}_4$ for odd n (the center is related to the conjugacy classes of representations). The conjugate representation in the group theoretic sense coincides with the complex conjugate iff A is block diagonal, i.e. iff p is even.

$SO(p,q)$ consists of two connected components if $pq>0$ (this is easy to see if $p=1$: the first line of $v^b = {\Omega_0}^b$ must have length 1, hence $(v^0)^2 = 1+\vec{v}^2 > 1$ and ortochronous transformations with $v^0>0$ cannot be continuously connected to transformations that change the time direction; in the general case one has to consider the determinant of Ω restricted to a positive definite subspace of maximal dimension).[5] Transformations that are not in the component $SO^0(p,q)$ of the unit element thus cannot be obtained by exponentiation of the Lie algebra. The above proof of the existence of Λ and the fixing of its normalization by the intertwining equations nevertheless apply, even if we consider general orthogonal transformations $O(p,q)$. In the latter case the normalized Λ's represent the double covering group $Pin(p,q)$ of $O(p,q)$. All elements of Pin can be written as $\Lambda \in Spin$ with a possible additional factor of γ^0, which implements the parity transformation in Minkowski space.[6] Another group that is sometimes considered is the Clifford group, which consists of all invertible elements Λ of the Clifford algebra. Since $\Lambda\gamma^a\Lambda^{-1}$ always corresponds to an orthogonal transformation the Clifford group is a (twisted) product of Pin and the abelian group that corresponds to multiplication by $\pm\lambda \in \mathbb{C}^*/\mathbb{Z}_2$.

It was discovered in 1956 that parity P and charge conjugation symmetry C are violated in weak interaction. In 1964 it was observed in kaon decay that even the product CP is not a symmetry of nature. CPT, however, is respected by any local quantum field theory and, so far, it seems also by nature. The proof of this theorem uses the transformation behavior of the building blocks of the Lagrangian. Our interwiners A, B and C are, indeed, related to just these three discrete transformations: A changes sign of space-like directions, C corresponds to charge conjugation, and their product B comes with complex conjugation, which is related to a change of the time direction in the Schrödinger equation.

If we want to have spinor fields in curved space we need a vielbein field ${e_m}^a$, which provides an orthonormal basis $e^a = dx^m {e_m}^a$ of cotangent space and whose inverse ${E_a}^m$ allows us to define the Dirac operator $\partial\!\!\!/ := \gamma^a {E_a}^m \partial_m$. Globally we also need a **spin structure**: On an (orientable) manifold the structure group of the tangent bundle can always be reduced to the (special) orthogonal group. For a spin structure we need to lift this to a bundle with structure group (S)Pin. The (s)pin structure, if it exists, need not be unique: On compact orientable Riemann surfaces, for example, there are 2^{2g} spin structures (which are combined into two classes, the even and the odd spin structures, whose elements are mixed by modular transformations). The Klein Gordon operator for spinors in torsion-free curved space is $(iD\!\!\!/ + m)(iD\!\!\!/ - m) = -\Box - m^2$, where we defined the Laplace operator on spinors as $\Box := D\!\!\!/^{\,2} = D^2 - \frac{1}{4}R$.
Proof: Since (γ^m) is a Lorentz-invariant tensor $D\!\!\!/^{\,2} = (\gamma^{(m}\gamma^{n)} + \gamma^{[m}\gamma^{n]})D_mD_n$ and $D_{[m}D_{n]} = \frac{1}{2}[D_m, D_n]$. Inserting the Lorentz generator in the spin representation we find $\gamma^{[m}\gamma^{n]}[D_m, D_n] = \gamma^{[m}\gamma^{n]}\frac{1}{2}{R_{mn}}^{ij}\Sigma_{ij} = -\frac{1}{2}R$ because $\gamma^{[m}\gamma^{n]}\gamma^{[i}\gamma^{j]} = A\Gamma^{mnij} - B(\Gamma^{mi}\eta^{nj} - \Gamma^{ni}\eta^{mj} - \Gamma^{mj}\eta^{ni} + \Gamma^{nj}\eta^{mi}) - C(\eta^{mi}\eta^{nj} - \eta^{mj}\eta^{ni})$ with $A = B = C = 1$, as can be seen by inserting $mnij = 0123,\ 0112$ and 0110 into this ansatz. □
With this definition one finds for integrals over compact manifolds that $\langle \Box\phi, \psi\rangle = \langle \phi, \Box\psi\rangle$.

[5] It can be shown that $Spin^0(p,q)$ is simply connected for $D > 2$. $SO(2,0)$ is the infinitely connected circle and the components of the hyperbola $SO(1,1)$ already are simply connected.

[6] Any orthogonal transformation can be generated by at most D reflections on hyperplanes orthogonal to vectors n, which can be realized as $\psi\!\!\!/ \to n\!\!\!/\psi\!\!\!/ n\!\!\!//n^2$; $SO(p,q)$ transformations correspond to an even number of reflections.

Chapter 2

Supersymmetry

We call a symmetry that (linearly) transforms physical Bose fields into physical Fermi fields a supersymmetry (SUSY). Because of the spin–statistics theorem this implies that a SUSY generator Q_α transforms as a spinor under Lorentz transformations and is an odd element of a $\mathbb{Z}_2$–graded symmetry algebra (assuming that Q and its hermitian conjugate act on a positive definite Hilbert space it can be shown that Q must have spin 1/2 in 4 dimensions [WE83]).

The main constraint on the form of the supersymmetry algebra comes from the Coleman-Mandula theorem [co67], which states that the most general Lie algebra of symmetries of the S-matrix is the direct sum of the Poincaré algebra and a reductive compact Lie algebra if
1. the S-matrix is based on a local relativistic quantum field theory in 4 dimensions,
2. there are only a finite number of particles associated with one particle states of a given mass,
3. and there is a mass gap between the vacuum and one particle states.
In other words: Space-time symmetries and internal symmetries don't mix. When it was realized that this no-go result can be circumvented by supersymmetries (which are not part of a Lie algebra), Haag, Lopuszanski and Sohnius [ha75, WE83] used the Coleman-Mandula theorem to analyze the general structure of *graded* symmetry algebras. The bosonic part of such algebras must be of the form that is predicted for Lie algebras of symmetries.

To take advantage of our knowledge about irreducible spin representations we will mostly use a special type of Weyl basis for our analysis of SUSY algebras: Restricting our attention to Minkowski signature $p = 1$ we can construct a Weyl representation in even dimensions using a set $\{\sigma^i\}$ of γ matrices that generate an irreducible representation of $\mathcal{C}(D-1,0)$, i.e. the dimension of this representation is $2^{D/2-1}$ and σ^{D-1} is proportional to the product $\sigma^1 \ldots \sigma^{D-2}$. Then we define

$$(\gamma^a)_{\underline{\alpha}}{}^{\underline{\beta}} = \begin{pmatrix} 0 & (\sigma^a)_{\alpha\dot\beta} \\ (\overline{\sigma}^a)^{\dot\alpha\beta} & 0 \end{pmatrix}, \qquad \gamma^0 = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \tag{2.1}$$

so that $\overline{\sigma}^0 = \sigma^0 = \mathbf{1}$ and $\overline{\sigma}^i = -\sigma^i$. It is easily checked that these matrices obey $\{\gamma^a, \gamma^b\} =$

$-2\eta^{ab}$. Changing, for example, the sign of σ^a (but not of $\overline{\sigma}^a$) we obtain an analogous basis for the other sign convention in the Clifford algebra, as is used in the book of Wess and Bagger [WE83]. In the Majorana–Weyl case $D \equiv 2(8)$ the matrices σ^i can be chosen to be real, so we automatically obtain a real Weyl basis (which could be modified into a Majorana–Weyl basis by $\sigma^a \to i\sigma^a$ and $\overline{\sigma}^a \to -i\sigma^a$).

Symmetries can always be assumed to be real, because real and imaginary part of a symmetry transformation acting on the *real* action must vanish individually. SUSYs are therefore often written in terms of Majorana spinors, which may cause some headache if we think about 6 dimensions, where no Majorana spinors exist. What matters, however, is that there always exist real representations (take $R \oplus R^*$ for a Weyl representation R). As far as counting of SUSYs is concerned there is thus no difference between 4 and 6 dimensions. What is different is that complex conjugation does not flip chirality in 6 dimensions, so that $R \oplus R^*$ cannot be written in terms of a single Dirac spinor. In 2, 6 and 10 dimensions extended SUSYs thus exist with various distributions of chiralities: For $N = 2$ theories in 6D and 10D the names IIA and IIB are used for the non-chiral and the chiral case, respectively; in 2D one talks about (p,q) SUSYs (where $N = p + q$).

To make chiralities more explicit it is useful to rewrite complex conjugates in terms of Dirac adjoints (intertwiners can be used to write everything with indices α and $\dot\alpha$, which correspond to the two inequivalent spinor representations). To write $\overline{\psi} = A^T\psi^*$ in terms of $(\psi^{\underline{\alpha}})^*$ we observe that complex conjugation of $AC = bcB$ gives $aA^TC^* = bc(B^T)^{-1}$, where $A^* = aA^T$. To pull down the index of $\overline{\psi}$ we multiply by C and use $BC^* = BB^*A = a^*bA^{-1}$ to arrive at

$$\overline{\psi}^{\underline{\alpha}} = \psi^*_{\underline{\beta}} A^{\overline{\beta}\underline{\alpha}} = a^*bc\,\psi^{\underline{\beta}*} B^{\dagger}_{\overline{\beta}}{}^{\underline{\alpha}}, \qquad \overline{\psi}_{\underline{\alpha}} = B_{\underline{\alpha}}{}^{\overline{\beta}}\psi^*_{\underline{\beta}} = a^*bA^{\dagger}_{\underline{\alpha}\overline{\beta}}\,\psi^{\underline{\beta}*}. \tag{2.2}$$

In Minkowski space $bc = -1$ is independent of the dimension and $A = \gamma_0$ implies $a = 1$ as $(\gamma^0)^2 = \mathbf{1}$. Putting the pieces together we obtain the complex conjugation formula

$$(\xi^{\underline{\alpha}}Q_{\underline{\alpha}})^* = abc(-)^{\xi Q}\;\overline{\xi}^{\underline{\alpha}}\overline{Q}_{\underline{\alpha}} = ab(-)^{\xi Q}\;\overline{\xi}_{\underline{\alpha}}\overline{Q}^{\underline{\alpha}} \tag{2.3}$$

(recall that $\xi^{\underline{\alpha}}Q_{\underline{\alpha}} = \xi^{\underline{\alpha}}C^T_{\underline{\alpha\beta}}C^{-1\underline{\beta\gamma}}Q_{\underline{\gamma}}$ and $C^TC^{-1} = c\mathbf{1}$, with $c = (-)^{[\frac{D/2+1}{2}]}$ being negative in $2, 4, 10, 12, \ldots$ dimensions, so that the position of indices is relevant in exactly these dimensions). One might expect that the intertwiner C should transform covariantly, i.e. that its indices can be raised with C^{-1}. The fact is that $(C^{-1})^{\underline{\alpha\gamma}}(C^{-1})^{\underline{\beta\delta}}C_{\underline{\gamma\delta}} = (C^{-1})^{\underline{\beta\alpha}} = (C^{-1T})^{\underline{\alpha\beta}}$. This motivated to two different conventions for the charge conjugation matrix with upper indices: Penrose and Rindler [PE84], among others, insist that $C^{\underline{\alpha\beta}}$ should be the result of pulling indices up, so that C with upper indices is equal to C^{-1T}. Wess and Bagger [WE83], however, and most other books on SUSY insist that $C^{\underline{\alpha\beta}}$ and $C_{\underline{\alpha\beta}}$ should be inverse, so that all indices are shifted by left multiplication with C, except for the indices of C itself in the dimensions where $c = -1$.

To compare our results with the usual spinor gymnastics in 4 dimensions we define the proper position for index contractions to be $\xi^\alpha Q_\alpha$ if Q is chiral and $\xi_{\dot\alpha}Q^{\dot\alpha}$ if it is antichiral. B must be proportional to the product of all $\gamma_*\gamma^a$ with γ^a real. In 4D and with σ^i being the Pauli matrices we thus find $B\propto\gamma^2$. Choosing a phase of B such that $C=BA^*=\pm i\gamma^2\gamma^{0*}$ is real the charge conjugation matrix becomes

$$\begin{pmatrix}\varepsilon_{\alpha\beta} & 0\\ 0 & \varepsilon^{\dot\alpha\dot\beta}\end{pmatrix} = C = -i\gamma^2\gamma^{0*}=\begin{pmatrix}-i\sigma_2 & 0\\ 0 & i\sigma_2\end{pmatrix}, \qquad \varepsilon\sim i\sigma^2=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} \tag{2.4}$$

Using the conventions of Penrose and Rindler this is consistent with $\varepsilon_{21}=\varepsilon^{21}=1=\varepsilon_{\dot1\dot2}=\varepsilon^{\dot1\dot2}$, and $\varepsilon_{\dot\alpha\dot\beta}=-\varepsilon_{\alpha\beta}$ with $\varepsilon_{\alpha\beta}\varepsilon^{\beta\gamma}=-\delta_\alpha^\gamma$. To recover those of **Wess and Bagger**,

$$\psi_\alpha=\varepsilon_{\alpha\beta}\psi^\beta, \qquad \overline\psi_{\dot\alpha}=\psi^*_\alpha=\varepsilon_{\dot\alpha\dot\beta}\overline\psi^{\dot\beta}, \qquad \varepsilon^{12}=\varepsilon_{21}=1=\varepsilon^{\dot1\dot2}=\varepsilon_{\dot2\dot1}, \qquad \varepsilon_{\alpha\beta}\varepsilon^{\beta\gamma}=\delta_\alpha^\gamma, \tag{2.5}$$

we have to change the sign of $\varepsilon^{\alpha\beta}$ and of $\varepsilon_{\dot\alpha\dot\beta}$.

For our choice of basis and with $A=\gamma^0$ we find in even dimensions

$$\psi^*_\alpha=\overline\psi_{\dot\alpha}, \qquad (\psi^\alpha)^*=b\overline\psi^{\dot\alpha}, \qquad \psi^*_{\dot\alpha}=b\overline\psi_\alpha, \qquad (\psi^{\dot\alpha})^*=\overline\psi^\alpha. \tag{2.6}$$

Note that in $D=2,6,10,\ldots$ dimensions with Minkowski signature charge conjugation preserves chirality and $C_{\underline{\alpha\beta}}$ is offdiagonal, so that $(\xi^\alpha Q_\alpha)^*=(-)^{\xi Q}b\,\overline\xi^{\dot\alpha}\overline Q_{\dot\alpha}$ is somewhat misleading and should rather be written as $(-)^{\xi Q}bc\,\overline\xi^\alpha\overline Q_\alpha=-(-)^{\xi Q}\,\overline\xi^\alpha\overline Q_\alpha$ (in this case the parameter ξ^α for a chiral SUSY transformation Q_α is antichiral). We thus obtain the real BRST transformations

$$\begin{aligned} D=4,8,\ldots: &\qquad (\xi^\alpha Q_\alpha)^*=(-)^{\xi Q}b\,\overline\xi^{\dot\alpha}\overline Q_{\dot\alpha}=(-)^{\xi Q}bc\,\overline\xi_{\dot\alpha}\overline Q^{\dot\alpha},\\ &\qquad s=\xi^{\underline\alpha}Q_{\underline\alpha}=\xi^\alpha Q_\alpha-\overline\xi_{\dot\alpha}\overline Q^{\dot\alpha}, \end{aligned} \tag{2.7}$$

$$\begin{aligned} D=2,6,\ldots: &\qquad (\xi^\alpha Q_\alpha)^*=(-)^{\xi Q}bc\,\overline\xi^\alpha\overline Q_\alpha, \qquad (\xi_{\dot\alpha}Q^{\dot\alpha})^*=(-)^{\xi Q}bc\,\overline\xi_{\dot\alpha}\overline Q^{\dot\alpha}\\ &\qquad s_{c.}=\xi^\alpha Q_\alpha-\overline\xi^\alpha\overline Q_\alpha, \qquad s_{a.c.}=\xi_{\dot\alpha}Q^{\dot\alpha}-\overline\xi_{\dot\alpha}\overline Q^{\dot\alpha}. \end{aligned} \tag{2.8}$$

In $D=2\bmod 8$ dimensions the negative terms in s_{chical} and $s_{antichiral}$ should be omitted with $Q_\alpha=Q^c_\alpha=B_\alpha{}^\beta Q^*_\beta$ and $\xi^{\alpha*}=\xi^\beta B_\beta{}^\alpha$, i.e. Q_α and $i\xi^{\dot\alpha}=iC^{\dot\alpha}{}_\beta\xi^\beta$ should be Majorana spinors since $\xi^{\alpha*}=\xi^\beta B_\beta{}^\alpha$ implies $\xi^{\dot\alpha}=C^{\dot\alpha}{}_\beta\xi^\beta=(CB^{-1T}C^{-1*})^{\dot\alpha}{}_{\dot\beta}\xi^{\dot\beta*}$ with $CB^{-1T}C^{-1*}=C(BC^{-1})^*=C\,bcB^{-1}C=bc\,CA^{-1T}=bcCA^*=abcCA^T=abcB$ and $abc=-1$. In case of extended supersymmetries we have a sum of such terms. There are, of course, terms for all other symmetries to be added to the BRST transformation, which at this point is only a convenient device to collect all symmetries into a single real operator. Here the $\xi^{\underline\alpha}$ are just commuting parameters. In supergravity the *superghosts* $\xi^{\underline\alpha}$ will become coordinate dependent bosonic fields that are essential for covariant quantization.

2.1 SUSY algebra and central charges

We make the following ansatz for the anti-commutator of supercharges in flat space:

$$\{Q^I_{\underline{\alpha}}, Q^J_{\underline{\beta}}\} = 2\delta^{IJ}(\gamma^a C)_{\underline{\alpha\beta}} P_a + \sum_{p\geq 0} (\Gamma^{a_1\ldots a_p} C)_{\underline{\alpha\beta}} Z^{IJ}_{a_1\ldots a_p}, \tag{2.9}$$

where the indices $I, J = 1, \ldots, N$ label the spinor representations of the supercharges in case of extended SUSY. The term proportional to the momentum on the r.h.s. is characteristic for the algebra of supercharges, which are sort of square roots of momenta. The degrees and symmetries of p-form charges $Z^{IJ}_{a_1\ldots a_p}$ that may occur depend on the dimensions and chiralities of the spinors in the anti-commutator.

In 4 dimensions only scalar central charges $Z^{IJ}_{a_1\ldots a_p}$ are allowed by the Coleman–Mandula theorem,[1] so that the sum over p is restricted to 0 and 4. In both cases Z^{IJ} must be antisymmetric because $c(-)^{[\frac{r+1}{2}]} = (-)^{[\frac{D/2+1}{2}]+[\frac{r+1}{2}]}$ is negative. This allows for $N(N-1)$ independent real central charges. Writing the algebra in terms of Weyl spinors and using the Jacobi identities it can be shown [ha75, WE83, ly96] that[2]

$$\{Q^A_\alpha, \overline{Q}_{\dot\beta B}\} = 2\sigma^m_{\alpha\dot\beta} P_m \delta^A_B, \qquad \{Q^A_\alpha, Q^B_\beta\} = \varepsilon_{\alpha\beta} a^{i[AB]} T_i, \qquad [Q^L_\alpha, T_l] = S_l{}^L{}_M Q^M_\alpha, \tag{2.10}$$

$$[T_l, T_m] = i f_{lm}{}^k T_k, \qquad \{\overline{Q}_{\dot\alpha A}, \overline{Q}_{\dot\beta B}\} = \varepsilon_{\dot\alpha\dot\beta} a^*_{i[AB]} T^i, \qquad [T^l, \overline{Q}_{\dot\alpha L}] = S^{*l}{}_L{}^M \overline{Q}_{\dot\alpha M} \tag{2.11}$$

is the most general form of the symmetry algebra, where a^l intertwines the representations S_l and $-S^{*l}$ in which Q and $\overline{Q}$ transform under the Lie algebra generators T^l (all other graded commutators vanish except for those with the angular momenta, which are fixed by the Lorentz transformation properties).

To arrive at this result (see chapter 1 of [WE83] for details) one expands the central charges $Z \pm i\varepsilon^{abcd} Z_{abcd}$ in terms of the internal bosonic generators T_l with complex coefficients $a^{l[AB]}$. Odd (even) form degrees, i.e. momenta (central charges), occur in the tensor products of spinors with opposite (equal) chirality. Since $[Q^A_\alpha, T_l]$ is a chiral fermionic symmetry generator it must be proportional to Q^B_β, i.e. $[T, Q^A_\alpha] = -S^A{}_B Q^B_\alpha$. and $[T, \overline{Q}_{\dot\alpha A}] = S^*{}_A{}^B \overline{Q}_{\dot\alpha B}$. This fixes the form of eqs. (2.10) and (2.11).

Evaluation of the Jacobi identities (JI) implies the hermiticity and intertwining properties of a and S: Since $[P_m, T_l] = 0$ the JI for $\{T, Q, \overline{Q}\}$ implies that $S^{*l}{}_M{}^L = S_l{}^L{}_M$, i.e. $S = S^\dagger$. The JI for $\{T, Q, Q\}$ implies that $[T, Z]$ is proportional to Z's, so that because of the JI for $\{Q, Q, \overline{Q}\}$ implies $[\overline{Q}, Z] = [Z, Z] = 0$. Since a reductible Lie algebra is a direct product of simple and abelian factors, the Coleman Mandula theorem now implies $[T, Z] = 0$, which, when inserted back into the JI for $\{T, Q, Q\}$ implies that a intertwines S and $-S^*$.

[1] In principle the Lorenz generators could provide a 1-form contribution on the r.h.s., but this is inconsistent with translational invariance of the SUSY charges [WE83].

[2] Here we use the somewhat cryptic convention of [WE83] that complex conjugation shifts the index position.

The simplest example of a solution to these constraints is the case $N = 2$, where ${f_{lm}}^k$ are the structure constants of an SU_2 in whose fundamental representation the supercharges transform, the representation matrices are $S_l = \sigma_l$ and $-S_l^* = -\sigma_l^*$, and ε^{AB} is the l-independent intertwiner so that

$$\{Q_\alpha^A, Q_\beta^B\} = \varepsilon_{\alpha\beta}\varepsilon^{AB}(c_1 Z_1 + ic_2 Z_2). \tag{2.12}$$

This $N = 2$ algebra in 4 dimensions can be obtained by dimensional reduction from $D = 6$ dimensions, in which case the momenta in the two 'internal' dimensions can be interpreted as the two central charges.

An interpretation of central charges in terms of 'internal momenta' is, however, not possible for the $N = 4$ and $N = 8$ algebras, whose versions without central charges can be obtained by reduction of $N = 1$ theories in $D = 10$ and $D = 11$: In these cases there can be 12 and 56 independent charges, with a maximum of 6 and 7 internal dimensions, respectively. The missing pieces cannot come from central charges in the $N = 1$ algebra in 10- dimensions, which does not allow for (scalar) central charges. It has been observed, however, a long time ago [ho82] that this algebra allows, in addition to the momentum, for a self-dual 5-form charge Z^+_{abcde} in the anticommutator of chiral supercharges (only odd forms occur in $s \otimes s$ and since $c = -1$ the matrices $\Gamma^{a_1 \cdots a_p} C$ are symmetric only for $p = 1, 2, 5, 6, 9$).

The way in which the central charges arise in this context is quite non-trivial [ab91]. The problem with the p-form charges, and the reason why they are forbidden by the Coleman Mandula theorem, is that they cannot be carried by point particles but only by extended objects. Recall that the classical coupling of a 1-form gauge field to a 0-form charge can be written as an integral of the 1-form over the $0 + 1$ dimensional world line of the particle. In a similar way strings may carry a 1-form charge, with the coupling to the corresponding 2-form gauge field given by its integral over the world sheet, and a p-form charge can be carried by a p-brane, an object that extends into 1 time-like and p space-like directions. Point particles, strings and membranes are thus 0-branes, 1-branes and 2-branes, respectively.

These ideas suggest that 10-dimensional super Yang-Mills theory should allow for a solitonic 5-brane solution that could carry the 5-form charge. Such a solution to the non-linear field equations indeed exists: We just have to recall that there is an instanton[3] solution with self-dual field strength to the 4-dimensional YM/Higgs equations. With a constant extension of this instanton into 1 time-like and 5 space-like directions this provides us with a 5-brane in 10 dimensions. If we then compactify down to 5 dimensions on a torus we can wrap the 5 trivial space-like dimensions of the 5-brane around the 5-torus and thus obtain a point-like particle in

[3] Instantons are localized in Euclidian time and their action is related to quantum mechanical tunneling probabilities, whereas solitons are localized wave packets that evolve in time without decay. Instantons in D dimensions thus give rise to solitons in $D + 1$. The existence of such solutions is due to non-linearities in the field equations and they are stable because they carry topological charges (instanton/winding numbers) [eg80].

$4+1$ dimensions that looks like a magnetic monopole. The maximal dimension in which this mechanism can provide extra central charges is thus $D = 5$.

To get down to 4 dimensions we also need to compactify one of the non-trivial dimensions of the instanton on a circle and deform the selfdual soliton solution to a selfdual configuration on $\mathbb{R}^3 \times S^1$. Alltogether we thus have a compactification on the 6-torus $T^6 = T^5 \times S^1$. The $N = 4$ SUSY multiplet contains a vector field and 6 scalars (these multiplets will be analysed in the next section; the scalars can be interpreted as the 6 compactified components of the vector field in $D = 10$ super-YM). This allows for 6 different types of magnetic charges by using one of these scalars as the Higgs field for the monopole construction. All of these 6 different charges can be realized by the different choices of the non-trivial direction in the compactification on T^6. In this way we find a realization of the maximal number 12 of central charges for the $N = 4$ algebra in $D = 4$. Only half of these charges are electric with the remaining half being magnetic, so that they cannot be seen in perturbation theory.

The simplest way to understand the non-perturbative nature of magnetic charges is the Dirac quantization condition: If we couple the field strength to both types of charges we have to modify the BI $dF = 0$ into an equation of motion $dF = *j_{(m)}$, where $j_{(m)}$ is the magnetic current 1-form. Of course F can then be written as $F = dA$ only locally and only in source-free regions of space-time. For a point-like magnetic source the gauge connection A can be defined outside a ray from the source to infinity, the so-called **Dirac string**, because the Poincaré lemma is valid for star-shaped regions.

The charge of a magnetic monopole can be measured by the integral $g = \int_{S^2} F$ of the magnetic flux over a 2-sphere containing the source, just as the electric charge is given by the integral $e = \int_{S^2} *F$ over the electric flux. Since F is continuous we can take out an infinitesimal disk Δ_ε of the sphere around the intersection with the Dirac string and let $\varepsilon \to 0$ to compute the magnetic charge: $g = \int_{S^2} F \approx \int_{S^2-\Delta_\varepsilon} dA = -\int_{\partial\Delta_\varepsilon} A$. The Aharonov-Bohm phase that the wave function of an electrically charged particle of charge e receives when carried around the Dirac string is $\exp(2\pi i e \oint A)$. Vanishing of this phase thus implies that the product $ge \in 2\pi\mathbb{Z}$ must be quantized for all electric charges e and magnetic charges g in a consistent quantum-mechanical system [$\mathtt{ha}_2\mathtt{96}$].[4] Therefore, as the electric charge goes to 0 in a perturbative expansion the magnetic one goes to infinity, and so does the mass of any solitonic solution that provides a magnetic source, for example, in a spontaneously broken GUT.

If we increase the electric coupling the monopoles become lighter and may eventually dominate the physics in the strong coupling regime. This suggests that there may be a dual description in which the elementary degrees of freedom are magnetic and the electrically charged

[4] For *dyons* that have both types of charges this generalizes to the Dirac–Zwanziger quantization condition $e_1 g_2 - e_2 g_1 \in 2\pi\mathbb{Z}$ with e_i and g_i being the electric and magnetic charge of the the i^{th} particle.

particles are solitons. Such a naive duality is spoiled, however, by quantum corrections (one of the obstacles is the running of the coupling constant). In supersymmetric theories the central charges imply Bogomol'nyi bounds [bo76] for the masses of charged particles (see below), so that many classical results are protected against quantum corrections (monopoles that saturate this bound are called BPS monopoles [pr75]). This lead to the Montonen–Olive conjecture of a duality in $N = 4$ super-YM theory. Such a duality that relates a strongly coupled (electric) theory to a weakly coupled (magnetic) theory is called S duality. This picture has then been extended to a group of $SL(2\mathbb{Z})$ transformations for $\tau = \frac{\theta}{2\pi} + i\frac{4\pi}{e^2}$, where θ is the coefficient of the topological term $\int F \wedge F$ in the YM action [ol78]. In the $N = 2$ supersymmetric case the duality is conjectured to apply to the low energy effective theory of a spontaneously broken SYM model [se94, bi96].

The analogous story for $N = 8$ is complicated by the necessity to include gravity since the minimal $N = 8$ multiplet in 4 dimensions contains a spin 2 particle. All the allowed central charges can be realized by compactifications of type IIA or IIB $N = 2$ supergravities in 10 dimensions or of the $N = 1$ supergravity in $D = 11$ [to95]. In the following table we list the allowed p-form charges and the dimensions of the existing solitonic p-branes [du95, to95], including their multiplicities in the IIB case. With $\binom{10}{2} = 45$, $\binom{10}{5} = 252$, $\binom{10}{6} = 210$ one finds

$D:\ N$	$Q_{\underline{\alpha}}$	degrees	Z^{IJ}	p-form	branes	#(charges)
10: 1	s	odd	sym.	$1,\ 5^+$	1, 5	$136 = 10 + \frac{1}{2}\binom{10}{5}$
10: IIA	Dirac	all	sym.	1,2,5,6,9,10	0,1,2,4,5,6	528=10+45+252+210+10+1
10: IIB	$s \oplus s$	odd	any	$1, 3, 5^+_3, 7, 9$	$1_2, 3, 5_2$	$528 = 3\times 10 + 120 + 3\times 126$
11: 1	Dirac	all	sym.	1,2,5	2,5	$528 = 11 + \binom{11}{2} + \binom{11}{5}$

As in the $N = 4$ case only half of the central charges are electrical, so that $28 = 56/2$ is the maximal number of charges that can be carried by perturbative states. This 'maximally democratic' situation is realized, for example, by the heterotic string [to95].

Note that the electric–magnetic duality applies to field strengths $F_{p+2} = *F'_{(D-p-4)+2}$, so that p branes that may carry the cooresponding p-form charges are dual to $D - p - 4$ branes [ne85, te86]. In particular, point particles are dual to point particles in 4 dimensions and strings are dual to strings in 6 dimensions and to 5-branes in 10 dimensions. After compactification $D - p - 4$ only gives a lower bound since some of the dimensions of the extended object may be wrapped around non-trivial cycles of the internal manifold.

The wrapping modes of string solitons [st90, du95] have a beautiful application to Calabi–Yau compactifications on singular manifolds: It has been known for some time that conifold singularities occur at finite distance (with respect to the Zamolodchikov metric of the σ model CFT) in the moduli spaces of Calabi–Yau spaces. At these singluarities correlation functions of the CFT diverge, which leads to conceptual problems. Strominger observed that these singularities can be explained by taking into account non-perturbative states that are due to the wrapping of 5-branes around cycles whose size shrinks to 0 as we approach the singular locus in moduli space [st95, gr95]. It is conjectured that all moduli spaces of Calabi–Yau compactification are connected via singular limits [ca89, av95], and that the physics is smooth at the transition points due to this *black hole condensation* mechanism. The change in the Hodge numbers, and in the corresponding numbers of massless matter multiplets, can indeed be reproduced by a counting of the non-perturbative states that become massless at the singularity.

2.2 Spontaneous SUSY breaking and the Witten index

One of the important motivations for SUSY is the hope for an explanation of the small cosmological constant. In nature SUSY cannot be realized as an unbroken symmetry since that would imply, for example, the existence of a scalar particle with the same mass and electric charge as the electron (Q_α commutes with the momentum P_m). It is thus important to investigate whether SUSY can be broken spontaneously and how this is related to the vacuum energy. For massive momentum eigenstates we find in the rest frame

$$\{Q_\alpha^A, \overline{Q}_{\dot\beta B}\} = 2M\delta_{\alpha\dot\beta}\delta_B^A \geq 0 \tag{2.13}$$

which is positive in a unitary theory, i.e. for a Hilbert space with positive norm. This implies that all SUSY generators vanish on (physical) states with vanishing momentum, like a vacuum with vanishing energy density. If, in turn, SUSY is not spontaneoulsy broken, then the l.h.s. vanishes and so does the vacuum energy.

The real puzzle is therefore why the cosmological constantant vanishes after SUSY breaking. Positivity of the vacuum energy is no longer required in SUGRA models, which may lead to *explicit* 'soft' SUSY breaking terms in the flat limit, but in any case we loose a prediction for the value of Λ [ni84, la87]. Recently Witten had an idea how SUSY could still keep Λ at 0 without enforcing the mass-degeneracy in 4 dimensions: In the context of string dynamics non-SUSY models in 4 dimensions can be related to supersymmetric ones in 3D, where SUSY can then keep $\Lambda = 0$ and at the same time avoid the troublesome implications in 4 dimensions [$\mathtt{wi}_1\mathtt{95}$].

An important quantity for the investigation of spontaneous SUSY breaking is the Witten index [wi82]

$$\mathrm{Tr}\,(-)^{N_F} = \mathrm{Tr}\; e^{2\pi i J_z}, \tag{2.14}$$

where N_F is the fermion number operator and the trace extends over the Hilbert space of a quantum mechanical system. This index only receives contributions from states with vanishing energy and momentum, as can be seen by evaluating the trace

$$2\sigma^m_{\alpha\dot\beta}P_m\delta^A_B\,\mathrm{Tr}(-)^{N_F}=\mathrm{Tr}(-)^{N_F}\{Q^A_\alpha,\overline{Q}_{\dot\beta B}\}=\mathrm{Tr}(-)^{N_F}Q^A_\alpha\overline{Q}_{\dot\beta B}-\mathrm{Tr}\,\overline{Q}_{\dot\beta B}(-)^{N_F}Q^A_\alpha=0 \quad (2.15)$$

for a subspace of fixed total momentum. For $P_m\neq 0$ all representations thus need to have an equal number of bosonic and fermionic degrees of freedom. Note that the Hamilton operator may not exist as an operator in the Hilbert space in a supersymmetric QFT with positive vacuum energy density. The Witten index can, however, be computed reliably at finite volume, i.e. in a box with periodic boundary conditions that do not spoil SUSY, since then the energy eigenvalues are discrete and finitely degenerate.[5]

Obviously, if the Witten index is non-vanishing then there are states with vanishing energy and SUSY cannot be spontaneously broken. Being an integer, we may expect that a continous change in the parameters of a model should not be able to change this index. From quantum mechanics we know that the low-lying energy spectrum indeed cannot have a discontinuous behaviour if the asymptotic form of the potential is not changed (discontinuities can arise, however, for example at $\lambda\to 0$ for $V=\frac12\phi^2+\lambda\phi^4$). The Witten index can thus be used to check whether SUSY can be broken by non-perturbative effects [wi82].

This index also makes explicit the relation between SUSY and topology: In the zero-momentum sector of the Hilbert space the Hamiltonian can be written as an anti-commutator of two supercharges,

$$H=\overline{Q}Q+Q\overline{Q},\qquad Q^2=\overline{Q}^2=0, \quad (2.16)$$

which is similar to the situation in Hodge theory, where the Laplacian is the anticommutator of d and its adjoint $\delta=(-)^p*^{-1}d*$; in 4 dimensions we can use $Q=Q_1$ or $Q=Q_2$ with $\overline{Q}=Q^*$. For a nilpotent operator all linear represention spaces decomposes into singlets $Q|\phi\rangle=0$ with $|\phi\rangle\neq Q|\phi'\rangle$ and doublets $(|\psi\rangle,Q|\psi\rangle$. For positive energy $Q|\phi\rangle=0$ implies $|\phi\rangle=Q(\overline{Q}|\phi\rangle/E)$ so that all positive energy states are doublets

$$|\phi_-\rangle=\tfrac{1}{\sqrt{E}}\,\overline{Q}|\phi_+\rangle,\qquad |\phi_+\rangle=\tfrac{1}{\sqrt{E}}\,Q|\phi_-\rangle, \quad (2.17)$$

whose contributions to the index cancel as we already know. For $E=0$, on the other hand, we already know that positivity of the Hilbert space norm implies that all states must be Q invariant and thus must be singlets (Q commutes with H, so that a Q-exact state would have to be Q applied to a zero energy state). The states with $E=0$ thus coorespond to the cohomology classes of Q. Singlet states with opposite statistics can in principle pair up and become doublets with positive energy under perturbations of the model. If $\mathrm{Tr}(-)^{N_F}$ is

[5] The definition $(-)^{N_F}=\exp(2\pi iJ_z)$ can still be used: If the Lattice has the appropriate symmetry $(-)^{N_F}$ is, for example, the 4^{th} power of a rotation by 90 degrees. An example where it is important to worry about the existence of operators is the 'theorem' that $N=2$ SUSY cannot be broken to $N=1$ in flat space [ce84], which may be spoiled by contact terms in the current algebra [hu86].

non-zero, however, then there is an excess of bosonic or fermionic states that can't pair up and thus have to stay at zero energy, and SUSY cannot be broken spontaneously. There is a unique 'Hodge decomposition' of the ($\vec{P} = 0$)–sector of the Hilbert space into a sum of 'exact' states $|\phi_+\rangle = Q|\phi_1\rangle$, 'coexact' states $|\phi_-\rangle = \overline{Q}|\phi_2\rangle$, and 'harmonic' states $|\phi_0\rangle$ with $Q|\phi_0\rangle = \overline{Q}|\phi_0\rangle = 0$. For supersymmetric σ models on Riemann surfaces the Witten index turns out to be given by the Euler characteristic [wi82].

2.3 4D SUSY multiplets and BPS bounds

Before we construct QFTs that actually feature SUSY we first study the representations of SUSY that can occur for momentum eigenstates in 4 dimensions. We have to treat separately the cases of massive and massless particles, since in the former case we can go to the restframe and assume $p_m = (M, 0, 0, 0)$, whereas for massless particles we can only use spatial rotations to achieve $p_m = (E, 0, 0, E)$. The respective *stability groups* (also called *little groups*) that leave this choice invariant are $SO(3) \cong SU(2)/\mathbb{Z}_2$ and $SO(2) \cong U(1)$, respectively.[6]

We first consider the case of massive particle with vanishing central charges. In the rest frame the anticommutation relations

$$\{Q^A_\alpha, \overline{Q}_{\dot\beta B}\} = 2M\delta_{\alpha\dot\beta}\delta^A_B \geq 0, \qquad \{Q, Q\} = \{\overline{Q}, \overline{Q}\} = 0, \tag{2.18}$$

then define the Clifford algebra $\mathcal{C}_\mathbb{C}(4N)$, as can be seen by computing the anticommutators for the set $\{\gamma^a\} = \{(Q^A_\alpha + \overline{Q}_{\dot\alpha A})/\sqrt{2M}\} \cup \{i(Q^A_\alpha - \overline{Q}_{\dot\alpha A})/\sqrt{2M}\}$. The irreducible faithful representation of this algebra can be constructed by declaring, for example, that $a^A_\alpha = Q^A_\alpha/\sqrt{2M}$ must annihilate a Clifford vacuum Ω, with the creation operators $(a^A_\alpha)^\dagger = \overline{Q}_{\dot\alpha A}/\sqrt{2M}$ generating the representation freely up to the anticommutation relations.

The state with the highest spin in a multiplet is found by symmetrizing in as many spinor indices as possible. If Ω is an SU_2 singlet then we obtain the *fundamental massive multiplet* whose dimension is 2^{2N}. Since the creation operators anti-commute the index pairs $(A, \dot\alpha)$ have to be anti-symmetrized. Symmetrization in $\dot\alpha$ thus requires anti-symmetrization in A, so that we can symmetrize at most N spinor indices and the maximal spin in the fundamental multiplet is $N/2$. If Ω itself transforms as a spin j representation then the dimension of the representation becomes $(2j+1)2^{2N}$ and its maximal spin is $j+N/2$. The detailed spin content of such a SUSY multiplet is obtained by decomposing the tensor products of the spin j representation Ω with the components of the fundamental massive multiplet into irreducible SU_2 representations. The result of this straightforward exercise is given in the table below.

[6] It was observed by Wigner that the difference between the representation theories for on-shell states and for local Lorentz–invariant fields is the reason for the necessity of gauge invariance for a covariant description of massless fields with spin 1 (and 2) [wi39, bi82].

Spin	$N=1$	$\|0\rangle$	$\|\frac{1}{2}\rangle$	$\|1\rangle$	$\|\frac{3}{2}\rangle$	$N=2$	$\|0\rangle$	$\|\frac{1}{2}\rangle$	$\|1\rangle$	$N=3$	$\|0\rangle$	$\|\frac{1}{2}\rangle$	$N=4$	$\|0\rangle$
0		2	1				5	4	1		14	14		42
$\frac{1}{2}$		1	2	1			4	6	4		14	20		48
1			1	2	1		1	4	6		6	15		27
$\frac{3}{2}$				1	2			1	4		1	6		8
2					1				1			1		1
$n_b{=}n_f$		2	4	6	8		8	16	24		32	64		128

Table: Spin content of massive SUSY multiplets for extended SUSY in 4D.

Since we are dealing with representations of a Clifford algebra we have a compact $SO(4N)$ symmetry acting on these multiplets (the relevant symmetry groups are compact since they have to respect the norms of the Hilber space states). Under $SO(4N)$ the fundamental massive multiplet decomposes into two irreducible representations, which consist of the bosonic states and the fermionic states, respectively. The $SO(4N)$ transformations do not change the masses of particles, but they transform among states of different spins, so that it is useful to consider smaller invariance groups that are direct products with $SU(2)$. An obvious invariance group of the N extended SUSY algebra is $U(N)$.[7] Whereas Q and $\overline{Q}$ transform under inequivalent representations of $SO(1,3)$, they are equivalent as $SO(3)$ representations (in arbitrary dimensions the same arguments apply to Q_α and $Q_{\dot\alpha}$, i.e. the representations s and c). This suggests that it may be possible to extend $U(N)$ to a group that also transforms Q's into $\overline{Q}$'s. Since ε intertwines the $SO(3)$ representations[8] Q and $\overline{Q}=Q^\dagger$ we choose a basis

$$q_a^A=a_\alpha^A,\qquad q_a^{N+A}=\sum_\beta \varepsilon_{a\beta}(a_\beta^A)^\dagger\qquad\Rightarrow\qquad (q_a^A)^\dagger=\varepsilon^{\alpha\beta}q_\beta^{N+A},\qquad (q_a^{N+A})^\dagger=-\varepsilon^{\alpha\beta}q_\beta^A \tag{2.19}$$

With $\Lambda=\varepsilon\otimes\mathbf{1}$ the operator algebra can be written in the compact form

$$(q_\alpha^r)^\dagger=\varepsilon^{\alpha\beta}\Lambda^{rs}q_\beta^s,\qquad \{q_\alpha^r,q_\beta^s\}=-e_{\alpha\beta}\Lambda^{rs},\qquad \Lambda=\begin{pmatrix}0&\mathbf{1}\\-\mathbf{1}&0\end{pmatrix}, \tag{2.20}$$

which is manifestly invariant under[9] $USp(2N)\otimes SU(2)$. It can be shown that states of equal spin transform irreducibly under this $USp(2N)$.

[7] In the case $N=1$ this is the R symmetry that corresponds to multiplication of Q and $\overline{Q}$ by complex conjugate phases, which plays some role in the construction of the SUSY standard model.

[8] As we go to the little group we leave away γ^0. In $4D$ $\varepsilon^{\alpha\beta}$ intertwines $\vec\sigma$ and $-\vec\sigma^*$ since

$$\begin{pmatrix}0&-\mathbf{1}\\\mathbf{1}&0\end{pmatrix}\begin{pmatrix}A&B\\C&D\end{pmatrix}=\begin{pmatrix}D&-C\\-B&A\end{pmatrix}\begin{pmatrix}0&-\mathbf{1}\\\mathbf{1}&0\end{pmatrix}.$$

Thus Q_α and $\overline{Q}_{\dot\alpha}$, transforming in complexe conjugate representationsof $SO(3)$, are intertwined by ε.

[9] $USp(2N)$ is the compact real form of the classical Lie group $C_N=Sp(2N)$.

The massless representations can now be analyzed in a similar way. We start with the operator algebra

$$\{Q^A_\alpha, \overline{Q}_{\dot\beta B}\} = 2\begin{pmatrix} 2E & 0 \\ 0 & 0 \end{pmatrix}_{\alpha\dot\beta} \delta^A_B, \qquad \{Q^A_\alpha, Q^B_\beta\} = \{\overline{Q}_{\dot\alpha A}, \overline{Q}_{\dot\beta B}\} = 0. \tag{2.21}$$

The operators Q_2 and $\overline{Q}_2$ are totally anticommuting and must be represented by 0 (just as totally commuting operators of an operator algebra can be replaced by numbers in an irreducible representation). The representation spaces can therefore be built on some Clifford vacuum of helicity h_- by acting with the N creation operators $a^\dagger_A = \frac{1}{2\sqrt{E}}\overline{Q}_{\dot 1 A}$. They are of dimension 2^N and range up to helicity $h_+ = h_- + N/2$; at helicity $h_- + n/2$ the degeneneracy is $\binom{N}{n}$. In general it is, however, necessary to add a representation with the CPT conjugate states, which have opposite helicity. The representations with $h_+ = -h_- = N/4$ are CPT complete.

For the multiplets of states that carry central charges [na78, fe81] we denote the antisymmetric matrix of eigenvalues of the charges by Z^{AB}. This matrix can be block-diagonalized by a unitary transformation $Q \to UQ$, $Z \to UZU^T = \varepsilon \otimes D$ with a diagonal positive matrix D and blocks of anti-symmetric 2×2 matrices (if N is odd then we have to append a row and a line with entries 0 to $\varepsilon \otimes D$). The invariant information thus consists of $[N/2]$ positive real eigenvalues Z_i, and the algebra in the block-diagonal basis reads

$$\{Q^{am}_\alpha, (Q^{bn}_\beta)^\dagger\} = 2M\delta^\beta_\alpha\delta^b_a\delta^m_n, \quad \{Q^{am}_\alpha, Q^{bn}_\beta\} = \varepsilon_{\alpha\beta}\varepsilon^{ab}\delta^{mn}Z_n, \quad \{(Q^{am}_\alpha)^\dagger, (Q^{bn}_\beta)^\dagger\} = \varepsilon^{\alpha\beta}\varepsilon_{ab}\delta_{mn}Z_n \tag{2.22}$$

with $a, b = 1, 2$ and $m, n = 1, \ldots, N/2$. The operators Q^{am}_α and $(Q^{am}_\alpha)^\dagger$ can now be linearly combined to operators

$$a^m_\alpha = \frac{1}{\sqrt{2}}(Q^{1m}_\alpha + \varepsilon_{\alpha\beta}(Q^{2m}_\beta)^\dagger), \quad b^m_\alpha = \frac{1}{\sqrt{2}}(Q^{1m}_\alpha - \varepsilon_{\alpha\beta}(Q^{2m}_\beta)^\dagger) \tag{2.23}$$

that satisfy the following algebra:

$$\{a^m_\alpha, (a^n_\beta)^\dagger\} = \delta_{\alpha\beta}\delta^{mn}(2M + Z_n), \qquad \{b^m_\alpha, (b^n_\beta)^\dagger\} = \delta_{\alpha\beta}\delta^{mn}(2M - Z_n), \tag{2.24}$$

$$\{a^m_\alpha, (b^n_\beta)^\dagger\} = 0 \quad \{a^m_\alpha, a^n_\beta\} = \{a^m_\alpha, b^n_\beta\} = \{b^m_\alpha, b^n_\beta\} = 0. \tag{2.25}$$

Positivity of norms implies that $2M \geq Z_n$ for all eigenvalues Z_n of central charges. This yields a Clifford algebra with $2(N-r)$ creation and annihilation operators if $0 \leq r \leq N/2$ of the central charges are equal to $2M$. If $r = N/2$, i.e. all if charges are equal to $2M$, then the size of the representation is the same as for a massless multiplet. The stability group is $SU(2)_{spin} \otimes USp(2N-2r)_{int.}$ in the massive case and $U(1)_{hel.} \otimes SU(N)_{int.}$ in the massless case.

As an example we consider the case $N = 4$: Here the *short multiplet* with $Z_1 = Z_2 = 2M$ consists of 1 spin=1, 4 spin=1/2 and 5 spin=0 representations with $1\cdot 3 + 4\cdot 2 + 5 = 16$ degrees of freedom. In the massless limit this turns into the *massless multiplet*, which has the same size, but this time decomposing into helicities $h = \pm 1$, 4 times $h = \pm\frac{1}{2}$ and 6 times $h = 0$. Giving, in turn, a vacuum expectation value (VEV) to a scalar in the massless multiplet we must arrive at a short multiplet via the Higgs mechanism that saturates the so-called Bogomol'nyi bound $Z = 2M$; such states are called BPS states and their mass formula is protected under (sufficiently well-behaved) deformations by the same mechanism that requires the pairing of all massless states to break SUSY spontaneously. For $N = 4$ we can also have an *intermediate multiplet* with $Z_1 < Z_2 = 2M$ and 1 spin=3/2, 6 spin=1, 14 spin=1/2 and 14 spin=0 representations. The size of this SUSY representation is $4 + 6 \cdot 3 + 14 \cdot 2 + 14 = 64$, in coincidence with the fundamental massive $N = 3$ multiplet; the fundamental massive $N = 4$ multiplet is also called *long multiplet* in this context.

2.4 Supersymmetric field theories

Having discussed the multiplets of momentum eigenstates that we can expect in a physical Hilbert space we now turn to the construction of local quantum field theories whose symmetry algebras contain SUSY generators. What we actually want to construct is local actions depending on some set of elementary fields that transform into total derivatives under SUSY transformations. In this context it is useful to think in terms of jet bundles, which means that we consider the fields ϕ^i and their formal partial derivatives $[\phi^i] = \{\phi^i, \partial_m\phi^i, \partial_m\partial_n\phi^i, \ldots\}$ as independent variables; local functionals like actions are then (formal) space-time integrals over analytic functions in the $[\phi^i]$ that are polynomials in $[\partial_m\phi^i]$.

Conceptually it is important to distinguish between the supercharge Q_α, the supersymmetry transformation D_α that should act linearly on the elementary fields, the implementation $\mathbf{Q}_\alpha$ of SUSY transformations in terms of a superspace differential operator acting on superfields, and the covariant derivative $\mathbf{D}_\alpha$ that also acts in superspace. Denoting the canonical coordinates by q^i and a symmetry transformation by $\delta_I q^i = f_I{}^i(q, \dot{q})$ the time derivative of the Noether charge is $\dot{Q}_I = \delta_I q^i\ (\delta L/\delta q^i)$. Using the Poisson brackets

$$\{A, B\}_{PB} := (-)^{iA}\left(\frac{\partial A}{\partial q^i}\frac{\partial B}{\partial p_i} - (-)^i\frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q^i}\right), \qquad \{p_i, q^j\}_{PB} = -\delta_i{}^j \tag{2.26}$$

with the Noether charge we can, in turn, recover the symmetry transformation

$$\delta_I A := \{Q_I, A\}_{PB}, \qquad \{Q_I, H\}_{PB} = 0, \qquad \dot{Q}_I = \delta_I q^i\ \frac{\delta L}{\delta q^i}. \tag{2.27}$$

If the elementary fields transform in some linear representation then it is natural to consider right multiplication with the representation matrices

$$\delta_I\phi^i = \{Q_I, \phi^i\}_{PB} = (-)^{Ij}\phi^j T_{Ij}{}^i = -(-)^j (T_I^c)^i{}_j\phi^j, \tag{2.28}$$

where the matrices T_I describe the right-action of the group and $T_I^c = -T_I^T$ denote the corresponding contragradient representation matrices. This is only possible if the structure functions $f_{IJ}{}^K([\phi])$ of the symmetry algebra $[\delta_I, \delta_J] = f_{IJ}{}^K \delta_K$ are constants, and we find $[T_I, T_J] = f_{IJ}{}^K T_K$ for the graded commutator. (With a left action of the representation matrices we would have obtained a negative sign for the structure constants.)

Upon quantization Poisson brackets are replaced by $i/\hbar$ times commutators

$$i\hbar\{p_i, q^j\}_{PB} \;\rightarrow\; [P_i, Q^j] = -i\hbar\delta_i{}^j, \qquad\qquad \delta_I A = \tfrac{i}{\hbar}\,[Q_I, A]. \tag{2.29}$$

The Schrödinger equation $i\partial_t\psi = H\psi$ implies the time evolution $\dot{\mathcal{O}} = i[H, \mathcal{O}]$ of Heisenberg operators. This is consistent with

$$\{P_m, \phi\}_{PB} = -\partial_m\phi, \qquad [P_m, \phi] = -i\partial_m\phi, \qquad \{Q_\alpha, \overline{Q}_{\dot\beta}\} = 2\sigma^m_{\alpha\dot\beta}P_m, \tag{2.30}$$

$$\{Q_\alpha, \phi\}_{PB} = -D_\alpha\phi, \qquad [Q_\alpha, \phi] = -iD_\alpha\phi, \qquad \{D_\alpha, \overline{D}_{\dot\alpha}\} = 2i\partial\!\!\!/_{\alpha\dot\alpha} := 2i\sigma^m_{\alpha\dot\alpha}\partial_m, \tag{2.31}$$

$$\{\overline{Q}_{\dot\alpha}, \phi\}_{PB} = -\overline{D}_{\dot\alpha}\phi, \qquad [\overline{Q}_{\dot\alpha}, \phi] = -i\overline{D}_{\dot\alpha}\phi, \qquad [D_\alpha, \partial_m] = [\overline{D}_{\dot\alpha}, \partial_m] = 0 \tag{2.32}$$

because $\{D_\alpha, \overline{D}_{\dot\alpha}\}\phi = i[Q_\alpha, i[\overline{Q}_{\dot\alpha}, \phi\}\} + i[\overline{Q}_{\dot\alpha}, i[Q_\alpha, \phi\}\} = -[\{Q_\alpha, \overline{Q}_{\dot\alpha}\}, \phi] = 2i\sigma^m_{\alpha\dot\alpha}\partial_m$, where $[A, B\} := AB - (-)^{AB}BA$ denotes the graded commutator. We also use the abbreviation $v_{\alpha\dot\alpha} := v\!\!\!/_{\alpha\dot\alpha} := \sigma^m_{\alpha\dot\alpha}v_m$ to write vectors in terms of spinor indices.

In order to construct a field theory with a linear realization of supersymmetry we next have to find representations of the algebra $\{D_\alpha, \overline{D}_{\dot\alpha}\} = 2i\sigma^m_{\alpha\dot\alpha}\partial_m$ and then constructions of invariant actions depending on those fields. The most natural representation is obtained by declaring $\overline{D}_{\dot\alpha}$ to be 'annihilation operators' on some elementary scalar field ϕ. Since $D_\alpha D_\beta D_\gamma = 0$ the resulting (scalar) *chiral multiplet* consists of ϕ, the Weyl spinor $\chi_\alpha := D_\alpha\phi/\sqrt{2}$ and the auxiliary field $F := -D^2\phi/4$ (F is not dynamical in a renormalizable theory since it has mass dimension 2 if ϕ has its canonical dimension 1). Note that ϕ must be a complex field since reality of ϕ would imply that it is also antichiral $D_\alpha\phi = 0$ and thus, because of the SUSY algebra, constant. The action of $\overline{D}_{\dot\alpha}$ on χ_α and F is fixed by the SUSY algebra and the definition of these fields. Denoting the SUSY transformation with constant commuting parameters $\xi^{\underline{\alpha}}$ by $s = \xi^\alpha D_\alpha + \overline{\xi}^{\dot\alpha}\overline{D}_{\dot\alpha}$ we find

$$\overline{D}_{\dot\alpha}\phi = 0, \qquad \chi_\alpha = \tfrac{1}{\sqrt{2}}D_\alpha\phi, \qquad F = -\tfrac{1}{4}D^2\phi, \tag{2.33}$$

$$s\phi = \sqrt{2}\xi\chi, \qquad s\chi_\alpha = \sqrt{2}(\xi_\alpha F + i\sigma^a_{\alpha\dot\alpha}\overline{\xi}^{\dot\alpha}\partial_a\phi), \qquad sF = \sqrt{2}i\sigma^a_{\alpha\dot\alpha}\overline{\xi}^{\dot\alpha}\partial_a\chi^\alpha. \tag{2.34}$$

as is easily checked using the identities

$$D^2 = -\varepsilon^{\alpha\beta}D_\alpha D_\beta, \quad D_\alpha D_\beta = \tfrac{1}{2}\varepsilon_{\alpha\beta}D^2, \quad [D^2, \overline{D}_{\dot\alpha}] = 4iD^\alpha\partial\!\!\!/_{\alpha\dot\alpha}, \quad [D_\alpha, \overline{D}^2] = 4i\partial\!\!\!/_{\alpha\dot\alpha}\overline{D}^{\dot\alpha}, \tag{2.35}$$

which follow from our conventions $D^\alpha = \varepsilon^{\alpha\beta}D_\beta,\ \varepsilon_{\alpha\beta}\varepsilon^{\beta\gamma} = \delta^\gamma_\alpha,\ D^2 := D^\alpha D_\alpha,\ \overline{D}^2 := D_{\dot\alpha}D^{\dot\alpha}$.

Invariant Lagrangians can be constructed by observing that $D^2\overline{D}^2$ acting on any (composite) field and D^2 acting on a (composite) chiral field always give expressions that transform into total derivatives. It can be shown [br92] that the most general supersymmetric lagrangian $\mathcal{L}$ that depends on a set $\{\phi^i\}$ of chiral fields and the corresponding hermitian conjugate anti-chiral fields $\{\overline{\phi}^i\}$ is of the form[10]

$$\mathcal{L} = -\tfrac{1}{4}D^2L + h.c., \qquad L = \tfrac{3}{8}\overline{D}^2K([\phi, D\phi, D^2\phi], [\overline{\phi}, \overline{D\phi}, \overline{D}^2\overline{\phi}]) + g(\phi), \tag{2.36}$$

where g is called superpotential and K is called Kähler potential. Note that the superpotential can be chosen not to contain any derivatives (no ∂'s and no D's). A redefinition $K \to K + f(\phi)+f^*(\overline{\phi})$, which changes the action only by total derivatives, is called Kähler transformation. Such a transformation together with a suitable normalization of the chiral fields can be used to bring an analytic Kähler potential into the form $K = -\frac{1}{3}\sum_i \overline{\phi}^i\phi^i + \ldots$ if the kinetic energies are positive. The dots denote terms of dimension 3 or higher. If we demand renormalizability such terms are forbidden and the superpotential must be cubic, so that

$$\mathcal{L} = -\tfrac{1}{4}D^2\left(-\tfrac{1}{8}\overline{D}^2\overline{\phi}\phi + g(\phi)\right) + h.c., \qquad g = \gamma + \lambda_i\phi^i + \tfrac{1}{2}m_{ij}\phi^i\phi^j + \tfrac{1}{6}\kappa_{ijk}\phi^i\phi^j\phi^k. \tag{2.37}$$

ε intertwines $\vec{\sigma}$ and $-\vec{\sigma}^{\,*}$, hence also σ^a and $(\overline{\sigma}^a)^T$, so that $\varepsilon^{\alpha\beta}\varepsilon^{\dot{\alpha}\dot{\beta}}\sigma^a_{\beta\dot{\beta}} = \overline{\sigma}^{a\dot{\alpha}\alpha}$ and

$$[D^2, \overline{D}^2] = 4i\partial\!\!\!/_{\alpha\dot{\alpha}}[D^\alpha, \overline{D}^{\dot{\alpha}}] = 8iD\partial\!\!\!/\overline{D} - 16\Box = 16\Box - 8i\overline{D}^{\dot{\alpha}}\partial\!\!\!/_{\alpha\dot{\alpha}}D^\alpha \tag{2.38}$$

because $\{D^\alpha, \overline{D}^{\dot{\alpha}}\} = 2i\overline{\partial\!\!\!/}^{\dot{\alpha}\alpha}$ and $\partial\!\!\!/\overline{\partial\!\!\!/} = -\Box\mathbf{1}$ with $\overline{\partial\!\!\!/} := \overline{\sigma}^m\partial_m$ and $\mathrm{tr}\,\mathbf{1}\delta^\alpha_\alpha = 2$. Evaluation of $\mathcal{L} = (\overline{D}^2\overline{\phi}D^2\phi + 2D\phi D\overline{D}^2\overline{\phi} + D^2\overline{D}^2\overline{\phi}\phi)/32 - (D^2\phi^i\partial_i g + D\phi^iD\phi^j\partial_i\partial_j g)/4 + h.c.$ thus yields

$$\mathcal{L} = -\tfrac{1}{2}\Box\phi^i\overline{\phi}^i - i\chi^i\sigma^a\partial_a\overline{\chi}^i + F^i\overline{F}^i + F^i\partial_i g - \tfrac{1}{2}\chi^i\chi^j\partial_i\partial_j g + \overline{F}^i\partial_i g^* - \tfrac{1}{2}\overline{\chi}^i\overline{\chi}^j\partial_i\partial_j g^*, \tag{2.39}$$

where the kinetic terms and $F\overline{F}$ come from the Kähler potential. Integrating out the auxiliary fields by inserting their equations of motion $\overline{F}_i = -\partial_i g$ we find the potential

$$V(\phi, \phi^*) = \sum_i |\partial_i g|^2 = |F(\phi)|^2 \tag{2.40}$$

[10] It is easy to see that all terms are of the form $D^2X + c.c$ and that all terms containing chiral and antichiral fields can be written as $D^2\overline{D}^2Y$: We define the operator t^α by $t^\alpha D_\beta\phi = \delta^\alpha_\beta\phi^i$, $t^\alpha D^2\phi = -2D^\alpha\phi^i$ and $t^\alpha\phi^i = t^\alpha\overline{\phi}^i = \{t^\alpha, \overline{D}_{\dot{\alpha}}\} = [t^\alpha, \partial_a] = 0$ so that $\{t^\alpha, D_\beta\} = \delta^\alpha_\beta\mathcal{E}(\phi^i, \chi^i, F^i)$, where $\mathcal{E}$ is the Euler operator that counts the degree of homogeneity in the component fields of chiral multiplets (formally one may write '$t^\alpha = \partial/\partial(D_\alpha)$' when acting on chiral fields). As t and D act linearly we may decompose the action into terms $\mathcal{L}_n$ of definite degree n in (ϕ^i, χ^i, F^i). Since $[D^2, t^\alpha] = 2\mathcal{E}D^\alpha$ and $[D^2, t^2] = 4\mathcal{E}(tD - \mathcal{E})$ a supersymmetric action with $D_\alpha\mathcal{L}_n = \partial_a X^a_\alpha$ can be written as $\mathcal{L}_n = -\frac{1}{4n^2}D^2(t^2\mathcal{L}_n) + \frac{1}{4n^2}\partial_a(t^2D^\alpha X^a_\alpha + 4nt^\alpha X^a_\alpha)$ for $n > 0$, i.e. $\mathcal{L}_n$ can be written as D^2 acting on some local function up to total derivatives. Similarly it can be shown that terms depending on antichiral fields can be written as $\overline{D}^2(-\overline{t}^2\mathcal{L}_n/4n^2)$ and terms that depend on both, chiral and antichiral fields, are of the form $D^2\overline{D}^2K$.

To show that X can be assumed to depend only on ϕ (without derivatives) is more involved and this result depends on the 'QDS-structure' of the SUSY representation on chiral fields [br92]; note that the linear SUSY representations on local fields are infinite dimensional because $\{D, \overline{D}\}$ contains the partial derivative.

for the scalar fields. The terms $-\frac{1}{2}\chi^i\chi^j\partial_i\partial_j g$ and their hermitian conjugates are the Yukawa couplings. (2.38) implies that we can define projection operators

$$\Pi_+ = \frac{\overline{D}^2 D^2}{16\Box}, \quad \Pi_- = \frac{D^2\overline{D}^2}{16\Box}, \quad \Pi_T = -\frac{D\overline{D}^2 D}{8\Box} = -\frac{\overline{D}D^2\overline{D}}{8\Box}, \qquad \Pi_+ + \Pi_- + \Pi_T = \mathbf{1}, \quad (2.41)$$

where Π_+ and Π_- project onto chiral and anti-chiral fields, respectively (to see this, evaluate $D\overline{D}^2 D = D[\overline{D}^2, D] + D^2\overline{D}^2$ and $\overline{D}D^2\overline{D} = \overline{D}[D^2, \overline{D}] + \overline{D}^2 D^2$). Π_T is called transversal projector.

2.5 Superspace

In the *superspace* approach SUSY transformations are interpreted as motions in a space with anticommuting coordinates θ_α and $\overline{\theta}_{\dot\alpha}$ in addition to the space-time coordinates x^m. Complete SUSY multiplets like (ϕ, χ, F) are combined into a single *superfield* $\Phi(x, \theta, \overline{\theta})$. The supersymmetry transformation acting on a superfield is then represented by a linear combination of an ordinary partial derivative and a derivative with respect to the anticommuting coordinates. With an appropriate ansatz we find the operators

$$\mathbf{Q}_\alpha = \frac{\partial}{\partial\theta^\alpha} + i\partial_{\alpha\dot\beta}\overline{\theta}^{\dot\beta}, \qquad \overline{\mathbf{Q}}_{\dot\alpha} = -\frac{\partial}{\partial\overline{\theta}^{\dot\alpha}} - i\theta^\beta\partial_{\beta\dot\alpha}, \qquad \{\mathbf{Q}_\alpha, \overline{\mathbf{Q}}_{\dot\alpha}\} = -2i\sigma^a_{\alpha\dot\alpha}\partial_a \qquad (2.42)$$

that obey the appropriate algebraic relations (since $\sigma^{m*} = \sigma^{mT}$ and $\partial/\partial\psi^* = (-)^{|\psi|}(\partial/\partial\psi)^*$ we have $\mathbf{Q}^* = \overline{\mathbf{Q}}$). A superfield Φ is then a function in superspace that satisfies

$$\mathbf{Q}_\alpha\Phi = D_\alpha\Phi, \qquad \overline{\mathbf{Q}}_{\dot\alpha}\Phi = \overline{D}_{\dot\alpha}\Phi, \qquad (2.43)$$

where D and $\overline{D}$ act on the component fields.[11]

Lemma: Any superfield $\mathcal{F}$ can be written in the form $\mathcal{F} = \mathcal{F}(\Phi) = \exp(\theta D + \overline{\theta D}) f(\phi)$, where $f(\phi)$ is the $\theta^{\underline{\alpha}}$-independent part of $\mathcal{F}$.

Proof: First we show that $\exp(\theta D + \overline{\theta D}) f(\phi)$ satisfies (2.43). Evaluating $Be^A = e^A(B - [A, B] + \frac{1}{2}[A, [A, B]] - \ldots)$ with $B = \mathbf{Q}_\alpha - D_\alpha$ and $A = \theta D + \overline{\theta D}$ we find $[A, B] = -D_\alpha + 2i\partial\!\!\!/_{\alpha\dot\alpha}\overline{\theta}^{\dot\alpha}$ and $[A, [A, B]] = 2i\partial\!\!\!/_{\alpha\dot\alpha}\overline{\theta}^{\dot\alpha}$. Putting everything together $\mathbf{Q}\mathcal{F} = D\mathcal{F}$ follows from $\mathbf{Q}_\alpha f = i\overline{\theta}^{\dot\alpha}\partial\!\!\!/_{\alpha\dot\alpha} f$ and $\overline{\mathbf{Q}} = \overline{D}$ by complex conjugation.

Next we show that any superfield must have a non-vanishing θ-independent part: Splitting $\mathcal{F} = \sum \mathcal{F}_{mn}$ into terms $\mathcal{F}_{mn}$ of degree m in θ and n in $\overline{\theta}$ the equations (2.43) imply recursion relations that allow to express all $\mathcal{F}_{mn}$ linearly in $f = \mathcal{F}_{00}$. Since the difference of the superfields $\exp(\theta D + \overline{\theta D}) f$ and $\mathcal{F}$ is again a superfield, this difference must vanishes, which completes the proof of the lemma. $\Box$

[11] $\{\mathbf{Q}, \overline{\mathbf{Q}}\} = -\{D, \overline{D}\}$ is consistent with this equation because $\mathbf{Q}\Phi = D\Phi$ is no superfield.

$\mathbf{Q}$ does not map superfields to superfields since $\{\mathbf{Q},\overline{D}\}=0$ but $\{\mathbf{Q},\overline{\mathbf{Q}}\}\neq 0$. To impose the chirality condition on superfields we thus need another differential operator in superspace, the *covariant derivative*

$$\mathbf{D}_\alpha=\frac{\partial}{\partial\theta^\alpha}-i\partial_{\alpha\dot\beta}\overline{\theta}^{\dot\beta},\qquad \overline{\mathbf{D}}_{\dot\alpha}=-\frac{\partial}{\partial\overline{\theta}^{\dot\alpha}}+i\theta^\beta\partial_{\beta\dot\alpha} \tag{2.44}$$

which satisfies

$$\{\mathbf{D}_\alpha,D_\beta\}=\{\mathbf{D}_\alpha,\mathbf{Q}_\beta\}=\{\mathbf{D}_\alpha,\overline{D}_{\dot\alpha}\}=\{\mathbf{D}_\alpha,\overline{\mathbf{Q}}_{\dot\alpha}\}=0,\qquad \{\mathbf{D}_\alpha,\overline{\mathbf{D}}_{\dot\alpha}\}=2i\sigma^a_{\alpha\dot\alpha}\partial_a, \tag{2.45}$$

so that it preserves the superfield property. Indeed, the chirality condition $\overline{\mathbf{D}}_{\dot\alpha}\Phi=0$ for a superfield is equivalent to the chirality of its θ-independent part since

$$\mathbf{D}_\alpha e^{\theta D+\overline{\theta D}}=e^{\theta D+\overline{\theta D}}(\mathbf{D}_\alpha+i\partial\!\!\!/_{\alpha\dot\alpha}\overline{\theta}^{\dot\alpha}+D_\alpha)=e^{\theta D+\overline{\theta D}}(\partial_{\theta^\alpha}+D_\alpha) \tag{2.46}$$

The components of a chiral superfield are easily evaluated using the formulas

$$e^{\theta D+\overline{\theta D}}=e^{-i\theta\partial\!\!\!/\overline{\theta}}e^{\theta D}e^{\overline{\theta D}}=e^{i\theta\partial\!\!\!/\overline{\theta}}e^{\overline{\theta D}}e^{\theta D}, \tag{2.47}$$

which follow from $[\theta D,\overline{\theta D}]=2i\theta^\alpha\partial\!\!\!/_{\alpha\dot\alpha}\overline{\theta}^{\dot\alpha}$ and the Baker–Campbell–Hausdorff formula

$$e^Ae^B=e^{A+B+\frac12[A,B]+\frac1{12}([A,[A,B]]-[B,[A,B]])+\text{ multiple commutators}}. \tag{2.48}$$

We thus obtain

$$\Phi(x,\theta,\overline{\theta})=e^{-i\theta\partial\!\!\!/\overline{\theta}}e^{\theta D}\phi(x)=\phi(y)+\theta D\phi(y)-\tfrac14\theta^2D^2\phi(y),\qquad y^m=x^m-i\theta\sigma^m\overline{\theta} \tag{2.49}$$

and the analogous formula for anti-chiral fields by complex conjugation. To obtain the $\overline{\theta}$-dependent components explicity we just have to formally Taylor-expand $\phi(y)$, $\chi(y)$ and $F(y)$ in $y-x$.

The advantage of the superspace formulation is that we can rewrite the action as a superspace integral and extend the Feynman rules to a supergraph calculus [WE83,br96]. To this end we define superspace integration with $\{z^M\}=\{x^m,\theta^\alpha,\overline{\theta}_{\dot\alpha}\}$ and δ-functions by

$$\textstyle\int d\theta^{\underline{\alpha}}=\frac{\partial}{\partial\theta^{\underline{\alpha}}},\qquad \int d^2\theta=\int d\theta^2d\theta^1,\qquad \int d^2\overline{\theta}=\int d\overline{\theta}^{\dot 1}d\overline{\theta}^{\dot 2},\qquad \int d^4\theta=\int d^2\theta d^2\overline{\theta}, \tag{2.50}$$

$$\textstyle\int d^6z=\int d^4xd^2\theta,\qquad \int d^6\overline{z}=\int d^4xd^2\overline{\theta},\qquad \int d^8z=\int d^4xd^4\theta \tag{2.51}$$

$$\delta^2(\theta-\theta')=-\tfrac12(\theta-\theta')^2,\qquad \delta^6(z-z')=\delta^2(\theta-\theta')\delta^4(x-x'), \tag{2.52}$$

$$\delta^2(\overline{\theta}-\overline{\theta}')=-\tfrac12(\overline{\theta}-\overline{\theta}')^2,\qquad \delta^6(\overline{z}-\overline{z}')=\delta^2(\overline{\theta}-\overline{\theta}')\delta^4(x-x'), \tag{2.53}$$

$$\delta^8(z-z')=\delta^2(\theta-\theta')\delta^2(\overline{\theta}-\overline{\theta}')\delta^4(x-x') \tag{2.54}$$

Up to total derivatives the action (2.36) can then be rewritten in terms of θ-integrations using

$$\int d^2\theta\exp(\theta D+\overline{\theta D})f(\phi)=\tfrac12D^2\exp(\overline{\theta D})\,f(\phi)+tot.div., \tag{2.55}$$

$$\int d^4\theta\exp(\theta D+\overline{\theta D})f(\phi)=\tfrac14D^2\overline{D}^2f(\phi)+tot.div. \tag{2.56}$$

As usual, propagators are most easily obtained by solving the equations of motion for the sources via evaluation of all possible projections.

2.6 Supersymmetric Yang–Mills theory

There are two apparently independent approaches to a supersymmetric generalization of Yang-Mills theory. The first is to look for a superfield containg the gauge fields: We might think about a real superfield whose highest component is the gauge field. For such a field, however, we would have to impose complicated constraints to get rid of higher spin components. It is much easier to start from a superfield that is based on a real scalar field

$$V=V^\dagger=C+\theta \not{A}\overline{\theta}+\tfrac{1}{2}\theta^2\overline{\theta}^2(D-\tfrac{1}{2}\Box C)+\Big((\theta\chi+\theta^2M+\overline{\theta}^2\theta(\lambda-\tfrac{i}{2}\not{\partial}\overline{\chi}))+h.c.\Big), \qquad (2.57)$$

which already contains a real vector field A as its $\theta\overline{\theta}$-component. The linear SUSY representation that comes with the *real scalar superfield* is therefore called *vector multiplet*. To find the multiplet structure of the component fields we use $\mathcal{F}(\Phi)=\exp(\theta D+\overline{\theta D})f(\phi)$. Expanding $e^{\theta D+\overline{\theta D}}=\frac{1}{2}(e^{-i\theta\not{\partial}\overline{\theta}}e^{\theta D}e^{\overline{\theta D}}+c.c.)$ we find

$$\begin{aligned}
\tfrac{1}{2}(\theta D+\overline{\theta D})^2 &= \tfrac{1}{2}\theta^\alpha\overline{\theta}^{\dot\alpha}[D_\alpha,\overline{D}_{\dot\alpha}]-\tfrac{1}{4}(\theta^2D^2+\overline{\theta}^2\overline{D}^2), && (2.58)\\
\tfrac{1}{3!}(\theta D+\overline{\theta D})^3 &= \theta^2(\tfrac{1}{4}D^2\overline{D}_{\dot\alpha}-\tfrac{i}{2}D^\alpha\not{\partial}_{\alpha\dot\alpha})\overline{\theta}^{\dot\alpha}+\overline{\theta}^2\theta^\alpha(-\tfrac{1}{4}\overline{D}^2D_\alpha-\tfrac{i}{2}\not{\partial}_{\alpha\dot\alpha}\overline{D}^{\dot\alpha}), && (2.59)\\
\tfrac{1}{4!}(\theta D+\overline{\theta D})^4 &= \tfrac{1}{4}\theta^2\overline{\theta}^2(\tfrac{1}{8}(D^2\overline{D}^2+\overline{D}^2D^2)-\Box). && (2.60)
\end{aligned}$$

so that the SUSY representation defined by a superfield V^i is

$$\not{A}^i_{\alpha\dot\alpha}=\tfrac{1}{2}[D_\alpha,\overline{D}_{\dot\alpha}]C^i=(D_\alpha\overline{D}_{\dot\alpha}-i\not{\partial}_{\alpha\dot\alpha})C=(i\not{\partial}_{\alpha\dot\alpha}-\overline{D}_{\dot\alpha}D_\alpha)C, \qquad (2.61)$$

$$\chi^i_\alpha=D_\alpha C^i,\qquad \lambda^i_\alpha = -\tfrac{1}{4}\overline{D}^2D_\alpha C^i,\qquad M^i=-\tfrac{1}{4}D^2C^i,\qquad D^i=\tfrac{1}{16}\{D^2\overline{D}^2+\overline{D}^2D^2\}C^i. \qquad (2.62)$$

The component fields χ^i_α, M^i and λ^i_α are complex. The real fields D^i transforms into a total derivative under SUSY (such terms are called *Fayet–Iliopoulos* or D-terms; they are gauge invariant and thus can contribute to the action only for abelian factors of the gauge group). The gauge invariant field strength $F^i_{mn}=\partial_mA^i_n-\partial_nA^i_m$ of the real gauge connection A_m is contained in $D_\alpha\lambda^{i\beta}=\not{F}^{i\ \beta}_{\ \alpha}+i\delta^\beta_\alpha D^i$ (see below).

Out of a chiral superfield Λ with lowest component L we can construct a special real superfield by adding its complex conjugate. This suggests the following supersymmetrization of gauge transformations:

$$\delta V=\Lambda+\Lambda^\dagger,\qquad \begin{array}{cc} \delta C=2\,\mathrm{Re}\,L, \quad \delta\chi=DL, & \delta M=D^2L,\\ \delta A_m=-2\,\mathrm{Im}\,\partial_mL, & \delta\lambda=\delta D=\delta F_{mn}=0.\end{array} \qquad (2.63)$$

Note that the transversal projector Π_T in (2.41) projects onto the gauge invariant content of the real superfield. For a chiral superfield of charge q the gauge transformation and a gauge invariant kinetic energy may thus be defined by

$$\Phi\to e^{-q\Lambda}\Phi,\qquad V\to V+\Lambda+\Lambda^\dagger,\qquad K(\Phi,\Phi^\dagger,V)=\Phi^\dagger e^{qV}\Phi. \qquad (2.64)$$

To complete the action of supersymmetric QED we need to choose a gauge-neutral superpotential and add the kinetic terms for the gauge fields A_m and the gauginos λ via the Kähler potential $K(V)=\lambda^\alpha\chi_\alpha+h.c.$,

$$D^2\overline{D}^2\left(\lambda^\alpha\chi_\alpha\right)+h.c.\sim D^2\left(\lambda^\alpha\lambda_\alpha\right)+h.c.\sim\left(-\tfrac{1}{4}F_{mn}F^{mn}-i\lambda\partial\!\!\!/\overline{\lambda}+\tfrac{1}{2}D^2\right). \tag{2.65}$$

In the superspace version polynomials and exponentials in the superfields are rather tedious to evaluate and we can use supergauge transformations to set $C=\chi_\alpha=M=0$. This is called the Wess–Zumino gauge, which is left invariant by ordinary gauge transformations, i.e. gauge transformations with Λ being θ-independent and imaginary (such a restricted Λ is no longer a superfield and its non-vanishing component is no linear SUSY representation, except for the trivial case where it is constant). In the Wess–Zumino gauge the gauge interaction is manifestly renormalizable.

The generalization to non-abelian gauge theories is now easy to guess: We let Φ become vectors that transform in some representation of the gauge group and $V=V^i\delta_i$. Then supergauge transformations are defined by [WE83]

$$\Phi'=e^{-\Lambda}\Phi,\qquad e^{V'}=e^{\Lambda^\dagger}e^Ve^\Lambda\quad\Rightarrow\quad V'=V+\Lambda+\Lambda^\dagger+O(\Lambda^2). \tag{2.66}$$

and supersymmetric gauge-covariant field strength can be defined by

$$W_\alpha=-\tfrac{1}{4}\overline{D}^2e^{-V}D_\alpha e^V\quad\Rightarrow\quad W'_\alpha=e^{-\Lambda}W_\alpha e^\Lambda, \tag{2.67}$$

which leads to the gauge-invariant supersymmetric action

$$\mathcal{L}=\frac{1}{4k}\operatorname{tr}\left(D^2(W^\alpha W_\alpha)+h.c.\right)+D^2\overline{D}^2(\phi^\dagger e^V\phi)+\left(D^2g_{inv}(\phi)+h.c.\right). \tag{2.68}$$

with additional D-terms $\mu_i^2D^i$ for abelian factors of the gauge group. Since the transformation law of V starts with the familiar V-independent term $\Lambda+\Lambda^\dagger$ the non-abelian theory also allows for a Wess–Zumino gauge with $V^3=0$.

2.7 Supercovariant derivatives and Bianchi identities

In an alternative approach to super Yang–Mills we start with the covariant derivatives

$$\mathcal{D}_a=\partial_a+A_a^i\delta_i\quad\rightarrow\quad\mathcal{D}_\alpha=D_\alpha+A_\alpha^i\delta_i \tag{2.69}$$

and try to impose reasonable constraints on the covariant field strengths F_{AB} defined by

$$[\mathcal{D}_A,\mathcal{D}_B\}=-T_{AB}{}^C\mathcal{D}_C+F_{AB}^i\delta_i,\qquad[\delta_i,\delta_j]=f_{ij}{}^k\delta_k,\qquad T_{\alpha\dot\beta}{}^c=2i\sigma^c_{\alpha\dot\beta} \tag{2.70}$$

with all other torsion components vanishing. The constrains must be consistent with the Bianchi identities

$$\sum_{ABC}(-)^{AC}\left(\mathcal{D}_A F^i_{BC}+T_{AB}{}^D F^i_{DC}\right)=0 \tag{2.71}$$

that follow from the Jacobi identity for commutators. (The first Bianchi identity, which arises as the coefficient of D_A in the Jacobi identity, is trivial in flat space with only internal symmetries).

In supersymmetry there are two types of constraints: The first type can be imposed by a mere redefinition of what we call the covariant derivative. Such **conventional constraints** are familiar from Riemannian geometry: There we can absorb the torsion $T_{ab}{}^c$ into a redefinition of the spin connection $\omega_m{}^{ab}$ that determines the covariant derivative and thus replace a general metric-compatible connection by the Christoffel connection. This is a mere change of basis of the covariant local coordinates of the jet bundle and the torsion then becomes a particular tensor field that may (or may not) be set to 0. Computing the field strengths in terms of the connections we find

$$F^i_{\alpha\dot\beta}=D_\alpha\overline{A}^i_{\dot\beta}+\overline{D}_{\dot\beta}A^i_\alpha+A^j_\alpha\overline{A}^k_{\dot\beta}f_{jk}{}^i-2iA\!\!\!/^i_{\alpha\dot\beta}, \tag{2.72}$$

so that $F^i_{\alpha\dot\beta}=0$ can be imposed as a conventional constraint.[12]

In order to construct gauge invariant interactions for matter fields we want to impose a covariant chirality condition $\overline{\mathcal{D}}_{\dot\alpha}\phi=0$. Covariantly chiral fields can, however, be charged under the gauge group only if $\{\mathcal{D}_\alpha,\mathcal{D}_\beta\}=F^i_{\alpha\beta}\delta_i$ vanishes. We thus impose the **standard constraints**

$$F^i_{\alpha\beta}=F^i_{\dot\alpha\dot\beta}=0,\qquad F^i_{\alpha\dot\beta}=0. \tag{2.73}$$

The general form of the gauge algebra, with the non-vanishing commutation relations

$$[\mathcal{D}_a,\mathcal{D}_b]=F^i_{ab}\delta_i,\qquad [\mathcal{D}_\alpha,\mathcal{D}_a]=i\sigma_{a\,\alpha\dot\beta}\overline{W}^{i\dot\beta}\delta_i,\qquad \{\mathcal{D}_\alpha,\overline{\mathcal{D}}_{\dot\beta}\}=2i\mathcal{D}\!\!\!/_{\alpha\dot\beta} \tag{2.74}$$

can then be obtained by solving the Bianchi identities, which also imply

$$\overline{\mathcal{D}}_{\dot\alpha}W=0,\qquad D^I:=\frac{1}{2}\mathcal{D}^\alpha W^I_\alpha,\qquad \mathcal{D}_\alpha W^{I\beta}=\sigma^{ab\beta}_\alpha F^I_{ab}+i\delta^\beta_\alpha D^I. \tag{2.75}$$

To derive this result we should first analyse the identities with contributions from torsions:

$$\sum_{\alpha\beta\dot\gamma}(...)=\sigma^a_{\alpha\dot\gamma}F_{a\beta}+\sigma^a_{\beta\dot\gamma}F_{a\alpha}=0\quad\Rightarrow\quad\sigma^a_{\alpha\dot\gamma}F_{a\beta}=\varepsilon_{\alpha\beta}\overline{W}_{\dot\gamma},\qquad F_{a\alpha}=\sigma_{a\alpha\dot\beta}\overline{W}^{\dot\beta} \tag{2.76}$$

i.e. $F_{a\alpha}$ contains no spin 3/2 component. Except for the complex conjugate of the above the only other BI with contributions from torsions is

$$\sum_{\alpha\dot\beta c}(...)=\mathcal{D}_\alpha F_{c\dot\beta}+\mathcal{D}_{\dot\beta}F_{c\alpha}+2i\sigma^a_{\alpha\dot\beta}F_{ab}=0\quad\Rightarrow\quad F_{ab}=-\frac{1}{4}(\overline{\mathcal{D}}\overline{\sigma}_a\sigma_b\overline{W}-\mathcal{D}\sigma_a\overline{\sigma}_bW) \tag{2.77}$$

[12] Then the gauge potential A^i_m can be written in terms of (covariant derivatives of) A^i_α and $\overline{A}^i_{\dot\alpha}$, which are therefore called prepotentials. This is similar to the fact that we can express the spin connection in terms of the vielbein if we impose $T_{ab}{}^c=0$.

Antisymmetry of F_{ab} thus implies

$$\overline{\mathcal{D}W} = \mathcal{D}W, \qquad F_{ab} = \frac{1}{2}(\mathcal{D}\sigma_{ab}W - \overline{\mathcal{D}}\overline{\sigma}_{ab}\overline{W}). \tag{2.78}$$

The only remaining Bianchi identity that contains new information is

$$\sum_{\dot\alpha\dot\beta c} (...) = \overline{\mathcal{D}}_{\dot\alpha}F_{c\dot\beta} + \overline{\mathcal{D}}_{\dot\beta}F_{c\dot\alpha} = 0 \quad \Rightarrow \quad (\sigma^c_{\alpha\dot\alpha}\overline{\mathcal{D}}_{\dot\beta} + \sigma^c_{\alpha\dot\beta}\overline{\mathcal{D}}_{\dot\alpha})W^\alpha = 0, \qquad \overline{\mathcal{D}}_{\dot\alpha}W_\alpha = 0, \tag{2.79}$$

i.e. W_α is covariantly chiral (use $\sigma^c_{\alpha\dot\alpha}\overline{\sigma}_c^{\dot\beta\beta} = 2\delta_\alpha^\beta\delta_{\dot\alpha}^{\dot\beta}$ after contraction with $\overline{\sigma}_c^{\gamma\dot\gamma}$). We can make contact with our previous results by relating gauge covariant derivatives to ordinary ones via [dr87, WE83]

$$\mathcal{D}_\alpha = D_\alpha + A^i_\alpha\delta_i = e^{iV}D_\alpha e^{-iV}, \qquad A^i_\alpha = e^{iV}[D_\alpha, e^{-iV}], \tag{2.80}$$

where V^i is a real scalar superfield. Covariantly chiral fields are related to chiral fields by multiplication with e^{iV}. It can be shown that $V = V^i\delta_i$ parametrizes the most general solution to the constraints, so that the real scalar superfield saves us all the work with the Bianchi identities in super-YM theory.

In supergravity, however, no such nice magic is known and we have to do it the hard way by solving the Bianchi identities with constraints. We can either work in superspace with the supervielbein and super-spin connection, and eventually use a superspace coordinate transformation to go to a Wess–Zumino gauge when life becomes to tedious, or we may avoid to introduce the redundant fields that are eliminated by that gauge from scratch and work with the structure of the gauge algebra. This is the approach that we will follow in the next section.

Chapter 3

Supergravity

3.1 Symmetry algebras

In this section we analyse the general structure of closed irreducible symmetry algebras (or gauge algebras) for which the infinitesimal symmetry transformations are implemented by derivations ∇_M on tensor fields $\phi(\varphi)$ that are functions of some set of elementary fields φ (and their derivatives). Linear independence of $\nabla_M\phi^i$ and the Jacobi identity for graded commutators $[\nabla_M,\nabla_N\} := \nabla_M\nabla_N - (-)^{MN}\nabla_N\nabla_M$ imply the Bianchi identities

$$[\nabla_M,\nabla_N\} = \mathcal{F}_{MN}{}^P\nabla_P \quad\Rightarrow\quad \sum_{MNP}(-)^{MP}\left(\nabla_M\mathcal{F}_{NP}{}^Q - \mathcal{F}_{MN}{}^R\mathcal{F}_{RP}{}^Q\right) = 0. \tag{3.1}$$

The structure functions $\mathcal{F}_{MN}{}^P = -(-)^{MN}\mathcal{F}_{NM}{}^P$ are graded antisymmetric and their grading is given by $|\mathcal{F}_{MN}{}^P| = |M|+|N|+|P| \mod 2$. The signs in cyclic sums originate from the Jacobi identity and can be understood from the rule that $[A,.\}$ should act like a graded derivation on the 'commutator product', i.e. $[A,[B,C\}\} = [[A,B\},C\} + (-)^{AB}[B,[A,C\}\}$, which is equivalent to $\sum_{ABC}(-)^{AC}[A,[B,C\}\} = 0$.

In supergravity we split the covariant symmetry transformations ∇_M into *space-time* symmetries $\{\mathcal{D}_A\} = \{\mathcal{D}_a,\mathcal{D}_{\underline{\alpha}}\}$ and internal symmetries $\{\delta_I\} = \{l_{ab},\delta_i,\delta_W,\delta_R\}$, which generate Lorentz transformations, Yang–Mills group actions, dilatations and R symmetries (which act only on $\mathcal{D}_{\underline{\alpha}}$), respectively. This split implies that the structure functions receive different interpretations:

$$[\mathcal{D}_A,\mathcal{D}_B\} = -T_{AB}{}^C\mathcal{D}_C + F_{AB}{}^I\delta_I, \qquad [\delta_I,\mathcal{D}_A\} = -g_{IA}{}^B\mathcal{D}_B, \qquad [\delta_I,\delta_J\} = f_{IJ}{}^K, \tag{3.2}$$

with torsions $T_{AB}{}^C$, field strengths $F_{AB}{}^I$, representations matrices $(g_I)_A{}^B$, and structure constants $f_{IJ}{}^K$. Note that $F_{iA}{}^B = 0$ since space-time symmetries are inert to gauge transformations. The field strengths corresponding to Lorentz transformations $R_{AB}{}^{ab} := F_{AB}{}^{ab}$ are called curvatures. With

$\mathcal{F}_{AB}{}^C = -T_{AB}{}^C$	$\mathcal{F}_{IB}{}^C = -g_{IB}{}^C$	$\mathcal{F}_{IJ}{}^C = 0$
$\mathcal{F}_{AB}{}^K = F_{AB}{}^K$	$\mathcal{F}_{IB}{}^K = 0$	$\mathcal{F}_{IJ}{}^K = f_{IJ}{}^K$

(3.3)

the Bianchi identities thus become

$$\text{BI 1:} \qquad \sum_{ABC}\!\!\!\!\bigcirc\, (-)^{AC} \left(\mathcal{D}_A T_{BC}{}^D + T_{AB}{}^E T_{EC}{}^D - F_{AB}{}^I g_{IC}{}^D\right) = 0, \tag{3.4}$$

$$\text{BI 2:} \qquad \sum_{ABC}\!\!\!\!\bigcirc\, (-)^{AC} \left(\mathcal{D}_A F_{BC}{}^I + T_{AB}{}^D F_{DC}{}^I\right) = 0. \tag{3.5}$$

and the meaning of the remaining identities is that $T_{AB}{}^C$ and $F_{AB}{}^K$ transform as representations under δ_I according to their indices,

$$\delta_I F_{AB}{}^K = -g_{IA}{}^D F_{DB}{}^K + (-)^{AB} g_{IB}{}^D F_{DA}{}^K + (-)^{IA+IB} F_{AB}{}^J f_{JI}{}^K, \tag{3.6}$$

$$\delta_I T_{AB}{}^C = -g_{IA}{}^D T_{DB}{}^C + (-)^{AB} g_{IB}{}^D T_{DA}{}^C + T_{AB}{}^D g_{ID}{}^C, \tag{3.7}$$

and that the representations matrices g_I and the structure constants $f_{IJ}{}^K$ are invariant tensors $\delta_I g_{JA}{}^B = 0$ (the representation property of g) and $\delta_I f_{JK}{}^L = 0$ (the Jacobi identity for f).

We assume that the δ_I are linearly represented on tensor fields and that representation ∂_m of infinitesimal translation is a linear combination of the covariant derivatives

$$\partial_m \phi = -\mathcal{A}_m{}^N(\varphi) \nabla_N \phi. \tag{3.8}$$

To specify the field content we assume that the connection one forms $\mathcal{A}^N = dx^m \mathcal{A}_m{}^N$ and their (symmetrized) derivatives can be chosen to be the only non-covariant variables of the jet bundle. (The formalism can be extended to the case of p-form gauge fields and reducible gauge algebras, as well as to algebras that only close off-shell [$\texttt{br}_1\texttt{96}$]). With $e_m{}^a := -\mathcal{A}_m{}^a$ and $e_m{}^a E_a{}^n = \delta_m^n$ we define

$$\{\mathcal{A}_m{}^M\} = \{-e_m{}^a, \mathcal{A}_m{}^\mu\} = \{-e_m{}^a, \psi_m{}^{\underline{\alpha}}, \mathcal{A}_m{}^I\} = \{-e_m{}^a, \psi_m{}^{\underline{\alpha}}, \omega_m{}^{ab}, A_m{}^i + \ldots\}, \tag{3.9}$$

$$\mathcal{D}_a = E_a{}^m(\partial_m + \mathcal{A}_m{}^\mu \nabla_\mu) = E_a{}^m(\partial_m + \psi_m{}^{\underline{\alpha}} \mathcal{D}_{\underline{\alpha}} + \tfrac{1}{2}\omega_m{}^{ab} l_{ab} + A_m{}^i \delta_i + \ldots). \tag{3.10}$$

In these equations the vielbein $e_m{}^a$ is assumed to be invertible and vielbein and gravitino (Rarita–Schwinger field) are interpreted as connections for translations and SUSY transformations. Commutation of the partial derivatives $[\partial_m, \partial_n] = 0$ and independence of $\nabla_N \phi$ then imply

$$\partial_m \mathcal{A}_n{}^P - \partial_n \mathcal{A}_m{}^P - \mathcal{A}_m{}^M \mathcal{A}_n{}^N \mathcal{F}_{NM}{}^P = 0, \tag{3.11}$$

which can be solved for the field strengths with bosonic indices

$$e_m{}^a e_n{}^b \mathcal{F}_{ab}{}^N = \partial_m \mathcal{A}_n{}^N - \partial_n \mathcal{A}_m{}^N - e_m{}^c \mathcal{A}_n{}^\mu \mathcal{F}_{\mu c}{}^N + e_n{}^c \mathcal{A}_m{}^\mu \mathcal{F}_{\mu c}{}^N + \mathcal{A}_n{}^\nu \mathcal{A}_m{}^\mu \mathcal{F}_{\mu\nu}{}^N. \tag{3.12}$$

This equation could again be split into equations for field strengths and torsions in terms of the various connections to obtain the usual lengthy formulas (the last term with $\nu = j, \mu = i, N = k$, for example, gives the A^2-term in YM).

It is straightforward to set up the BRST formalism for symmetry algebras of this type. The BRST transformations of the matter fields is defined by replacing the gauge parameters by ghost fields of opposite grading $|C^I| \equiv |\nabla_I| + 1 \bmod 2$, i.e. $s\phi^i = C^N \nabla_N \phi^i$. For any closed and irreducible gauge algebra one may check that $s^2\phi^i = 0$ uniquely fixes the BRST transformations of the ghost fields.

$$s\phi^i = C^N \nabla_N \phi^i \quad \Rightarrow \quad sC^P = \tfrac{(-)^M}{2} C^M C^N \mathcal{F}_{NM}{}^P. \tag{3.13}$$

$s^2 C^P = 0$ is then equivalent to the Bianchi identity (3.1).

Anti-commutativity of s and d, which follow from $[s, \partial_m] = \{s, dx^m\} = 0$, may then be used to define a new nilpotent operator $\tilde{s} := s + d$ and $\tilde{C}^N = C^N + \mathcal{A}^N$ so that $s + d = \tilde{C}^N \nabla_N$ on tensor fields. (3.13) implies because of formal identity of the algebras that

$$(s + d)\, \tilde{C}^P = \tfrac{1}{2}(-)^N \tilde{C}^N \tilde{C}^M \mathcal{F}_{MN}{}^P \tag{3.14}$$

whose split into parts with ghost number 0, 1 and 2 yields

$$\begin{aligned} sC^P &= \tfrac{1}{2}(-)^N C^N C^M \mathcal{F}_{MN}{}^P, && (3.15)\\ s\mathcal{A}^P + dC^P &= C^M \mathcal{A}^N \mathcal{F}_{NM}{}^P, && (3.16)\\ d\mathcal{A}^P &= \tfrac{1}{2} \mathcal{A}^M \mathcal{A}^N \mathcal{F}_{NM}{}^P. && (3.17) \end{aligned}$$

The first two equations define the BRST transformations of connections and ghost fields. Consistency of the last equation with the tensor transformation law of the field strengths can be checked by a straightforward computation.

To obtain the more conventional form of this transformation law we use the reparametrization

$$\xi^a := C^m e_m{}^a, \qquad \xi^\mu := C^\mu + C^m \mathcal{A}_m{}^\mu = C^\mu + i_C \mathcal{A}^\mu. \tag{3.18}$$

ξ^m corresponds to the vector field entering the Lie derivative and we thus obtain

$$\begin{aligned} s\,\phi &= (\xi^m \partial_m + \xi^\mu \nabla_\mu)\, \phi, && (3.19)\\ s\, e_m{}^a &= \xi^n \partial_n e_m{}^a + (\partial_m \xi^n)\, e_n{}^a + \xi^\mu \mathcal{A}_m{}^N \mathcal{F}_{N\mu}{}^a, && (3.20)\\ s\, \mathcal{A}_m{}^\mu &= \xi^n \partial_n \mathcal{A}_m{}^\mu + (\partial_m \xi^n)\, \mathcal{A}_n{}^\mu + \partial_m \xi^\mu + \xi^\nu \mathcal{A}_m{}^N \mathcal{F}_{N\nu}{}^\mu, && (3.21)\\ s\, \xi^m &= \xi^n \partial_n \xi^m + \tfrac{1}{2}(-)^\mu \xi^\mu \xi^\nu \mathcal{F}_{\nu\mu}{}^a E_a{}^m, && (3.22)\\ s\, \xi^\mu &= \xi^n \partial_n \xi^\mu + \tfrac{1}{2}(-)^\nu \xi^\nu \xi^\rho (\mathcal{F}_{\rho\nu}{}^\mu - \mathcal{F}_{\rho\nu}{}^a E_a{}^m \mathcal{A}_m{}^\mu). && (3.23) \end{aligned}$$

The C^N are called covariant ghosts: The necessity of a redefinition of ghost variables in covariant equations can already be observed in Riemannian geometry: Since the Lie derivative maps tensors into tensors it should be possible to write it in terms of covariant derivatives. But this works out only if we combine it with a Lorentz transformation and redefine the parameter Λ:

$$\mathcal{L}_\xi + \tfrac{1}{2}\Lambda_{ab}l^{ab} = \xi^l D_l - (D_i\xi^k + \xi^l T_{li}{}^k)\Delta_k{}^i + \tfrac{1}{2}\hat\Lambda_{ab}l^{ab}, \qquad \hat\Lambda_{ab} = \Lambda_{ab} - \xi^l\omega_{lab}. \tag{3.24}$$

($\Delta_i{}^j$ and l^{ab} are the GL_n and Lorentz generators; for simplicity we avoid any world indices on tensors by contraction with the vielbein, which is a connection in the present context, or with differentials in case of field strengths). Using $\hat\Lambda$ we also find $s\omega_{na}{}^b = -D_n\hat\Lambda_a{}^b - \xi^l R_{lna}{}^b$, in analogy with the tensorial property of the variation of the connection coefficients $s\Gamma_{nl}{}^m = D_nD_l\xi^m + D_n(\xi^k T_{kl}{}^m) + \xi^k R_{knl}{}^m$. Of course these results are contained in their above extension to more general algebras of covariant derivatives if world indices are avoided.

Returning to the construction of supergravity theories, the next step is to impose constraints since the connections we introduced so far yield highly reducible theories that, furthermore, usually do not allow for matter fields obeying equation of motion of the type that we expect. First one ones redefinitions $\nabla_M \to X_M{}^N\nabla_N$ with $X_M{}^N = \delta_M^N + H_M{}^N(\mathcal{F})$ of the covariant derivatives to bring the gauge algebra into a standard form, where we have the conventional constaints

$$T_{\alpha\dot\beta}{}^a = 2i\sigma^a_{\alpha\dot\beta}, \qquad T_{ab}{}^c = T_{\alpha\beta}{}^\gamma = T_{\dot\alpha\dot\beta}{}^{\dot\gamma} = T_{a\beta}{}^{\underline{\gamma}} = 0, \qquad F^i_{\alpha\dot\beta} = 0. \tag{3.25}$$

To allow for chiral matter multiplets one extends this to the following collection of standard constraints:

$$T_{ab}{}^c = 0, \qquad T_{\underline{\alpha\beta}}{}^a = 2i\gamma^a_{\underline{\alpha\beta}}, \qquad F^i_{\underline{\alpha\beta}} = 0, \qquad T_{\alpha\beta}{}^{\dot\gamma}. \tag{3.26}$$

(which of these constraints are conventional slightly depends on whether we gauge R and Weyl symmetries).

Consistency of the constraints requires that the Bianchi identities are fulfilled, the check of which is the crucial (and most tedious) step in the construction of a SUGRA theory. These identities usually imply additional constraints and the general parametrization of the allowed curvatures and torsions requires the introduction of auxiliary fields that, together with the vielbein $e_m{}^a$ and the gravitino $\psi_m{}^{\underline{\alpha}}$, constitute the (off-shell) graviton multiplet. In some complicated cases, like 10-dimensional SUGRA and $N = 4$–extended SUGRA in 4 dimension, it has be shown that our approach cannot lead to a satisfactory theory. In these cases on must extend our framework and admit open and reducible gauge algebras.

The standard constraints are usually not sufficient and finding a useful complete set of constraints (i.e. obtaining an irreducible SUGRA theory) requires some experience (educated guesses and tedious evaluation of the consequences). In 4-dimensions, for example, there are 3 known sets of solutions, called old minimal, new minimal and non-minimal SUGRA. Non-minimal SUGRA has some ugly features as far as allowed matter couplings are concerned and new minimal SUGRA is the one that automatically comes out of superstring theory.

It turns out that not all of the BIs are independent. Some of them can be solved explicitly for the curvatures in terms of torsions. Inserting these solutions with curvature and torsion defined in terms of the gauge connections as above, the second BIs become redundant. This is the content of the following

Theorem (Dragon): The second BI follows from the Ricci identity $\mathcal{D}^2 = R$ and the first set of BIs [dr79,MU89].

To find the most general local action that is invariant under a given gauge algebra the BRST formalism can be used to derive the descent equations, which reduce to problem to the computation of cohomologies of (super) Lie algebra [br92].

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