Reflexive Polyhedra

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• Supported by

– Austrian Research Funds FWF Projekt Nr. P 10641
Conformal field theory and algebraic geometry

– Austrian National Bank ÖNB Projekt Nr. 5674
Geometrical interpretation of fermionic string models

• Publications

– M. Kreuzer, H. Skarke
On the classification of quasihomogeneous functions
Commun. Math. Phys. 150 (1992) 137 – 147

– M. Kreuzer, H. Skarke
No mirror symmetry in Landau-Ginzburg spectra!
Nucl. Phys. B388 (1992) 113 – 130

– M. Kreuzer, H. Skarke
On the classification of reflexive polyhedra
TUW 95/26 preprint, hep-th/9512204

– H. Skarke
Weight systems for toric Calabi–Yau varieties and reflexivity of Newton polyhedra
TUW 96/04 preprint, alg-geom/9603007

• Some relevant references

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[2] E. Witten, Nucl. Phys. **B403** (1993) 159

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Catastrophes, manifolds & strings

• $N = 2$ SUSY and geometry

– $N = 2$ **world sheet superconformal symmetry** (which is required by space-time SUGRA) implies the existence of a subalgebra of the operator algebra with non-singular operator products. This subalgebra is called **'chiral ring'** [1].

– In order to obtain a 4-dimensional string theory the heterotic string has to be compactified with **6 dimensions living on a Calabi-Yau manifold** (a compact complex Kähler manifold admitting a Ricci-flat metric).

– The **topology** of this Calabi–Yau manifold (more precisely: its cohomology ring) is **isomorphic to** the **chiral ring** of the corresponding superconformal σ model.

• Calabi–Yau's and singularities

– Quasi-homogeneous **polynomials** with **isolated critical points** at the origin **define Calabi–Yau hypersurfaces** in weighted projective spaces.

– These polynomials can also be interpreted as superpotentials of $N = 2$ supersymmetric 2-dimensional field theories, which presumably flow to superconformal theories (Landau-Ginzburg models) in the infrared.

• Gauged linear σ models

– Assuming the existence of conformally invariant IR fixed points we can smoothly interpolate between the Calabi–Yau phase and the Landau–Ginzburg phase [2].

Calabi–Yau & mirror symmetry

• What about 'mirror symmetry' ?

– From the field theory point of view 'mirror symmetry' is nothing but charge conjugation.

– From the geometrical point of view it is quite exciting, because it relates manifolds with different topologies.

– Equivalence of the formulas for Yukawa couplings allows non-perturbative calculations (weak and strong σ model coupling are exchanged; strings living on manifolds with different topology become equivalent in the full quantum theory).

– Mirror symmetry relates type IIA and type IIB strings.

• Calabi–Yau constructions

– Complete Intersection Calabi Yau manifolds (CICYs) in products of projective spaces (these are all manifolds with negative Euler characteristic!).

– Transversal hypersurfaces in weighted projective spaces (this set of manifolds is almost mirror-symmetric, but not exactly!).

– CICYs in products of weighted projective spaces: Little is known about this class of manifolds.

– Hypersurfaces in toric varieties (see below): This large class of manifolds contains all (abelian orbifolds of) tranversal hypersurfaces in weighted $\mathbf{P}_4$, and it is mirror symmetric by construction (Batyrev showed [3] that the generic hypersurface of a toric variety has vanishing first Chern class iff the polyhedron that defines the variety is reflexive).

Toric varieties

• Polytopes, fans, and manifolds

– A toric variety V_Σ is a generalization of a projective space.

– Like $\mathbb{P}_n$ it can either be written as a (holomorphic) quotient,

$$\mathbb{P}_n = (\mathbb{C}^n - \{0\})/\mathbb{C}^* \qquad \longrightarrow \qquad V_\Sigma = (\mathbb{C}^N - F_\Sigma)/(\mathbb{C}^*)^{N-n},$$

or it can be glued together using affine coordinated patches [4].

– $\mathbb{C}^* = \mathbb{C} - \{0\}$ is called (algebraic) *torus*; the name toric geometry derives form the torus action on V_Σ.

– A fan Σ is a collection of cones such that each face of a cone is a cone of Σ. The combinatorial data describing the linear relations among integral generators of the cones encode the torus action in the quotient construction (F_Σ is the union of interesections of coordinate hyperplanes consisting of fixed points of the action of $(\mathbb{C}^*)^{N-n}$). Equivalently, they encode how the affine coordinate patches are glued together.

– Given a polytope Δ we consider the fan Σ of cones over its faces and write $V_\Delta = V_\Sigma$.

• Combinatorics and topology

– The combinatorial properties of a polytope Δ determine the topological properties of the associated complex manifold/variety:

– V_Σ is compact if Σ is complete, i.e. if 0 is in the interior of Δ. Interior points of faces of Δ correspond to cohomology classes.

• Reflexivity

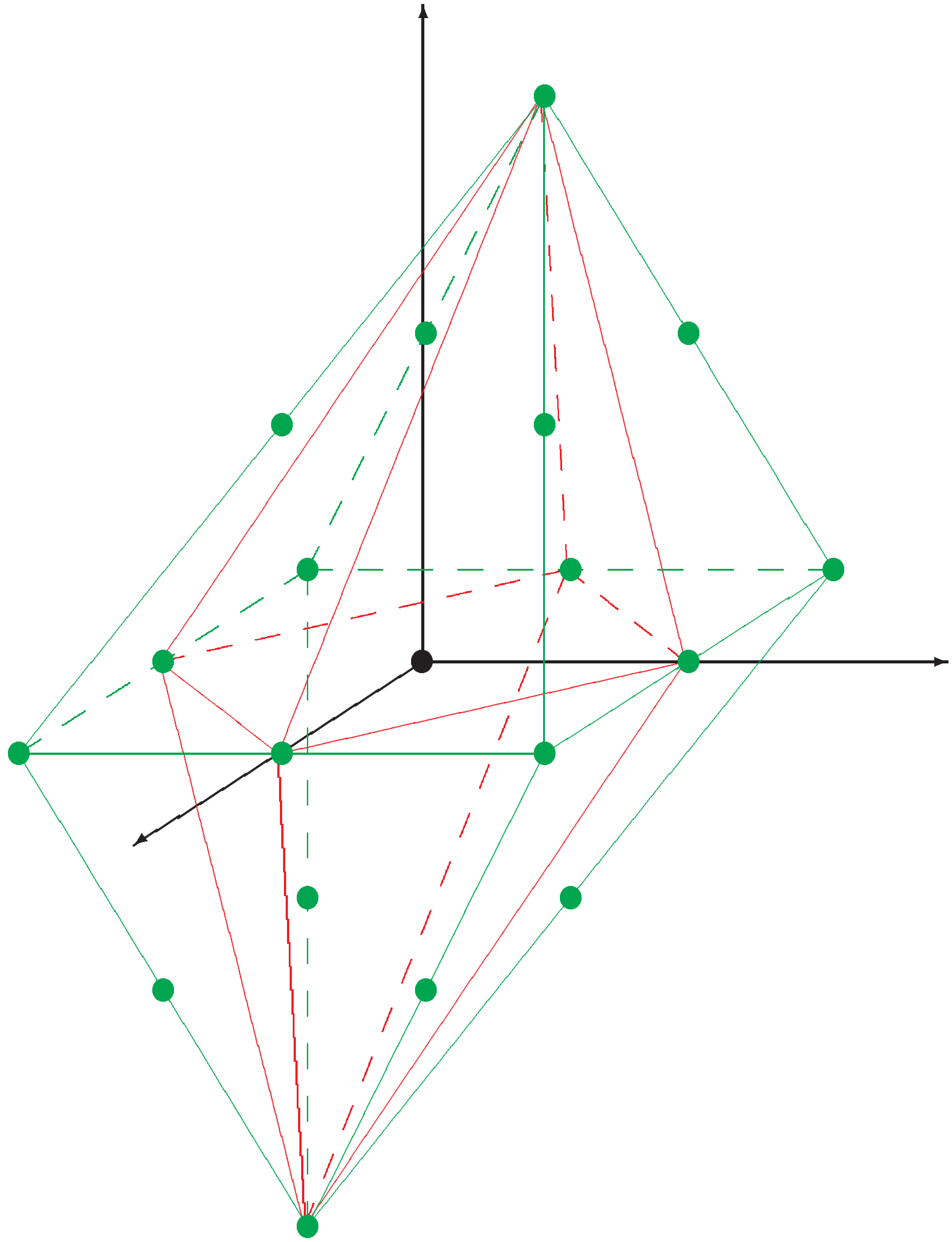
– The distance of a lattice hyperplane H to a point P is 1 plus the number of parallel lattice hyperplanes between H and P.

– If $0 \in \Delta$ then we define the dual $\Delta^* := \{y : \langle y, x\rangle \geq -1 \forall x \in \Delta\}$.

– An integral polytope with $0 \in \Delta$ is called reflexive if its dual is also integral. Equivalently, all facets have distance 1 to the origin.

– For reflexivity it is necessary that 0 be the only integral interior point of Δ. In 2 dimensions this condition is also sufficient. In 3 dimensions this is no longer true (see the example on the next page).

Examples of 3-dimensional polytopes



For the red polytope the equations of the 8 bounding planes are

$$\pm(x-y)\pm\tfrac{1}{2}z=1,\qquad \pm(x+y)+(1\pm\tfrac{1}{2})z=1,$$

i.e. they all have distance 2 from the single interior point. The larger green polytope, on the other hand, is reflexive.

Classifying reflexive polytopes

• Minimal polytopes, simplices, weights

- The dual of a subpolytope contains the dual polytope.
- Minimal polytopes that contain 0 in their interior can be spanned by simplices that contain 0 in their interior.
- The barycentric coordinates of 0 with respect to a simplex are called 'weights'. The combination of weights corresponding to a minimal subpolytope bounds the dual polyhedron.

• Classification

- Theorem: For reflexive polytopes of fixed dimension there are only finitely many (combinations of) weight systems.
- Vertex pairing matrices determine Δ in a coordinate independent way, possibly up to a finite freedom in the choice of a lattices.
- A complete classification of toric Calabi–Yau hypersurfaces can thus be achieved by an enumeration of vertex pairing matrices (in some normal form).

• Simplex decompositions for $n \leq 4$

- n=2: 3; 2+2.
- n=3: 4; 2+3, $\underbrace{3+3}$.
- n=4: 5; 4+2, 3+3, $\underbrace{4+3}$, $\underbrace{4+4}$;

  3+2+2, $\underbrace{3+3}$+2, $\underbrace{3+3}\underbrace{+3}$; 2+2+2+2,

  where the numbers denote the numbers of edges of the simplices, and brackets indicate common points. In $n=4$ dimensions there are thus 9 different cases with up to $2n=8$ vertices.

Reflexive polygons

All reflexive polygons are subpolygons of:

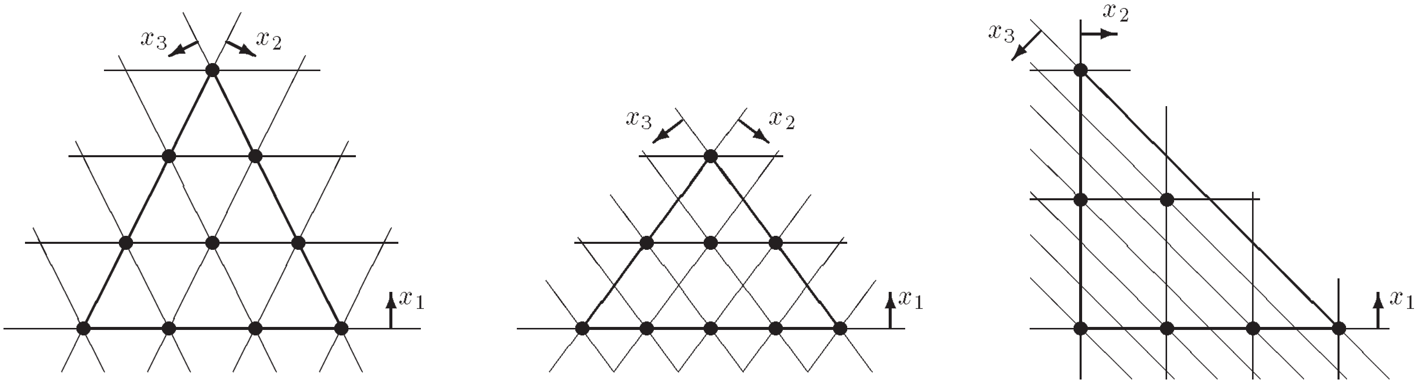


Fig. 1: The bounding simplices Q for the weight systems $(\frac{1}{3},\frac{1}{3},\frac{1}{3})$, $(\frac{1}{2},\frac{1}{4},\frac{1}{4})$ and $(\frac{1}{2},\frac{1}{3},\frac{1}{6})$.

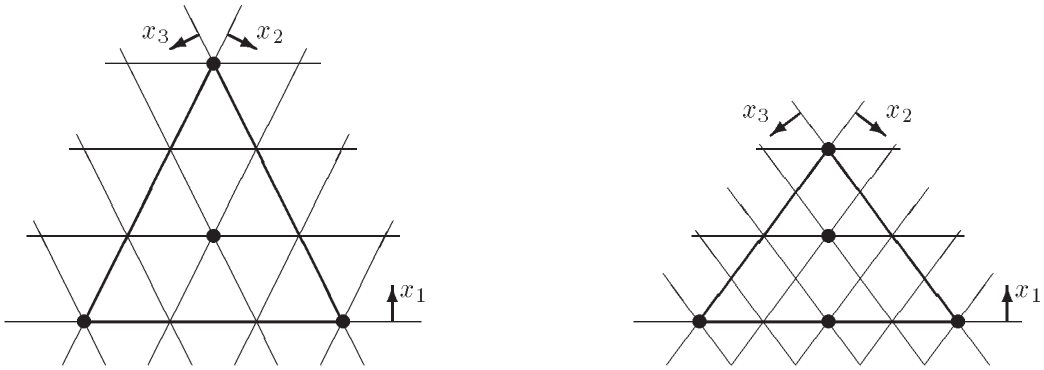


Fig. 2: Alternative lattices for the weight systems $(\frac{1}{3},\frac{1}{3},\frac{1}{3})$ and $(\frac{1}{2},\frac{1}{4},\frac{1}{4})$.

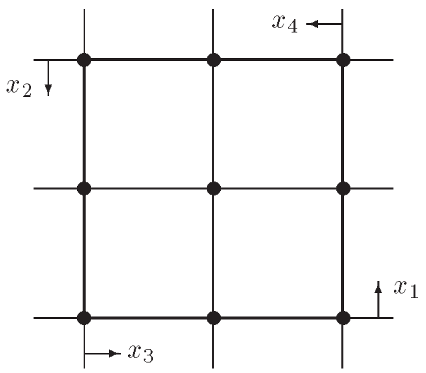


Fig. 3: $2+2$ full lattice

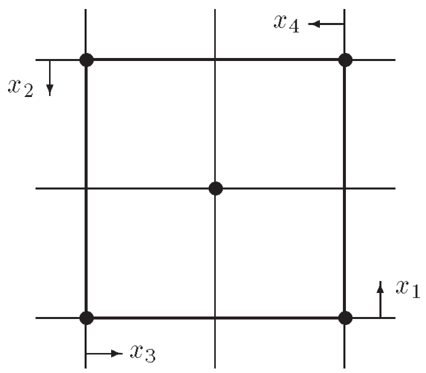


Fig. 4: $2+2$ alternative lattice

• General results on weight systems

- A maximal Newton polyhedron is the set of all nonnegative integer points in a lattice allowed by a weight system or a combination of weight systems.
- Example: The weight systems $(2,1,1,0,0)/4$ and $(2,0,0,1,1)/4$ define a $\mathbb{Z}^3$ sublattice of $\mathbb{Z}^5$. The maximal Newton polyhedron is an Egyptian pyramid of height 2 and baselength 4.
- **Theorem:** Four or lower dimensional maximal Newton polyhedra with an interior point are reflexive.
- **Lemma:** Maximal Newton polyhedra corresponding to weighted $\mathbb{P}^n$'s or abelian orbifolds of weighted $\mathbb{P}^n$'s that allow transverse polynomials have an interior point.
- For the classification we need only those weight systems whose maximal Newton polyhedra span all coordinate hyperplanes

• Weight systems in up to 4 dimensions

- Results of the classification of weight systems:

	$n=2$ ($\mathbb{P}_2$)	$n=3$ ($\mathbb{P}_3$)	weighted $\mathbb{P}_4$	$n=4$ toric
span	3	58	4169	38730
total	3	95	7555	184026

- In 2 and 3 dimensions we exactly recover the known weights, but in 4 dimensions – the case most relevant for strings in 4 dimensions – we find a huge number of non-transversal weights that lead to reflexive polytopes!
- In spite of a large redundancy it seems that the number of topologically distinct toric Calabi–Yau's is larger than expected.

Connectedness of moduli spaces

• Singularities and moduli spaces

- It has been checked that the moduli spaces of all CICYs and of all hypersurfaces in weighted $\mathbb{P}_4$ are connected at singular points [6].
- It is believed that physics is smooth at these singularities (with the change in the massless spectra being related to 'black hole condensation' [5]).

• Perspectives

Our classification approach leads to some 10^5 reflexive bounding pairs. It should be straigthforward to check their connectedness, which implies connectedness of all toric Calabi–Yau hypersurfaces.