Lecture notes

Geometry, Topology and Physics II

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Chapter 1

Vector bundles

1.1 Vielbein and Lorentz connection

So far we described the tangent space with local coordinates (x^m, v^n), where the components v^n of a vector at a point with coordinates x^m refer to the holonomous basis ∂_{x^m} of $T\mathcal{M}$. As a consequence the transition functions for v^n on the overlap of two charts is given by the Jacobi matrix $\frac{\partial y^m}{\partial x^l}$, which is a general linear transformation. But the general linear group only has tensor representations and therefor we cannot describe spinors in this formalism. This problem can be solved by introducing an orthonormal basis $e^a = dx^m e_m{}^a$ for the cotangent space and the dual basis $E_a = E_a{}^m \partial_m$ with $g(E_a, E_b) = \eta_{ab}$ for the tangent space, so that

$$g_{mn} E_a{}^m E_b{}^n = \eta_{ab}, \qquad e_m{}^a E_a{}^n = \delta_m^n, \qquad g_{mn} = \eta_{ab} e_m{}^a e_n{}^b. \tag{1.1}$$

We will denote *Lorentz indices* by letters from the beginning of the alphabet and *world indices* by $k, l, m, \ldots$; for a Riemannian metric we have $\eta_{ab} = \delta_{ab}$ and in the pseudo-Riemannian case the diagonal matrix η_{ab} has entries ± 1. In physics $e_m{}^a$ is called *vielbein*,[1] and we can use it to write down a Dirac operator $\partial\!\!\!/ = \gamma^m \partial_m \gamma^a E_a{}^m \partial_m$ and an action $\int e\, \overline{\psi}(iD\!\!\!/ - m)\psi$ for spinor fields in curved space in terms of the usual γ-matrices satisfying $\{\gamma^a, \gamma^b\} = 2\eta^{ab}$. The volume element can be written as $e = |\det(e_m{}^a)| = \sqrt{|g|}$. The x-dependent γ-matrices in curved space are linear combinations $\gamma^m = \gamma^a E_a{}^m$ of constant representation matrices γ^a of the Clifford algebra, where $E_a{}^m$ is determined only up to a local (i.e. x-dependent) Lorentz transformation on its Lorentz index a.

In order to define the covariant derivative $D\psi$ of spinor fields we introduce the *Lorentz connection* (or *spin connection*) $\omega_a{}^b = dx^m \omega_{ma}{}^b$ for objects, like spinors and tensors with Lorentz indices, that transform in some representation under local Lorentz transformations. Metricity of

[1] In mathematics $e^a = dx^m e_m{}^a$ is called *soldering form* since the vielbein provides a soldering of the (principal bundle of the) cotangent bundle with an orthonormal frame bunde (see below).

the connection tranlates into antisymmetry in the last two indices $\omega_{ab}+\omega_{ba}=0$, which preserves orthonormality of tangent vectors under parallel transport. If we want to use holonomous and orthogonal bases (i.e. tensors with both types of indices) at the same time we may do so by defining the total covariant derivative as

$$D = d + \Gamma_l{}^m \Delta_m{}^l + \tfrac{1}{2}\,\omega_{ab}\, l^{ab}, \qquad l_{ab}v_c = \eta_{ac}v_b - \eta_{bc}v_a. \tag{1.2}$$

with $D = dx^n D_n$, $\Gamma_l{}^m = dx^i \Gamma_{il}{}^m$ and $\omega_{ab} = dx^i \omega_{iab}$. The constant flat metric η_{ab} is invariant under l_{ab} so that upper and lower Lorentz indices transform with the same sign. On spinors the algebra

$$[l_{ab}, l_{cd}] = \eta_{ac}l_{bd} - \eta_{bc}l_{ad} - \eta_{ad}l_{bc} + \eta_{bd}l_{ac} \tag{1.3}$$

of Lorentz transformations is represented as $l_{ab}\psi = \frac{1}{4}[\gamma_a, \gamma_b]\psi$.

Cartan invented a very efficient calculus by introducing differential forms and orthonormal bases e^a of contangent space: On (co)vector fields $v = e^a v_a$ the spin-connection term of the covariant derivative acts by matrix multiplication $Dv_a = dv_a + \omega_a{}^b v_b$. Curvature and torsion can then be defined by Cartan's structure equations (CSEq)

$$R_a{}^c = d\omega_a{}^c + \omega_a{}^b \omega_b{}^c, \qquad T^a = de^a + \omega^a{}_b e^b, \tag{1.4}$$

or, in an even more compact symbolic form,[2] $R = D^2$ and $T = De$. The Bianchi identities

$$dT + \omega \wedge T = R \wedge e, \qquad dR + \omega \wedge R - R \wedge \omega = 0. \tag{1.5}$$

are now a trivial consequence of $d^2 = 0$. The total antisymmetrization that is implicit in these 3-forms replaces the cyclic sum over the respective indices.

In order to relate the spin connection ω to the affine connection Γ we impose that parallel transport should not depend on which basis we use for tangent space. The vielbein thus has to be covariantly constant,

$$D_m e_n{}^a = \partial_m e_n{}^a - \Gamma_{mn}{}^l e_l{}^a + \omega_m{}^a{}_b e_n{}^b = \partial_m e_n{}^a - \Gamma_{mn}{}^a - \omega_{mn}{}^a = 0, \tag{1.6}$$

which provides a relation between Γ and the spin connection (here we use the vielbein and its inverse to convert the second index of ω and the third index of Γ to the appropriate basis). Contraction of this equation with $dx^m dx^n$ leads to $de^a - dx^m dx^n \Gamma_{mn}{}^a - dx^m dx^n \omega_{mb}{}^a e_n{}^b = de^a + \omega^a{}_b e^b - dx^m dx^n \Gamma_{mn}{}^a = 0$, which shows that the definition of the torsion via the CSEq agrees with our previous definition. In order to establish the equivalence of the definitions of the curvature we introduce the symbod $(D^\omega)_a{}^b = \delta_a^b d + \omega_a{}^b$ for the covariant derivative that only acts on Lorentz indices. Then Cartan's definition amounts to $R^\omega = (D^\omega)^2$ and

$$(R^\omega)_a{}^b v_b = \tfrac{1}{2} dx^i dx^j (D_i^\omega D_j^\omega v)_a = \tfrac{1}{2} dx^i dx^j (D_i D_j v + \Gamma_{ij}^l D_l v)_a = E_a^l (R^\Gamma)_i{}^j \Delta_j{}^i v_l, \tag{1.7}$$

[2] Derivatives of the vector field components drop out in $D^2 v = d^2 v + d(\omega v) + \omega dv + \omega^2 v = (d\omega + \omega^2)v$.

where $[D_i, D_j] = -T_{ij}{}^l D_l + R^\Gamma_{ijk}{}^l \Delta_l{}^k$. In a sense, the torsion contribution to $[D_i, D_j]$ is thus taken into account in Cartan's calculus by contracting the form index of the spin connection with a differential and having the connection only act on the Lorentz indices.

A formula for the spin connection as a function of vielbein and torsion can be obtained by using $\omega_{ma}{}^b = E_a{}^n \partial_m e_n{}^b - \Gamma_{ma}{}^b$, $g_{mn} = e_m{}^a e_n{}^b \eta_{ab}$ and our formula for $\Gamma(g,T)$. The same result can also be obtained directly from the structure equation $T^a = de^a + \omega^a{}_b e^b$ if we use the formula $2\omega_{[mn]r} = \sum_{mnr} \omega_{mnr} - \omega_{rmn}$ that allows to compute a tensor that is antisymmetric in its last 2 indices from its antisymmetrization in the first two indices:[3]

$$-\omega_{mnl} = e_{ma}\partial_{[n}e_{l]}{}^a + e_{la}\partial_{[n}e_{m]}{}^a + e_{na}\partial_{[m}e_{l]}{}^a + \tfrac{1}{2}(T_{mnl} + T_{lmn} - T_{nlm}) \tag{1.8}$$

For fixed vielbein the components of spin connection and torsion are thus related by invertible linear equations, so that it is equivalent to use $\{e_m{}^a, \omega_{ma}{}^b\}$ or $\{e_m{}^a, T_{mn}{}^a\}$ as independent of fields. Usually it is more convenient to work with the former set.

Example: The vielbein calculus is very useful for the evaluation of the curvature tensor. As an example we derive the Schwarzschild metric for a spherically symmetric black hole. Parametrizing the time dilatation with $T(r)$ and the length contraction in the radial direction with $R(r)$, where $4\pi r^2$ is the surface of the sphere at fixed r:

$$ds^2 = T^2 dt^2 - R^{-2} dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\varphi^2). \tag{1.9}$$

A convenient choice for the vielbein is therefore

$$e^0 = T\,dt, \qquad e^1 = R^{-1}\,dr, \qquad e^2 = r\,d\theta, \qquad e^3 = r\sin\theta\,d\varphi. \tag{1.10}$$

$$de^0 = T'\,dr\,dt, \qquad de^1 = 0, \qquad de^2 = dr\,d\theta, \qquad de^3 = \sin\theta\,dr\,d\varphi + r\cos\theta\,d\theta\,d\varphi. \tag{1.11}$$

Vanishing torsion $T^a = de^a + e^b\omega_b{}^a = 0$ thus yields the nonvanishing connection coefficients

$$\omega_0{}^1 = -T'R\,dt, \qquad \omega_1{}^2 = -R\,d\theta, \qquad \omega_1{}^3 = -R\sin\theta\,d\varphi, \qquad \omega_2{}^3 = -\cos\theta\,d\varphi. \tag{1.12}$$

and the curvatures

$$R_0{}^1 = (T'R)'\,dt\,dr, \qquad R_0{}^2 = T'R^2\,dt\,d\theta, \qquad R_0{}^3 = T'R^2\sin\theta\,dt\,d\varphi, \tag{1.13}$$

$$R_1{}^2 = -R'\,dr\,d\theta, \qquad R_1{}^3 = -R'\sin\theta\,dr\,d\varphi, \qquad R_2{}^3 = (1-R^2)\sin\theta\,d\theta\,d\varphi. \tag{1.14}$$

The vacuum Einstein equations in the orthonormal basis are $\mathcal{R}_{ab} = R_{acb}{}^c = E_a{}^m E_c{}^n R_{mnb}{}^c = 0$. The Ricci tensor turns out to be diagonal with

$$\mathcal{R}_{00} = R(T'R)'/T + 2T'R^2/rT, \qquad \mathcal{R}_{11} = -R(T'R)'/T - 2R'R/r, \tag{1.15}$$

$$\mathcal{R}_{22} = \mathcal{R}_{33} = -T'R^2/rT - R'R/r + (1-R^2)/r^2. \tag{1.16}$$

[3] The cyclic sum gives the total antisymmetrization of such a tensor, so that both, $\omega_{[mn]r}$ and $\sum_{mnr}\omega_{mnr}$, are functions of vielbein and torsion. We thus obtain $\omega_{mab} = E_a{}^n E_b{}^r(\omega_{[mn]r} - \omega_{[nr]m} + \omega_{[rm]n})$ with $\omega_{[mn]r} = \eta_{ab} e_r{}^a \partial_{[m} e_{n]}{}^b - \frac{1}{2}T_{mnr}$.

Summation of $\mathcal{R}_{00}=0$ and $\mathcal{R}_{11}=0$ implies that R and T are proportional and by a rescaling of t we can set $T=cR$, where c is the speed of light. $\mathcal{R}_{22}=0$ then implies $\partial_r(\ln(1-R^2))=-1/r$ and $R^2=1-\frac{2MG}{rc^2}$, where the choice of the integration constant is parametrized by M times Newton's gravitational constant G. We thus obtain the Schwarzschild geometry

$$ds^2=(1-\tfrac{2MG}{rc^2})\,c^2\,dt^2-\frac{1}{1-\frac{2MG}{rc^2}}\,dr^2-r^2(d\theta^2+\sin^2\theta\,d\varphi^2). \tag{1.17}$$

Exercise 1: Check the above result for the Ricci tensor and clarify the meaning of the integration constant M by comparison of the acceleration that follows from the geodesic equation in the Schwarzschild geometry for a test particle that is initially at rest at some large distance r (i.e. $\dot{x}^m=(c,0,0,0))$ at $t=0$) with Newton's law $F=-Gm_1m_2/r^2$ for the attractive force $F_i=m_i\,\ddot{x}$ between two masses m_i at distance r. Note that the force can be attributed to the variation of the time dilatation factor $T=\sqrt{g_{00}}$. Why is the acceleration decreasing as we approach the horizon?

Since the Lie derivative of a tensor field is again a tensor field it should be possible to rewrite it in terms of covariant derivatives. If Lorentz indices are involved this is, however, only possible if we combine local coordinate and Lorentz transformations to a total transformation

$$\delta:=\mathcal{L}_\xi+\tfrac{1}{2}\Lambda_{ab}l^{ab}=\xi^lD_l-(D_i\xi^k+\xi^lT_{li}{}^k)\Delta_k{}^i+\tfrac{1}{2}\hat{\Lambda}_{ab}l^{ab},\qquad \hat{\Lambda}_{ab}:=\Lambda_{ab}-\xi^l\omega_{lab}. \tag{1.18}$$

The same ξ-dependent redefinition of the Lorentz transformation has to be used if we want to write the variation of the spin connection under such a transformation in a manifestly covariant form:

$$\delta\Gamma_{nl}{}^m=D_nD_l\xi^m+D_n(\xi^kT_{kl}{}^m)+\xi^kR_{knl}{}^m,\qquad \delta\omega_{na}{}^b=D_n\hat{\Lambda}_a{}^b+\xi^lR_{lna}{}^b. \tag{1.19}$$

To derive these formulas we can use $[\delta,D_n]v^j=(\delta\Gamma_{nl}{}^m)\Delta_m{}^lv^j$ and $[\delta,D_n]v^c=\frac{1}{2}(\delta\omega_{nab})l^{ab}v^c$.

1.2 Fiber bundles

The tangent space of a manifold can itself be regarded as a manifold, which may have an interesting and non-trivial topological structure. There exists, for example, no smooth non-vanishing tangent vector field on a sphere ('you cannot comb the hair on a sphere'). This is the motivation for defining vector bundles and more general fiber bundles.

A **bundle** is a triple (E,B,π) consisting of two topological spaces E and B and a continuous surjective map $\pi:E\to B$. B is called **base** space, π is the **projection** of the total space E to the base space, and $F_x=\pi^{-1}(x)$ is the **fiber** at x. Usually we will be interested in the situation where all fibers F_x are homeomorphic, in which case we call $F\cong F_x$ the typical fiber. In the case of vector bundles the typical fiber is a vector space. Locally the tangent bundle looks like a product space $\mathcal{M} \times V$ with $V \cong T_x\mathcal{M}$, but globally there may be a twist: For a trivial bundle $S^2 \times V$ over the sphere, for example, any constant vector would provide a non-vanishing 'vector field'. Similarly, the Möbius band is a twisted line bundle over S^1, whose topology differs from the trivial bundle $S^1 \times \mathbb{R}$. The twist is generated by the way the fibers are glued together globally, but locally these bundles look like a product. The gluing involves a smooth group action on the fiber. The corresponding topological group is called the structure group of the bundle. This leads to the following definition of *fiber bundles*, which are also called *twisted products*.

A **fiber bundle** (E, B, π, F, G) is a bundle (E, B, π) with typical fiber F and a covering of B by a family of open sets $\{U_i\}$ such that there are homeomorphisms $\varphi_i : \pi^{-1}(U_i) \to U_i \times F$ with $\pi = pr_1 \circ \varphi_i$, where pr_1 projects to the first component of $(x, f) \in U_i \times F$, i.e. a point in F_x with $x \in U_i$ has to be mapped to (x, f) with $f \in F$. The maps φ_i are called **local trivializations**. The gluing data are given by maps $g_{ij} : U_i \cap U_j \to G$ into the structure group such that the transition functions $t(g)_{ij}(x) = \varphi_i \circ \varphi_j^{-1}$ correspond to the action $g(x) \to t_g(x) : F \to F$ of $g_{ij}(x)$ on the typical fiber F for each point in $x \in U_i \cap U_j$.

The essential data of a fiber bundle are the transition functions, which need to satisfy the **cocycle condition** $g_{ij} \circ g_{jk} = g_{ik}$ on the overlap of three coordinate patches. The gluing data can be used to reconstruct the fiber bundle $E = X/\sim$ as the disjoint union $X = \bigcup (U_i \times F)$ with the identification $(x_i, f_i) \sim (x_j, f_j)$ if $x_i \in U_i$ and $f_j = t_{ij}(x_i) f_i$. The cocycle condition guarantees that the gluing of the patches $U_i \times F$ of E is well defined.

If E and M are both C^k manifolds then we may consider C^k fiber bundles, in which case we require that π is a C^k map and that φ_i are C^k diffeomorphisms. In the following we will consider C^∞ bundles, which implies that the structure group is then a Lie group. [4] A (real or complex) **vector bundle** is a fiber bundle whose typical fiber is a (real or complex) vector space V and whose local trivializations φ_i act as linear isomorphims on the fibers, i.e. $\varphi_i : \pi^{-1}(x) \to V$ is an invertible linear map and $g_{ij}(x) : V \to V$ is a general linear transformation. If all transition function belong to a subgroup $G \subseteq GL(n)$ then we call G the **structure group** of the vector bundle. A **line bundle** is a vector bundle whose fiber is 1-dimensional.

Another typical group action is the (left) action of a group onto itself. Accordingly, the other important example of the fiber bundle is a **principal bundle** $P(M, G)$, which is a fiber bundle $E = P$ whose typical fiber is a Lie group G and whose transition functions correspond to a left action of the structure group on itself. Since left and right actions commute there is a canonical right action of G on the total space P of a principal bundle which acts free and

[4] The transition function always belong to the diffeomorphism group of the fibers, but this is an infinite dimensional group.

transitive on each fiber, so that the base space can be identified with P/G.

Given a vector bundle we can use the transition function g_{ij} to construct the corresponding principal bundle with the same structure group. A particular incarnation of this principle bundle is the frame bundle, whose elements correspond to the space of frames, i.e. over each point of the base the fiber consists of a certain set of vector space bases and an element of structure group can be identified with a change of the basis. In the case of the tangent bundle we can use any metric to construct orthonormal bases so that the structure group can always be reduced from $GL(n)$ to $O(n)$. If M is orientable we can, in addition, use oriented bases and thus further restrict the structure group to $SO(n)$.

For a principlal G bundle we can, in turn, use any manifold V with a left action $t : G\times V \to V$ of the group we can construct an **associated bundle** as the orbit space $P\times V/G$, where G acts on $P\times V$ by its right action on P combined with the left action of V, i.e. $(u, v) \mapsto (ug, t_g^{-1}(v))$. For $V = G$ with its left action on itself we get back the principal bundle. For V a vector space that carries a representation t, i.e. a linear left action, of the structure group G we obtain an associated vector bundle E, whose principal bundle is P.

Principal bundles can be defined more abstractly using the G action that is inherited from the right action of the structure group G on itself. Thus, a principal G bundle is a bundle P with a fiberwise free and transitive right-action of G. If we cover the base by contractible patches $\mathcal{U}_i$ then there exists a global section $s_i : \mathcal{U}_i \to P$ over each patch, which trivializes $\pi^{-1}(\mathcal{U}_i)$ as $\mathcal{U}_i \times G \ni (x, g) \xrightarrow{\varphi_i^{-1}} s_i(x)g \in P$, i.e. $\varphi_i(s_i(x)g) = (x, g)$. The transition functions $g_{ij} = \varphi_i \circ \varphi_j^{-1}$ are then determined by $s_i(x)g_{ij} = s_j(x)$ on the overlap of two fibers because, for example, the point $s_j(x) \in \pi^{-1}(x)$ can be written as $s_j(x)e = s_i(x)g_{ij}e$ so that the fibers are glued by left multiplication with g_{ij}. For associated bundles $P \times V$ we can use the specified sections to select a representative $(s_i(x), v(x))$ in the G-orbit $(ug^{-1}(x), t_g v(x))$. On the overlap of two patches the change of representative from s_j to s_i requires a right-multiplication with g_{ij}^{-1}, which shows that we need a group action $t(g_{ij})$ on the fiber of the associated bundle to transform the representative on the patch $U(j)$ to the appropriate representative on $U(i)$.

A (global) **section** of a bundle is a map $f : B \to E$ with $\pi \circ f = \mathbf{1}$. A local section satisfies the same condition, but it is only defined on some subset of the base manifold. A vector field can thus be defined to be a (smooth) section of a vector bundle. A vector bundle always admits global section (for example, the 0-secion). But it is easy to show that a principal bundle that admits a global section is trivial. A simple example is the Möbius strip, whose structure group is $\mathbb{Z}_2$. An example with a continuous structure group is the tangent bundle of the sphere S^2, whose frame bundle has no global section because there is no non-vanishing vector field. This bundle is related to the Hopf fibration $S^3 \to S^2$, whose fiber is $S^1 \sim U(1) \sim SO(2)$. More generally we can define a projection of any odd-dimensional sphere S^{2n+1} onto $\mathbb{CP}^n$ with fiber

S^1 by $(x_0, \ldots, x_n) \mapsto (x_0 : \ldots : x_n)$.

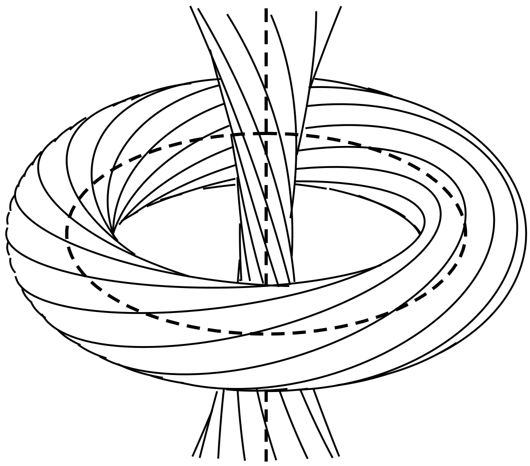


Fig. 1: S^1 fibers for $\varphi \in \frac{\pi}{6}\mathbb{Z}$ on the tori with $\theta = \frac{\pi}{6}$ and $\theta = \frac{5\pi}{6}$ of the Hopf fibration viewed with an angle $\alpha = 0.62$, i.e. with $(y, z) = (x_2,\ x_3 \cos\alpha - x_1 \sin\alpha)$.

The case $S^3 \to S^2$ can be visualized by stereographic projection of the sphere $S^3 = \{(z_0, z_1) : |z_0|^2 + |z_1|^2 = 1\}$ to $\mathbb{R}^3 \cup \{\infty\}$ with $x = (\mathrm{Re}\, z_1, \mathrm{Im}\, z_1, \mathrm{Im}\, z_0)/(1 - \mathrm{Re}\, z_0)$. The 'north pole' $(1+0i, 0+0i) \in S^3$ is thus mapped to the point ∞ that compactifies $\mathbb{R}^3$. If we relate the sphere with angles $(\theta = 2\theta', \varphi)$ to the affine patch $(z_0/z_1 : 1) \in \mathbb{P}^1$ by another stereographic projection $(z_0 : z_1) = e^{i\lambda}(\sin\theta', \cos\theta'/e^{i\varphi})$ we find

$$\vec{x} = \frac{(\mathrm{Re}\, z_1, \mathrm{Im}\, z_1, \mathrm{Im}\, z_0)}{1 - \mathrm{Re}\, z_0} = \frac{(\cos\theta' \cos(\lambda - \varphi), \cos\theta' \sin(\lambda - \varphi), \sin\theta' \sin\lambda)}{1 - \sin\theta' \cos\lambda} \tag{1.20}$$

with $(\theta = 2\theta', \varphi)$ parametrizing S^2 and λ parametrizing the S^1 fibers. For fixed $0 < \theta < \pi$ the angles φ and λ parametrize a torus. The decomposition of S^2 into an upper and a lower hemisphere around $\theta = 0$ and $\theta = \pi$ therefore decomposes S^3 into two solid tori $\theta \le \theta_0$ and $\theta \ge \theta_0$, respectively. With the above stereographic map to $\mathbb{R}^3 \cup \infty$ the S^1 fibers wind around the circle $\theta = 0$ in the 12-plane and around the x_3 coordinate line $\theta = \pi$. The fact that this principal $U(1) \sim S^1$ bundle has no global section and thus is nontrivial can be visualized by noting that S^2 cannot be embedded into S^3 by a choice of a smooth global section $\lambda(\theta, \varphi)$: If we drop the north pole $\theta = 0$, for example, then $S^2 - N$ can be mapped to a disc that is bounded by the circle $\theta = 0$, i.e. the fiber over N. But for an embedding of S^2 all points of that circle have to be identified. The Hopf bundle can be interpreted in many different ways. In particular, it is the principal bundle of the spin bundle over the sphere.

Exercise 2: Use orthonormal frames on the northern and on the southern hemisphere to compute the transition functions of the tangent bundle of S^2 and show that the Hopf bundle over S^2 correspond to the "square root" of the (co)tangent bundle.

Homomorphisms among fiber bundles $E \to M$ and $F \to N$ are smooth maps $f : E \to F$ that map fibers into fibers homomorphically. Hence $f \circ \pi_E = \pi_F \circ f$ defines an action of f on the base, which we denote by the same symbol, and $f : E_x \to F_{f(x)}$ is linear for vector bundles and a group homomorphism for principal bundles. Generalizing the construction of tensor fields we can now use the operations of linear algebra to construct many new vector bundles from elementary building blocks:

- The fiber of the *Whitney sum* $E \oplus F$ of two vector bundles over the same base is the direct sum of the fibers. The transition functions are block-diagonal matrices $g_{ij} = \begin{pmatrix} g^E_{ij} & 0 \\ 0 & g^E_{ij} \end{pmatrix}$, i.e. the Whitney sum is associated with direct sum representation.

- The transition functions of the *tensor product bundle* $E \otimes F$ are given by the tensor product of the representations. (This differs from the product bundle, whose base would be $M \times M$.)

- The fiber of the *dual bundle* $E^\vee$ is the dual vector space E^*_x, which carries the contragradient representation $g^\vee_{ij} = g^T_{ji}$, i.e. the transpose of the inverse matrix.

- If $F \subset E$ is a subbundle of E we can choose a basis of E_x whose first elements are a basis of F_x so that the transition functions are of the form $g_{ij} = \begin{pmatrix} g^F_{ij} & h_{ij} \\ 0 & f_{ij} \end{pmatrix}$. The transition functions f_{ij} then define the *quotient bundle* E/F.

- The *exterior power* bundle $\Lambda^k E$ arises by antisymmetrization of the k-fold tensor product of E with itself. For the maximal value $k = \mathrm{rank}(E) \equiv \dim(E_x)$ we get the *determinant bundle* $\det(E)$, which has has rank 1 and thus is a line bundle.

- Given a bundle $E \to N$ and a smooth map $f : M \to N$ we define the *pull-back bundle* f^*E over M by the trivializations $f^{-1}(U_i)$ and the pull-back of the transition functions.

- The tangent bundle of a submanifold $M \subset N$ of dimension m is a subbundle of the restriction of TN to M, i.e. of the pull back of the embedding $i_M : M \to N$. The quotient $\mathcal{N}(N/M) = i^*_M TN/TM$ is called *normal bundle* of M in N. Its rank is $n - m$.

- If a bundle map $f : E \to F$ from $E \to M$ to $F \to N$ has constant rank on the fibers then the fiber-wise kernel and image of f define subbundles $\mathrm{Ker}(f) \subset E$ and $\mathrm{Im}(f) \subset F$. A sequence of vector bundle homomorphisms
$$E' \xrightarrow{f} E \xrightarrow{g} E'' \tag{1.21}$$
is called exact if $\mathrm{Im}(f) = \mathrm{Ker}(g)$. A sequence $0 \to E' \xrightarrow{f} E \xrightarrow{g} E'' \to 0$ is exact iff f is injective, g is surjective and $\mathrm{Im}(f) = \mathrm{Ker}(g)$.

- Spin bundles are vector bundles whose structure group is the spin group $Spin(n)$, i.e. the double cover of $SO(n)$. The structure group can be reduced to $SO(n)$ iff the manifold is orientable. An orientable manifold is said to admit spin structures if the orthogonal transition functions can be lifted to $Spin(n)$. Since there is a $\mathbb{Z}_2$ ambiguity in the lift $SO(n) \to Spin(n)$ on each patch, there may be a global topological obstruction to the existence of such a lift, which, more or less by definition, is the second Stiefel-Whitney class. A problem can only occur with compatibility on tripple overlaps so that spin structures always exist in two dimensions.

- A complex vector bundle is a vector bundle whose fibers are complex vector spaces and whose transition functions are in $GL(k, \mathbb{C})$. A *holomorphic vector bundle* is a complex vector bundle whose total space is a complex manifold with bi-holomorphic trivializations $\varphi_i : \pi^{-1}(U) \to U \times \mathbb{C}^k$. The base space of a homolorphic vector bundle is a complex manifold. Since tensor products and duals of line bundles are again line bundles the equivalence classes of holomorphic line bundles form an abelian group, which is called the Picard group $\mathrm{Pic}(M)$ of a complex manifold.

1.3 Lie groups and Lie algebras

A Lie algebra $\mathfrak{h}$ is a (non-associative) algebra, i.e. a ring with a vector space structure and a bilinear multiplication, whose product is antisymmetric and satisfies the Jacobi dentity

$$[X, Y] = -[Y, X], \qquad \sum\!\!\!\!\!\!\bigcirc [X, [Y, Z]] := [X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0 \tag{1.22}$$

A super Lie algebra is defined by the $\mathbb{Z}_2$-graded version of these conditions.

A Lie group $\mathcal{G}$ comes with two natural actions on itself, namely the left multiplication L_g and the right multiplication R_g, whose differential maps L_{g*} and R_{g*} define two different global transports of tangent vectors,

$$L_g : h \to gh, \qquad R_g : h \to hg, \qquad L_{g*} : T_h\mathcal{G} \to T_{gh}\mathcal{G}, \qquad R_{g*} : T_h\mathcal{G} \to T_{hg}\mathcal{G} \tag{1.23}$$

with $g, h \in \mathcal{G}$. The tangent vectors $X(e) \in T_e\mathcal{G}$ at the unit element $e \in \mathcal{G}$ thus are in one-to-one correspondence with left-invariant vector fields $X \in T\mathcal{G}$, which by definition satisfy $X(g) = L_{g*}X(e)$. The Lie bracket commutes with the push-forward und thus closes on the linear space $\mathfrak{g}$ of left-invariant vector fields. This gives $\mathfrak{g}$, which is naturally identified with the tangent space at the unit element $\mathfrak{g} \cong T_e\mathcal{G}$, the structure of a Lie algebra. Choosing some basis $T_a \subset \mathcal{G}$ we conclude that

$$[T_a, T_b] = f_{ab}{}^c T_c, \qquad \sum\!\!\!\!\!\!\bigcirc_{abc} f_{ab}{}^d f_{dc}{}^e = 0. \tag{1.24}$$

Of course there is an analogous construction with right-invariant vector fields, which would merely invert the sign of the structure constants $f_{ab}{}^c$. Dual to the left-invariant vector fields T_a

there is a basis θ^a of left-invariant one-forms, $\theta^a(T_b) = \delta^a_b$. The formula $d\omega(X,Y) = X(\omega(Y)) - Y(\omega[X]) - \omega([X,Y])$ for the exterior derivative of a one-form thus implies $(d\theta^c)_{ab} = d\theta^c(T_a,T_b) = -f_{ab}{}^d\theta^c(T_d) = -f_{ab}{}^c$. With the coordinate independent Lie-algebra valued Maurer–Cartan form $\theta = \theta^c T_c$ this yields the Maurer–Cartan equation

$$d\theta + \tfrac{1}{2}[\theta,\theta] = 0, \qquad [\theta,\theta] = [T_a,T_b]\,\theta^a\wedge\theta^b \tag{1.25}$$

where θ can be interpreted as a global transport that maps a tangent vector at g to a tangent vector at the origin. Every group manifold is thus parallelizable, which implies that the curvature of the respective affine connection vanishes (for a parallel basis $DT_a = 0$ of tangent space $R(T_a,T_b)T_c = [D_{T_a},D_{T_b}]T_c - D_{[T_a,T_b]}T_c = 0$). The torsion of the left-invariant connection on a group manifold is also easily evaluated,

$$T_{ab}{}^c = \theta^c(D_{T_a}T_b - D_{T_b}T_a - [T_a,T_b]) = -f_{ab}{}^c. \tag{1.26}$$

The connection coefficients are obtained by writing the structure equation in terms of the cobasis, $d\theta^c + \frac{1}{2}f_{ab}{}^c\theta^a\wedge\theta^b = 0$. For matrix groups we can write $\theta = g^{-1}dg$, which obviously satisfies (1.25) because $d(g^{-1}dg) = -(g^{-1}dgg^{-1})dg$.

A Lie group can be reconstructed from its Lie algebra (at least) locally by the exponential map

$$\exp : \mathfrak{g} \to \mathcal{G}, \qquad X \to x(1) \quad \text{with} \quad x(0) = e, \quad \dot{x}(t) = L_{x(t)*}X \tag{1.27}$$

which is given by the integral curve through e of the left-invariant vector field $X \in \mathfrak{g}$ at curve parameter $t = 1$. For matrix groups the exponential map clearly coincides with the exponential function. The Baker-Campbell-Hausdorff formula

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]+\frac{1}{12}([A,[A,B]]-[B,[A,B]])+\text{ multiple commutators}} \tag{1.28}$$

shows that the group structure can be reconstructed from the commutators. It can be shown that the exponential map is surjective for compact connected groups.

Example: For the non-compact group $SL(2,\mathbb{R})$ the exponential map is not surjective. Its Lie algebra $sl(2,\mathbb{R})$ consists of traceless real matrices X, whose eigenvalues λ_i must be imaginary (complex conjugate) or real. X is diaganolizable except for $\lambda_1 = \lambda_2 = 0$. The possible eigenvalues of $\exp(X)$ are therefore real positive or complex conjugate on the unit circle. The exponential map therefore misses all $SL(2,\mathbb{R})$ matrices with negative eigenvalues $\lambda_i \neq -1$ and the non-diagonalizable matrices with Jordan normal form $\begin{pmatrix} -1 & a \\ 0 & -1 \end{pmatrix}$.

For connected "simple" groups like $SL(2,\mathbb{R})$ one can show that each group element can be written as a product of two exponentials. The image of the exponential map is thus not necessarily a subgroup, but it generates the group. Since BCH formally allows us to express any product $e^A e^B$ by a single exponential we conclude that in general the BCH formula cannot have infinite radius of convergence. For infinite-dimensional Lie algebras it may happen that there exists no corresponding Lie group.

Lie groups and Lie algebras can be represented on themselves by the adjoint actions $\mathrm{ad_g} = L_g R_g^{-1} : \mathcal{G} \to \mathcal{G}$ and $\mathrm{ad_X} : \mathfrak{g} \to \mathfrak{g}$ with $\mathrm{ad_X}\, Y = [X, Y]$. Moreover, the restriction of the tangent map of $\mathrm{ad_g}$ to the tangent space $T_e\mathcal{G}$ at the unit e defines an adjoint action $\mathrm{Ad_g} : \mathfrak{g} \to \mathfrak{g}$ of $\mathcal{G}$ on its Lie algebra $\mathfrak{g}$,

$$\mathrm{ad}_g\, h = ghg^{-1}, \qquad \mathrm{ad}_X\, Y = [X, Y], \qquad \mathrm{Ad}_g\, X = L_{g*} X R_{g*}^{-1} \tag{1.29}$$

For $g = \exp(X)$ we find $\mathrm{Ad}_g\, Y = \exp(\mathrm{ad}_X)Y$ because left- and right-multiplication commute.

1.4 Connections on fiber bundles

A connection on a vector bundle $E \to M$ is a linear map $\nabla : C^\infty(E) \to C^\infty(E \otimes T^*M)$ that satisfies

$$\nabla(fv) = f\nabla(v) + v \otimes df \tag{1.30}$$

for smooth functions $f \in C^\infty(M)$ and smooth sections v of E. When evaluated on a tangent vector field $X \in TM$ a connection thus defines a pointwise linear map on the fibers, the covariant derivative

$$D_X := i_X \nabla, \qquad \Rightarrow \qquad D_{fX} v = f D_X v, \qquad D_X(fv) = X(f)v + f D_X v. \tag{1.31}$$

It can be checked that $[D_{fX}, D_{gY}](hv) - D_{[fX,gY]}(hv) = fgh([D_X, D_Y](v) - D_{[X,Y]}(v))$ so that

$$R(X, Y)v = D_X D_Y v - D_Y D_X v - D_{[X,Y]} v \tag{1.32}$$

defines an $\mathrm{End}(E)$-valued 2-form, i.e. a 2-form of linear maps on the fibers, which is called curvature R (or field strength $F \equiv R$) of the connection. Given local coordinates x^i on the base M and a local basis $e_a(x)$ of the fibers we can define the covariant derivative of the component functions $v = v^a e_a$ in terms of the connection coefficients

$$\nabla e_a = A_a{}^b e_b, \qquad A_a{}^b = dx^i A_{ia}{}^b, \qquad D_i v^a = \partial_i v^a + A_{ia}{}^b v^a. \tag{1.33}$$

and the components of the curvature can be written as $R(\partial_i, \partial_j) = R_{ij}{}^I \delta_I$ where δ_I is a basis for the linear transformations of the fibers. We can also expand $A_a{}^b = A^I (\delta_I)_a{}^b$ in such a basis. In physics the one-forms A^I are called gauge connections.

It is often useful to define connections on principal bundles, from which connections on the associated bundles can be deduced. To motivate the abstract definition of such a connection we observe that the kernel of the push forward $\pi_* : TE \to TM$ defines a natural vertical subbundle

$V(E) \subset TE$ of the tangent space TE of (the total space of) a vector bundle. The fibers of $V(E)$ can be identified with the fibers of the vector bundle E by linearity. Locally E can be parametrized by (x^i, v^a) and for each tangent vector, i.e. at each point in E, there are $\dim M$ tangent vectors $D_i = \partial_i + A_{ia}{}^b v^a e_b$ that project onto a basis ∂_i of TM. These vectors span a subbundle $H(E) \subset TE$, called the horizontal subbundle, and $TE = H(E) \oplus V(E)$. It is easy to see that the connection data can be uniquely recovered from a choice of such a horizontal subbundle and that this correspondence is independent of a choice of the basis, which was only used to write down the geometrical objects more explicitly. Note that the horizontal subbundle $H(E) \subset TE$ has to satisfy some linearity condition along the fibers of E in order that the map $(X, v) \to D_X v$ with $D_X v$ defined as the unique vector in $H(E)_{|v} \cap \pi_*^{-1}(X)$ for $X \in T_{\pi(v)}M$ is a connection on E.

On a principal bundle $P(M, G)$ the vertical subbundle $V(P) \subset TP$ is defined as the image of the Lie algebra $\mathfrak{g}$ under the right action of the structure group on the bundle space, $V(P) = R_{g*}\mathfrak{g} \subset TP$. At each point $p \subset P$ we can thus identify vertical vectors $Y_p^\# \in V_p$ with vectors $Y \in \mathfrak{g}$, and thus generate $V(P)$ by right-invariant vector fields $Y^\#$ that are naturally identified right-invariant vector fields Y on the structure group G.

A connection on a principal bundle $P(M, G)$ is now defined as a choice of a G-invariant horizontal subbundle $H(P) \subset TP$ such that $TP = H(P) \oplus V(P)$. This data can equivalently be encoded in the Ehresman connection ω, which is a right-invariant Lie algebra valued 1-form on P that corresponds to the identity on $V(P)$ via the identification with $\mathfrak{g}$ [NA90, eg80]

$$R_g^* w_{ug}(X) = w_{ug}(R_{g*}X) = g^{-1} w_u(X) g, \qquad \omega(A^\#) = A \quad \forall A \in \mathfrak{g}. \tag{1.34}$$

Choosing a local section $\sigma_i : U_i \to P$ we can define the gauge connection $A_i = \sigma_i^* \omega$ on U_i as the pull-back of ω to the coordinate chart along σ. The Ehresmann connection can be recovered as the pull-back to the bundle space along projection π plus the Maurer-Cartan form on the fiber, which is isomorphic to the structure group:

$$A_i = \sigma_i^* \omega, \qquad \omega = \gamma_i^{-1}\, \pi^*(A_i)\, g_i + \gamma_i^{-1}\, d\, g_i, \tag{1.35}$$

where g_i parametrizes the fiber in the local trivialization $U_i \times G$ defined by the section s_i, i.e. $\sigma_i g_i = p \in P$, and d is the exterior derivative on P. The gauge transformation

$$A_j = g_{ij}^{-1}(A_i + d) g_{ij} \tag{1.36}$$

for a change $\sigma_j = \sigma_i g_{ij}$ of the local trivialization is the compatibility condition $\omega_{|U_i} = \omega_{|U_j}$ for a consistent definition of ω.

A curve $\tilde{\gamma} : [0, 1] \to P$ is a *horizontal lift* of a curve γ in the base space M if it projects onto that curve $\pi(\tilde{\gamma}) = \gamma$ and if all tangent vectors to the lift $\tilde{\gamma}$ belong to the horizontal subspace

$H(P)$. Since tangent vectors to $\gamma \in M$ are lifted to vectors in $H(P)$ the lifts are integral curves to the lifted tangent vectors and unique for each choice of an initial point on the fiber $\pi^{-1}\gamma(0)$.

Horizontal lifts of closed curves define a map $\gamma \to G$ from loops in M to the structure group because $\tilde\gamma(0)$ and $\tilde\gamma(1)$ belong to the same fiber and hence differ by a right-action of a unique group element $g \in G$, i.e. $\tilde\gamma(1) = R_g\tilde\gamma(0)$. For each connection ∇ on a fiber bundle over M the image of the space of all loops in the base space with base point x forms a subgroup $\mathrm{Hol}_x(\nabla) \subseteq G$ of the structure group, called the *holonomy group* of the connection. The holonomy group thus describes the possible action of parallel transport along closed loops on the fibers of a bundle. If M is arcwise connected then $\mathrm{Hol}(\nabla)$ is independent of the choice of the base point (up to conjugation by some group element). If a loop in the base M is contractible then its image is continuously connected to the identity in $\mathrm{Hol}(\nabla)$. It can be shown that the image $\mathrm{Hol}^0_x(\nabla) \subset \mathrm{Hol}_x(\nabla)$ of all contractible loops, which forms a subgroup, is equal to the connected component of the identity in $\mathrm{Hol}(\nabla)$.

The Riemannian holonomy group $\mathrm{Hol}(g)$ of a Riemannian manifold (M, g) is the holonomy group of the Levi-Civita connection on the tangent bundle. Since this connection is torsion free the curvature tensor has extra symmetries. This can be used to classify all possible holonomy groups $\mathrm{Hol}(g)$, which turn out to characterize different types of geometries:

Theorem (Berger): Let (M, g) be a simply connected irreducible (i.e. no product) and not locally symmetric (i.e. DR is not identically 0) Riemannian manifold. Then the possible holonomy groups are contained in the following table [J000]:

(i)	$\mathrm{Hol}(g) = SO(n)$	$\dim(M)$	generic
(ii)	$\mathrm{Hol}(g) = U(m) \subset SO(2m)$	$n = 2m$	complex Kähler
(iii)	$\mathrm{Hol}(g) = SU(m) \subset SO(2m)$	$n = 2m$	Calabi–Yau (Ricci flat Kähler)
(iv)	$\mathrm{Hol}(g) = Sp(m) \subset SO(4m)$	$n = 4m$	hyperkähler (RF=Ricci flat)
(v)	$\mathrm{Hol}(g) = Sp(m) \cdot Sp(1) \subset SO(4m)$	$n = 4m$	quaternionic–Kähler (Einstein)
(vi)	$\mathrm{Hol}(g) = G_2 \subset SO(7)$	$n = 7$	RF, related to imaginary octonions
(vi)	$\mathrm{Hol}(g) = \mathrm{Spin}(7) \subset SO(8)$	$n = 8$	RF, related to octonions

Note that quaternionic–Kähler manifolds are not Kähler. They are Einstein spaces $\mathcal{R}_{mn} \sim g_{mn}$, but not Ricci flat. (Locally) symmetric spaces are defined, for example, on p. 50-55 in [J000].

Chapter 2

Lie algebras and representations

An ideal $\mathfrak{h}$ of an algebra $\mathfrak{g}$ is a subalgebra $[\mathfrak{h},\mathfrak{h}] \subseteq \mathfrak{h} \subseteq \mathfrak{g}$ for which $[\mathfrak{g},\mathfrak{h}] \subseteq \mathfrak{h}$, where operations on sets are defined as the set of results of the respective operations on elements of the sets. Ideals of algebras are important because the quotient space $\mathfrak{g}/\mathfrak{h}$, which consists of the cosets $x+\mathfrak{h}$ for $x \in \mathfrak{g}$, again form an algebra (the product is independent of the representative x of the class $[x] = x + \mathfrak{h}$). Ideals thus can be used to decompose algebras into building blocks.

The *derived algebra* $\mathfrak{g}' = [\mathfrak{g},\mathfrak{g}]$, the *derived series* $\mathfrak{g}^{\{i\}} = [\mathfrak{g}^{\{i-1\}},\mathfrak{g}^{\{i-1\}}]$ and the *lower central series* $\mathfrak{g}_{\{i\}} = [\mathfrak{g},\mathfrak{g}_{\{i-1\}}]$ with $\mathfrak{g}^{\{1\}} = \mathfrak{g}_{\{1\}} = \mathfrak{g}'$ are all ideals of $\mathfrak{g}$. A Lie algebra (LA) is called *solvable* (*nilpotent*) if the derived (lower central) series terminates with $\{0\}$. Nilpotency implies solvability (one can think of elements of a solvable Lie algebra as upper triangular matrices and of the nilpotent ones as strictly upper triangular, i.e. with zeros on the diagonal). The *radical* $\mathfrak{g}_{rad}$ of $\mathfrak{g}$ is the maximal solvable ideal which exists (and is unique) because the sum of two solvable ideals is again a solvable ideal.

The center of a Lie algebra is the ideal consisting of all elements that commute with all others, $\mathcal{Z}(\mathfrak{g}) = \{x \in \mathfrak{g} \mid [x,y] = 0 \ \forall y \in \mathfrak{g}\}$. The centralizer in $\mathfrak{g}$ of a subalgebra $\mathfrak{h}$ is the subalgebra $C_{\mathfrak{g}}(\mathfrak{h}) = \{x \in \mathfrak{g} | [x,\mathfrak{h}] = 0\}$ of elements that commute with all elements of $\mathfrak{h}$. The normalizer $\mathcal{N}_{\mathfrak{g}}(\mathfrak{h}) = \{x \in \mathfrak{g} | [x,\mathfrak{h}] \subseteq \mathfrak{h}\}$ of $\mathfrak{h}$ in $\mathfrak{g}$ is the largest subalgebra of $\mathfrak{g}$ that contains $\mathfrak{h}$ as an ideal.

A Lie algebra is called *simple* if it is nonabelian and has no proper ideal. It is called *semisimple* if it contains no solvable ideal. Semisimple Lie algebras can be shown to be direct sums of simple Lie algebras. A Lie algebra is *reductive* iff its radical is equal to its center $\mathcal{Z}(\mathfrak{g})$. A *reductive* Lie algebra therefore is a direct sum of a semisimple and an abelian part. For any Lie algebra the quotient $s = \mathfrak{g}/\mathfrak{g}_{rad}$ is semisimple, which implies the *Levi decomposition* of $\mathfrak{g}$ into the semidirect sum $\mathfrak{g} \cong s \ltimes \mathfrak{g}_{rad}$ of a solvable ideal and a semisimple quotient.

The Cartan-Killing form $\kappa(x,y) = \mathrm{tr}(\mathrm{ad}_x \circ \mathrm{ad}_y)$ is a symmetric bilinear form on $\mathfrak{g}$ that is

ad_x-invariant due to the Jacobi identity, i.e. $\mathrm{ad}_x(\kappa([y,z]) := \kappa([x,y],z) + \kappa(y,[x,z]) = 0$ or

$$\kappa([x,y],z) = \kappa(x,[y,z]). \tag{2.1}$$

It is common to rescale the components of κ with respect to a basis T^a of $\mathfrak{g}$ by a factor I_{ad}, called the Dynkin index of the adjoint representation, which will be defined later:

$$\kappa_{ab} := \frac{1}{I_{\mathrm{ad}}}\,\mathrm{tr}(\mathrm{ad}_{T_a}\circ \mathrm{ad}_{T_b}) = \frac{1}{I_{\mathrm{ad}}} f_{ae}{}^c f_{bc}{}^e. \tag{2.2}$$

The *Cartan criterion* states that a Lie algebra is solvable iff $\kappa(x,x) = 0\ \forall x \in \mathfrak{g}'$ and that it is semisimple iff κ is non-degenerate. The Killing form can then be used to define $f_{abc} = f_{ab}{}^d\kappa_{dc}$, which is proportional to $\kappa([T_a,T_b],T_c)$ and hence antisymmetric in all indices. While no canonical form of the structure constants is known for the solvable case, the (semi)simple Lie algebras have been enumerated completely by E. Cartan.

The first step in the classification of simple Lie algebras is the choice of a Cartan subalgebra (CSA) $\mathfrak{g}_0$, which is a maximal abelian subspace consisting of ad-diagonalizable elements. (We first work over the complexified Lie algebra. Possible real forms can be classified straightforwardly once we know the complete list of complex algebras. Likewise, the independence of the various choices that will be made is easily varified a posteriori, and we will then be able to parametrize these choices in a useful way.) A basis of $\mathfrak{g}_0$ is denoted by H^i with $i = 1,\ldots,r$. The dimension $r = \dim(\mathfrak{g}_0)$ of the CSA is called the *rank* of the Lie algebra. Since the H^i commute their adjoint actions can be diagonalized simultaneously and we can choose a basis $\{T_a\}$ of the form

$$[H^i,E^\alpha] = \alpha^i E^\alpha, \qquad \{T_a\} = \{H^i |\, i\le r\}\cup\{E^\alpha |\ \alpha\in\Phi\}, \tag{2.3}$$

which is called a Cartan-Weyl basis or a root decomposition of $\mathfrak{g}$. The vectors α^i of eigenvalues are called root vectors. The set of all roots is denoted by Φ. We will see that the generators $E^{\pm\alpha}$, which are called step operators or ladder operators, are uniquely determined by the roots α up to a choice of their normalization.

Ad-invariance of the Killing form for H^i implies that $\kappa(E^\alpha,E^\beta)$ vanishes if $\alpha+\beta\neq 0$. Non-degeneracy therefore implies that Φ is symmetric with respect to the origin, i.e. for each root E^α there is a root $E^{-\alpha}$. For the same reason κ is blockdiagonal and restricts to a nondegenerate metric $\kappa^{ij} = \kappa(H^i,H^j)$ on the CSA. Since each E^α defines a linear map $\mathfrak{g}_0 \ni h \to \mathrm{ad}_h E^\alpha = h_i \mathrm{ad}_{H^i}E^\alpha = \alpha^i E^\alpha \to h_i\alpha^i \in \mathbb{C}$ we can identify the root space with the dual space $\mathfrak{g}_0^*$ of the CSA, which is thus naturally equipped with the dual metric $(\alpha,\beta) = \alpha^i\kappa_{ij}\beta^j$.

Since the H^i eigenvalues add up for the entries of a commutator we know that $[E^\alpha,E^{-\alpha}] = \tilde\alpha_i H^i$ is an element of $\mathfrak{g}_0$. The coefficient vector $\tilde\alpha_i$ can be computed using the Killing form

$$\kappa([E^\alpha,E^{-\alpha}],H^j) = \kappa(\tilde\alpha_i H^i,H^j) = \alpha^j\kappa(E^\alpha,E^{-\alpha}) = \tilde\alpha_i\kappa^{ij}, \tag{2.4}$$

$$H^\alpha = [E^\alpha,E^{-\alpha}] = \kappa(E^\alpha,E^{-\alpha})\alpha^j\kappa_{ji}H^i, \qquad [H^\alpha,E^{\pm\alpha}] = \pm\kappa(E^\alpha,E^{-\alpha})\alpha^j\kappa_{ji}\alpha^i E^{\pm\alpha}. \tag{2.5}$$

We thus observe that every root vector E^α yields an $sl(2)$ subalgebra spanned by $E^{\pm\alpha}$ and H^α, which decomposes $\mathfrak{g}$ into representations of that subalgebra. We will reduce everything to the representation theory of $sl(2)$, whose generators are often denoted by $J_\pm$ and J_3. Using the notation of quantum mechanics (but with $\hbar = 1$) all finite dimensional irreducible represetations of dimension $2j+1$ are labeled by $j \in \mathbb{Z}/2$. In a basis $|j,m\rangle$ with $-j \le m \le j$ and $j-m \in \mathbb{Z}$

$$\begin{array}{l} [J_3\,, J_\pm] = \pm J_\pm \\ [J_+, J_-] = \; 2J_3 \end{array} \qquad \Rightarrow \qquad \begin{array}{l} J_3|j,m\rangle = m\,|j,m\rangle \\ J_\pm|j,m\rangle = \sqrt{j(j+1)-m(m\pm 1)}\,|j,m\pm 1\rangle \end{array}. \tag{2.6}$$

We want to identify $E^{\pm\alpha}$ with $J_\pm$ and H^α with $2J_3$, so that the eigenvalues of H^α will be integer. This can be done if we normalize the generators of the root spaces such that $\kappa(E^\alpha, E^{-\alpha}) = 2/(\alpha,\alpha)$. It is convenient to define the *coroots vectors* $(\alpha^\vee)^i = 2\alpha^i/(\alpha,\alpha)$ so that our result takes the form $H^\alpha = (\alpha^\vee, H) = \alpha^{\vee i}\kappa_{ij}H^j = \alpha^\vee_i H^i$.

For any root α we can now use the fact that the adjoint action of $E^{\pm\alpha}$ decomposes the Lie algebra into representations of $sl(2)$, which are called root strings $\{E^{\beta+n\alpha}\}$ through E^β. The eigenvalues of H^α on these strings are given by the scalar products $(\alpha^\vee, \beta+n\alpha)$ for $n_- \le n \le n_+$, which are integers. The spin of the representation is $j = (n_+ - n_-)/2$. In particular, we conclude that $(\alpha^\vee, \beta) = 2(\alpha,\beta)/(\alpha,\alpha) \in \mathbb{Z}$. Moreover, if we choose $\kappa(E^\beta, E^\beta) > 0$ positivity holds for the complete root string and, due to simplicity of the algebra, for all root vectors.

The next step is to split the root space into a positive part Φ_+ and a negative part Φ_-, which is done by a choice of a linear hyperplane that does not contain any roots. The direct sum of the positive root spaces $\mathfrak{g}_+ = \mathrm{lin\,span}(\{E^\alpha | \alpha \in \phi_+\})$ is a nilpotent subalgebra because a non-zero commutator of positive roots is again a positive root.[1] The *simple roots* are now defined as the minimal set of *algebra* generators $E^i_+ \equiv E^{\alpha(i)}$ of $\mathfrak{g}_+$ or, equivalently, as positive roots $\alpha(i)$ that cannot be represented as a sum of other positive roots (with respect to the fixed choice of Φ_+) with nonnegative integer coefficients. The properties of root strings imply that the simple roots are linearly independent so that their number is equal to the rank r, as our use the labels i suggests. A *Chevalley basis* of a simple Lie algebra consists of $E^i_\pm$ and $H^i = [E^i_+, E^i_-]$ where $E^i_- = E^{-\alpha(i)}$ and $E^i_\pm$ are normalized as above, i.e. $[H^i, E^i_\pm] = \pm 2E^i_\pm$.

Restricting our attention to the root string of a simple root $\alpha^{(i)}$ through a simple root $\alpha^{(j)}$ it is clear that $n_- = 0$ and that there are two alternatives: Either $n_+ = 0$ so that $[E^i_+, E^j_+] = 0$ and $(\alpha^{(i)}, \alpha^{(j)}) = 0$, i.e. the roots commute and the root vectors are orthogonal, or the corresponding off-diagonal entry of the *Cartan matrix*

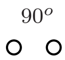
$$A^{ij} := (\alpha^{(i)\vee}, \alpha^{(j)}) = 2\frac{(\alpha^{(i)},\, \alpha^{(j)})}{(\alpha^{(i)},\, \alpha^{(i)})} \tag{2.7}$$

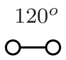
is a negative integer. The diagonal entries are $A^{ii} = 2$ and the symmetric matrix $A^{ij}(\alpha^{(i)},\, \alpha^{(i)})$

[1] Its direct sum with the CSA $\mathfrak{g}_0 \oplus \mathfrak{g}_+$ is a Borel subalgebra, i.e. a maximal solvable subalgebra of $\mathfrak{g}$. The decomposition $\mathfrak{g} = \mathfrak{g}_+ + \mathfrak{g}_0 + \mathfrak{g}_-$ is called triangular (or Gauss) decomposition of $\mathfrak{g}$.

of scalar products of simple roots has to be positive definite. We will now show that these conditions are sufficient to determine all possible Cartan matrices of simple Lie algebras. The Cartan classification is then completed by the reconstruction of the complete algebra from A^{ij}.

Clearly any submatrix of a Cartan matrix obtained by removing the i^{th} line and column again fulfills the defining property that $A^{ii} = 2$, $A^{ij} \le 0$ for $i \ne j$ with $A^{ij} < 0$ iff $A^{ji} < 0$, and positive definiteness of the corresponding scalar product (which implies $\det(A) > 0$). We therefore commence with the case of rank $r = 2$ Cartan (sub)matrices. For $A^{ij} = \begin{pmatrix} 2 & -a \\ -b & 2 \end{pmatrix}$ with $\det(A) = 4 - ab > 0$ we find four inequivalent integer solutions, where we can choose $|\alpha^{(1)}|^2/|\alpha^{(2)}|^2 = a/b \ge 1$ for $a, b \ne 0$. It is convenient to encode the data of A^{ij} in a (Coxeter) graph, where we draw a node for each simple root and connect two simple roots by a lines (laces) if $a \ge b > 0$. In addition, we need to encode the relative lengths of the roots, which is conventionally done by adding an arrow pointing to the shorter roots and/or by drawing the long roots with circles and the shorter ones with full discs. The resulting graphs are called *Dynkin diagrams*. For rank two we thus find

$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \equiv$ 

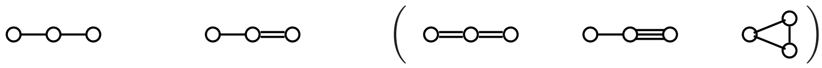
$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \equiv$ 

$\begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix} \equiv$ 

$\begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix} \equiv$ 

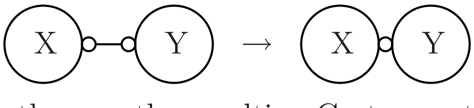
with angles of 90^o, 120^o, 135^o and 150^o between the simple roots. The corresponding Lie algebras will turn out to be the classical Lie algebras[2] $D_2 \equiv A_1 \oplus A_1 \cong so(4) \cong su(2) \oplus su(2)$, $A_2 \cong su(3)$, $B_2 \equiv C_2 \cong so(5) \cong sp(4)$ and the exceptional G_2, respectively.

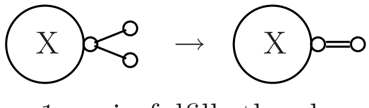
The Dynkin diagram of a simple Lie algebra is connected because disconnected parts correspond to commuting subsets of generators and thus to a direct sum of the Lie algebras. For rank $r = 3$ we find two connected Coxeter graphs, which amount to the three Dynkin diagrams called $A_3 = D_3$, B_3 and C_3 in the table below. The latter differ by the direction of the arrow. Other diagrams like the ones with two double lines, one tripple line, or a closed loop are forbidden because the angles among the roots would add up to 360^o so that the root vectors become linearly dependent (and the determinant of the Cartan matrix vanishes):



From the result for $r = 3$ we can proceed with the following observation:

Lemma: A simple lace (i.e. a single line) in a Dynkin diagram of a simple Lie algebra with rank r can be contracted to a point and two nodes connected by simple laces to the same node can be replaced by a single node connected with a double line according to the graphs





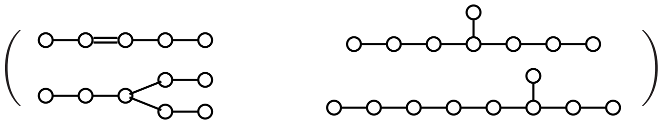
(2.8)

In both cases the resulting Cartan matrix of rank $r - 1$ again fulfills the above critieria [GE82].

[2]At small rank there are coincidences between the member of otherwise inequivalent series of classical Lie algebras, which explain the well-known isomorphisms (see the table below).

Corollary: There can be no closed loops. There can be at most one double line. And there can be at most one branching point where one node is connected to three other nodes. Since the only diagram with a tripple line is the rank 2 Lie algebra G_2 this also implies that simple roots can have at most two different lengths.

This reduces the classification to the considerations of the exceptional cases F_4, E_6, E_7 and E_8 (cf. the table of simple Lie algebras), whose Dynkin diagrams cannot be extended because the following diagrams can be checked to correspond to a Cartan matrix with determinant zero:

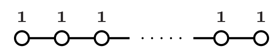
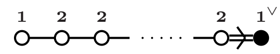
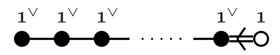
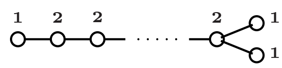
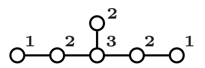
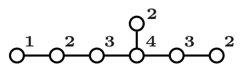
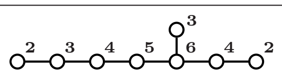
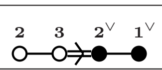
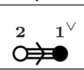


(2.9)

Instead of computing the determinant if is faster to directly compute the null Eigenvectors.

Exercise 3: Show that the moves (2.8) generate valid Cartan matrices from valid Cartan matrices and compute the zero-Eigenvectors of the Cartan matrices corresponding to (2.9).

Putting everything together we are thus left with four infinite series and five exceptionals cases. The Dynkin diagrams are listed in the following table, which contains additional information on the resulting algebras that will be discussed below.

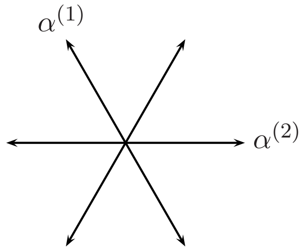
$\mathfrak{g}$	dual Coxeter labels	dim	$g^\vee$	center	Weyl group	exponents + 1
A_r		r^2+2r	$r+1$	$\mathbb{Z}_{r+1}$	$\mathcal{S}_{r+1}$	$2,3,4,\ldots,r+1$
B_r		$2r^2+r$	$2r-1$	$\mathbb{Z}_2$	$\mathcal{S}_r \ltimes (\mathbb{Z}_2)^r$	$2,4,6,\ldots,2r$
C_r		$2r^2+r$	$r+1$	$\mathbb{Z}_2$	$\mathcal{S}_r \ltimes (\mathbb{Z}_2)^r$	$2,4,6,\ldots,2r$
D_r		$2r^2-r$	$2r-2$	$\mathbb{Z}_2{\times}\mathbb{Z}_2$ r_{even} $\mathbb{Z}_4$ r_{odd}	$\mathcal{S}_r \ltimes (\mathbb{Z}_2)^{r-1}$	$2,4,6,\ldots,2r-2,r$
E_6		78	12	$\mathbb{Z}_3$	$\|W\| = 3\cdot 4!\cdot 6!$	2,5,6,8,9,12
E_7		133	18	$\mathbb{Z}_2$	$\|W\| = 4!\cdot 4!\cdot 7!$	2,6,8,10,12,14,18
E_8		248	30	–	$\|W\| = 4!\cdot 6!\cdot 8!$	2,8,12,14,18,20,24,30
F_4		52	9	–	$\mathcal{S}_3 \ltimes \mathcal{S}_4 \ltimes (\mathbb{Z}_2)^3$	2,6,8,12
G_2		14	4	–	$\mathcal{D}_6$	2,6

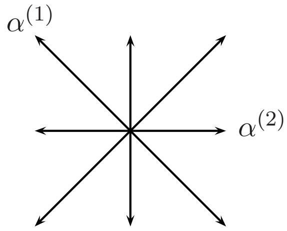
Positive definiteness of the Cartan matrices for the exceptional cases are easily verified. The existence of the infinite series follows from their representation in terms of the classical Lie algebras $A_r \sim \mathfrak{sl}(r+1)$, $B_r \sim \mathfrak{so}(2r+1)$, $C_r \sim \mathfrak{sp}(2r)$ and $D_r \sim \mathfrak{so}(2r)$.

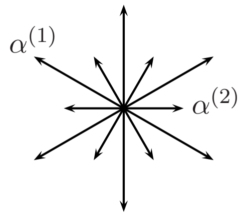
The classification of simple Lie algebras is now completed by the reconstruction of $\mathfrak{g}$. The Cartan matrix encodes the structure constants for the commutators of the CSA elements and the simple roots, while all other roots can be obtained as multiple commutators. The Serre relations $(ad_{E^i_\pm})^{1-A^{ij}}E^j_\pm = 0$ describing the termination of the root strings follow from the properties of $\mathfrak{sl}(2)$ representations. Everything is summarized by the Chevalley–Serre relations

$$[H^i,H^j]=0, \qquad [H^i,E^j_\pm]=\pm A^{ij}E^j_\pm, \qquad [E^i_+,E^j_-]=\delta_{ij}H^j, \qquad (ad_{E^i_\pm})^{1-A^{ij}}E^j_\pm=0 \quad (2.10)$$

(in some books, like [FU97], the transposed convention for the Cartan matrix is used). As an example we consider the rank 2 cases, for which the root systems look as follows,







$A_2 \sim \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ $\qquad B_2 \sim \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}$ $\qquad G_2 \sim \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$

For the last case G_2 the root $2\alpha^{(1)}+3\alpha^{(2)}$ does not belong to a root string through a simple root, so that the Serre relations are not sufficient to obtain the shape of the diagram. While a straightforward analysis of further root strings could solve this issue, there is a much simpler approch: Due to the properties of the corresponding $\mathfrak{sl}(2)$ representations the root system is invariant under any reflection S_α on the hyperplane that is orthogonal to a root α. These automorphisms

$$S_\alpha\beta = \beta - (\alpha^\vee,\beta)\alpha. \quad (2.11)$$

of the root system are called Weyl reflections. They generate the Weyl group, which is a Coxeter group, i.e. a finite group generated by reflections (the classification of these groups is closely related to the Cartan classification; but recall the difference between Coxeter graphs and Dynkin diagrams). It can be shown that the Weyl group is generated by Weyl reflections for simple roots $s_i = S_{\alpha^{(i)}}$ (called *fundamental* or *simple* reflextions). These generators obey the relations

$$s_i^2=1, \qquad s_is_j=s_js_i \ \text{ if } A_{ij}=0, \qquad (s_is_j)^{m_{ij}}=1 \quad m_{ij}=\frac{\pi}{\pi-\theta_{ij}}\in\{2,3,4,6\}, \quad (2.12)$$

where θ_{ij} denote the angles between the simple roots. It can be shown that the Weyl group generates the complete root system from the simple roots and that different choices of a set of simple roots (up to a permutation within the set) are related by a Weyl group element. In particular, the exchange of positive and negative roots corresponds to $w=s_1\ldots s_r$, and the rank r is the maximal number of generators that is required to represent an element of the Weyl group.[3] We can introduce a grading among the roots where we assign height 1 to the

[3] The minimal number of factors in a representation of $w=\prod s_i$ is called the lengths $l(w)$ of the Weyl group element.

simple root and height $\sum c_i$ to a root $\alpha = \sum c_i \alpha^{(i)}$. By now we know that all roots can have at most two different lengths. It can be shown that there is a unique *highest root* θ, which always is a long root. It is standard to normalize the metric such that the hightest root has $(\theta, \theta) = 2$. The highest root can be written as a linear combination $\theta = \sum g_i \alpha^{(i)} = \sum g_i^\vee \alpha^{(i)\vee}$ of the simple (co)roots. The unique coefficients are called (dual) Coxeter labels g_i ($g_i^\vee$). The (dual) Coxeter number is one plus the sum of all (dual) Coxeter labels. The dual Coxeter labels and the dual Coxeter number of all simple Lie algebras are given in the table.

We now turn to the discussion of finite-dimensional representations, which clearly have to decompose into the well-known $\mathfrak{sl}(2)$ representations for any $\mathfrak{sl}(2)$ subalgebra $H^i, E^i_\pm$ generated by a simple root. Let us introduce the root lattice L and the coroot lattice $L^\vee$,

$$L(\mathfrak{g}) = \mathrm{span}_\mathbb{Z}\{\alpha(i)\}, \qquad L(\mathfrak{g})^\vee = \mathrm{span}_\mathbb{Z}\{\alpha(i)^\vee\}, \tag{2.13}$$

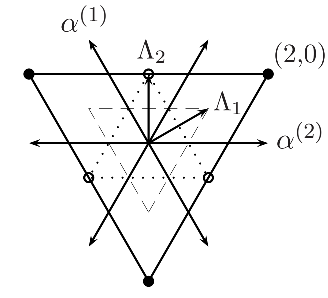
as the set of integer linear combination of simple root and coroots, respectively. For the simply laced Lie algebras ADE, i.e. if the Coxeter graph has simple laces and hence all roots have equal length, we have $L = L^\vee$. Since $H^\alpha = \alpha_i^\vee H^i$ the pairings $(\alpha^\vee, \lambda) = \lambda^i$ of Eigenvectors $|\lambda\rangle$ of CSA elements $h|\lambda\rangle = \lambda|\lambda\rangle$ must be integers. The weight vectors λ therefore belong to the weight lattice

$$\Lambda_w(\mathfrak{g}) = (L(\mathfrak{g})^\vee)^* = \mathrm{span}_\mathbb{Z}(\Lambda_i), \qquad (\alpha_i^\vee, \Lambda_j) = \delta_{ij}, \tag{2.14}$$

which is generated by the *fundamental weights* Λ_i. The *Dynkin basis* $\{\Lambda_i\}$ which is dual to the basis $\alpha_i^\vee$ of the coroot lattice. The coefficients λ^i of a weight $\lambda = \lambda^i \Lambda_i$ are called *Dynkin labels.*

All finite dimensional irreducible representations of simple Lie algebras are so-called highest weight representations, i.e. they are uniquely characterized by a highest weight $|\lambda\rangle$ with $E^i_+|\lambda\rangle = 0$ and the complete representation is generated by the lowering operators E^i_-. The set of eigenvalues of such a representation form a convex lattice polytope in the weight lattice. λ is called the dominant weight. The shape of the representation could be worked out from the properties of its $\mathfrak{sl}(2)$ building blocks, but, again, the Weyl group helps by generating all the vertices of the polytope from the fundamental weights. The Weyl reflextions decompose the weight space into *Weyl chambers*, and the dominant weight clealy has to be chosen from the fundamental chamber, which is the cone generated by the fundamental weights.

Example: For $A_2 \equiv SU(3)$ the Dynkin labels $(1, 0)$, $(0, 1)$ and $(1, 1)$ correspond to the fundamental 3, the conjugate $\overline{3}$ and the adjoint reprensentation 8, respectively. The decomposition $3 \otimes \overline{3} = 8 \oplus 1$ is quite obvious. Since weights at the boundary of an irreducible representation have multiplicity 1 and $\Lambda_2 = \rho + \sigma = \sigma + \rho$ with $\rho = \Lambda_1$ and $\sigma = \Lambda_1 - \alpha^{(2)}$ we find $3 \otimes 3 = 6 \oplus \overline{3}$, where 6 denotes the representation with highest weight $(2, 0)$. $3 \simeq \{\rho, \sigma, -\Lambda_2\}$ and $\overline{3}$ are indicated by dashed and dotted lines. 6 is the symmetric tensor representation and $\overline{3}$ is the antisymmetric tensor. For $\mathfrak{su}_n \simeq A_{n-1}$ all HWR are (anti)symmetrizations of tensors (Young tableaux).



Since the root system of $\mathfrak{g}$ corresponds to the adjoint representation the root lattice $L(\mathfrak{g})$ is a sublattice of the weight lattice $\Lambda_w(\mathfrak{g})$ and thus decomposes the weight lattice into a finite set of equivalence classes $\lambda + L(\mathfrak{g})$, called *conjugacy* (or congruence) *classes*. The conjugacy classes form a finite abelian group $\Lambda_w(\mathfrak{g})/L(\mathfrak{g})$ that can be shown to be isomorphic to the center $\mathcal{Z}(G)$ of the universal covering group $\tilde{G}$ (the Lie group $\tilde{G}$ is unique if we choose the compact real form). The relation to the center can be obtained by comparing the exponentiations in the adjoint representation and in a representation that contains weights generating all conjugacy classes, respectively. Since the shift operators $E^j_\pm$ change the weights by roots all weights in a hightest weight representation (HWR) belong to the same conjugacy class. Moreover, all highest weights of representations that are contained in the tensor product are in the conjugacy class of the sum of the highest weights of the factors. Together with the dimensions of the representations this information is often sufficient to uniquely determine the decomposition.

For each HWR $\mathcal{R}_\Lambda$ there is the conjugate representation $\mathcal{R}^+_\Lambda$ which contains $-\Lambda$ as its lowest weight and whose tensor product with $\mathcal{R}_\Lambda$ always contains the trivial representation $\mathbf{1}$ as a direct summand (in physics this implies the existence of an invariant bilinear term for fields in conjugate representations). The corresponding highest weight can be obtained by Weyl reflection at the hightest root S_θ, i.e. $\Lambda^+ = -S_\theta\Lambda$. A representation is self-conjugate if $\Lambda^+ = \Lambda$ (the adjoint representation is always self-conjugate because its hightest weight is $\Lambda_{ad} = \theta$).

Another important quantity is the Weyl vector $\rho = \frac{1}{2}\sum_{\alpha>0}\alpha$, which is half the sum of the positive roots and can be shown to be equal to

$$\rho = \frac{1}{2}\sum_{\alpha>0}\alpha = \sum_{i=1}^{r}\Lambda_i \tag{2.15}$$

It is related to the dimension of the Lie algebra by the so-called *strange formula*

$$(\rho,\rho) = \frac{1}{24} g^\vee(\theta,\theta)\dim\mathfrak{g} \tag{2.16}$$

and it enters the Freudenthal recursion formula

$$\mathrm{mult}_\Lambda(\lambda) = \frac{2}{(\Lambda+\rho,\Lambda+\rho)-(\lambda+\rho,\lambda+\rho)}\sum_{\alpha>0}\sum_{\substack{m>0\\ \lambda+m\alpha\in R_{(\Lambda)}}}(\lambda+m\alpha,\alpha)\,\mathrm{mult}_\Lambda(\lambda+m\alpha) \tag{2.17}$$

for the multiplicity of a weight λ in a representation $R_{(\Lambda)}$. These multiplicities are encoded in the character χ_Λ of $R_{(\Lambda)}$, which is a functional on $\mathfrak{g}_0$ defined by

$$\chi_\Lambda(h) = \mathrm{tr}\exp(R_{(\Lambda)}(h)) = \sum_{\lambda\in R_{(\Lambda)}}\mathrm{mult}_\Lambda(\lambda)e^{(\lambda,h)} \tag{2.18}$$

with $R_{(\Lambda)}|\lambda\rangle = (\lambda,h)|\lambda\rangle$. They can be computed efficiently in terms of sums over Weyl group elements $\sigma\in W$ by the Weyl-Kac character formula

$$\chi_\Lambda(h) = \frac{\sum_{\sigma\in W}\mathrm{sign}(\sigma)\exp[(\sigma(\Lambda+\rho),h)]}{\sum_{\sigma\in W}\mathrm{sign}(\sigma)\exp[(\sigma(\rho),h)]} \tag{2.19}$$

where $\mathrm{sign}(\sigma) = \pm 1$ for elements σ of even/odd length. For conjugate representations $\chi_{\Lambda^+}(\mu) = \chi_\Lambda(-\mu)$ and for direct sums and products

$$\chi_{\Lambda_1 \oplus \Lambda_2} = \chi_{\Lambda_1} + \chi_{\Lambda_2}, \qquad \chi_{\Lambda_1 \otimes \Lambda_2} = \chi_{\Lambda_1} \cdot \chi_{\Lambda_2}. \tag{2.20}$$

This can be used to construct algorithms for the reduction of tensor products (see, for example, section 1.7 in [FU92]).

The universal enveloping algebra $U(\mathfrak{g})$ is the associative algebra that is generated by $\mathfrak{g}$ with the Lie algebra relations. Its center is related to numerically invariant tensors with adjoint indices. It can be shown that the center $Z(U(\mathfrak{g}))$ is generated by r Casimir operators $\mathcal{C}_l = c^{a_1 \ldots a_l} T_{a_1} \ldots T_{a_l}$, corresponding to the invariant tensors $c^{a_1 \cdots a_l}$. The numbers of indices of these tensors by definition differ by 1 from the exponents of the Lie algebra, as listed in the last column of the table. All invariant tensors can be obtained as symmetrized traces in some representations of the Lie algebra. The Harish-Chandra theorem [HU72] implies that HW representations are uniquely characterized by the eigenvalues of the Casimir invariants. The Dynkin index I_Λ of a representation is related to the second Casimir $C_\Lambda = \kappa^{ab} T_a T_b$ in a representation R_Λ of dimension d_Λ by $I_\Lambda = \frac{d_\Lambda}{d} C_\Lambda$.

For each simple Lie algebra there are two standard real forms: The normal real form is the real span of the Cartan–Weyl basis (which is called the normal real form), while the compact form is generated by $\{iH^i, \frac{\sqrt{(\alpha,\alpha)}}{2}(E^\alpha - E^{-\alpha}), i\frac{\sqrt{(\alpha,\alpha)}}{2}(E^\alpha - E^{-\alpha})\}$ so that again all structure constants are real and $\kappa^{ab} = -\delta^{ab}$. The signatures σ of other possible real forms $\mathfrak{g}_{d|\sigma}$ are restricted by the condition that the number $\frac{1}{2}(d-\sigma)$ of negative eigenvalues must be the dimension of a simple Lie algebra. (E_6, for examples, has real forms with five different signatures: $E_{6|-78}$ $E_{6|-26}$ $E_{6|-14}$, $E_{6|2}$, $E_{6|6}$. The lists of all possible real forms can be found in section 8.4 of [FU97].)

Chapter 3

Complex manifolds

3.1 Vector bundles on complex manifolds

Since any complex manifolds M also has the structure of a real manifold we can consider different types of tangent spaces. Using local complex coordinates $z^j = x^j + iy^j$ we define the *complex tangent bundle*, TM whose fibers are complex linear combinations of ∂_{x^i} and ∂_{y^i}. It is the direct sum

$$TM = T'M \oplus T''M \tag{3.1}$$

of the *holomorphic tangent bundle* $T'M$, whose fibers are spanned by $\partial_i \equiv \partial_{z^i}$, and the anti-holomorphic tangent bundle $T''M$ that is generated by $\overline{\partial}_i \equiv \partial_{\bar{\imath}} \equiv \partial_{\bar{z}^i}$,

$$\tfrac{\partial}{\partial z^i} = \tfrac{1}{2}(\tfrac{\partial}{\partial x^i} - i\tfrac{\partial}{\partial y^i}), \qquad \tfrac{\partial}{\partial \bar{z}^i} = \tfrac{1}{2}(\tfrac{\partial}{\partial x^i} + i\tfrac{\partial}{\partial y^i}). \tag{3.2}$$

The analogous decomposition of the complex cotangent bundle $T^*M = T^{*\prime}M \oplus T^{*\prime\prime}M$ leads to a decomposition of the exterior algebra into (p, q) forms

$$\Lambda(M) = \oplus\Lambda^{(p,q)}, \qquad \Lambda^{(p,q)} = T^{*(p,q)} = \Lambda^p T^{*\prime} \wedge \Lambda^q T^{*\prime\prime} \tag{3.3}$$

The *canonical bundle* $K_M = \det T'^*M = \Lambda^{(n,0)}$ of a complex manifold M is the determinant bundle of the holomorphic cotangent bundle. Its dual is the anti-canonical bundle K_M^*.

The exterior derivative d decomposes into a sum of ∂ and $\overline{\partial}$,

$$d = \partial + \overline{\partial}, \qquad \partial = dz^i \partial_{z^i}, \qquad \overline{\partial} = d\bar{z}^i \partial_{\bar{z}^i}, \qquad \partial^2 = \overline{\partial}^2 = \{\partial, \overline{\partial}\} = 0, \tag{3.4}$$

which increase the holomorphic and the anti-holomorphic degree, respectively. Let $Z^{p,q}_{\overline{\partial}}(M)$ and $B^{p,q}_{\overline{\partial}}(M)$ be the space of $\overline{\partial}$-closed and $\overline{\partial}$-exact (p, q) forms, respectively.
The **Dolbeault cohomology** groups are defined as the quotient groups

$$H^{p,q}_{\overline{\partial}}(M) = Z^{p,q}_{\overline{\partial}} / B^{p,q}_{\overline{\partial}}. \tag{3.5}$$

For polydiscs Δ, i.e. products of open discs $\Delta_i \subset \mathbb{C}$, they are trivial except for $H^{p,0}$, which consists of holomorphic p-forms with holomorphic coefficient functions. This follows from the

$\overline{\partial}$ **Poincaré lemma:** $H^{p,q}(\Delta) = 0$ for $q \geq 1$ and polydiscs Δ [GR78].

As a warm-up for the proof of this lemma we derive the Cauchy integral formula for smooth functions $f \in C^\infty(\overline{\Delta})$ on the closure of some open disc $\Delta = \Delta^0 = \overline{\Delta} - \partial\Delta \subset \mathbb{C}$. To simplify notation we drop the explicit $\overline{z}$-dependence of smooth function $f(z,\overline{z}) \cong f(\mathrm{Re}\, z, \mathrm{Im}\, z) \cong f(z)$, i.e. $f(z)$ is now not assumed to be holomorphic. For $z \in \Delta^0$ and with the parametrization $w - z = re^{i\theta}$ Stokes' theorem for the 2-form $d\left(\frac{f(w)dw}{w-z}\right) = \frac{\partial f}{\partial \overline{w}} \frac{d\overline{w} \wedge dw}{w-z}$ implies

$$\int\limits_{\Delta'=\Delta-\Delta_\varepsilon(z)} \frac{dw\, d\overline{w}}{2\pi i} \frac{\partial_{\overline{w}} f(w)}{w-z} + \oint\limits_{\partial\Delta} \frac{dw}{2\pi i} \frac{f(w)}{w-z} = \oint\limits_{\partial\Delta_\varepsilon(z)} \frac{dw}{2\pi i} \frac{f(w)}{w-z} = \int_0^{2\pi} \frac{de^{i\theta}}{2\pi i} \frac{f(\varepsilon e^{i\theta})}{e^{i\theta}} = f(z) + \mathcal{O}(\varepsilon). \qquad (3.6)$$

$\Delta_\varepsilon(z) \subset \Delta$ is a disk of radius ε around z so that $r = \varepsilon$ on its boundary $\partial\Delta_\varepsilon(z)$. Since $\frac{d\overline{w}\, dw}{z-w} = \frac{2ir\, dr\, d\theta}{re^{i\theta}}$ the surface integral is regular at $w = z$. Hence we can take the limit $\varepsilon \to 0$ and obtain the **Cauchy integral formula**

$$f(z) = \oint_{\partial\Delta} \frac{dw}{2\pi i} \frac{f(w)}{w-z} + \int_\Delta \frac{dw \wedge d\overline{w}}{2\pi i} \frac{\overline{\partial} f(w)}{w-z} \qquad (3.7)$$

In the holomorphic case $\overline{\partial} f = 0$ this yields $f(z)$ in terms of its boundary values. Next we observe that $1/z$ is a Green's function for $\partial_{\overline{z}}$, i.e. $\partial_{\overline{z}}(1/z) = 2\pi\delta^2 z$, which follows from (3.7) if we interpret f as test function and take Δ large enough to contain the support of f. This implies the $\overline{\partial}$ Poincaré lemma in one variable:

$$\partial_{\overline{z}}\left(\tfrac{1}{z}\right) = 2\pi\delta^2(z) \equiv \pi\delta(\mathrm{Re}\, z)\delta(\mathrm{Im}\, z), \qquad g(z) = \int \frac{dw\, d\overline{w}}{2\pi i} \frac{f(w)}{w-z} \quad \Rightarrow \quad \partial_{\overline{z}} g(z) = f(z), \qquad (3.8)$$

i.e. every smooth function f on Δ can be written as a $\overline{\partial}$-derivative.

Now we are ready to prove the $\overline{\partial}$ Poincaré lemma: To simplify notation we use multi-indices like $I = i_1 \ldots i_p$ with $|I| = p$ and the symbol $\varepsilon_I(j)$ that is 0 if $j \notin I$ and ± 1 according to $dz^I = \varepsilon_I(j) dz^j \wedge dz^{I'_j}$ with $|I'_j| = |I| - 1$ otherwise. $\varepsilon_I(j)$ therefore removes the terms for which dz^j is not a factor of dz^I and takes care of the sign that is necessary to pull out dz^j on the left hand side otherwise. For $q = |J| > 0$ the cocycle condition $\overline{\partial}\omega$ with $\omega = \omega_{IJ} dz^I \wedge d\overline{z}^J$ now implies $\overline{\partial}\Omega = \omega$ for $\Omega = \frac{1}{q}\sum_{j \leq n} t_j \omega$ with

$$t_j \omega(z_1, \ldots, z_j, \ldots, z_n) = \sum_{\substack{|I|=p \\ |J|=q}} (-)^I \varepsilon_J(j) \int \frac{dw\, d\overline{w}}{2\pi i} \frac{\omega_{IJ}(z_1, \ldots, w, \ldots, z_n)}{w - z_j} dz^I \wedge d\overline{z}^{J'_j} \qquad (3.9)$$

because $d\overline{z}^l \partial_{\overline{z}^l}$ anticommutes with t_j for $l \neq j$ and vanishes on $\omega_{IJ} dz^I \wedge d\overline{z}^J$ if $\varepsilon_J(j) \neq 0$ so that only $\overline{\partial}\frac{1}{w-z_j}$ contributes to $\overline{\partial}\Omega$, which proves the lemma. □

It can be shown that the $\overline{\partial}$ Poincaré also holds for $\Delta = \mathbb{C}^n$ [GR78]. Note that the Dolbeault cohomology groups $H^{p,0}$ are not even finite-dimensional in the non-compact case. In the compact case, however, the vector bundles $\Lambda^(p,0)$ have only a finite number of linearly independent holomorphic sections so that the Hodge numbers $h^{p,q} = \dim H^{p,q}_{\bar\partial}(M)$ are finite.

An important set of holomorphic bundles can be constructed on $\mathbb{P}^n$. Every homogeneos polynomial $p(z)$ of degree d defines functions $p_i(z) = p(z)/z_i^d$ on the affine patches $\mathcal{U}_i = \{(z_0 : \ldots : z_n) \mid z_i = 1\}$. These functions provide global sections of a line bundle $\mathcal{O}(d)$ that is defined by the transition functions[1] $g_{ij} = (z_j/z_i)^d$. While global holomorphic sections only exist for nonnegative degrees d the line bundles $\mathcal{O}(d)$ are defined for arbitrary integers d. It can be shown that every line bundle on $\mathbb{P}^n$ is isomorphic to $\mathcal{O}(d)$ [GR78], i.e. the Picard group of $\mathbb{P}^n$ is generated by $\mathcal{O}(1)$ and hence isomorphic to $\mathbb{Z}$. The line bundle $\mathcal{O}(1)$ is called universal bundle.

Another special case is the tautological line bundle, which is constructed as follows: The projection $(z_0, \ldots, z_n) \to (z_0 : \ldots : z_n)$ from $\mathbb{C}^{n+1} - \{0\}$ to $\mathbb{P}^n$ defines a $\mathbb{C}^*$-bundle over $\mathbb{P}^n$ with $\mathbb{C}^* \equiv \mathbb{C} - \{0\}$. The tautological bundle is obtained by extending the fibers to $\mathbb{C}^n$. In other words, the tautological bundle is the subbundle of the trivial $\mathbb{C}^{n+1}$ bundle over $\mathbb{P}^n$ whose fiber over $(z_0 : \ldots : z_n)$ is the line $(\lambda z_0, \ldots, \lambda z_n)$ with $\lambda \in \mathbb{C}$.

Exercise 4: Show that the tautological bundle is $\mathcal{O}(-1)$ and that the canonical bundle is equal to $\mathcal{O}(-n-1)$. The tangent bundle of $\mathbb{P}^1$ is thus equal to the anti-canonical bundle $\mathcal{O}(2)$. Show that the Hopf fibration of S^3 is dual to the principal bundle of $\mathcal{O}(1)$, which is the spin bundle because its square is the tangent bundle.
Hint: Use the local parametrization $\frac{\lambda^{(i)}}{z_i}(z_0, \ldots, z_n)$ for the fibers of the tautological bundle. A convenient choice of coordinates on patches $\mathcal{U}_i$ with $z_i \neq 0$ is $x_j = \left\{\begin{array}{rl} -z_{j-1}/z_i & \text{for } j \le i \\ z_j/z_i & \text{for } j > i \end{array}\right.$. Expressing the coordinates x_i on $\mathcal{U}_i$ in terms of, say, the coordinates $y_j = z_j/z_0$ on $\mathcal{U}_0$ it is easy to compute the determinant of the Jacobian matrix $\partial x/\partial y$ and thus obtain the transition functions g_{0i} for the determinant bundle. The other transition functions follow from the cocycle condition. The transition functions for the Hopf fibration along the equator $|z_0| = |z_1|$ are easily found by representing the $U(1)$ fiber as $z_i/|z_i|$ on the patch $\mathcal{U}_i$.

Let M be a compact complex manifold and $L \in \mathrm{Pic}(M)$ a holomorphic line bundle. Then it can be shown that the space $\Gamma(L)$ of global holomorphic sections is a vector space V of finite dimension, say, $m+1$ (in the language of sheaf cohomology the space of global sections $\Gamma(\mathcal{O}(L))$ of the 'sheaf' $\mathcal{O}(L)$ is denoted by $H^0(M, \mathcal{O}(L))$ [GR78], which coincides with the space of holomorphic functions $H^{00}_{\bar\partial}$ in the case of the trivial line bundle). It may happen that there are points $p \in M$ for which all sections $v \in V$ vanish. Then p is called a *base point* of L, and the set of all such points is the base locus. If a line bundle has no base point then, by choosing some basis v_i of V, we can define a map $M \ni p \to (v_0(p) : \ldots : v_m(p)) \in \mathbb{P}^m$. We need to

[1]Strictly speaking, $\mathcal{O}(d)$ denotes the sheaf of local holomorphic section of the line bundle that is defined by these transition functions.

consider the projectivisation $P(V) \cong \mathbb{P}^m$ because v_i are sections of a line bundle and only ratios $v_i(p) : v_j(p)$ are meromorphic *functions* on M. A base-point free line bundle L is called *very ample* if this map defines an embedding of M into $\mathbb{P}^m$. L is called *ample* if some positive power L^k is very ample.[2] *Chow's theorem* states that every compact complex submanifold of $\mathbb{P}^n$ is algebraic, i.e. the vanishing set of a finite number of homogeneous polynomials.[3]

Our final step in this section into the realm of algebraic geometry is the relation between *divisors* and *line bundles*. We first define a *hypersurface* in a complex manifold M as a subset N such that for each $p \in N$ there is a neighbourhood U and a nonzero holomorphic function f on U with $N \cap U = \{u \in U | f(u) = 0\}$. A hypersurface is thus a possibly singuar 'submanifold' of codimension 1. It is called *irreducible* if it is not the union of two distinct hypersurfaces. Irreducible hypersurfaces are also calle *prime divisors*, and a *divisor* is a locally finite formal sum

$$D = \sum a_i N_i, \qquad a_i \in \mathbb{Z} \tag{3.10}$$

of prime divisors N_i. D is an *effective divisor* if all $a_i \geq 0$. Any prime divisor N is thus locally defined by equations $f_\alpha = 0$ with $g_{\alpha\beta} = f_\alpha / f_\beta$ nonvanishing on the overlap $\mathcal{U}_\alpha \cap \mathcal{U}_\beta$. The $g_{\alpha\beta}$ obviously satisfy the cocycle condition and thus can be used to define a line bundle $L = [N]$.

In turn, the collection $f = \{f_\alpha\}$ of local defining functions of the hypersurface N evidently defines a global holomorphic section of the corresponding line bundle $[N]$. We can thus associate a prime divisor, denoted by (f), to any holomorphic section of a line bundle that has only single zeros. More generally, if we take into account multiplicities of zeros and poles, we can associate a divisor $D = (f)$ to every meromorphic section f of a given line bundle L: Locally the meromorphic functions f_α can be written as quotients $f_\alpha = g_\alpha / h_\alpha$ of holomorphic functions, where g_α has zeros of constant order ord_g on its vanishing set V_g and h_α has zeros of constant order ord_h on its vanishing set V_h. The associated divisor (f) is then locally defined by

$$(f)_\alpha = (f)_0 - (f)_\infty = \mathrm{ord}_g \, V_g - \mathrm{ord}_h \, V_h, \qquad f = g/h \text{ on } \mathcal{U}_\alpha. \tag{3.11}$$

$(f)_0$ and $(f)_\infty$ are called divisor of zeros and divisor of poles, respectively. The ratio of two nonzero meromorphic sections of a line bundle is a meromorphic section of the trivial bundle, i.e. a meromorphic function. The corresponding divisors are called *principal divisors*. The map $D \to [D]$ from the linear group of divisors to the Picard group is a surjective group homomorphism because every line bundle can be reconstructed from one of its meromorphic sections. Since the ratio of two sections is a meromorphic function on M the kernel of this map is given by the subgroup of principal divisors Div_0 and we obtain an isomorphism

$$\mathrm{Div} \,/\, \mathrm{Div}_0 \ni D \bmod D_0 \;\leftrightarrow [D] \in \mathrm{Pic}(M), \qquad D \sim D + D_0 \tag{3.12}$$

[2]The *Kodeira embedding theorem* gives a simple criterion for this: L is ample iff it is positive, i.e. first Chern class $c_1(L)$ is a positive element of H^2_{DR} [GR78].

[3]Such theorems are called GAGA theorems, named after an article by J.-P. Serre: 'Géometrie Algébrique et Géometrie Analytique' (1956).

between the Picard group and group of divisor classes that consists of the equivalence class of divisors modulo principal divisors. Divisors in the same class are called *linearly equivalent*.

The above correspondence gives a one-to-one relation between meromorphic sections of line bundles and divisors. In particular, the effecitive divisors correspond to holomorphic sections. As an example consider hyperplanes H in $\mathbb{P}^n$, which are solutions of linear equations and thus correspond to sections of $\mathcal{O}(1)$. As divisors all hyperplanes are linearly equivalent. The line bundle $\mathcal{O}(1) = [H]$ it therefore called *hyperplane bundle*. The line bundle $[V_D]$ of a hypersurfaces $V_d \subset \mathbb{P}^n$ that is defined by a homogeneous polynomial equation of degree d is $\mathcal{O}(d) = [d\,H]$. The *Bertini theorem* implies that a generic section of $\mathcal{O}(d)$ defines a smooth hypersurface in $\mathbb{P}^n$, and, more generally, r generic polynomials of degrees d_i intersect transversally and so that the common zeros define a smooth *complete intersection* submanifold of codimension r [GR78].

An important application of the above is a formula that relates the canonical bundle of a hypersurface to the one of the ambient space. For complex manifolds we define the (holomorphic) normal bundle of a hypersurface V as the quotient bundle

$$\mathcal{N}_V = T'M_{|V}/T'V \qquad \Leftrightarrow \qquad 0 \to T'V \to T'M_{|V} \to \mathcal{N}_V \to 0, \tag{3.13}$$

where we also gave the equivalent definition of the quotient by a linear subspace $A \subset B$ in terms of the 'short exact sequence' $0 \to A \to B \to B/A \to 0$ (the first two maps are the obvious embeddings and the projection onto B/A makes the sequence exact). For a smooth hypersurface defined by $f_\alpha = 0$ the differentials df_α vanish on all tangential vectors $v \in T'V$ and thus locally are elements of the conormal bundle $\mathcal{N}_V^*$. On the overlap $\mathcal{U}_\alpha \cap \mathcal{U}_\beta$ of two patches

$$df_\alpha = d(g_{\alpha\beta} f_\beta) = f_\beta\, dg_{\alpha\beta} + g_{\alpha\beta}\, df_\beta, \tag{3.14}$$

where $g_{\alpha\beta}$ are the transition functions of the line bundle $[V]$. Since $f_\beta = 0$ on V we observe that df_α transforms like f_α when restricted to V. The collection $\{df_\alpha\}$ thus provides a section of $\mathcal{N}_V^* \otimes [V]$ that is nonvanishing because the equations $f_a = 0$ have single zeros. This implies that $\mathcal{N}_V^* \otimes [V]$ is a trivial line bundle and $\mathcal{N}_V^* = [-V]$. In terms of local coordinates $(f, \vec{x})_\alpha$, with $\vec{x}_\alpha$ parametrizing $V \cap \mathcal{U}_\alpha$, we arrive at the same result by computing the relevant block of the Jacobi matrix

$$\begin{pmatrix} \partial_{f_\beta} \\ \partial_{x_\beta} \end{pmatrix} = \begin{pmatrix} \frac{\partial(g_{\alpha\beta} f_\beta)}{\partial f_\beta} & \frac{\partial(g_{\alpha\beta} f_\beta)}{\partial x_\alpha} \\ 0 & \frac{\partial x_\beta}{\partial x_\alpha} \end{pmatrix} \begin{pmatrix} \partial_{f_a} \\ \partial_{x_\alpha} \end{pmatrix} \tag{3.15}$$

for the transition functions of $T'M$ in the holonomic basis. The vector components transform contragradient to the basis. Hence the transition functions for $g_{\alpha\beta}^{(\mathcal{N})} : \mathcal{U}_\beta \to \mathcal{U}_\alpha$ of $\mathcal{N}_V$ are $\partial(g_{\alpha\beta} f_\beta)/\partial f_\beta$ and agree with those of $[V]$ on the hypersurface $f_\beta = 0$. Using the canonical bundle formula $K_V = K_{M|_V} \otimes \det \mathcal{N}(M/V)$ we thus derive the
Adjunction formula:

$$K_V = (K_M \otimes [V])_{|V} \tag{3.16}$$

for a smooth hypersurface $V \subset M$. For a hypersurface in $\mathbb{P}^n$ this implies that the canonical bundle is trivial iff the defining equation has degree $n+1$. A trivial canonical bundle is one of the possible definitions of a Calabi–Yau manifold. The first non-trivial example is a cubic in $\mathbb{P}^2$, which defines a torus. A genic cubic in 3 variables has $\binom{5}{3} = 10$ coefficients, but these can be redefined by 3^2 independent linear transformations without changing the hypersurface. This leaves one relevant parameter for the deformation of the complex structure of the torus. The simplest family of K3 surfaces are quartics in $\mathbb{P}^3$ with $\binom{7}{4} - 4^2 = 19$ relevant parameters that deform the complex structure. The most prominent Calabi–Yau 3-fold is the quintic in $\mathbb{P}^4$. In the 3-dimensional case it can be shown that the Hodge numbers h_{11} and h_{12} are related to (Kähler) metric and complex structure deformations, respectively, and that the Euler number is $\chi = 2(h_{11} - h_{12})$. In applications to the simplest case of supersymmetric string compactifications the Hodge numbers are related to the numbers of massless generations and anti-generations, whose net number thus becomes[4] $|\chi|/2$. The quintic in $\mathbb{P}^4$ has $h_{12} = \binom{9}{5} - 5^2 = 101$ complex structure deformation and one Kähler parameter, which comes from the size of the ambient $\mathbb{P}^4$. These data are conveniently summaries as $\mathbb{P}^4[5]^{1,101}_{-200}$ with the degree of the equation in brackets, the Hodge data as superscripts and the Euler number as subscript. The generation number 100 is way too large to be realistic, but it can be cut down by taking free quotients, which reduce the Euler number by a factor of the group order. For the Fermat quintic $z_0^5 + z_1^5 + \ldots + z_4^5 = 0$ there exist two (projectively) commuting free $\mathbb{Z}_5$ actions, namely the phase symmetry $z_i \to \rho^i z_i$ with ρ a 5^{th} root of unity and the cyclic permutation $z_0 \to z_1 \to \ldots \to z_4 \to z_0$ (for both group actions the only fixed point that satisfies the equation is $z_i = 0$, which is not in $\mathbb{P}^n$). We thus obtain a 'four generation model' with $\chi = -8$.

For complete intersections in $\mathbb{P}^n$ the Calabi–Yau condition is obtained by iterating the adjuction formula: The degrees of the equations have to add up to the degree $n+1$ of the anticanonical bundle. We thus find four more examples [HU92]

$$\mathbb{P}^4[5]^{1,101}_{-200}, \qquad \mathbb{P}^5\,[4|2]^{1,89}_{-176}\,, \qquad \mathbb{P}^5\,[3|3]^{1,73}_{-144}\,, \qquad \mathbb{P}^6\,[3|2|2\,]^{1,73}_{-144}\,, \qquad \mathbb{P}^7\,[2|2|2|2]^{1,65}_{-128}\,. \tag{3.17}$$

The series terminats with 4 quadrics in $\mathbb{P}^7$ because a linear equation would only reduce the dimension of the projective ambient space $\mathbb{P}^n[1] \cong \mathbb{P}^{n-1}$. Soon after the seminal paper by Candelas, Horowitz, Strominger and Witten on Calabi–Yau compactification [ca85] Tian and Yau came up with the first three generation model, which they constructed as a free $\mathbb{Z}_3$ quotient of a codimension 3 complete intersection in a product space $\mathbb{P}^3 \times \mathbb{P}^3$ and the somewhat more useful example of a free $\mathbb{Z}_3 \times \mathbb{Z}_3$ quotient of a codimension two complete intersection in $\mathbb{P}^3 \times \mathbb{P}^2$ was then analyzed by Schimmrigk. The respective data are

$$\begin{array}{c} \mathbb{P}^3 \\ \mathbb{P}^3 \end{array}\!\left[\begin{array}{c|c|c} 3 & 0 & 1 \\ 0 & 3 & 1 \end{array}\right]^{(14,23)}_{-18}, \qquad \begin{array}{c} \mathbb{P}^3 \\ \mathbb{P}^2 \end{array}\!\left[\begin{array}{c|c} 3 & 1 \\ 0 & 3 \end{array}\right]^{(8,35)}_{-54}, \tag{3.18}$$

[4]It is assumed that only chiral fermion stay approximately massless because their masses are protected [GR87].

where the colums denote the degrees of the equations and Calabi–Yau condition is that the lines have to add up to the degrees of the anticanonical bundles. In Schimmrigk's example a possible choice of parameters that is compatible with a free $\mathbb{Z}_3 \times \mathbb{Z}_3$ quotient (the group action is analogous to the above $\mathbb{Z}_5 \times \mathbb{Z}_5$ action on the quintic) leads to the equations

$$W^{(3,0)} = x_0^3 + x_1^3 + x_2^3 + x_3^3 = 0, \qquad W^{(1,3)} = x_0 y_0^3 + x_2 y_1^3 + x_2 y_2^3 = 0. \tag{3.19}$$

The Calabi–Yau complete intersections (CICYs) in products of projective spaces were completely enumerated by Candelas and his collaborators.

Exercise 5: Check that the Hodge numbers $h_{12} = 23$ and 35 in the examples (3.18) agree with the numbers of relevant parameters in the homogeneous equations of appropriate degrees.

Hint: For the Tian–Yau manifold the number of monomial is 56. In addition to the 32 coefficients of linear coordinate transformations we have to subtract 1 for an overall scaling of the bilinear equation.

Compact complex manifolds with trivial canonical bundle have a holomorphic n form Ω that is unique up to normalization. For the above examples Ω can be constructed with the **Poincaré residue map:** Consider a (local) meromorphic n form $\omega = g(z)dz^I/f(z)$ with a single pole along the irreducible hypersurface V defined by $f(z) = 0$. We define a map to a local $n-1$ form ω' by

$$\omega = \frac{g(z)\, dz^1 \wedge \ldots \wedge dz^n}{f(z)} = \frac{df}{f} \wedge \omega' \tag{3.20}$$

with $df = dz^i \partial_{z^i} f$ (mind the correspondence to the adjuction formula). The relation to the residue of an integral encirling the hypersurface becomes clear if we think about f as a coordinate for the transversal direction. Due to the transversality condition $df \neq 0$ for a smooth hypersurfaces an explicit formula for ω' can be given by

$$\omega' = (-)^j \, \frac{g(z) dz^{I'_j}}{\partial f / \partial z^j}, \tag{3.21}$$

where we have chosen a coordinate z_j with $\partial_j f \neq 0$. The holomorphic n-form on a Calabi–Yau hypersurface (and, by iteration, also on a complete intersection) in $\mathbb{P}^n$ can now be constructed as follows: We start with an n-form

$$\omega = \sum_j (-)^j \, \frac{z^j dz^{I'_j}}{f(z)} = \frac{z^0 dz^1 \wedge \ldots \wedge dz^n - \ldots + (-)^n dz^0 \wedge \ldots \wedge dz^{n-1} z^n}{f(z)} \tag{3.22}$$

which is a nonvanishing meromorphic section of the canonical bundle $K(\mathbb{P}^n)$ if f is a section of $\mathcal{O}(n+1)$. For $f = z^0 \ldots z^n$ the n form has single poles at all coordinate hyperplanes, so that we recover the result $K(\mathbb{P}^n) = [(n+1)H] = \mathcal{O}(-n-1)$. For a generic polynomial the hypersurface $f(z) = 0$ is a smooth Calabi–Yau and we can use the Poincaré residue map to construct the holomorphic $n-1$ form ω'. Because of the Euler formula $\sum z^i \partial_i f(i) = (n+1) f(z)$ transversality implies that on each patch $z_i = 1$ there is at least one other nonvanishing gradient $\partial_j f \neq 0$ at each point of the hypersurface. Locally we thus find the formula

$$\omega' = -(-)^{i+j} \frac{dz^0 \ldots \widehat{dz^i} \ldots \widehat{dz^j} \ldots dz^n}{\partial_j f}, \qquad z^i = 1, \tag{3.23}$$

where the hats mark the differentials that have to be omitted. For a complete intersection of codimension r we set $f = f_1 \ldots f_r$ and we have to take r residues to obtain the holomorphic $n-r$ form on the Calabi–Yau. In string theory the period integrals of Ω play an important role because they contain, via mirror symmetry, nonperturbative information on instanton corrections to Yukawa couplings and on 'quantum volumes' of D-branes. In mathematics these are related to the Gromov–Witten invariants, which count holomorphic curves and are topological invariants of the manifold.

Basics of **Sheaves and Čech cohomology.** Here we only collect some definitions to get a flavor of the concepts. More can be found, for example, in chapters 0 and 1 of the standard reference [GR78] for algebraic geometry and in chapters 1 and 2 of the very useful notes by Chris Peters on "*Complex Surfaces* (long version, for PhD-students mainly)" that is available on the internet at his home page `http://www-fourier.ujf-grenoble.fr/~peters/`.

We start with the definition of a sheaf, which captures the essence of what is necessary for defining cohomology groups on a topological space, but is way too general for most applications, where more managable special cases like coherent sheaves are used (coherent sheaves are, roughly speaking, quotients of possibly singular vector bundles on possibly singular varieties).

Definition: A sheaf $\mathcal{F}$ on a topological space X associates to each open set $U \subset X$ an abelian group $\mathcal{F}(U)$, whose elements are called sections of $\mathcal{F}$ over U and for each pair $U \subset V$ of open sets a restriction map $r_{V,U} : \mathcal{F}(V) \to \mathcal{F}(U)$ that satisfies the following compatibility conditions:

- For $U \subset V \subset W$ restrictions are compatible $r_{W,U} = r_{V,U} r_{W,V}$. We thus write $\sigma_{|_V} = r_{V,U}(\sigma)$.
- For sections $\sigma \in \mathcal{F}(U)$ and $\tau \in \mathcal{F}(V)$ that agree on a non-empty intersection $\sigma_{|_{U\cap V}} = \tau_{|_{U\cap V}}$ there exists a section $\rho \in \mathcal{F}(U \cup V)$ with $\rho_{|_U} = \sigma$ and $\rho_{|_V} = \tau$.
- If $\sigma_{|_U} = \sigma_{|_V} = 0$ for $\sigma \in \mathcal{F}(U \cup V)$ then $\sigma = 0$.

Examples: $\mathcal{O}(U)$ denotes the sheaf of holomorphic functions on U, $\Omega^p(U)$ is the sheaf of holomorphic p forms, and $\mathcal{O}^*(U)$ consists of nonvanishing holomorphic functions with the multiplicative group structure.

Definition: Let $\mathcal{F}$ be a sheaf on M and $\underline{U} = \{U_\alpha\}$ a locally finite open cover. The degree p cochains $C^p(\underline{U}, \mathcal{F})$ are defined as

$$C^0(\underline{U}, \mathcal{F}) = \prod_\alpha \mathcal{F}(\mathcal{U}_\alpha), \qquad C^1(\underline{U}, \mathcal{F}) = \prod_{\alpha\neq\beta} \mathcal{F}(\mathcal{U}_\alpha \cap \mathcal{U}_\beta),$$
$$C^p(\underline{U}, \mathcal{F}) = \prod_{\alpha_0\neq\alpha_2\neq\ldots\neq\alpha_p} \mathcal{F}(\mathcal{U}_{\alpha_0} \cap \ldots \cap \mathcal{U}_{\alpha_p}) \tag{3.24}$$

and the coboundary operator $\delta : C^p(\underline{U}, \mathcal{F}) \to C^{p+1}(\underline{U}, \mathcal{F})$ is

$$(\delta\sigma)_{i_0 \ldots i_{p+1}} = \sum_{j=0}^{p+1} (-)^j \sigma_{i_0 \ldots i_{j-1} i_{j+1} \ldots i_{p+1}}, \tag{3.25}$$

which is easily checked to satisfy $\delta^2 = 0$. The idea now is to make the open sets small enough to have trivial local cohomology so that all information is contained in the 'transition functions' on the operlaps. Then the cohomology becomes independent of the chosen cover, where, like for the simplicial complex, different covers can be related by a common refinement. More precisely, the Čech cohomology groups $H^p(M, \mathcal{F})$ are defined as a direct limit of $H^p(\underline{U}, \mathcal{F}) = Z^p(\underline{U}, \mathcal{F})/\delta C^{p-1}(\underline{U}, \mathcal{F})$ over all appropriate covers $\underline{U}$ [GR78].

Examples: For the sheaf $\mathcal{O}(L)$ of holomorphic sections of a line bundle L over M the group $H^0(M, \mathcal{O}(L))$ consists of collections of local sections $\sigma = \{\sigma_U\} \in C^0(\underline{U}, \mathcal{O}(L))$ that satisfy $(\delta\sigma)_{UV} = \sigma_V - \sigma_U = 0$ on $U \cap V$ and thus is the space $\Gamma(\mathcal{O}(L))$ of global holomorphic sections. For the cohomology group $H^1(M, \mathcal{O}^*)$ the group structure should be written multiplicatively so that $(\delta g)_{UVW} = g_{VW} g_{UV}/g_{UV} = 1$ on $U \cap V \cap W$ is the cocycle condition for a holomorphic line bundle. Multiplication of the transition data g_{UV} by a δ-exact term $(\delta t)_{UV} = t_V/t_U$ corresponds to a change $\varphi_U \to \varphi_U t_U$ of a local trivializations $\varphi_U : L_U \to U \times \mathbb{C}$ of the bundle, so that $H^1(M, \mathcal{O}^*) = \mathrm{Pic}(M)$ is the Picard group of M.

A basic tool for the computation of cohomology groups is the Mayer–Vietoris sequence: The sequence of inclusions $U \cap V \to U, V \to U \cup V$ gives rise to a short exact sequence of differential complexes

$$0 \to \Omega^*(U \cup V) \to \Omega^*(U) \oplus \Omega^*(V) \to \Omega^*(U \cap V) \to 0 \tag{3.26}$$

where the second map is defined by the obvious restrictions and the third map yields the difference of the restrictions to $U \cap V$ of p-forms in the direct sum.

Now let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be a short exact sequence of differential complexes with f and g chain maps, i.e. f and g commute with the differentials $f d_A = d_B f$ and $g d_B = d_C g$ where $A^q \xrightarrow{d_A} A^{q+1}$ etc., and we will frequently omit the index of the differentials $d_.$. To every such sequence we can associate a long exact sequence of cohomology groups

$$\ldots \to H^{q-1}(C) \xrightarrow{d^*} H^q(A) \xrightarrow{f^*} H^q(B) \xrightarrow{g^*} H^q(C) \xrightarrow{d^*} H^{q+1}(A) \to \ldots \tag{3.27}$$

where f^*, g^* are the natural induced maps of cohomology classes and $d^*[c] = [a]$ for some $a \in H^*(A)$ with $f(a) = db$ and $g(b) = c$, which exists because g is surjectiv and $g(db) = dg(b) = dc = 0$. It can be shown that d^* is independent of the representatives c and a and turns (3.27) into a long exact sequence. Applying this result to the Mayer–Vietoris sequence we can relate, for example, the cohomology of a union $U \cup V$ to the cohomology groups of its subsets. Another application uses the exponential sequence $\mathbb{Z} \to \mathbb{C} \xrightarrow{\exp} \mathbb{C}^*$ to relate sheaf cohomology groups with coefficients in the respective abelian groups (note that the exponential maps sums to products).

In order to prove the equality of Čech and de Rham cohomology $H_{DR}(M) \equiv H(\mathcal{U}, \mathbb{R})$ for a 'good cover' $\mathcal{U}$ of M one can apply the above constructions to the doubly graded Čech–de Rham complex with differential $D = \delta + d_{DR}$ (see chapter II of Bott and Tu [BO82]). Similarly, the $\overline{\partial}$ Poincaré lemma can be used to prove the Dolbeault theorem $H^q(M, \Omega^p) = H^{p,q}_{\overline{\partial}}(M)$ [GR78].

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