Lecture notes

String Theory II

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Chapter 1

Conformal field theory

Conformal field theory is concerned with quantum field theories that are invariant under conformal coordinate transformations. Most of the known results refer to two dimensions, where the conformal group is infinite dimensional so that Ward identities strongly constrain the structure of quantum fields and correlation functions. Much of the interest in conformal field theory comes from string theory, but there are also important applications in statistical mechanics and solid state physics, like second order phase transitions of two-dimensional systems and the (fractional) quantum Hall effect [DI97, ga99, cr99].

We first discuss the conformal group in arbitrary dimensions and the Wick rotation with the field content of string theory, which provides the basic examples of free fields. We evaluate the 2-point correlations and discuss the appropriate vacuum states and then proceed to more general concepts and techniques of Euclidean conformal field theory.

1.1 The conformal group

A conformal transformation is an angle-preserving diffeomorphism (or coordinate transformation, if one prefers a 'passive' point of view) of a Riemannian manifold. In two-dimensional Minkowski space such transformations arise as the residual gauge symmetry of general coordinate and Weyl invariance in the 'conformal gauge' $g_{mn} = \eta_{mn}$, i.e. as coordinate transformations whose effect on the metric can be compensated by a Weyl transformation. We consider a constant pseudo-metric η_{mn} with arbitrary signature. In order to obtain the component of the identity of the conformal group in flat space we consider infinitesimal transformations $x^m \to x^m + \xi^m$, for which we have to solve the conformal Killing equation

$$h_{mn} := \partial_m \xi_n + \partial_n \xi_m + 2\Lambda \eta_{mn} = 0. \tag{1.1}$$

Contracting with the inverse metric η^{mn} and taking the double divergence we obtain

$$\eta^{mn}h_{mn} = 2(\partial\xi + d\Lambda) = 0 \qquad \Rightarrow \qquad \partial^m\partial^n h_{mn} = \left(2 - \tfrac{2}{d}\right)\Box\partial\xi = 0. \tag{1.2}$$

For $d > 1$ dimensions this implies $\Box\Lambda = \Box\partial\xi = 0$. (In one dimension there is, of course, no restriction on ξ.) Now we compute the symmetrized derivative of the divergence of h_{mn},

$$\partial_l\partial^m h_{mn} + \partial_n\partial^m h_{ml} = \Box(\partial_l\xi_n + \partial_n\xi_l) + 2\partial_l\partial_n\partial\xi + 4\partial_l\partial_n\Lambda = 2\left(1 - \tfrac{2}{d}\right)\partial_l\partial_n\partial\xi = 0, \tag{1.3}$$

where we used $\Box(\partial_l\xi_n + \partial_n\xi_l) = -2\eta_{ln}\Box\Lambda = 0$. In more than two dimensions this implies that all second derivatives of Λ vanish, i.e.

$$d > 2 \qquad \Rightarrow \qquad \Lambda = -\tfrac{1}{d}\partial\xi = 2\,bx - \lambda \tag{1.4}$$

for some constants λ and b_m. In order to solve for ξ we still need the antisymmetric part of $\partial_m\xi_n$, whose derivative is

$$\partial_l(\partial_m\xi_n - \partial_n\xi_m) = \partial_m\partial_l\xi_n - \partial_n\partial_l\xi_m = 2(\eta_{lm}\partial_n\Lambda - \eta_{ln}\partial_m\Lambda) = 4(\eta_{lm}b_n - \eta_{ln}b_m). \tag{1.5}$$

Integrating this equation we find

$$\tfrac{1}{2}(\partial_m\xi_n - \partial_n\xi_m) = \omega_{mn} + 2x_m b_n - 2x_n b_m \tag{1.6}$$

with an antisymmetric integration constant $\omega_{mn} = -\omega_{nm}$. Putting the pieces together

$$\partial_m\xi_n = \omega_{mn} + 2x_m b_n - 2x_n b_m + \eta_{mn}(\lambda - 2bx) \tag{1.7}$$

and thus

$$\xi^n = a^n + x^m\omega_m{}^n + \lambda\, x^n + x^2 b^n - 2bx\, x^n. \tag{1.8}$$

a, ω, λ and b generate translations, Lorentz transformations, dilatations and 'special conformal transformations', respectively.

The finite form of the special conformal transformations is $x^n \to y^n = (x^n + x^2\, b^n)/(1 + 2bx + b^2x^2)$. They form a subgroup, as can be seen by writing the transformation as a combination of two inversions and a translation: Since $x^2/y^2 = 1 + 2bx + b^2x^2$ we find $\vec y/y^2 = \vec x/x^2 + \vec b$. The inversion $\vec x \to \vec x/x^2$ itself is also a conformal map, but it changes the orientation (the radial direction is reversed) and hence is not continuously connected to the identity. The Jacobi determinant for a special conformal transformation is $|\frac{\partial y}{\partial x}| = (\frac{y^2}{x^2})^d$ and for the scale factor we find $\eta^{mn}\frac{\partial y^i}{\partial x^m}\frac{\partial y^j}{\partial x^n} = \eta^{ij}/(1 + 2bx + b^2x^2)^2$. Note that a special conformal transformation has a singular point $\vec x = -\vec b/b^2$. A proper representation of the conformal group thus requires an extension of space-time that adds points at infinity. Evaluation of the Lie brackets of all generating vector fields shows that the conformal group is isomorphic to $SO(p+1, q+1)$ for a space with signature (p, q) [DI97].

The situation in 2-dimensional Minkowski space is best analyzed in light-cone coordinates $x^\pm = x^0 \pm x^1$ where the metric is off-diagonal. Then $h_{++} = 2\partial_+\xi_+ = 0$ shows that $\xi^\pm$ is independent of $x^\mp$ and $h_{+-} = 0$ only fixes Λ in terms of $\partial\xi$. The conformal group thus consists of arbitrary reparametrizations of the light cone. In string theory the topology of space-time is a cylinder and a complex basis for 2π-periodic infinitesimal transformations with vector fields $\xi_n^+ = e^{inx^+}\partial_+$ yields the Lie brackets $[\xi_m^+, \xi_n^+] = i(n-m)\xi_{m+n}^+$. The quantum version of this infinite dimensional Lie algebra is the Virasoro algebra

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}, \tag{1.9}$$

which has a central extension in case of conformal anomaly $c \neq 0$, with a second copy of the same algebra for the second light cone coordinate.

In Euclidean space we use complex coordinates $z = x+iy$ and the conformal Killing equation turns into the Cauchy-Riemann equation $\partial_{\bar z}\xi(z,\bar z) = 0$ so that the conformal transformations correspond maps $z \to \xi(z)$ that are holomorphic (or anti-holomorphic, if we admit a change of orientation). Riemann's mapping theorem states that every simply connected complex one-dimensional domain with at least two boundary points can be mapped holomorphically onto the interior of the unit disc $|z| < 1$, which in turn is conformally equivalent to the upper half plane $\operatorname{Im} z > 0$.[1] The case of one boundary point corresponds to the complex plane $\mathbb{C}$ and Riemann's number sphere, the complex projective space $\mathbb{P}^1 = \{z : w\}$, has no boundary point.

Global conformal transformations of the compactified complex plane $\mathbb{P}^1$ coincide with the structure of the conformal group in $d > 2$, i.e. with finite-dimensional group generated by translations $z \to z + c$, rotations and dilatations $z \to \lambda z$ and special conformal transformations $z \to z/(z + d)$ because regularity of a holomorphic vector field at infinity restricts $\xi = \xi_1 + \xi_0 z + \xi_{-1}z^2$ to a quadratic polynomial.[2] The resulting group is the group

$$z \to \frac{az+b}{cz+d}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2,\mathbb{C}) \tag{1.10}$$

of Möbius transformations. Its Lie algebra corresponds to the subalgebra $\{L_n\}$ with $|n| \leq 1$ of the Virasoro algebra.

1.2 Wick rotation and Euclidean fields

In quantum field theory it is often useful to make an analytic continuation to Euclidean time. The direction of the Wick rotation in the complex time plane is fixed by convergence requirements: The field operators (in the Heisenberg picture) at time τ are given by

[1] $|w| < 1$ implies $\operatorname{Im} u > 0$ for $u = i(1+w)/(1-w)$. This inverse map is $w = (u-i)/(u+i)$.

[2] $\xi(z)\partial_z = \sum \xi_n z^{1-n}\partial_z = \xi(w)\partial_w$, with $w = 1/z$ we find $\partial_w = -z^2\partial_z$, i.e. regularity at $z = 0$ for $n \leq 1$ and at $w = 0$ for $n \geq -1$.

$\mathcal{O}(\tau,\sigma)=e^{i\tau H}\mathcal{O}(0,\sigma)e^{-i\tau H}$. Important quantities, like scattering amplitudes, can be expressed in terms of time ordered correlation functions, which are of the form

$$\langle \mathcal{O}_n(\sigma_n,0)e^{-i(\tau_n-\tau_{n-1})H}\mathcal{O}_{n-1}(\sigma_{n-1},0)\ldots\mathcal{O}_2(\sigma_2,0)e^{-i(\tau_2-\tau_1)H}\mathcal{O}_1(\sigma_1,0)\rangle. \tag{1.11}$$

For a positive Hamiltonian this is a convergent expression if the time differences $\tau_i-\tau_{i-1}$ have negative imaginary part. Thus the time evolution should go into the direction of negative imaginary time and we set $\tau=-it$, so that $\sigma^\pm=\tau\pm\sigma=-i(t\pm i\sigma)$ with σ the space coordinate and t the Euclidean time.

In string theory the world sheet has the topology of a cylinder and the fields are 2π-periodic in σ. Instead of considering the complex variables $\xi=i\sigma^+=t+i\sigma$ and $\overline{\xi}=i\sigma^-=t-i\sigma$ it is thus useful to map the world sheet onto the punctured complex plane: The map $\xi\to z=\exp(\xi)$ automatically implements 2π-periodicity in σ and thus is one-to-one. Hence we define

$$z=e^{\xi}=e^{i\sigma^+},\qquad \overline{z}=e^{\overline{\xi}}=e^{i\sigma^-},\qquad \sigma^\pm=\tau\pm\sigma=-i(t\pm i\sigma). \tag{1.12}$$

This transforms the left (right) movers $\phi(\sigma^\pm)$ to (anti) holomorphic fields on the punctured plane. The puncture at the origin corresponds to the asymptotic past.

Important examples of free conformal fields are provided by the field content of the superstring with flat target space. For convenience we set the string tension $T=\frac{1}{2\pi\alpha'}=\frac{1}{4\pi}$. In light cone coordinates the Lagrangian thus becomes

$$4\pi\mathcal{L} \;=\; \partial_+X\cdot\partial_-X \;+\; b_{++}\partial_-c^+ + b_{--}\partial_+c^- \;+\; i\psi\partial_-\psi+i\tilde\psi\partial_+\tilde\psi \;+\; \beta\partial_-\gamma+\tilde\beta\partial_+\tilde\gamma, \tag{1.13}$$

which contains the D coordinate fields X^μ (free bosons), the ghosts c and the anti-ghosts b of the bosonic string, the fermions ψ and the superconformal ghosts β,γ.

The general solution of the equations of motion is parametrized by

$$X(\sigma,\tau)=X_L(\sigma^+)+X_R(\sigma^+),\qquad X_L^\mu=\tfrac12 x^\mu+p^\mu\sigma^++\sum_{n\neq0}\tfrac{i}{n}\alpha_n^\mu e^{-in\sigma^+}, \tag{1.14}$$

and

$$c^+=\sum_{n=-\infty}^{\infty}c_ne^{-in\sigma^+},\qquad b_{++}=2i\sum_{n=-\infty}^{\infty}b_ne^{-in\sigma^+} \tag{1.15}$$

for the bosonic string and analogous expressions for the additional fields of the superstring. Canonical quantization leads to the (anti)commutation relations

$$[\alpha_m^\mu,\alpha_n^\nu]=n\delta_{m+n}\eta^{\mu\nu},\qquad [P^\mu,x^\nu]=i\eta^{\mu\nu},\qquad \{b_m,c_n\}=\delta_{m+n}, \tag{1.16}$$

$$\{\psi_r^\mu,\psi_s^\nu\}=\eta^{\mu\nu}\delta_{r+s},\qquad\qquad [\gamma_r,\beta_s]=\delta_{r+s} \tag{1.17}$$

with $p^\mu = \alpha_0^\mu = \tilde\alpha_0^\mu$ for the zero mode. This operator algebra can be used to construct a Fock space representation and the Hilbert space of physical states is given by the cohomology of the BRST operator on that space. It is convenient to also perform a Wick rotation in target space, which amounts to $\eta^{\mu\nu} \to -\delta^{\mu\nu}$ in our conventions.

Time ordering on the world sheet now corresponds to radial ordering on the complex plane.

$$\mathcal{R}\,A(z)B(w) \;:=\; \theta(|z|-|w|)A(z)B(w) + (-)^{AB}\theta(|w|-|z|)B(w)A(z). \tag{1.18}$$

For free bosons $X^\mu(z,\overline{z}) = X_L^\mu(z) + X_R^\mu(\overline{z})$, with the Fubini-Veneziano field

$$X^\mu(z) = \tfrac{1}{2}x^\mu - ip^\mu \log z + \sum_{n\neq 0} \tfrac{i}{n}\alpha_n^\mu z^{-n}, \tag{1.19}$$

the two-point function $\langle \mathcal{R}\,X^\mu(z,\overline{z})X^\nu(w,\overline{w})\rangle$ has contributions from the oscillator modes and from the 0-modes x^μ and p^μ. Assuming $|z| > |w|$ the left-movers yield

$$\Big\langle \sum_{m>0} \tfrac{i}{m}\alpha_m^\mu z^{-m} \sum_{n<0} \tfrac{i}{n}\alpha_n^\nu w^{-n} \Big\rangle = \delta^{\mu\nu}\sum_{m>0}\tfrac{1}{m}(\tfrac{w}{z})^m = -\delta^{\mu\nu}\log(1-\tfrac{w}{z}) \tag{1.20}$$

The definition of the complete two-point function for free bosons is somewhat subtle because of an infrared divergence. It is convenient to represent the zero mode algebra on momentum Eigenstates $p_\mu|k\rangle = k_\mu|k\rangle$, which are normalized as $\langle k'|k\rangle = (2\pi)^D\delta^D(k'-k)$. Our normal ordering amounts to taking p as annihilation operator and x as creation operator. Then the vacuum is a non-normalizable state $|k=0\rangle$, for which the expectation value of x^μ is ill-defined. If we evaluate $\langle x^\mu x^\nu\rangle$ for a Gaussian distribution this expectation value becomes proportional to $(\Delta X)^2\delta^{\mu\nu}$ with $\Delta X \to \infty$ for momentum eigenstates. A rigorous treatment requires an IR cutoff and only physical correlation functions can be expected to be IR finite (see [GR87, p.139-149]). Quantities like the Hamiltonian and the Virasoro constraints only depend on derivatives of X, which are safe. The interpretation of correlation functions along the usual lines of scattering theory also leads to cutoff-independent results, as will be discussed below.

We can avoid this problem by noting that the vacuum expectation value amounts to a subtraction of the normal ordered operator product. Normal ordering of the zero modes yields $-\delta^{\mu\nu}\log(z\overline{z})$, which combines with the contributions from the oscillators to

$$\mathcal{R}X^\mu(z,\overline{z})X^\nu(w,\overline{w}) - \;:X^\mu(z,\overline{z})X^\nu(w,\overline{w}):\; = -\delta^{\mu\nu}\log|z-w|^2 \tag{1.21}$$

Because of the symmetry under the exchange of z and w the same result holds for $|z|<|w|$.

So far we defined our Euclidean (conformal) quantum field theory by analytic continuation of the quantized fields. An alternative approach is the direct definition of correlation function in terms of the Euclidean path integral. Here the starting point is the path integral representation of time-ordered vacuum expectation values

$$\langle\phi(x_1)\ldots\phi(x_n)\rangle \equiv \langle 0|T\hat\phi(x_1)\ldots\hat\phi(x_n)|0\rangle = \frac{1}{Z[0]}\int \mathcal{D}\phi\,\phi(x_1)\ldots\phi(x_n)e^{\frac{i}{\hbar}S} \tag{1.22}$$

where $Z[J]$ is the generating functional of the Greens functions $\langle\phi(x_1)\ldots\phi(x_n)\rangle$. Wick rotation leads to a negative definite action so that it is convenient to define the Euclidean action as $S_E=\int dt_E\mathcal{L}_E$ with $\mathcal{L}_E=-\mathcal{L}(\tau\to -it_E)$. Euclidean field theory is then defined by

$$\langle\phi(x_1)\ldots\phi(x_n)\rangle=\frac{1}{Z[0]}\prod\frac{\delta}{\delta J(x_i)}Z[J]_{|_{J=0}},\qquad Z[J]=\int\mathcal{D}\phi\, e^{-S_E/\hbar+\int J\phi}. \tag{1.23}$$

This formula is reminiscent of correlation functions in statistical mechanics, where $Z=\sum e^{-\beta H}$ is the partition function. S_E is thus the analog of the Hamilton operator in the Boltzmann factor $e^{-\beta H}$ and $\hbar$ is the analog of the temperature $T=1/\beta$. The free energy $F=-T\log Z$ is the analog of the generating functional for connected Greens functions. Euclidean quantum field theory is thus closely related to statistical mechanics and it is quite common to use the respective terminology.

From the path integral representation it is easily seen that the 2-point correlation, or propagator, is the Greens function for the differential operator A that occurs in the free action $S=\frac{1}{2}\int d^2x\,\phi A\phi$ because, with a shift $\phi\to\phi+A^{-1}J$ of the integration variable, we find

$$Z[J]=Z[0]e^{\frac{1}{2}\int d^2xd^2y\,J(x)K(x,y)J(y)},\qquad \langle\phi(x)\phi(y)\rangle=K(x,y) \tag{1.24}$$

with $A\,K(x,y)=\delta(x-y)$. For the free boson the Euclidean action is

$$S=\frac{1}{4\pi}\int d^2z\,\partial X(z,\overline{z})\overline{\partial}X(z,\overline{z}),\qquad \partial:=\partial/\partial z,\quad \overline{\partial}:=\partial/\partial\overline{z} \tag{1.25}$$

with $dxdy=d^2z/2$ and $A=-\frac{1}{2\pi}\partial\overline{\partial}$. The Green function for this operator indeed (up to a constant) is $-\log|z-w|^2$, because

$$\partial\overline{\partial}\log|z|^2=\pi\delta(x)\delta(y)=2\pi\delta^{(2)}(z),\qquad z=x+iy. \tag{1.26}$$

This can be seen, for example, with the regularization

$$\partial\overline{\partial}\log(|z|^2+\varepsilon)=\partial\,\frac{z}{z\overline{z}+\varepsilon}=\frac{\varepsilon}{(|z|^2+\varepsilon)^2}\to\pi\delta(x)\delta(y)=2\pi\delta^{(2)}(z), \tag{1.27}$$

since, with $|z|=\sqrt{\varepsilon}r$ and for the test function 1, we find $dxdy=rdrd\varphi$ and $\int_0^\infty\frac{r\,dr}{(r^2+1)^2}=\frac{1}{2}$;

$$\begin{aligned}\partial&:=\partial_z=\tfrac{1}{2}(\partial_x-i\partial_y), & \partial_x&=\partial+\overline{\partial}, & d^2z&:=2dxdy=idzd\overline{z},\\ \overline{\partial}&:=\partial_{\overline{z}}=\tfrac{1}{2}(\partial_x+i\partial_y), & \partial_y&=i\partial-i\overline{\partial}, & \delta^2(z)&:=\tfrac{1}{2}\delta(x)\delta(y)\end{aligned} \tag{1.28}$$

are the relations between real and complex coordinates; formally, $\delta(x)\delta(y)=-2i\delta(z)\delta(\overline{z})$. A direct evaluation of the propagator from its Fourier transform $1/k^2$ shows the IR divergence, which could be regularized with a mass term. The form of the propagator $-\log z-\log\overline{z}$ is quite typical for CFT correlations because it decomposes into purely holomorphic and anti-holomorphic contributions, which are, however, not single valued on the complex plane. The individual contributions can be analysed with holomorphic operator product techniques, which we are going to develop next. Eventually these building blocks (the conformal blocks) have to be combined to yield the single-valued correlations of the complete theory.

1.3 Tensors, energy–momentum, and correlations

A general conformal tensor field has h holomorphic and $\overline{h}$ antiholomorphic covariant indices so that the $(h, \overline{h})$ form $\phi_{h,\overline{h}} = \phi(z,\overline{z})(dz)^h(d\overline{z})^{\overline{h}}$ is coordinate independent,

$$\phi(z,\overline{z}) = (\tfrac{\partial\xi}{\partial z})^h(\tfrac{\partial\overline{\xi}}{\partial\overline{z}})^{\overline{h}}\phi(\xi,\overline{\xi}). \tag{1.29}$$

Vector indices are counted negative and spinor representations amount to half-integral h. For a pure left-mover with Fourier expansion $\phi = \sum \phi_n e^{-in\sigma^+}$ the conformal map $\xi \to z = e^\xi$ from the cylinder to the punctured plane has $dz = zd\xi$ and thus leads to the Laurent expansion

$$\phi(z) = \sum_{n=-\infty}^{\infty} \frac{\phi_n}{z^{n+h}}, \qquad \phi_n = \oint \frac{dz}{2\pi i} z^{n+h-1}\phi(z), \tag{1.30}$$

which we use as the definition of a meromorphic field $\phi(z)$ in terms of the modes ϕ_n. (In order to simplify the formulas we drop the factors i^h from the Jacobian for $\sigma^+ = -i\xi$.) For general conformal fields that depend on z and $\overline{z}$ we need a double expansion

$$\phi(z,\overline{z}) = \sum_{m,n} \phi_{m,n}\, z^{-m-h}\,\overline{z}^{-n-\overline{h}}, \qquad \phi_{m,n} = \oint \frac{dz}{2\pi i}\oint \frac{d\overline{z}}{2\pi i} z^{m+h-1}\,\overline{z}^{n+\overline{h}-1}\phi(z,\overline{z}), \tag{1.31}$$

Our $\overline{z}$ contour will be oriented clockwise so that $\oint d\overline{z}/\overline{z} = 2pi$, but we will mostly suppress the $\overline{z}$-dependence to make the formulas more transparent and rather explicitly mention the restriction to meromorphic fields when this is not clear from the context.

In order to take full advantage of the conformal symmetry we want to construct a Fock space based on a vacuum state that is invariant under Möbius transformations, i.e. $L_0|0\rangle = L_{\pm1}|0\rangle = 0$. Actually, $L_n|0\rangle = 0$ for $n \geq -1$ would still be consistent with a central term in the Virasoro algebra, but we don't need this yet. Recall that $L_0 \pm \overline{L}_0$ are the generators of time and space translations on the cylinder. After the conformal map $z = e^{t+i\sigma}$ to the complex plane $L_0 \pm \overline{L}_0$ generate dilatations (a shift in t) and rotations (a shift in σ). This is why $h \pm \overline{h}$ are called conformal weight and spin, respectively. For meromorphic fields, which have $\overline{h} = 0$, spin and conformal weight coincide.

We normalize the Euclidean energy momentum tensor $T(z) = \sum L_n z^{-n-2}$, and analogously all other Noether charges, such that a conformal transformation δ_ξ is generated by the commutator with $\oint \frac{dz}{2\pi i}\xi(z)T(z) = \sum \xi_{-m}L_m$. Infinitesimal global conformal transformations are generated by the global vector fields $\xi(z) = \sum \xi_n z^{1-n} = \xi_1 + \xi_0 z + \xi_{-1}z^2$. Infinitesimal translations $z \to z + \xi_1$ and special conformal transformations $z \to z + \xi_{-1}z^2$ are therefore generated by L_{-1} and L_1, respectively. For finite transformations we find

$$e^{\lambda L_{-1}}\phi(z) = \phi(z+\lambda), \qquad e^{\lambda L_0}\phi(z) = \phi(e^\lambda z), \qquad e^{\lambda L_1}\phi(z) = \phi(\tfrac{z}{1-\lambda z}), \tag{1.32}$$

with the adjoint action of L_n (more explicitly, $e^{\lambda L_{-1}}\phi(z)e^{-\lambda L_{-1}}=\phi(z+\lambda)$ etc.).

The energy momentum tensor itself transforms with an anomalous inhomogeneous term

$$\delta_\xi T(w) = [T_\xi,T] = \sum_{m,n}\xi_{-n}[L_n,L_m]w^{-m-2} \tag{1.33}$$

$$= \sum_{m,n}\xi_{-n}L_{m+n}w^{-m-2}(n-m)+\frac{c}{12}\sum_n\xi_{-n}w^{n-2}n(n-1)(n+1) \tag{1.34}$$

$$= \xi(w)\partial T(w)+2\partial\xi(w)T(w)+\tfrac{c}{12}\partial^3\xi(w). \tag{1.35}$$

with $T_\xi=\sum\xi_{-m}\,L_m$; in the double sum we used $n-m=-(m+n+2)+2(n+1)$. For finite transformations $z\to w(z)$ this leads to

$$T(z)=\left(\tfrac{\partial w}{\partial z}\right)^2T(w)+\tfrac{c}{12}S(w,z),\qquad S(w,z):=\frac{\partial w\;\partial^3w-\frac32(\partial^2w)^2}{(\partial w)^2}. \tag{1.36}$$

$S(w,z)$ is called *Schwartzian derivative* of w with respect to z. As usual the result for the finite transformation can be established by checking the group property and the correct infinitesimal form. Indeed, $S(w,z)$ can be shown to be the unique object of conformal weight $h=2$ such that $S(w(f(z)),z)=(\partial_z f)^2\,S(w,f)+S(f,z)$ [Gi89]. For the transformation of $T(z)$ from the cylinder to the complex plane this leads to a shift of the zero mode[3] $L_0^{plane}=L_0^{cyl.}+c/24$.

Returning to the discussion of the vacuum we note that a local operator $\phi(z)$ should be a regular function of z except for singularities at the position of other operators. For a Möbius invariant vacuum state there should be nothing special about $z=0$. Hence $\phi(z)|0\rangle$ should be analytic for small times $|z|\to0$. Briefly, the vacuum state should be a state with 'nothing at the origin'. The contour integrals for fixed times $t=\log\varepsilon$,

$$\oint_{|z|=\varepsilon}\frac{dz}{2\pi i}z^m\phi(z)|0\rangle=\phi_{m-h+1}|0\rangle, \tag{1.37}$$

thus have to vanish for $m\ge0$. We conclude that $\phi_n|0\rangle=0$ for $n\ge1-h$, i.e. ϕ_n should be annihilation operators for these values of n.

Local fields $\phi(z,\overline{z})$ in a CFT that have a tensorial transformation law $(dz)^h(d\overline{z})^{\overline{h}}\phi(z,\overline{z})=(dw)^h(d\overline{w})^{\overline{h}}\phi(w,\overline{w})$ under conformal transformations $z\to w(z)$ are also called *primary fields.* Quasi-primary fields are local fields that transform as tensors under global conformal transformations. The advantage of the SL(2)–invariant vacuum is that all correlation functions of quasi-primary fields transform covariantly under the Möbius group. For 2-point functions of quasi-primary fields covariance implies

$$\langle\phi_i(z)\phi_j(w)\rangle=\frac{C_{ij}}{(z-w)^{h_i+h_j}}, \tag{1.38}$$

[3] The central charge can thus be related to the Casimir energy due to the finite size of the cylinder.

because invariance under translations and rotations implies that the l.h.s. only depends on $z-w$ and dilatations fix the exponent. Under a special conformal transformation $z' = z/(1-\lambda z)$ the coordinate difference becomes $z'-w' = \frac{z-w}{(1-\lambda z)(1-\lambda w)}$. Covariance of (1.38) requires

$$(\tfrac{\partial z'}{\partial z})^{h_i}(\tfrac{\partial w'}{\partial w})^{h_j} = (\tfrac{z'-w'}{z-w})^{h_i+h_j} \tag{1.39}$$

with $\frac{\partial z'}{\partial z} = \frac{1}{(1-\lambda z)^2}$, which implies that C_{ij} can be different from zero only if $h_1 = h_2$.

An analogous calculation for 3–point correlations yields

$$\langle \phi_i(z_i)\phi_j(z_j)\phi_k(z_k)\rangle = \frac{C_{ijk}}{r_{ij}^a r_{jk}^b r_{ik}^c}, \qquad r_{ij} = z_i - z_j \tag{1.40}$$

with $a+b+c = h_i+h_j+h_k$. Invariance under special conformal transformations further implies $a = h_1+h_2-h_3$, $b = h_2+h_3-h_1$ and $c = h_1+h_3-h_2$, i.e.

$$\langle \phi_i(z_i)\phi_j(z_j)\phi_k(z_k)\rangle = \frac{C_{ijk}}{r_{ij}^{h_1+h_2-h_3} r_{jk}^{h_2+h_3-h_1} r_{ik}^{h_1+h_3-h_2}}. \tag{1.41}$$

It is no surprise that we need 4 operator insertions to get a non-trivial coordinate dependence because 3 points can always be fixed to, say, $z_1 = 0$, $z_2 = 1$ and $z_3 = \infty$ by a Möbius transformation. For N-point functions we hence expect a parametrization by an analytic function depending on $N-3$ independent complex coordinates. 4-point functions, for example, can be parametrized by

$$\langle \phi_1(z_1)\phi_2(z_2)\phi_3(z_3)\phi_4(z_4)\rangle = F(\tfrac{r_{12}\,r_{34}}{r_{13}\,r_{24}}) \prod_{i<j} (r_{ij})^{h/3-h_i-h_j} \tag{1.42}$$

with $h = \sum h_i$, where the Möbius invariant combination $\frac{r_{12}\,r_{34}}{r_{13}\,r_{24}} = 1 - \frac{r_{14}\,r_{23}}{r_{13}\,r_{24}}$ is called cross ratio or anharmonic ratio. Since we only used global conformal invariance analogous results are valied for conformal field theories in arbitrary dimensions: N-point functions with $N > 3$ can be shown to depend on $N(N-3)/2$ independent cross ratios $(r_{ij}r_{kl})/(r_{ik}r_{jl})$ [Gi89,DI97]. In two dimensions the number of independent functions is smaller because any 4 points have to lie on a common plane. Before we discuss the implications of the full conformal invariance in two dimensions we will first have to develope some powerful operator product techniques.

1.4 First order systems and ghosts

Except for the coordinate fields X^μ all fields of the superstring have a first order action of the form

$$S = \tfrac{1}{2\pi}\int d^2z\; (B\bar\partial C + \overline{B}\partial\overline{C}), \qquad T_{bc} = -hB\partial C + (1-h)\partial BC \tag{1.43}$$

where $h = h_B$ is the conformal weight of B and $h_C = 1-h$. For ghosts $(B,C) = (b,c)$ and superghosts $(B,C) = (\beta,\gamma)$ we thus have $h = 2$ and $h = 3/2$, respectively. For real free fermions $B = C = \psi^\mu$ there is an extra factor 1/2 in the action. The Euclidean convention for the normalization of $T(z)$ will be discussed presently.

For the first order (B,C) system in eq. (1.43) we thus obtain $C_m|0\rangle = B_n|0\rangle = 0$ for $m \geq h$ and $n \geq 1-h$ and consistency with the (anti)commutation relations

$$[C_m, B_n]_\pm = \delta_{m+n,0} \tag{1.44}$$

requires that the operators $C_{m\leq h-1}$ and $B_{n\leq -h}$ are creation operators. This fixes the definition of the Fock vacuum for single valued tensor fields. For fermions and for supersymmetry ghosts in the Ramond sector, however, the action of ψ_0^μ, β_{-1} and γ_1 is not yet specified. The complete definition of the Ramond vacua is therefore more subtle and will be discussed later.

In order to evaluate correlation functions with insertions of B and C we need to normal order the product $C(z)B(w) \equiv \mathcal{R}C(z)B(w)$. We first use a general ansatz for the normal ordering by declaring C_m and B_n to be annihilation operators for $m \geq q$ and $n \geq 1-q$ with $q-m \in \mathbb{Z}$, i.e. q is half-integral for spinors in the Neveu-Schwarz sector and integral for tensors and for spinors in the Ramond sector. With $Q \equiv q-h$ and $\theta_{z/w} \equiv \theta(|z|-|w|)$ we find

$$\begin{aligned}C(z)B(w) - {:}C(z)B(w){:} &= \theta_{z/w}C(z)B(w) + \theta_{w/z}(-)^{BC}B(w)C(z) - {:}C(z)B(w){:}\\ &= \theta_{z/w}\sum_{m=n\geq q}[C_m, B_{-n}]_\pm z^{h-1-m}w^{n-h} + (-)^{BC}\theta_{w/z}\sum_{m=n\leq q-1}[B_{-n}, C_m]_\pm z^{h-1-m}w^{n-h}\\ &= \theta_{z/w}\sum_{n\geq q}\tfrac{1}{z}(\tfrac{w}{z})^{n-h} - \theta_{w/z}\sum_{n\leq q-1}\tfrac{1}{w}(\tfrac{z}{w})^{h-n-1} = (\tfrac{w}{z})^Q\frac{1}{z-w}.\end{aligned} \tag{1.45}$$

If the fields $B(z)$ and $C(w)$ are single valued, i.e. for tensors and for spinors in the NS sector, we can take $Q=0$ and observe that the result obeys the cluster decomposition property, i.e. it goes to 0 if the distance between the positions of the operators goes to infinity. For other values of Q the correlations blow up at the origin and at infinity, which perfectly matches with our previous result for the vacuum state with 'nothing at the origin'. The states that are annihilated by C_m and B_n for $m \geq Q+h$ and $n \geq 1-Q-h$ are called Q-vacua. The normal ordering term $(\frac{w}{z})^Q/(z-w)$ is a Greens function of $\overline{\partial}$ for all values of Q, but with boundary conditions that do not obey cluster decomposition for $Q \neq 0$. Correlators involving B and C would thus blow up if one of the operators approaches the origin, which we interpret as a signal for the presence of an operator at $z=0$ that creates a state from the true vacuum. For the ghost system of the bosonic string $Q=1$ and $Q=2$ correspond to the down-vacuum $|\downarrow\rangle = |1\rangle$ and the up-vacuum $|\uparrow\rangle = |2\rangle$, respectively, and $|0\rangle = b_{-1}|\downarrow\rangle$. Note that the conformal vacuum $|Q=0\rangle$ is not a state with minimal energy $H = L_0 + \overline{L}_0$ for $h \neq 0$.

In the Ramond sector there are two possible 'vacua' $|Q = \pm\frac{1}{2}\rangle$ that obey cluster decomposition and one usually takes the symmetric superposition for the definition of the normal ordering subtraction

$$: C(z)B(w) :^{(\mathrm{R})} \;=\; C(z)B(w) - \tfrac{1}{2}\frac{\sqrt{\frac{z}{w}} + \sqrt{\frac{w}{z}}}{z-w} \tag{1.46}$$

In any case there is a singularity at the origin and the 'true vacuum' is the one in the NS sector. We can think of the Ramond vacua as being generated from $|0\rangle$ by some operator Σ which is called spin field and which generates a branch cut in the complex plane.

The Virasoro generators for the BC system (1.43) are easily evaluated from the Laurent expansions of B and C,

$$L_n = :\sum (r-n+nh)B_{n-r}C_r: , \qquad L_0 = \sum_{r\geq h} rB_{n-r}C_r + (-)^{BC}\sum_{r<h} rC_rB_{n-r} \tag{1.47}$$

Invariance of the vacuum under dilatations L_0 and special conformal transformations L_1 is now obvious. For translational invariance we observe that the only term without annihilators in the formula for L_{-1} has $r = h-1$ and its coefficient $r-n+nh$ vanishes.

Comparison of our result with the normal ordering contribution on the cylinder gives us an indirect possibility to determine the central charge. From the Casimir effect one gets a contribution $-(-)^{BC}\sum_{r\geq h} r$ to L_0 on the cylinder, whose renormalized (or ζ-regularized) value has to be compensated by the shift $c/24$ due to the conformal map to the complex plane. The generalized Riemann ζ-function is

$$\zeta(s,q) = \sum_{n=0}^{\infty}\frac{1}{(n+q)^s}, \qquad \zeta(-n,q) = -\frac{B_{n+1}(q)}{n+1}, \tag{1.48}$$

The Bernoulli polynomials $B_n(q)$ are defined by their generating function $\frac{te^{xt}}{e^t-1} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}$ and

$$B_2(x) = x^2 - x + \tfrac{1}{6}, \qquad \zeta(-1,q) = \tfrac{6q(1-q)-1}{12} \tag{1.49}$$

The Bernoulli numbers are $B_n = B_n(1)$ and with $B_2(1) = \frac{1}{6}$ and $B_2(\frac{1}{2}) = -\frac{1}{12}$ we recover our previous results for free bosons and free fermions. Putting the pieces together we find $c/24 = -(-)^{BC}\zeta(-1,h-1)$ and

$$\frac{c}{2} = (-)^{BC}(1-6h(1-h)) \tag{1.50}$$

We will soon derive this result regorously by a direct operator product computation. Inserting the values $h = 1/2,\ 3/2$ and 2 we obtain $c_\psi = 1/2$, $c_{bc} = -26$ and $c_{\beta\gamma} = 11$ (mind the factor 1/2 for a single Majorana fermion). The central charge of the superstring thus becomes $c = \frac{3}{2}D - 15$ and the critical dimension is 10, as was required by Lorentz invariance.

In our analysis we only considered meromorphic fields, i.e. periodic boundary conditions on the punctured plane. We already know that the Casimir energy is shifted by 1/8 in the

Ramond sector, so that the Ramond vacua are not scale invariant and the spin field $\Sigma(z)$ gets a contribution $h = 1/8$ from each Majorana fermion.

In Euclidean field theory there are different kinds of conjugations that one can define. Complex conjugation $z \to \bar{z}$ would exchange left- and right-movers and thus differs from its Minkowski space interpretation. It is therefore natural to define Hermitian conjugation of a conformal field $\phi(z) = \sum \phi_n/z^{n+h}$ by $\phi_n^\dagger = \phi_{-n}$, which is the action of complex conjugation on the Fourier modes of periodic fields on the Minkowski space cylinder. From a slightly different point of view, complex conjugation in Minkowski space reverses the sign of the Euclidean time and amounts to $z \to 1/\bar{z}$. To avoid an exchange of left- and right-movers we combine this with a Hermitian conjugation on the 'real surface' $z^\dagger = \bar{z}$ and define the *BPZ conjugation* [be84, zw93] of a conformal field by

$$\phi^T(z,\bar{z}) = z^{-2h}\bar{z}^{-2\bar{h}}\left(\phi(1/\bar{z},1/z)\right)^\dagger\Big|_{z^\dagger=\bar{z}}, \tag{1.51}$$

which implies $\phi^T_{mn} = (-)^{h+\bar{h}}\phi_{-n,-m}$ because $(\frac{-1}{\bar{z}^2})^h(\frac{-1}{z^2})^{\bar{h}}\phi(\frac{1}{\bar{z}},\frac{1}{z}) = \phi(\bar{z},z) = \sum \phi_{mn}\bar{z}^{-m-h}z^{-n-\bar{h}}$.

Given some Fock space basis of states $|\phi_r\rangle$ the conjugate states $\langle\phi_r^c|$ w.r.t. the canonical scalar product are defined by $\langle\phi_s^c|\phi_r\rangle = \delta_r^s$. In the ghost sector the conjugate of the SL(2) invariant vacuum $\langle 0^c|$ differs from the hermitian conjugate $\langle 0| \equiv (|0\rangle)^\dagger$ because $|0\rangle = b_{-1}|\downarrow\rangle$ implies $\langle\downarrow|b_1 = \langle 0|$ so that

$$1 = \langle\downarrow|\uparrow\rangle = \langle 0|c_{-1}c_0c_1|0\rangle \qquad \Rightarrow \qquad \langle 0^c| = \langle 0|c_{-1}c_0c_1 \tag{1.52}$$

A nonvanishing vacuum expectation value therefore requires a ghost number violation by 3 units (or 6 units, if we include right-movers), which is related via an index theorem to an anomaly in the conservation law of the ghost number current and to the number of conformal symmetries of the sphere (see below). For a general BC system the 'ghost number' $\#(C)-\#(B)$ is violated by $2h-1$ units and therefore is conserved exactly for the case $h=\frac{1}{2}$ of free fermions.

1.5 Operator-state correspondence and vertex operators

An important property of conformal field theory is the one-to-one correspondence between states and *local* operators [PO98],

$$\phi(z,\bar{z}) \qquad \longleftrightarrow \qquad |\phi\rangle = \lim_{z\to 0}\phi(z,\bar{z})|0\rangle. \tag{1.53}$$

Note that locality is crucial for the isomorphism: Adding a pure annihilation operator to ϕ would not change its value on the vacuum, but it can be shown that an operator without any creation part cannot be a local operator.

For free bosons it is convenient to work in momentum space for a number of reasons. The operator that corresponds to a momentum eigenstate $|k\rangle$ is the normal ordered exponential of X and it is called (the) vertex operator,

$$V_k(z,\overline{z}) = :e^{ikX(z,\overline{z})}:, \qquad |k\rangle = \lim_{z\to 0} V_k(z,\overline{z})|0\rangle \tag{1.54}$$

Since $e^{k_\mu p^\mu \log(z\bar{z})} = (z\bar{z})^{kp}$, the use of the momentum eigenstates

$$:e^{ikX(z,\bar{z})}: \;=\; e^{ik_\mu \sum\limits_{n<0} \frac{i}{n}\alpha_n^\mu z^{-n} + a.h.} e^{ik_\mu x^\mu} e^{k_\mu p^\mu \log(z\bar{z})} e^{ik_\mu \sum\limits_{n>0} \frac{i}{n}\alpha_n^\mu z^{-n} + a.h.} \tag{1.55}$$

($a.h.$ denotes the antiholomporphic contributions) instead of the coordinate functions turns logarithmic short distance singularites into poles. At the massless level of closed strings

$$h_{\mu\nu} :\partial X^\mu \bar{\partial} X^\nu e^{ikX}(z,\bar{z}): \qquad \longleftrightarrow \qquad h_{\mu\nu}\alpha_{-1}^\mu \tilde{\alpha}_{-1}^\nu |k\rangle \tag{1.56}$$

is the vertex operator that creates an incoming asymptotic state with polarization $h_{\mu\nu}$.

Vertex operators are composite operators, and we need the *Wick theorem* for the evaluation of singularities in operator products. For free fields we therefore define the contraction $\underbrace{A(z)B(w)} = \mathcal{R}\,A(z)B(w) \;-\; $:A(z)B(w):. Then the radially ordered product can be written as a sum over normal ordered expressions with all possible contractions like

$$\begin{aligned} \mathcal{R}\,ABCD \;\;&=\;\; :ABCD: + \underbrace{AB} :CD: + \underbrace{AC} :BD: + \underbrace{AD} :BC: + \underbrace{BC} :AD: \\ &\qquad + \underbrace{BD} :AC: + \underbrace{CD} :AB: + \underbrace{AB}\underbrace{CD} + \underbrace{AC}\underbrace{BD} + \underbrace{AD}\underbrace{BC} \end{aligned} \tag{1.57}$$

with appropriate signs for fermions. For radially ordered products of composite local operators contractions within normal ordered expressions have to be omitted:

$$\begin{aligned} &\mathcal{R}:AB::CD: \;-\; :ABCD: \\ &\quad = \underbrace{AC} :BD: + \underbrace{AD} :BC: + \underbrace{BC} :AD: + \underbrace{BD} :AC: + \underbrace{AC}\underbrace{BD} + \underbrace{AD}\underbrace{BC}. \end{aligned} \tag{1.58}$$

In the following applications the radial ordering symbols $\mathcal{R}$ will often not be written explicitly. In order evaluate the operator product of Vertex operators we use the series expansion

$$\begin{aligned} :e^{A(z)}::e^{B(w)}: \;&=\; \sum_{m,n\geq 0} \tfrac{1}{m!\,n!} \;:A^m(z)::B^n(w): \\ &= \sum_{0\leq l\leq m,n} \tfrac{l!}{m!n!}\tbinom{m}{l}\tbinom{n}{l}\big(\underbrace{A(z)B(w)}\big)^l :A^{m-l}(z)B^{n-l}(w): \;\; = e^{\overbrace{A(z)B(w)}} :e^{A(z)}e^{B(w)}:, \end{aligned} \tag{1.59}$$

and insert the contraction $\underbrace{X(z,\bar{z})^\mu X(w,\bar{w})^\nu} = -\delta^{\mu\nu}\log((z-w)(\bar{z}-\bar{w}))$ to arrive at

$$V_k(z,\bar{z})V_q(w,\bar{w}) = |z-w|^{2kq} :e^{ikX(z,\bar{z})+iqX(w,\bar{w})}: \tag{1.60}$$

Momentum conservation in 2-point functions implies that $\langle 0|V_{-k}$ and $V_k|0\rangle$ are conjugate states, with conformal dimensions $h(V_{\pm k}) = k^2/2$. With the contraction $\underbrace{\partial X^\mu(z) X^\nu(w)} = \frac{-\delta^{\mu\nu}}{z-w}$ we find

$$\partial X^\mu(z) V_k(w,\bar w) \;-\; :\partial X^\mu(z) V_k(w,\bar w): \;=\; \frac{-ik^\mu}{z-w}\, V_k(w), \tag{1.61}$$

and $\partial V_k(z,\bar z) = ik_\mu :\partial X^\mu V_k: (z,\bar z)$.

In string theory we have to be more restrictive because only BRST-invariant states and operators have a physical interpretation. Since $|\downarrow\rangle = c_1|0\rangle = \lim_{z\to 0} c(z)|0\rangle$ the physical states that are built on the down vacuum $\prod(\alpha^\dagger_{n_i})\,|\downarrow\rangle$ actually correspond to operators with ghost number one,

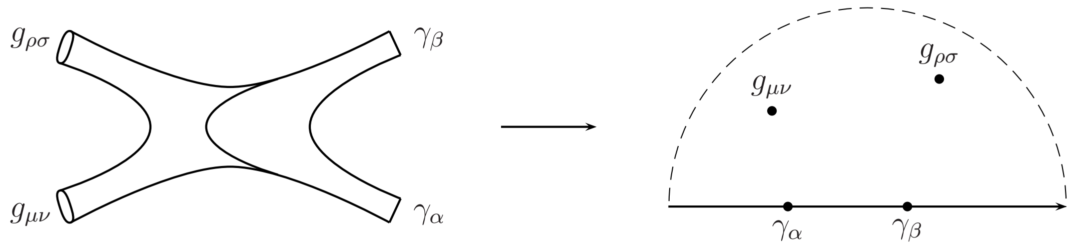
$$c(z)\mathcal{O}(z), \qquad \mathcal{O}(z) = :P(\partial X, \partial^2 X, \ldots) V_k(z): \tag{1.62}$$

where $P(\partial X, \partial^2 X, \ldots)$ is a polynomial in the derivatives of X^μ (to avoid clumsy notation we suppress anti-holomorphic dependencies). Since X^μ commutes with the ghosts no normal ordering is required. If we assume that $\mathcal{O}(z)$ is a primary field of conformal weight h, i.e. it transforms as a tensor $[Q,\mathcal{O}] = \mathcal{L}_c\mathcal{O} = c\partial\mathcal{O} + h\partial c\mathcal{O}$, then BRST-invariance requires

$$[Q, c(z)\mathcal{O}(z)] = c\partial c(z)\mathcal{O}(z) - c(z)\Big(c\partial\,\mathcal{O}(z) + h\partial c\,\mathcal{O}(z)\Big) = 0 \qquad \Leftrightarrow \qquad h = 1. \tag{1.63}$$

No normal ordering is necessary in the ghost sector as long as there are no antighosts. For the tachyon vertex operator $c(z)V_k(z)$ BRST-invariance thus implies the on-shell condition $k^2 = 2 = -m^2$ for the Euclidean momentum. Physical gravitions $t_{\mu\nu}\partial X^\mu \overline{\partial} X^\nu V_k$ and photons $: t_m \dot X^\mu V_k :$ have to be massless $k^2 = 0$ with transversal polarization $tk = 0$. We already know this from our analysis of the physical state condition $Q(\mathcal{O}|0\rangle) = 0$, and we will soon confirm this in the operator language because we will see that the normal ordered photon vertex transforms as a *primary field only if* $tk = 0$.

String scattering amplitudes correspond to a sum over all (gauge equivalence classes of) surfaces connecting the asymptotic states. For tree level amplitudes the worldsheet can be mapped conformally either to the sphere for closed strings and to upper half plane (which is more convenient for calculations but equivalent to the disk) for open strings. Asymptotic states are thereby mapped to punctures with vertex operator insertion where the vertex operator carries the quantum numbers (momentum, spin, polarization) of the corresponding particles. In the open string case the punctures are on the boundary, but since closed strings states are always required by unitarity (they can be regarded as bound states of open strings) one can always insert additional closed string vertex operators in the bulk (i.e. in the interior of the upper half plane). The string diagram for photon-graviton scattering, for example, is the upper half plane with two photon vertices $\dot X V_k = (\partial X + \overline{\partial} X)V_k$ on the real line and two graviton vertices with $\operatorname{Im} z_i > 0$.



Since all metrics are conformally equivalent on the sphere and on the disk the integral over world sheet geometries reduces to an integration over the positions of the operator insertions, some of which can be fixed by global conformal transformations. Classical gauge invariance thus translates into the independence of physical correlations of such a choice.

As a first example we consider the three tachyon vertex, which corresponds to the correlation function

$$\langle 0|\, cV_{k_1}(z_1)\, cV_{k_2}(z_2)\, cV_{k_3}(z_3)\, |0\rangle. \tag{1.64}$$

where a choice of z_i selects a unique Möbius transformation. This expression can either be interpreted as open string tachyon scattering or as the chiral factor for closed strings. In the former case the z_i are on the real axis and we have to sum over two inequivalent orderings like $z_1 < z_2 < z_3$ and $z_2 < z_1 < z_3$.[4] The correlation function of the vertex operators is easily evaluated using the Wick theorem,

$$\langle V_{k_1}(z_1)\ldots V_{k_n}(z_n)\rangle = (2\pi)^D\delta^D(k_1+\ldots k_n)\prod_{i<j}(z_i-z_j)^{k_ik_j}\prod_{i<j}(\bar z_i-\bar z_j)^{k_ik_j} \tag{1.65}$$

where the vacuum expectation value (VEV) referes to the vacuum of the free boson sector. The correlation function factorizes and its holomorphic structure is essentially determined by the short distance singularities: For the 3-point function the on-shell conditions imply $k_i^2 = 2$, hence

$$k_ik_j = \tfrac{1}{2}((k_i+k_j)^2 - k_i^2 - k_j^2) = -1, \qquad 1 \le i < j \le 3. \tag{1.66}$$

so that the kinematics leads to negative scalar products k_ik_j. The result still has to be multiplied with the ghost contribution.

It is not a coincidence that the number of ghost insertions is exactly what we need to compensate the anomaly in the ghost number current. Their contribution is the Vandermonde determinant

$$\langle 0|c(z_1)c(z_2)c(z_3)|0\rangle = \begin{vmatrix} 1 & 1 & 1 \\ z_3 & z_2 & z_1 \\ z_3^2 & z_2^2 & z_1^2 \end{vmatrix} = (z_1-z_2)(z_1-z_3)(z_2-z_3) \tag{1.67}$$

[4]For explicit calculations it is usually better to work with the upper half plane with one of the open string vertex operators inserted at infinity, but cyclical equivalence of orderings is easier to see if we map the world sheet to the disk.

because $c(z) = \ldots + c_1 + c_0 z + c_{-1}z^2 + \ldots$ and $\langle \frac{1}{2}\partial^2 c\partial c c\rangle = \langle c_{-1}c_0c_1\rangle = 1$. This factor can be interpreted as a finite Faddeev Popov determinant coming from the gauge fixing of the $SL(2,\mathbb{C})$ transformations due to the choice of z_i. Paramtetrizing $z \to z' = \frac{az+b}{cz+d}$ with $ad-bc=1$ by $a=1+\alpha/2$, $b=\beta$, $c=\gamma$, $d=1-\alpha/2$ the infinitesimal transformation becomes $z' \sim z+\beta+\alpha z-\gamma z^2$ and

$$\left|\frac{\partial(z_i,z_j,z_k)}{\partial(\alpha,\beta,\gamma)}\right| = (z_i-z_j)(z_j-z_k)(z_k-z_i) \tag{1.68}$$

is the functinonal derterminant for fixing the positions z_i, z_j and z_k. As it should be, the ghost contribution removes the coordinate dependence of the tachyon vertex. But only on-shell amplitudes are coordinate independent in string theory. The choice of the positions of the insertions can be interpreted as a gauge condition. We are thus reminded of the well-known fact that also in QFT only S-matrix elements are gauge independent.

The evaluation of the photon-tachyon vertex can be found in section 6.5 of [PO98] and we next turn to the three gauge boson vertex. We thus consider three massless vertex operators with polarization tensors $t^{(i)}_{\mu_i}$ and the correlation function

$$t^{(1)}_{\mu_1}\,t^{(2)}_{\mu_2}\,t^{(3)}_{\mu_3}\;\langle 0|\,{:}\partial X^{\mu_1}V_{k_1}(z_1){:}\,{:}\partial X^{\mu_2}V_{k_2}(z_2){:}\,{:}\partial X^{\mu_3}V_{k_3}(z_3){:}\,|0\rangle \tag{1.69}$$

The result can also be interpreted as the holomorphic contribution to the three graviton vertex in the closed string context. Since any surviving ∂X in a normal ordered expression annihilates the amplitude we only have to consider contractions that include all three ∂X. The overall factor $\prod_{i<j}(z_i-z_j)^{k_ik_j}$ can be dropped on shell because the kinematics now implies $k_ik_j=0$. We thus find a contribution with contractions of derivative terms $\underbrace{\partial X^{\mu_i}\partial X^{\mu_j}} = -\delta^{\mu_i\mu_j}/(z_i-z_j)^2$,

$$\sum t^{(1)}_{\mu_1}\,t^{(2)}_{\mu_2}\,t^{(3)}_{\mu_3}\;\frac{-\delta^{\mu_1\mu_2}}{(z_1-z_2)^2}\left(\frac{-ik_1^{\mu_3}}{(z_3-z_1)}+\frac{-ik_2^{\mu_3}}{(z_3-z_2)}\right), \tag{1.70}$$

and two types of terms that only involve the contractions $\underbrace{\partial X^{\mu_i}(z_i)V_{k_j}(z_j)} = -ik_j^{\mu_i}V_{k_j}/(z_i-z_j)$. If two contractions involve the same vertex we obtain

$$\sum \frac{-ik_3^{\mu_1}}{z_1-z_3}\frac{-ik_3^{\mu_2}}{z_2-z_3}\left(\frac{-ik_1^{\mu_3}}{z_3-z_1}+\frac{-ik_2^{\mu_3}}{z_3-z_1}\right) \tag{1.71}$$

and if all contractions go to different exponentials

$$\frac{-ik_1^{\mu_3}}{z_3-z_1}\frac{-ik_2^{\mu_1}}{z_1-z_2}\frac{-ik_3^{\mu_2}}{z_2-z_3}+\frac{-ik_2^{\mu_3}}{z_3-z_2}\frac{-ik_3^{\mu_1}}{z_1-z_3}\frac{-ik_1^{\mu_2}}{z_2-z_1}. \tag{1.72}$$

Defining $k^\mu_{ij} = \frac{1}{2}(k^\mu_i - k^\mu_j)$ and using transversality of the polarizations $t^{(i)}_\mu k^\mu_i = 0$, momentum conservation we observe $k_1^{\mu_3} = -k_2^{\mu_3} = k_{12}^{\mu_3}$. Since $\frac{1}{z_3-z_1} - \frac{1}{z_3-z_2} = \frac{z_1-z_2}{(z_3-z_1)(z_3-z_2)}$ the first contribution (1.70) yields the well-known 3-gluon vertex

$$i(\delta^{\mu_1\mu_2}k_{12}^{\mu_3}+\delta^{\mu_2\mu_3}k_{23}^{\mu_1}+\delta^{\mu_3\mu_1}k_{31}^{\mu_2}) \tag{1.73}$$

where the z-dependence again is cancelled by the ghost contribution. For the second type of contributions (1.71) we get an overall factor $-ik_{12}^{\mu_3}k_{23}^{\mu_1}k_{31}^{\mu_2}$ and a z-dependent expression

$$\sum \frac{1}{z_1-z_3}\frac{1}{z_2-z_3}\left(\frac{1}{z_3-z_1}-\frac{1}{z_3-z_2}\right)=\sum \frac{z_1-z_2}{(z_1-z_3)^2(z_2-z_3)^2}=\frac{\sum(z_1-z_2)^3}{(\prod_{i<j}(z_i-z_j))^2}. \quad (1.74)$$

But $\sum(z_1-z_2)^3=3\sum(z_1z_2^2-z_1^2z_2)$ is -3 times the Vandermonde determinant of the ghost contribution so that the contribution of (1.71) becomes $3ik_{12}^{\mu_3}k_{23}^{\mu_1}k_{31}^{\mu_2}$. Together with (1.72) the complete result becomes

$$i(k_{23}^{\mu_1}\delta^{\mu_2\mu_3}+k_{31}^{\mu_2}\delta^{\mu_3\mu_1}+k_{23}^{\mu_1}\delta^{\mu_2\mu_3})+i\tfrac{\alpha'}{2}k_{23}^{\mu_1}k_{31}^{\mu_2}k_{12}^{\mu_3} \quad (1.75)$$

times an overall kinematical factor $g_s(2\pi)^D\delta^D(k_1+k_2+k_3)$ (the coupling constant g_s appears linearly because of the Euler number of the disk. We thus find the gauge invariant renormalizable interaction term for gauge bosons plus an additional non-renormalizable, interaction for which we reinserted $\alpha'=2$ in order to display the mass dimension.

For closed strings we have to multiply our result with the contribution from the right-movers to obtain the 3-graviton vertex (plus dilaton and B-field interaction terms). For open strings we have to add the graph contribution with reversed cyclic ordering of the vertices. Since the 3-point function is antisymmetric the complete result vanishes. This should not come as a surprise: There is no 3-photon vertex because the photon has no charge. In open string theory there is, however, a simple way to introduce non-abelien gauge fields that has been known since the time when dual models were designed as a model for strong interactions. And this is even more natrual if we think about the colour strings of QCD: The end-points of open strings may carry a charge, which means that we asign to them so-called Chan Paton labels i and j. An open string state is therefore labelled by $|k,ij\rangle$ (possibly with additional quantum number) and can be expanded in some matrix basis T_{ij}^a. The corresponding vertex operator $\mathcal{O}_{ij}=\mathcal{O}_aT_{ij}^a$ has to be inserted at the boundary of the world sheet. We can think of the Chan Paton labels as labels for different boundary conditions with the open string vertex operators changing the boundary condition from i to j. An open string n-point function thus obtains an additional trace factor $\mathrm{tr}(T^{a_1}\ldots T^{a_n})$ that is due to the sum over all possible boundary conditions for the boundary segments that join the vertex operators. For the 3-point function the sum over cyclic orderings, together with the antisymmetry of (1.75), yields a nonvanishing result that is proportional to the structure constants $\mathrm{tr}\,T^{a_1}[T^{a_2},T^{a_3}]\sim f^{a_1a_2a_3}$. We thus recover the correct Yang-Mills interaction (plus a nonrenormalizable $\mathrm{tr}\,F^3$-interaction that should be reproduced by the non-abelian version of the Born-Infeld effective action).

If we think about the Chan Paton degrees of freedom as quarks sitting at the ends of the string then i and j should label the fundamental and complex conjugate representations of the gauge group $U(n)$. The ends of the strings thus are different and they can only join and split in an orientable way. If we consider unoriented strings then i and j must belong to the same representation and the orientation reversion operator $\Omega : X(\sigma) \to X(\pi - \sigma)$ has to act in an appropriate way on the matrices T^a_{ij}. It can be shown that the only consistent possibilities correspond to the orthogonal or the symplectic group, which are the possible gauge groups for unoriented strings [PO98]. Additional consistensy condidtions come from loop amplitudes: Anomaly cancellation for $D = 10$ $N = 1$ super Yang-Mills theory is know to work only for certain gauge groups with rank 16 and dimension 496. While $E_8 \times E_8$, which is realized by the so-called heterotic string, is also a solution, the only suitable Chan-Paton gauge group is $SO(32)$. Type I string theory, which is the unique consistent open string theory in 10 dimensions, thus has unoriented strings and gauge group $SO(32)$. This can also directly be derived from string theory by requiring cancellations of IR divergences among loop amplitudes like the annulus and the Möbius strip.

For amplitudes with more than 3 particles we expect an integration over the world sheet and the additional vertex insertions must not increase the ghost number because otherwise the correlation function would be zero. An additional possibile insertion would therefore be a so-called integrated vertex, whose BRST variation is an integral of a total derivative,

$$Q \int d^2z \; \mathcal{O}(z, \bar{z}) = 0, \qquad s\mathcal{O}^{(2)} = -d\mathcal{O}^{(1)} \tag{1.76}$$

where $\mathcal{O}^{(2)} = d^2z\mathcal{O}$ is a two-form whose BRST variation we denote by s. Since $s^2 = \{s, d\} = 0$ the Poincaré lemma implies that $s\mathcal{O}^{(1)} = -d\mathcal{O}^{(0)}$ is again a total derivative. This chain of descent equations

$$s\mathcal{O}^{(n)} + d\mathcal{O}^{(n-1)} = 0 \tag{1.77}$$

clearly terminates at form degree zero: For $s\mathcal{O}^{(0)}$ the Poincaré lemma would allow a constant r.h.s., which, however, has to be vanish in the field theory context. From our solution of the BRST cohomology we already know that all solution are given by the on-shell vertex operators $c\bar{c}\mathcal{O}$ with primary fields $\mathcal{O}$ of conformal weight $(1, 1)$. Since the generator $L_{-1} = \{b_{-1}, Q\}$ of translations is BRST–exact, b_{-1} can be used as a homotopy to integrate the descent equations (such a homotopy exists in every generally coordinate invariant theory [br90]). Introducing formal fermionic coordinates $\theta = dz$ and $\bar{\theta} = d\bar{z}$ the solution to the descent equations can now be written as a superfield

$$\mathcal{O}(z, \bar{z}, \theta, \bar{\theta}) = e^{\theta b_{-1} + \bar{\theta}\bar{b}_{-1}} \mathcal{O}^{(0)} = \mathcal{O}^{(0)} + \theta\mathcal{O}^{(1,0)} + \bar{\theta}\mathcal{O}^{(0,1)} + \theta\bar{\theta}\mathcal{O}^{(2)}, \tag{1.78}$$

(with $(s + d)\mathcal{O}(z, \bar{z}, \theta, \bar{\theta}) = 0$) or, in more mathematical terms, as an element of the exterior algebra. The integrated Vertex thus becomes

$$\mathcal{O}^{(2)} = d^2z \; b_{-1}\bar{b}_{-1}(c\bar{c}\mathcal{O}(z, \bar{z})). \tag{1.79}$$

Focusing on the holomorphic part we can insert the contour integral for b_{-1} to obtain

$$b_{-1}(c(z)\mathcal{O}(z)) = \oint \frac{dw}{2\pi i} b(w)c(z)\mathcal{O}(z) = \mathcal{O}(z). \tag{1.80}$$

Tree level amplitudes are therefore obtained by inserting 3 BRST-invariant vertex operators $c(z)\mathcal{O}(z)$ and BRST-invariant integrals $\int dz\mathcal{O}(z)$ for the remaining external legs. Since the amplitude is invariant under global conformal transformations we are free to fix 3 positions of the insertion to some arbitrary values.

For the simplest example of tachyon scattering, with $\mathcal{O} = V_k$ and $k^2 = 2$, the resulting n-point function

$$\langle \prod_{i=1}^{3} c(z_i)V_{k_i}(z_i) \prod_{i=4}^{n} V_{k_i}(z_i)\rangle \tag{1.81}$$

is proportional to

$$\prod_{i<j\le 3} (z_i - z_j) \prod_{i<j} (z_i - z_j)^{k_i k_j} \delta(\sum k_i), \tag{1.82}$$

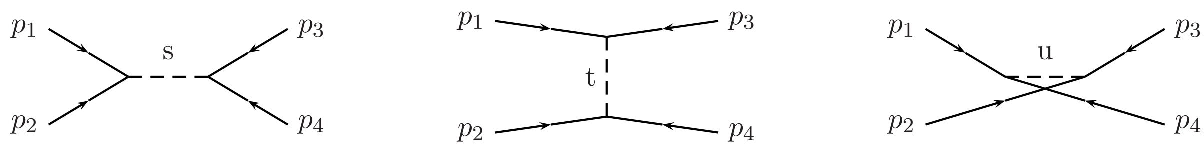
where the first factor comes from the ghost insertions. Fixing $z_1 = 0$, $z_2 = 1$, $z_4 = \infty$ and using the on-shell conditions we thus obtain the Virasoro–Shapiro amplitude

$$A = \int d^2z |1-z|^{2p_2p_3} |z|^{2p_3p_1} \tag{1.83}$$

for the scattering of two tachyons. For open strings the same calculation leads to the Veneziano amplitude

$$A = \int_0^1 dz (1-z)^{p_2p_3} z^{p_3p_1} \qquad + \qquad \text{non-cyclic permutations} \tag{1.84}$$

The on-shell kinematics of any 4-point function is a function of two of the three relativistic invariants $s = (p_1+p_2)^2$, $t = (p_1+p_3)^2$ and $u = (p_1+p_4)^2$, which are constrained by $s+t+u = -\sum m_i^2$ with $m_i^2 = -p_i^2$ in our Euclidean conventions. (This easily follows from momentum conservation $p_4^2 = (p_1+p_2+p_3)^2$, where we take all momenta as incomming for simplicity.) The kinematical invariants s, t, and u correspond to the energy of the exchanged particle in the s, t, and u (crossed) channel, respectively



The duality hypothesis, which lead to the discovery of string theory (then called 'dual models'), states that s, t and u channel contributions alone should yield the same scattering amplitude $A(s,t,u)$ via analytic continuation. If we follow Veneziano and Virosoro and postulate a dual

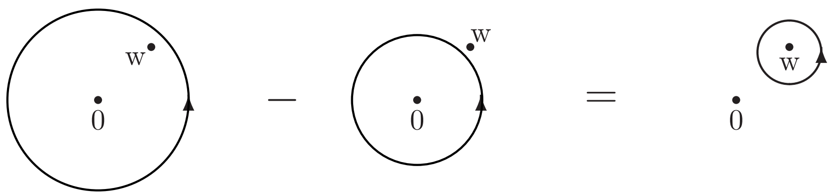


Fig. 3: Commutators and contour integration

amplitude, then the particle masses are recovered by the positions of the poles due to the exchange of on-shell particles in the respective channels. A string diagram clearly has duality built in. For a CFT that is not defined via an action integral over the world sheet but abstractly in terms of the operator algebra duality (called crossing symmetry in that context) is an important constraint.

1.6 Operator product expansion

For a conformal field theory with meromorphic quantum fields $\mathcal{O}_i(z)$ and conformal weights h_i we expect that radially ordered operator products can be expanded into a Laurent series

$$\mathcal{O}_i(z)\mathcal{O}_j(w) = \sum_k (z-w)^{h_k-h_i-h_j} \mathcal{C}_{ij}{}^k \mathcal{O}_k(w). \tag{1.85}$$

In general conformal fields depend on z and $\bar{z}$ and the operator product expansion (OPE) has a more complicated form, but we will mostly suppress antiholomorphic dependencies if they are not essential. Moreover, as we have seen in the calculation of the 2-point correlation, radial ordering is essential for obtaining well-defined analytic short distance singularities and we should think of the expansions as inserted into expectation values. With these caveats in mind, we can use these expansions as powerful computational tools: We will see that the full mode algebra is encoded in the short distance singularities. Deformation of integration contours thus enables simple and rigorous manipulations of infinite sums.

Consider, for example, a conformal tensor field $\phi(z)$ of weight h. The conserved quantities $T_\xi = \oint \frac{dz}{2\pi i}\, \xi(z)T(z)$ generate infinitesimal conformal transformations $z' = z+\xi(z)$ via the equal time commutator with ϕ,

$$[T_\xi, \phi(w)] = \oint \frac{dz}{2\pi i} \xi(z)[T(z), \phi(w)] = \delta_\xi \phi(w) = \xi\partial\phi + h\partial\xi\phi. \tag{1.86}$$

Since lines of equal time correspond to circles around the origin and as integration contours can be deformed as long as no singularities are encountered we can express the commutator in terms of a contour integral as shown in Fig. 3:

$$\oint \frac{dz}{2\pi i}[\xi(z)T(z),\phi(w)] = \oint_{|z-w|=\varepsilon} \frac{dz}{2\pi i}\,\xi(z)\mathcal{R}T(z)\phi(w). \tag{1.87}$$

Comparing the last two equations and expanding $\xi(z)$ around w we conclude that the short distance singularity of the OPE $\mathcal{R}T(z)\phi(w)$ must be given by

$$T(z)\phi(w) = \frac{h\phi(w)}{(z-w)^2} + \frac{\partial\phi(w)}{z-w} + \text{regular terms} \tag{1.88}$$

We will usually omit the radial ordering symbol and the symbol $\sim$ will mean equality up to regular terms. In order to obtain the OPE of $T(z)$ with itself we recall the conformal transformation (1.35). The non-tensorial contribution $\frac{c}{12}\partial^3\xi$ corresponds to a forth order pole in the OPE and we conclude that

$$T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}, \tag{1.89}$$

where the relative factor $2/12 = 1/(3!)$ in the anomalous term comes from the Taylor expansion of $\xi(z)$ around w.

The operator product expansion provides an efficient tool to define a normal ordered product also for interacting and for composite fields. Consider the product of local fields $A(z)$ and $B(w)$,

$$A(z)B(w) = \sum_{n=-n_0}^{\infty} [AB]_{-n}(w)(z-w)^n, \tag{1.90}$$

where the locality axiom assumes that the pole order is bounded by some (integer) number n_0. The singular part of this expansion is called the 'contraction' of A and B [FU92, bo93]:

$$\underbrace{A(z)B(w)} = \sum_{n=1}^{n_0} \frac{[AB]_n(w)}{(z-w)^n} \tag{1.91}$$

We further assume that the operator algebra is associative and closed in the sense that all coefficients $[AB]_n$ of the OPE, as well as the derivatives of all operators, belong to the algebra.

Now the normal ordered product (NOP) can be defined by subtracting the singularity,

$$[AB](w) \equiv\ :A(w)B(w):\ := \lim_{z\to w}\left(A(z)B(w) - \underbrace{A(z)B(w)}\right) = \oint_{|z-w|=\varepsilon} \frac{dz}{2\pi i}\frac{A(z)B(w)}{z-w}, \tag{1.92}$$

i.e. $[AB] = [AB]_0$. This method of defining a finite part is also called 'point splitting', because we first seperate the positions of the fields A and B by a small distance ε and then take the regular part of the operator product in the limit $\varepsilon \to 0$. In terms of modes this means

$$[AB](w) = \sum_n C_n w^{-n-h_A-h_B}, \qquad C_m = \sum_{n\le -h_A} A_n B_{m-n} + \sum_{n>-h_A} B_{m-n}A_n \tag{1.93}$$

as can be seen by inserting $A(z) = \sum A_n z^{-n-h_A}$ and $B(w) = \sum B_n w^{-n-h_B}$ and deforming the integration contour into the difference between the two circles $|z| = |w| \pm \varepsilon$ as in Fig. 3. Then $1/(z-w)$ has a convergent expansion in w/z and in z/w, respectively. In the first integral the lower limit $0 \leq n + h_A$ on n arises from the requirement that the pole orders must not be too high to produce a residue; in the second integral the condition is that we need a pole to get a non-zero contribution, which is the case for $n + h_A \geq 1$. For free fields we recover our previous definition of contraction and normal ordering. For general local fields the value on n for which A_n has to be put to the right agrees with what we found for annihilators of the translation invariant vacuum in (1.37). The NOP (1.93) is, however, not commutative: $[BA] \neq [AB]$. Obviously, the non-commutativity comes from the expansion of the operator product at w rather than at $\sqrt{zw}$. We can derive a formula that expresses $[BA]_n - [AB]_n$ in terms of derivatives:

$$[BA]_n = \sum_{l=0}^{n_0-n} \frac{(-1)^{n+l}}{l!}\, \partial^l [AB]_{n+l}. \tag{1.94}$$

Proof: Expand $RB(z)A(w) = \sum [BA]_n(w)(z-w)^{-n} = RA(w)B(z) = \sum [AB]_n(z)(w-z)^{-n}$ at w. $\square$

Note that the NOP is commutative if the contraction of A and B is a $\mathbb{C}$-number!

In general the NOP is also not associative. But the non-associativity problem can be administrated nicely with the **rearrangement lemma**

$$[A[BC]] - [[AB]C] = [B[AC]] - [[BA]C]. \tag{1.95}$$

Proof: We insert the partial fraction decomposition of $\frac{1}{x-z}\frac{1}{y-z}$ into the definition of $[A[BC]]$,

$$[A[BC]](z) = \int\limits_{|x-z|=2\varepsilon} \frac{dx}{2\pi i}\frac{A(x)}{x-z} \int\limits_{|y-z|=\varepsilon} \frac{dy}{2\pi i}\frac{B(y)C(z)}{y-z} = \iint\limits_{|x-z|>|y-z|} \frac{dx\,dy}{(2\pi i)^2}\left(\frac{1}{x-z} - \frac{1}{y-z}\right)\frac{A(x)B(y)C(z)}{y-x}. \tag{1.96}$$

For $[[AB]C]$ we deform the integral of x around y into the difference of two contours around z,

$$[[AB]C](z) = \int\limits_{|y-z|=2\varepsilon} \frac{dy}{2\pi i} \int\limits_{|x-y|=\varepsilon} \frac{dx}{2\pi i}\frac{A(x)B(y)}{x-y}\frac{C(z)}{y-z} = \Bigg(\iint\limits_{|x-z|>|y-z|} \frac{dx\,dy}{(2\pi i)^2} - \iint\limits_{|x-z|<|y-z|} \frac{dx\,dy}{(2\pi i)^2}\Bigg)\frac{A(x)B(y)C(z)}{(x-y)(y-z)}. \tag{1.97}$$

Then the difference $[A[BC]] - [[AB]C]$ is symmeytric under the exchange $A(x) \leftrightarrow B(y)$. $\square$

To memorize eq.(1.95) observe that the product of a commutator with another operator is associative. The commutator has to be on the l.h.s.; the field on the right cannot be moved away. For more than two fields we fix a default ordering by the recursive definition

$$[A_1 A_2 \ldots A_n] := [A_1[A_2 \ldots A_n]]. \tag{1.98}$$

Note that the contraction operation commutes with differentiation

$$\underbrace{\partial A(z)B(w)} = \partial_z \underbrace{A(z)B(w)}, \qquad \underbrace{A(z)\partial B(w)} = \partial_w \underbrace{A(z)B(w)}, \tag{1.99}$$

so that the first order pole in such contraction vanishes: $[\partial AB]_1 = [A\partial B]_1 = 0$. It is also easy to check that the Leibniz rule is valid for NOPs:

$$\partial[AB] = [\partial AB] + [A\partial B] \tag{1.100}$$

(inserting into the definition of $\partial[AB]$, the term $[\partial AB]$ arisies after partial integration).

With our formalism we can avoid manipulations with infinite normal ordered sums or with (operator valued) distributions by encoding everything in OPEs of (operator valued) meromorphic fields. To see explicitly how the OPE encodes the δ-functions consider the equal time anti-commutator

$$\{b(z), c(w)\} = \sum_{m,n} z^{n-2} w^{1-m} \{b_{-n}, c_m\} = \frac{1}{z} \sum_{n\in\mathbb{Z}} \left(\frac{z}{w}\right)^n \tag{1.101}$$

with $z = \exp(t + i\sigma)$ and $w = \exp(t + i\sigma')$. The sum on the r.h.s. of this equation is the Fourier representation of $\delta(\sigma - \sigma')$ (the factor $1/z$ comes from the transformation to the complex plane). More generally, the OPE poles in an expansion

$$\mathcal{O}_i(z)\mathcal{O}_j(w) = \sum_k \frac{C_{ij}{}^k}{(z-w)^\Delta} \mathcal{O}_k(w), \qquad \Delta = h_i + h_j - h_k \tag{1.102}$$

can be related to the equal time commutator by exploiting the radial ordering of the operator product. With $z_\pm = e^{t+i\sigma\pm\varepsilon}$ and $w = e^{t+i\sigma'}$ we thus find

$$[\hat{\mathcal{O}}_i(t+i\sigma), \hat{\mathcal{O}}_j(t+i\sigma')] \sim \lim_{\varepsilon\to 0} \left(\frac{(z_+/w)^{h_k - h_j}}{(1 - w/z_+)^\Delta} - \frac{(z_-/w)^{h_k - h_j}}{(1 - w/z_-)^\Delta} \right) i^\Delta C_{ij}{}^k \hat{\mathcal{O}}_k(t + i\sigma') \tag{1.103}$$

where we define $\hat{\mathcal{O}}_l(\sigma^+) := (iz)^{h_l}\mathcal{O}_l(z)$, which coincides with $\mathcal{O}_l(\sigma^+)$ for primary fields with our convention $z = e^{i\sigma^+}$. The "$\sim$"–symbol is used because the time ordering in the limit $\varepsilon \to 0$ will turn out to be somewhat delicate. Since $1 - w/z_\pm \approx i(\sigma - \sigma' \mp i\varepsilon)$ the formula

$$\lim_{\varepsilon\to 0} \left(\frac{1}{(x+i\varepsilon)^n} - \frac{1}{(x-i\varepsilon)^n} \right) = 2\pi i \frac{(-)^n}{(n-1)!} \partial_x^{n-1}\delta(x) \tag{1.104}$$

shows that the leading singularity in the coefficient of $C_{ij}{}^k\mathcal{O}_k(\sigma')$ is $-2\pi i\delta^{(\Delta-1)}(\sigma-\sigma')/(\Delta-1)!$. But for $\Delta > 1$ the limit $\varepsilon \to 0$ has to be evaluated more carefully. We thus define

$$\delta_{(\Delta,h_i)}(\sigma - \sigma') := \tfrac{(\Delta-1)!}{2\pi i^{1-\Delta}} \lim_{\varepsilon\to 0} \left(\tfrac{(w/z_-)^{\Delta-h_i}}{(1-w/z_-)^\Delta} - \tfrac{(w/z_+)^{\Delta-h_i}}{(1-w/z_+)^\Delta} \right) \tag{1.105}$$

with $\delta_{(1,h)} = \delta(\sigma - \sigma')$ independent of h. For $\Delta > 1$ we can use the recursion formula

$$\delta_{(\Delta+1,h)} = \partial_\sigma \delta_{(\Delta,h)} + i(\Delta - h)\delta_{(\Delta,h)} \tag{1.106}$$

that is obtained by differentiation of (1.105), where we used the definition $\partial_\sigma z_\pm = iz_\pm$. Hence

$$\delta_{(2,h)} = \delta' + i(1-h)\delta\,, \qquad \delta_{(3,h)} = \delta'' + i(3-2h)\delta' - 2\tbinom{h-1}{2}\delta\,,$$
$$\delta_{(4,h)} = \delta''' + 3i(2-h)\delta'' - (11 - 12h + 3h^2)\delta' + 6i\tbinom{h-1}{3}\delta\,, \tag{1.107}$$

which is consistent with $\delta_{(\Delta+1,h)} - \delta_{(\Delta+1,h+1)} = i\Delta\delta_{(\Delta,h)}$. We thus obtain

$$\tfrac{1}{2\pi i}[\hat{\mathcal{O}}_i(\sigma^+), \hat{\mathcal{O}}_j(\sigma'^+)]_{|t} = -\sum_k \tfrac{C_{ij}{}^k}{(\Delta-1)!}\delta_{(\Delta,h_i)}\hat{\mathcal{O}}_k(\sigma'^+) \tag{1.108}$$

The lower derivative terms in $\delta_{(\Delta,h_i)}$ should eventually go away on dimensional grounds for the proper conformal fields $\mathcal{O}(\sigma^+)$ on the cylinder. We check this for the energy momentum tensor, where $\Delta = h_i = 2$ so that $\delta_{(22)} = \delta' - i\delta$ and $\delta_{(4,2)} = \delta''' + \delta'$. Putting everything together, the shift $T(\sigma) = \hat{T}(\sigma) - \frac{c}{24}$ in L_0 by the Schwinger term in (1.36) compensates the δ'-term from $\delta_{(4,2)}$, but, alas, the imaginary and symmetric contribution from $\delta_{(22)}$ does not go away. Inserting the Fourier series and the Virasoro algebra one finds

$$\tfrac{1}{2\pi i}[T(\sigma^+), T(\sigma'^+)] = \tfrac{c}{12}\delta''' - 2T(\sigma'^+)\delta' - \partial T(\sigma'^+)\delta\,, \tag{1.109}$$

which is an odd function of $\sigma - \sigma'$. This differs from (1.108) by an even and imaginar term $2iT\delta$ of wrong scaling dimension. Presumably the problem is related to the precise definition of time ordering, which sometimes requires contact terms at equal times, where it is ill-defined, in order to produce Lorentz covariant contractions and normal ordered operator products in canonical quantization. This was well-known in the 50s, when the modified time ordering was called T^*-product (in [IT80], eq. (6-60), it is denoted by $\hat{T}$). In [BO59] renormalization was based on the interpretation of counterterms as contact terms in (time ordered) Greens functions. In any case, an OPE with bounded pole order corresponds to local equal time commutators and is thus called "locality axion" in some axiomatic approaches to CFT.

In (1.87)–(1.89) we observed that the information of the singular part of the OPE is equivalent to the commutation relations of the modes of the respective operators. This is true for the Laurent modes of arbitrary local fields:

$$[A_m, B_n] = \Big(\oint\limits_{|x|>|y|}\tfrac{dx}{2\pi i}\oint\tfrac{dy}{2\pi i} - \oint\limits_{|y|>|x|}\tfrac{dy}{2\pi i}\oint\tfrac{dx}{2\pi i}\Big)\; A(x)B(y)\; x^{m+h_A-1}\; y^{n+h_B-1} \tag{1.110}$$

We can, therefore, define an operator algebra of meromorphic fields by stating the contractions of a complete set of elementary fields. The integrand on the r.h.s. of eq. (1.110) has poles only at the origin and at $x = y$. Thus the total integration contour can be deformed into $\oint_0 dy \oint_y dx = -\oint_0 dx \oint_x dy$. We can describe this contour by a formal commutator $[\oint dx, \oint dy]$ with an implicit 'time ordering' of circles, i.e. the integral on the left encloses the origin at a later time. For conformal primary fields of weigth h we find

$$[L_n, \phi(z)] = z^n\,(z\partial + (n+1)h)\,\phi(z), \qquad [L_n, \phi_m] = (n(h-1) - m)\,\phi_{n+m} \tag{1.111}$$

for the transformation properties in terms of the Virasoro generators L_n. For non-tensorial fields (descendents) we have to expect additional non-linear terms like in

$$[L_n, T(z)] = \tfrac{c}{12}(n^3 - n)w^{n-2} + 2(n+1)w^nT(w) + w^{n+1}\partial T(w), \tag{1.112}$$

which is equivalent to the Virasoro algebra.

Consisteny of the operator algebra requires that the Jacobi identity for the commutators (1.110) is satisfied. This translates into an identity for integration contours in a tripple integral. Together with the associativity of the operator algebra[5] the following equation encodes the relevant criterion, called **associativity of the operator product algebra** [bo91]:

$$\begin{aligned}
\oint_0 \frac{dz}{2\pi i}\oint_z \frac{dy}{2\pi i}\oint_y \frac{dx}{2\pi i}\;\underbrace{\underbrace{A(x)B(y)}\,C(z)}\;f(x,y,z) &\;+\\
\oint_0 \frac{dx}{2\pi i}\oint_x \frac{dz}{2\pi i}\oint_z \frac{dy}{2\pi i}\;\underbrace{\underbrace{B(y)C(z)}\,A(x)}\;f(x,y,z) &\;+\\
\oint_0 \frac{dy}{2\pi i}\oint_y \frac{dx}{2\pi i}\oint_x \frac{dz}{2\pi i}\;\underbrace{\underbrace{C(z)A(x)}\,B(y)}\;f(x,y,z) &\;=\;0
\end{aligned} \tag{1.113}$$

for all functions $f(x,y,z)$ that are analytic on the punctured complex plane $\mathbb{C}^* = \mathbb{C} - \{0\}$. It is straightforward to check (1.113) for a given set of contractions: The first two integrals are evaluated by Taylor–expanding f to the appropriate order (we may assume that $f(x,y,z) = f(x)g(y)h(z)$). Then the integrand for the final integral must be a total derivative.

1.7 The Wick theorem

The important rule for computing OPEs of composite operators is the **Wick theorem**:

$$\underbrace{A(z)[BC](w)} = \oint_w \frac{dv}{2\pi i}\,\frac{\underbrace{A(z)B(v)}\,C(w)}{v-w} \;+\; [B(w)\,\underbrace{A(z)C(w)}] \tag{1.114}$$

Proof: The singularities of the operator product $A(z)B(v)C(w)$ as a function of z near v and w are given by the contractions of $A(z)$ with $B(v)$ and $C(w)$. Integrating $dv/(v-w)$ around w,

$$\underbrace{A(z)[BC](w)} = \oint_w \frac{dv}{2\pi i}\,\frac{\underbrace{A(z)B(v)}\,C(w) + (-)^{AB}B(v)\,\underbrace{A(z)C(w)}}{v-w}, \tag{1.115}$$

we obtain the Wick theorem. □

The last term in (1.115) can be simplified to give the normal product of B with the contraction of A and C, but the integral with the contraction of A and B has to be evaluated carefully: If $\underbrace{A(z)B(v)}$ and $C(w)$ have a short distance singularity then terms in the expansion of $1/(z-v)^n$ around w,

$$\frac{1}{(z-v)^n} = \frac{1}{(z-w)^n} + \binom{n}{1}\frac{(v-w)}{(z-w)^{n+1}} + \binom{n+1}{2}\frac{(v-w)^2}{(z-w)^{n+2}} + \ldots \tag{1.116}$$

[5] Note that the identity $R\,(A(x)B(y))\,C(z) = R\,(B(y)C(z))\,A(x)$ involves some analytic continuation.

can combine with poles $1/(v-w)^m$ to produce a residue in the v integration.

In terms of the operator product coefficients the Wick theorem thus reads

$$[A[BC]]_q = [B[AC]_q] + \sum_{l=0}^{q-1} \tbinom{q-1}{l} [[AB]_{q-l}C]_l \qquad q > 0 \tag{1.117}$$

(we always omit the obvious sign factors in case of fermions). For $q = 0$ the rearrangement lemma tells us that there is an additional normal ordered commutator on the r.h.s. of this expression: $[A[BC]] = [B[AC]] + [([AB] - [BA])C]$. If the contraction $\underbrace{A(z)B(w)}$ is a $\mathbb{C}$-number function, i.e. if all $[AB]_q$ are proportional to the identity for $q > 0$, so that only $l = 0$ contributes in the above sum, then the Wick theorem reduces to the usual expression for free fields: $\underbrace{A[BC]} = [\underbrace{AB}\,C] + [B\,\underbrace{AC}]$. In particular, by iteration of this equation,

$$\underbrace{A(z)B^n(w)} = n\,\underbrace{A(z)B(w)}\,B^{n-1}(w), \qquad \underbrace{A(z)e^{B(w)}} = \underbrace{A(z)B(w)}\,e^{B(w)}. \tag{1.118}$$

whenever $\underbrace{A(z)B(w)} \in \mathbb{C}$.

The situation is more complicated if there is a composite operator on the left. As an example we compute the central charge of a free boson. For the current $J = \partial X$ we have $\underbrace{J(z)J(w)} = -1/(z-w)^2$ and $T = -J^2/2$. Expanding $\underbrace{J(w)T(z)}$ about w we find

$$\underbrace{T(z)J(w)} = \frac{J(w)}{(z-w)^2} + \frac{\partial J(w)}{z-w}, \tag{1.119}$$

i.e. J is a primary field of weight 1. The contraction of T with itself becomes

$$\begin{aligned}
\underbrace{T(z)T(w)} &= -\frac{1}{2}\oint_w \frac{dv}{2\pi i}\left(\frac{J(v)}{(z-v)^2} + \frac{\partial J(v)}{z-v}\right)\frac{J(w)}{v-w} - \frac{1}{2}[J(w)\,\underbrace{T(z)J(w)}] && (1.120)\\
&= -\frac{1}{2}\oint_w \frac{dv}{2\pi i}\left(\frac{-1}{(v-w)^3} + \frac{J^2(w)}{v-w}\right)\left(\frac{1}{(z-w)^2} + 2\frac{v-w}{(z-w)^3} + 3\frac{(v-w)^2}{(z-w)^4}\right)\\
&\quad -\frac{1}{2}\oint_w \frac{dv}{2\pi i}\left(\frac{2}{(v-w)^4} + \frac{[J\partial J](w)}{v-w}\right)\left(\frac{1}{z-w} + \ldots + \frac{(v-w)^3}{(z-w)^4}\right)\\
&\quad -\frac{1}{2}\left(\frac{J^2(w)}{(z-w)^2} + \frac{[J\partial J](w)}{z-w}\right) && (1.121)\\
&= \frac{3/2-2/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}, && (1.122)
\end{aligned}$$

so that a free boson has central charge is $c = 1$. Note that the short distance singularity of $J(v)J(w)$, together with the expansion of $1/(z-v)$ around w, produces the central term. With the Wick theorem (1.57) for free fields the forth order pole comes from the double contraction. Simlarly, the OPE of $T(z)$ with $V_k(w) = {:}e^{ikX(w)}{:}$ can be evaluated with the result

$$\underbrace{T(z)V_k(w)} = \frac{k^2/2}{(z-w)^2}V_k(w) + \frac{\partial V_k(w)}{z-w} \tag{1.123}$$

confirming our expactation from (1.60) that the vertex operator is a primary field with $h = \frac{1}{2}k^2$. For the massless vertex operator we find

$$\underbrace{T(z)\,[\partial X^\mu V_k](w)} = \frac{-ik^\mu V_k(w)}{(z-w)^3} + \frac{\frac{k^2}{2}+1}{(z-w)^2}[\partial X^\mu V_k](w) + \frac{\partial[\partial X^\mu V_k](w)}{z-w} \tag{1.124}$$

whose contraction with a transversal polarization tensor is a primary field, as we anticipated from the operator-state correspondence.

Eventually we compute the central charge of a first order system with energy–momentum tensor

$$T_{bc} = (1-j)[\partial bc] - j[b\partial c], \qquad \underbrace{b(z)c(w)} = \varepsilon\,\underbrace{c(z)b(w)} = \frac{\varepsilon}{z-w} \tag{1.125}$$

with $\varepsilon = 1$ for fermions (like the ghosts bc with $j = 2$) and $\varepsilon = -1$ for bosons (like the superconformal ghosts $\beta\gamma$ with $j = \frac{3}{2}$). Then

$$\underbrace{T_{bc}(z)b(w)} = \frac{jb(w)}{(z-w)^2} + \frac{\partial b(w)}{z-w}, \qquad \underbrace{T_{bc}(z)c(w)} = \frac{(1-j)c(w)}{(z-w)^2} + \frac{\partial c(w)}{z-w}. \tag{1.126}$$

Trusting that the OPE of T_{bc} with itself is of the correct form for an energy–momentum tensor, we only need to compute the central term. So we use (1.117) to directly evaluate the 4^{th} order pole term. Only the last term on the r.h.s. of that equation can contribute, hence

$$\frac{c}{2} \;=\; [T_{bc}T_{bc}]_4 \;=\; (1-j)[T_{bc}[\partial bc]]_4 - j[T_{bc}[b\partial c]]_4 \tag{1.127}$$

$$= (1-j)\sum_{l=1}^{3}\tbinom{3}{l}[[T_{bc}\partial b]_{4-l}c]_l - j\sum_{l=2}^{3}\tbinom{3}{l}[[T_{bc}b]_{4-l}\partial c]_l \tag{1.128}$$

$$= 6(1-j)j\;[bc]_1 + 3(1-j^2)\;[\partial bc]_2 + (1-j)\;[\partial^2 bc]_3 - 3j^2\;[b\partial c]_2 - j\;[\partial b\partial c]_3 \tag{1.129}$$

$$= 6(1-j)j\;\varepsilon + 3(1-j^2)\;(-\varepsilon) + (1-j)\;2\varepsilon - 3j^2\;\varepsilon - j\;(-2\varepsilon) \tag{1.130}$$

$$= \varepsilon(6j(1-j)-1), \tag{1.131}$$

where we used

$$\underbrace{T_{bc}(z)\partial b(w)} = \frac{2jb(w)}{(z-w)^3} + \frac{(j+1)\partial b(w)}{(z-w)^2} + \frac{\partial^2 b(w)}{z-w}, \qquad \underbrace{b(z)c(w)} = \frac{\varepsilon}{z-w}. \tag{1.132}$$

The dependence on the statistics of b is only through an overall sign and we confirm the result (1.50), which implied the critical dimension $D = 10$ for the superstring.

With the Wick theorem it is straightforward to compute any OPE of composite operators in terms of elementary contractions that define the CFT, but in practice this can become very tedious and it is easy to make mistakes. For free fields it is often much faster to sum over multiple contractions. But fortunately there is the Mathematica package ‘OPEdefs.m’, written by K. Thielemans [th91], which does the job for us. The above calculation of the central charges for free bosons and for a bc system, for example, can be done on a computer by loading the package into a Mathematica session (with “<<OPEdefs.m”) and by typing the commands that are listed in table I.

Bosonic[dX]; OPE[dX,dX]=MakeOPE[{−One,0}]; T=−NO[dX,dX]/2; OPESimplify[OPE[T,T],Together]	Fermionic[b,c]; OPE[b,c]=MakeOPE[{One}]; T=(1−j) NO[b',c]−j NO[b,c']; Factor[OPEPole[4][T,T]]

Table I: Calculation of the central charge for the bosonic string with OPEdefs.m

To compute the central charge for a $\beta\gamma$ system, we just need to replace *Fermionic[b,c]* by *Bosonic[B,C]* and *OPE[b,c]=MakeOPE[One]* by *OPE[B,C]=MakeOPE[-One]*.

The zero mode $Q_J = J_0 = \oint \frac{dz}{2\pi i} J(z)$ of a primary field $J(z)$ with conformal weight $h_J = 1$ commutes with $T(z)$ (and hence with all Virasoro generators L_n) and thus provides a conserved charge because

$$\underbrace{T(z)J(w)} = \frac{J(w)}{(z-w)^2} + \frac{\partial J(w)}{z-w} = \partial_w \frac{J(w)}{z-w} \tag{1.133}$$

is a total derivative so that $[T(z), Q_J] = (\oint_{|z|>|w|} - \oint_{|z|<|w|}) \frac{dw}{2\pi i} T(z)J(w) = 0$.

Returning to the bosonic string, i.e. $j = 2$ and $D = 26$, we first consider the BRST current j_Q and the corresponding charge $Q_{BRST} = \oint j_Q$. Naively, we would take $j_Q = cT_x + \frac{1}{2}cT_{bc}$, whose OPE with $T = T_x + T_{bc}$ is

$$T(z)[cT_x + \tfrac{1}{2}cT_{bc}](w) \sim \frac{9c(w)}{(z-w)^4} + \frac{3\partial c(w)}{(z-w)^3} + \frac{[cT_x + \frac{1}{2}cT_{bc}](z)}{(z-w)^2}, \tag{1.134}$$

so that this expression is not a conformal field. The non-covariant terms, however, are the same as the ones in the OPE of $T(z)$ with $-\frac{3}{2}\partial^2 c(w)$. A Noether current is only defined up to total derivatives, so we can work with the covariant BRST-current

$$j_Q := cT_x + \tfrac{1}{2}cT_{bc} + \tfrac{3}{2}\partial^2 c = -\tfrac{1}{2}c\, \partial X_\mu\, \partial X^\mu + b\, c\, \partial c + \tfrac{3}{2}\partial^2 c, \tag{1.135}$$

which is a conformal field with weight 1.

The OPEs of j_Q with J_c and with b are

$$\underbrace{j_Q(z)J_c(w)} = \partial_w \frac{-2c(w)}{(z-w)^2} + \frac{j_Q(w)}{z-w}, \qquad \underbrace{j_Q(z)b(w)} = \frac{3}{(z-w)^3} + \frac{J_c(w)}{(z-w)^2} + \frac{T(w)}{z-w}. \tag{1.136}$$

The OPE of j_Q with itself is a total derivative in 26 dimensions:

$$\underbrace{j_Q(z)j_Q(w)} = \partial_w \frac{2[\partial c\ c](w)}{(z-w)^2} \tag{1.137}$$

implying nilpotency of the BRST charge. Note that $(\partial c)^2 = 0$, which follows from the identity[6]

$$[FF](w) = -\tfrac{1}{2}\sum_{l>0} \frac{(-)^l}{l!}\partial^l [FF]_l \qquad (F \text{ fermionic}) \tag{1.138}$$

for fermionic operators F. This identity, in turn, is a consequence of (1.94)).

[6] According to ref. [th91], OPEdefs.m uses the rules (1.94), (1.95), (1.100), (1.117) and (1.138).

1.8 Ghost number anomaly and topology

For each bc or $\beta\gamma$ system the ghost number current $J_c(z) = -[bc](z)$ is a classically conserved current with $h = 1$. We have seen in the case of the ghost system, however, that expectation values of operators sandwiched between $SL(2,\mathbb{C})$ vacua vanish unless the ghost number of the operator is 3 for each chirality. For general spin j this anomaly (i.e. quantum mechanical violation of the ghost number) will turn out to be proportional to the so-called 'background charge' $Q = \varepsilon(1-2j)$, which shows up in the OPE

$$T_{bc}(z)J_c(w) \sim \frac{Q}{(z-w)^3} + \frac{J(z)}{(z-w)^2}, \qquad Q = \varepsilon(1-2j). \tag{1.139}$$

Like $T(z)$ the quantum current $J_c(z)$ therefore transforms non-tensorial and non-linear under conformal transformations

$$\delta_\xi J_c = \xi\partial J_c + \partial\xi J_c + \tfrac{1}{2}Q\,\partial^2\xi, \qquad \overset{w\approx z+\xi}{\Longrightarrow} \qquad J_c(z) = \frac{\partial w}{\partial z}J_c(w) + \frac{Q}{2}\frac{\partial_z^2 w}{\partial_z w}, \tag{1.140}$$

where the finite transformation was obtained by an ansatz with a second derivative and an additional Jacobi matrix factor for the correct global scaling weight. We will use this formula to relate the background charge Q to the quantum field theoretic anomaly coefficient. The remaining OPEs of J_c are

$$J_c(z)c(w) \sim \frac{c(w)}{z-w}, \qquad J_c(z)b(w) \sim \frac{-b(w)}{z-w}, \qquad J_c(z)J_c(w) \sim \frac{\varepsilon}{(z-w)^2}. \tag{1.141}$$

In terms of Q the cental charge of the BC system is $c = \varepsilon(12j(1-j)-2) = \varepsilon(1-3Q^2)$.

The charge $Q_J = J_0$ still commutes with $T(z)$ because the third order pole does not contribute to the residue. But current conservation will turn out to be spoiled in curved space and hence on compact Riemann surfaces of genus $g \neq 1$. In quantum field theory the action principle and consistency conditions typically constrain the possible form of chiral anomalies to a topological densities, whose coefficients can be obtained from index theorems. In the present context

$$\partial_m J_N^m = \frac{q}{2\pi}\sqrt{g}\,R^{(2)}, \tag{1.142}$$

where $R^{(2)}$ is the curvature scalar on the world sheet. Due to the Gauß-Bonnet theorem

$$\frac{1}{2\pi}\int \sqrt{g}\,R^{(2)} = \chi = 2-2g \tag{1.143}$$

this yields a total ghost number violation by $q(2-2g)$ units on a Riemann surface of genus g. Our aim is to show that the anomaly coefficient coincides with the background charge $q = Q$.

The Euclidean analog of light cone coordinates are conformally flat coodinates $z = x+iy$,

$$g = e^\phi((dx)^2 + (dy)^2) = \tfrac{1}{2}e^\phi(dz\otimes d\bar z + d\bar z\otimes dz), \qquad \sqrt{g} = 2g_{z\bar z} = e^\phi. \tag{1.144}$$

The local existence of such coordinates can be shown by using the Beltrami parametrization[7]

$$\begin{aligned} e^z &= \lambda(dz+\mu d\bar z) & \lambda &= e_z{}^z & \mu &= \mu_{\bar z}{}^z = e_{\bar z}{}^z/\lambda \\ e^{\bar z} &= \bar\lambda(d\bar z+\bar\mu dz) & \bar\lambda &= e_{\bar z}{}^{\bar z} & \bar\mu &= \mu_z{}^{\bar z} = e_z{}^{\bar z}/\bar\lambda \end{aligned} \tag{1.146}$$

of the vielbein $e_m{}^a$ in some reference coordinate system $z = x + iy$. Now we want to find complex coordinates $Z = Z(z, \overline{z})$ such that

$$dZ = \lambda(dz+\mu d\bar z), \qquad d\bar Z = \bar\lambda(d\bar z+\bar\mu dz), \tag{1.147}$$

where the Beltrami differential μ is invariant under a holomorphic change of variables $Z \to Z'(Z)$ and thus paramatrizes the complex structure with holomorphic coordinates Z in terms of the reference coordinates z. Positivity of the metric requires $|\mu| < 1$ and integrability $d^2Z = (\partial(\lambda\mu) - \overline{\partial}\lambda)dz\,\overline{d}z = 0$ of the Z coordinate implies

$$(\overline\partial-\mu\partial)\lambda = (\partial\mu)\lambda \qquad\Rightarrow\qquad \ln\lambda = (\overline\partial-\mu\partial)^{-1}(\partial\mu), \tag{1.148}$$

which can be solved for the conformal factor $|\lambda|^2$ because $\overline{\partial} - \mu\partial$ is an elliptic operator [LE87]. The new coordinate Z is a solution to the Beltrami equation $(\overline{\partial} - \mu\partial)Z = 0$.

In terms of local conformally flat complex coordinates Z the transition functions are conformal and therefore holomorphic (or antiholomorphic, if we admit a change of orientation). Conformal equivalence classes of metrics on an orientable surface are therefore in one-to-one correspondence to complex structures. The curvature in conformally flat coordinates (1.144) is easily evaluated using the Weyl transformation formula $\sqrt{g}\,R = \sqrt{g'}(R' - \Delta\phi)$ for $g_{mn} = e^\phi g'_{mn}$. On the sphere $\mathbb{P}^1$ the round (Fubini–Study) metric is a Kähler metric,

$$g_{z\bar z} = \partial\overline\partial\log(1+z\bar z) = \tfrac{1}{(1+z\bar z)^2} = \tfrac12 e^\phi \qquad\Rightarrow\qquad R(x,y) = 2R(z,\overline z) = -e^{-\phi}\partial\overline\partial\phi = 1 \tag{1.149}$$

Integrating $\int dxdye^\phi = 4\pi$ we find agreement with Gauß-Bonnet. Under a conformal transformation $z \to f(z)$ the conformal factor ϕ transforms as

$$w = f(z) \qquad\Rightarrow\qquad \phi(z) = \phi(w) + \log|\tfrac{\partial w}{\partial z}|^2. \tag{1.150}$$

For a flat metric $\phi = 0$ on the complex plane all curvature is localized at infinity,

$$w = 1/z \qquad\Rightarrow\qquad \phi(w) = -2\log(w\overline w), \qquad (\sqrt g\,R)(w,\bar w) = 4\pi\delta^{(2)}(w) \tag{1.151}$$

again in agreement with Gauß-Bonnet.

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[7] In terms of μ the action of a free boson can be written as

$$S(X,\mu,\overline\mu) = \frac{1}{2\pi\alpha'}\int d^2z\frac{1}{1-\mu\overline\mu}(\partial-\overline\mu\overline\partial)X(\overline\partial-\mu\partial)X. \tag{1.145}$$

Beltrami differentials are used, for example, for cohomology calculations because the Becci ghosts $C = c+\mu\bar c$ lead to a holomorphic factorization of the (geometrical) gauge degrees of freedom with the BRST transformations $s\mu = (\overline\partial - \mu\partial)C + C\mu$ and $sC = C\partial C$.

We can now put the pieces together and first check the integrated anomaly on the sphere. With $w = 1/z$ and $\partial_z^2 w = 2w^3$ equation (1.140) implies $\oint \frac{dz}{2\pi i} J_c(z) + \oint \frac{dw}{2\pi i} J_c(w) + \oint \frac{dw}{2\pi i} \frac{Q}{w} = 0$, which shows that the (left-moving) charge is shift by Q units on $\mathbb{P}^1$ in accord with the value for the ghost number violation ($\oint \frac{dz}{2\pi i} J_c(z)$ and $\oint \frac{dw}{2\pi i} J_c(w)$ measure the ghost number on the two hemispheres).

In order to obtain the local anomaly expression (1.142) we need to take into account the relation between the standard normalization of the Noether current $J_N^m = \frac{1}{4\pi}\sqrt{g}\, g^{ml} b_{ln} c^n$ and the CFT normalization of holomorphic currents $J(z) \equiv J_z = -bc = -4\pi g_{z\bar z} J_N^{\bar z}/\sqrt{g}$, where J_N^m is a vector density and $\sqrt{g}\, g^{z\bar z} = 2$. Current conservation in complex coordinates thus reads $\partial_m J_N^m = -\frac{1}{2\pi}(\partial_{\bar z} J_z + \partial_z J_{\bar z}) = 0$, where J_z is a conformal primary field with $h = 1$. Since $\delta_\xi \phi = \xi \partial \phi + \partial \xi$ the anomaly in the ghost current J_c can be compensated by defining $\hat J_c(z) = J_c(z) - \frac{Q}{2}\partial\phi$, which transforms as a tensor and thus is conserved $\partial_m \hat J^m = 0$ for an arbitrary conformal factor ϕ. But this implies $\partial_{\bar z} J_z + \partial_z J_{\bar z} = Q\partial\bar\partial\phi = -Q\sqrt{g}\, R$ and we find the anomaly (1.142). Although $\hat J$ is related to J by a term that is local in ϕ quantum field theory tells us that we can preserve general coordinate invariance and it is not possible to remove the anomaly by a generally covariant local renormalization of the current density J^m.

With a similar calculation we can derive the Liouville action for the conformal ϕ factor in non-critical string theory $c = D - 26 \neq 0$. Here we observe that covariant conservation $D_m T^{mn}$ of the energy momentum tensor can be written as $\partial_{\bar z} T_{zz} + D_z T_{\bar z z} = 0$ with $\Gamma_{zz}{}^z = \partial\phi$. The anomalous term in $\delta_\xi T(z) = \frac{c}{12}\xi''' + \ldots$ can be removed by $\hat T_{zz} = T_{zz} - \frac{c}{12}(\partial^2\phi - \frac{1}{2}(\partial\phi)^2)$, where the contribution from $\frac{1}{2}(\partial\phi)^2$ is needed to cancel the spurious term $\partial^2\xi\partial\phi$ in $\delta_\xi(\partial^2\phi)$. Now $D_{\bar z}\hat T_{zz} = \partial_{\bar z}\hat T_{zz}$ vanishes for $\phi = 0$ and transforms covariantly, hence $\partial_{\bar z}\hat T_{zz} = 0$ and

$$D_z T_{\bar z z} = -\tfrac{c}{12}\partial_{\bar z}\Big(\partial^2\phi - \tfrac{1}{2}(\partial\phi)^2\Big) = -\tfrac{c}{12}(\partial_z - \partial\phi)\partial\bar\partial\phi \qquad \Rightarrow \qquad T_{\bar z z} = \tfrac{c}{12}\sqrt{g}\, R \tag{1.152}$$

Like in the case of the ghost current the Weyl anomaly $T_{\bar z z} \sim \operatorname{tr} T_{mn}$ can be compensated by a renormalization of the action with a local functional of ϕ, the Liouville action

$$\mathcal{L}_L = \tfrac{c}{48\pi}(\tfrac{1}{2}(\partial\phi\bar\partial\phi + \mu^2 e^\phi). \tag{1.153}$$

The Liouville potential $\mu^2 e^\phi$ comes from a cosmological term $\int \mu^2\sqrt{g}$, which can be added to the Polyakov action for $D \neq 26$ when conformal invariance is broken anyway). This action can, however, not be written as a local functional of the metric with general coordinate invariance preserved (the corresponding non-local Wess-Zumino action is of the form $\mathcal{L}_{WZ} \sim R\frac{1}{\Box}R$).

The ghost number violation that forces us to insert ghosts into physical correlation functions can be understood directly from the path integral: Recall that the total gauge fixed action is

$$\mathcal{L} = \mathcal{L}_P + \tfrac{T}{2}\int \sqrt{-g}\, b^{mn}(Pc)_{mn}, \qquad (Pc)_{mn} = D_m c_n + D_n c_m - g_{mn} Dc, \tag{1.154}$$

where lagrange multipliers, Weyl ghost and the trace part of the anti-ghost have been integrated out and the operator P maps vector fields into traceless symmetric tensors. The path integral over a fermionic variable vanishes if the integrand does not depend on that variable. Therefore we need to insert an extra ghost for any zero mode of P and an extra anti-ghost for any zero mode of $P^\dagger$. This implies that the total ghost number violation is equal to the index of P,

$$\operatorname{index} P \equiv \dim\ker P - \dim\ker P^\dagger = -Q\chi \tag{1.155}$$

i.e. the number of zero modes of P minus the number of zero modes of $P^\dagger$. The adjoint is defined with respect to the natural positive ultralocal scalar product[8]

$$||\delta g||^2 = \int \sqrt{g}\, g^{kl} g^{mn} \delta g_{km} \delta g_{ln}, \tag{1.156}$$

on the space of metric deformations, in terms of which the ghost action can be written as a scalar product

$$\mathcal{L}_c \sim \langle b|Pc\rangle = \int_\Sigma \sqrt{g}\;\, g^{kl} g^{mn}\, b_{km}(Pc)_{ln}, \tag{1.157}$$

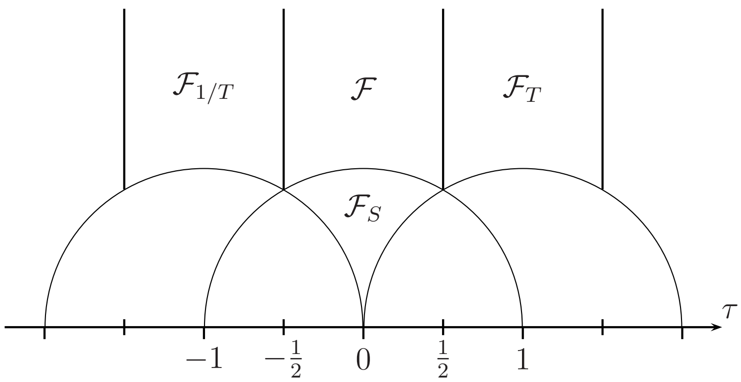
We thus derived the Riemann Roch theorem, which equates the ghost number violation (1.142) to the index of the operator P. The zero modes of P correspond to (global) conformal Killing vector fields and thus to symmetries of the Riemann surface. The zero modes of $P^\dagger$, on the other hand, are orthogonal to the gauge variations of the metric. Their number is thus equal to the number of non-trivial metric deformations and hence to the number of moduli (i.e. parameters of the complex structure) of the Riemann surface.

Counting the number of *complex* parameters Riemann-Roch thus implies that the number of conformal symmetries minus the number of moduli of a Riemann surfaces should be given by $3(1-g)$. These numbers can be computed directly in terms of the Schottky parametrization, which constructs a general Riemann surface of genus g by gluing g cylinders with the boundary compoments of a sphere with $2g$ holes. This can be done, for example, by choosing local complex coordinates z_i and w_i for which the holes are at $|z_i| = |w_i| = 1$ with the gluing prescription $z_i w_i = t_i$. This shows that the moduli space is itself a complex manifold, parametrized by t_i and the positions z_i and w_i of the holes, where $\log|t_i|$ corresponds to the lengths of the handles and the arguments of the moduli to a twisting of the cylinders by an angle $\operatorname{Im}\log t_i$. In order to increase the genus by one we have to add two holes and one cylinder. Together this adds three complex moduli per handle. The only exceptional cases are genus 0 and genus 1: The sphere has three (complex) conformal Killing symmetries, which can be used to fix the positions of the first two holes, and no moduli. Hence the torus has only one modulus, and one symmetry is left over (if we descirbe the torus as the complex plane modulo the lattice generated by the complex numbers 1 and τ, then τ is the modulus and the symmetry is the translation symmetry

[8] A possible additional trace term $g^{km}g^{ln}$ does not contribute to traceless deformations.

of the plane). For genus $g > 1$ we have used up all symmetries of the sphere, so there are no more symmetries and the number of moduli is $3g - 3$, in agreement with Riemann Roch.

There is another parametrization of the moduli space in terms of periods, which is mainly useful at small genera. Here one starts with a canonical basis of $2g$ homology cycles with intersection numbers $a_i \cap b_j = \delta_{ij}$. We can now introduce a set of g holomorphic 1-forms, called Abelian differentials, which are normalizes by their periods $\int_{a_i} \omega_j = \delta_{ij}$. The period matrix $\Omega_{ij} = \int_{b_i} \omega_j$ can be shown to be symmetric and to have positive definite imaginary part, i.e. they map the moduli space into the so-called Siegel upper half plane. Ω_{ij} has $g(g+1)/2$ entries, which is the correct number of moduli for $g \le 3$. For larger genus the entries of Ω_{ij} are therefore constrained in a complicated way. For genus 1 we can represent the torus by a double periodic lattice $\Gamma = \langle u, v\rangle$ and $\omega_1 = dz/u$ with $\operatorname{Im}\tau > 0$ for $\tau = \Omega_{11} = v/u$ because of the required orientation of the intersection of the two cycles. The complex structure is thus parametrized by τ in the upper half plane.



So far we only considered the local structure of the moduli space, whose global structure is quite complicated. We first consider the simplest case $g = 1$. The parameter space $\operatorname{Im}\tau > 0$ is called Teichmüller space, which is simple connected. There are, however, infinitely many different values of τ that parametrize the same torus. Two examples are given by the transformation T, which sends $\tau \to \tau + 1$, and S, which sends $\tau \to -1/\tau$. These transformation generate the infinite discrete group $PSL(2, \mathbb{Z})$,

$$\tau \to \frac{a\tau + b}{c\tau + d}, \qquad a, b, c, d \in \mathbb{Z}, \quad ad - bc = 1, \tag{1.158}$$

which is called the modular group. All conformally equivalent tori are related by (1.158), which can be used to choose a representative in the fundamental domain $\mathcal{F} = \{\tau | \operatorname{Im}\tau > 0 \wedge |\operatorname{Re}\tau| \le \frac{1}{2} \wedge |\tau| \ge 1\}$ or in one of its images under a modular transformation.

The case of genus one suggests that it is convenient to construct the moduli space in two steps. First we take the space $\{[g_{mn}]\}/\operatorname{Diff}^0$ of conformal equivalence classes of metrics modulo "small" diffeomorphisms, i.e. diffeomorphisms that are continuously connected to the identity. This is a simply connected complex manifold, called Teichmüller space (for $g = 1$ this is the upper half plane). In a second step we have to divide out the discrete mapping class group $\mathrm{MCG} = \mathrm{Diff} \,/\, \mathrm{Diff}^0$, which consists of homotopy classes of (oriented) "big" diffeomorphisms. Since such diffeomorphisms do not change intersection numbers this group is effectively represented by symplectic transformations on the homology lattice $H^1 \cong \mathbb{Z}^{2g}$. For genus 1 this representation is injective so that the MCG is isomorphic to $SL(2,\mathbb{Z})$. Since the MCG has fixed points the moduli space is not a manifold but rather has orbifold singularities. For genus one these are at $\tau = i$, which is a fixed point of S and thus a $\mathbb{Z}_2$ singularity, and at $\tau = \exp(i\pi/3) \sim \exp(2i\pi/3)$, the fixed point of TS, which has order 3 (the two neighboring wedges of the representatives in $\mathcal{F}$ cover an angle $2\pi/3$ for the holomorphic coordinate τ).

The moduli space is not compact, with its boundary points corresponding to the pinching of cycles. For genus 1 there is only one possible degeneration, for which $\mathrm{Im}\,\tau \to \infty$ so that the torus becomes infinitely long and, which is conformally equivalent, degenerates into a sphere with two points identified. For $g > 1$ there are two types of boundary points because we can either pinch a non-trivial homology cycle, which removes one handle and leads to a connected Riemann surface Σ_{g-1}, or we can pinch the surface into two disconnected surfaces Σ_{g_1} and Σ_{g-g_1}. In CFT we are actually interested in correlation function and thus should consider moduli spaces of surfaces with punctures, because the positions of operator insertions provide additional moduli. Inserting a complete set of states at the two punctures of Σ_{g-1} or $\Sigma_{g_1} \cup \Sigma_{g-g_1}$ that mark the position of the pinched cycle we can sew up the lower genus surfaces and thus reconstruct certain limits of higher genus correlation functions. Consistency condidtions of these sums with the correlation functions on Σ_g are called factorization constraints.

Another construction of Riemann surfaces uses the fact that the Weyl factor can be used to make the curvature constant on any Σ_g. The universal covering space has, of course, the same property. For genus 0 and one this yields the sphere and the plane with positive and zero curvature, respectivly. For higher genus the universal cover is the upper half plane with the Poincaré metric $d^2\tau/(\mathrm{Im}\,\tau)^2$, which has constant negative curvature and is invariant under the group $SL(2,\mathbb{R})$ of real Möbius transformations. For $g > 1$ Σ_g is thus obtained from the upper half plane as a quotient by a discrete subgroup of $SL(2,\mathbb{R})$, called Fuchsian group. This is the content of the uniformization theorem.

Non-orientable surfaces can be treated by going to the orientable double cover. $\mathbb{RP}^2$, for example, is obtained from the sphere as a quotient by the involution $\sigma : z \to -1/\bar{\zeta}$, which reverses the orientation and acts freely. A similar trick works for boundaries: If we think about mirror charges the boundary can be obtained as the fixed point set of an orientation reversing involution. In the case of the disk we can use, for example, $\sigma : z \to 1/\bar{\zeta}$, which has a fixed circle at $|z| = 1$. Every conformal equivalence class of metrics can thus be obtained as a quotient of a compact orientable Riemann surface, where the boundary corresponds to the fixed-point set of an involution. The moduli spaces are, however, restricted by consistency with this involution. At $\chi = 0$, for example, annulus, Möbius strip and the Klein bottle only have a real modulus, as is discussed in detail, for example, in [an02, PO98].

In string theory the higher genus correlation functions have to be integrated over the respective moduli space, where the measure can be obtained by the Faddeev Popov procedure. As usual the gauge fixing determinant is represented by a functional integral with ghost fields, and a careful analysis indeed leads to the insertions of ghost zero modes that are required for a non-vanishing result. For $g > 1$ only antighost insertions play a role. The correct measure is obtained from the ghost path integral if we insert $\frac{1}{2\pi}\int d^2z(\mu_{\bar z}^{(I)z}b_{zz} + \bar\mu_z^{(I)\bar z}\bar b_{\bar z\bar z})$, with the Beltrami differentials related to the relevant metric variations by $\mu_m^{(I)n} = \frac{1}{2}g^{nl}\partial_I g_{lm}$ (this yields the correct number of zero modes and can also be shown to provide a form of appropriate degrees to be integrated over the moduli space). At genus one the ghost and the anti-ghost zero modes are constant so that we can simply insert the left-moving and the right-moving ghost number currents (with an extra factor 1/2 coming from the symmetry of the torus). For a detailed derivation of these results see, for example, [PO98]. Some aspects are also discussed in [NA90].

1.9 Ward identities and conformal bootstrap

In the axiomatic approach one tries to construct conformal field theories by prescribing some set of data that is sufficient to construct, or at least uniquely define, all correlation functions. A number of axioms has to be imposed on allowed sets of data in order to guarantee a consistent and sensible definition of the correlators. In applications to statistical mechanics it is a priory not clear that higher genus surfaces are a necessary ingredient. But it turned out that all known models can be defined at arbitrary genus, and in string theory this is clearly indispensible. The general strategy is then as follows: A certain set of local fields, including the energy momentum tensor, and their OPEs (or some equivalent set of data) define the correlators on the sphere.

Since the operator product singularities of $T(z)$ with primary fields $\phi_i(w_i, \bar w_i)$ fixes all poles (as a function of z) of the correlation functions of primary fields ϕ_i with an additional insertion of an energy-momentum tensor, the correlations satisfy the *conformal Ward identity*

$$\langle T(z)\phi_1(w_1,\bar w_1)\ldots\phi_n(w_n,\bar w_n)\rangle = \sum_i\left(\frac{h}{(z-w_i)^2}+\frac{\partial_{w_i}}{z-w_i}\right)\langle\phi_1(w_1,\bar w_1)\ldots\phi_n(w_n,\bar w_n)\rangle \tag{1.159}$$

(A possible z-independent ambiguity is fixed by the cluster property for $z \to \infty$.) Similar identities can be derived for multiple insertions of $T(z_i)$. Considering various contour integrals of the Ward identity times meromorphic functions of z we can, therefore, compute correlation functions of all descendent fields, once the correlation functions of the primaries are known.

By the independent action of the left-moving Virasoro algebra $\mathcal{V}ir$ and of the right-moving $\overline{\mathcal{V}}ir$ and the corresponding states are organized into representation of $\mathcal{V}ir \otimes \overline{\mathcal{V}}ir$, which are called conformal families. Since energy should be bounded from below these families are highest weight representations, which can be labelled by the Eigenvalues $(h_i, h_{\bar{i}})$ of the zero modes. The Hilbert space thus decomposes into a sum of representations $\mathcal{H} = \oplus_{h_i,\bar{h}_{\bar{i}}}(V_i \otimes \overline{V}_{\bar{i}})$, where $V_i = V(c, h_i)$ are representations of the Virasoro algebra with central charge c and conformal weight h_i. A CFT is called *rational* if this sum is finite, i.e. if there is a finite number of conformal families.

If we define the character of a conformal family as the trace $\chi_i(\tau) = \mathrm{tr}_{V_i}\, e^{2\pi i\tau(L_0-c/24)}$ over the representation space then the torus partition function can be decomposed into a finite sum

$$Z(\tau) = \int_{\Sigma(\tau)} \mathcal{D}\phi\, e^{-S_E(\phi)} = \mathrm{tr}\, e^{2\pi i(P\,\mathrm{Re}\,\tau + iH\,\mathrm{Im}\,\tau)} = \mathrm{tr}\, e^{2\pi i(L_0^{cyl}\tau - \overline{L}_0^{cyl}\bar{\tau})} = \sum M_{ij}\chi_i(\tau)\overline{\chi}_j(\bar{\tau}), \quad (1.160)$$

where we represented the path integral as a trace over the Hilbert space of closed strings with length 2π and with the double-periodic torus boundray conditions implemented by insertion of a σ^1-translation by $2\pi\,\mathrm{Re}\,\tau$ (with momentum $P = L_0 - \overline{L}_0$) and an imaginary time evolution by $\sigma^0 = 2\pi\,\mathrm{Im}\,\tau$. The $c/24$ in the definition of the character is thus due to the shift in the Hamiltonian $H = L_0 + \overline{L}_0$ when transformed from the cylinder to the complex plain. The resulting sum is over all combinations ij of left-moving representations V_i with right-moving representations $\overline{V}_{\bar{j}}$, which are assumed to occur with a multiplicity M_{ij}. $Z(\tau)$ should be invariant under modular transformations. While the generator T acts diagonally on $\chi_i(\tau)$ because all members of a conformal family have the same weight h modulo integers, the S-transformation in general mixes the characters,

$$\begin{aligned} \chi_i(\tau+1) &= T_{ij}\chi_j(\tau) = e^{2\pi i\tau(h_i-c/24)}\chi_i(\tau), &\qquad T^\dagger &= T^{-1} = T^*, \\ \chi_i(-1/\tau) &= S_{ij}\chi_j(\tau), & S^\dagger &= S^{-1} = S^*. \end{aligned} \quad (1.161)$$

A necessary condition for modular invariance of the partition function is therefore that M_{ij} commutes with the representation matrices,

$$T^tMT^* = M, \qquad S^tMS^* = M \qquad \Leftrightarrow \qquad [M,T] = [M,S] = 0. \quad (1.162)$$

Matrices with non-negative integer elements that commute with S and T and for which $M_{00} = 1$ (i.e. the multiplicity of the vacuum is 1) are thus called *modular invariants*. The classification of such matrices is part of the reconstruction of a conformal field theory from its chiral data. The relations among the generators S and T of $PSL(2,\mathbb{Z})$ lift to relations among the representation matrices,

$$S^2 = (ST)^3 = C, \qquad C^2 = 1 \quad (1.163)$$

where $C_{ij} = \delta_{ij^+}$ is the charge conjugation matrix that maps a conformal family V_j to its conjugate V_{j^+}. This follows from the CPT theorem because S^2, while leaving τ invariant, maps the periodicities $(1,\tau)$ to $(-1,-\tau)$, i.e. to parity times time reversal, and thus may act nontrivially on the conformal fields [di88].

The basic building blocks of all correlations functions are the 3-point functions $\langle\phi_i\phi_j\phi_k\rangle$. The dimension of the corresponding space of conformal blocks are called fusion rule coefficients N_{ijk} because $N_{ijk}\neq 0$ requires that (the charge conjugate) of ϕ_k shows up in the operator product (fusion) of ϕ_i and ϕ_j. The charge conjugation matrix is $C_{ij}=N_{ij0}=N_{ij}{}^0$ with $N_{ij}{}^k=N_{ijl}C^{lk}$ describing the number of "independent" (i.e. not fixed by Ward identities) coefficients of elements of the conformal family of ϕ_k in the operator product $\phi_i(z)\phi_j(w)$. The fusion rules $N_{ij}{}^k$ can be interpreted as the structure constant of a commutative and associative algebra.

Inserting OPEs of operators we can reduce correlation functions to sums over more elementary building blocks. The geometrical picture of this process is the cutting of Riemann surfaces by insertion of a complete set of states. The inverse process is called sewing, and its consistency amounts to a number of sewing constraints. In a rational CFT the 4-point functions can be written as a finite sum

$$\langle\phi_i(z,\overline{z})\phi_j(0,0)\phi_k(1,1)\phi_l(\infty,\infty)=\sum_m C_{ij}{}^m C_{klm}\mathcal{F}^{(m)}_{ijkl}(z)\overline{\mathcal{F}}^{(m)}_{ijkl}(\overline{z}) \tag{1.164}$$

over chiral conformal blocks $\mathcal{F}^{(m)}(z)$. The chiral blocks are multivalued, i.e. sections of nontrivial bundles over the moduli spaces of punctured Riemann surfaces. Their monodromies are important characteristics of the conformal field theory (Moore–Seiberg data). The most elementary sewing constaint, the crossing symmetry or duality of the 4-point functions

$$\begin{aligned}\sum_m C_{ij}{}^m C_{klm}\mathcal{F}^{(m)}_{ijkl}(z)\overline{\mathcal{F}}^{(m)}_{ijkl}(\overline{z}) &= \sum_n C_{ik}{}^n C_{jln}\mathcal{F}^{(n)}_{ikjl}(1-z)\overline{\mathcal{F}}^{(n)}_{ikjl}(1-\overline{z}) && (1.165)\\ &= \tfrac{1}{z^{2h_i}\overline{z}^{2\overline{h}_i}}\sum_p C_{il}{}^p C_{kjp}\mathcal{F}^{(p)}_{ilkj}(\tfrac{1}{z})\overline{\mathcal{F}}^{(p)}_{ilkj}(\tfrac{1}{\overline{z}}) && (1.166)\end{aligned}$$

derives from the 3 different ways to reduce 4-point functions to 2-point functions by inserting OPEs for pairs of fields or, equivalently, of cutting the sphere such that each part is left with 2 punctures.

Once the chiral building blocks of a rational CFT are known, factorization or sewing constraints and modular invariance (invariance under the mapping class group) have to be imposed on the full correlation functions. While these are infinitely many constraints it has been shown by Moore and Seiberg that a finite number of conditions at $g\le 1$ is sufficient to guarantee consistency at all genera. The following list of axioms is essentially taken from their paper [mo89].

1. There is a unique $SL_2(R)\times SL_2(R)$ invariant vacuum with $h=\bar{h}=0$,
2. Operator-state correspondence: For each vector $\alpha\in\mathcal{H}$ in the Hilbert space there is a corresponding operator ϕ_α and its (charge) conjugate,

3. Primary fields and energy momentum tensor: The field content consists of primary fields $\alpha = i$, with tensorial transformation $[L_n, \phi_i(z, \overline{z})] = (z^{n+1}\partial_z + h_i(n+1)z^n)\phi_i$, and their descendents (which are obtained from ϕ_i by commutation with L_n's or with the Fourier modes of other chiral fields).
4. Locality: $\langle 0|\mathcal{O}_1(z_1, \overline{z}_1) \ldots \mathcal{O}_n(z_n, \overline{z}_n)|0\rangle$ exist for $|z_i| > |z_{i+1}|$ and have an analytic continuation to $\mathbb{C}^n$ for $z_i \neq z_j$. (The singularities at $z_i = z_j$ are thus poles of finite order.)
5. One loop correlation functions exist and are modular invariant (actually, duality of four-point functions on the sphere and modular invariance of the one-point function on the torus is sufficient).

An analysis of the 1-point functions on the torus lead to the Verlinde formula

$$N_{ijk} = \sum_n \frac{S_{in}S_{jn}S_{kn}}{S_{0n}} \tag{1.167}$$

which is a remarkable formula for the fusion rule coefficients (which have to be non-negative integers) in terms of the modular S matrices [ve88, mo88a]. Stated differently, $\lambda_i^{(j)} = \frac{S_{ij}}{S_{0j}}$ diagonalizes the fusion algebra $\lambda_i^{(n)}\lambda_j^{(n)} = N_{ij}{}^k\lambda_k^{(n)}$, which implies $S^\dagger = S^{-1} = S^*$.

Another interesting approch to the axiomatics can be found in [ga98, ga99]. The discovery of D-branes stimulated much recent interest in CFT on surfaces with boundaries, which lead to interesting new progress in the axiomatic approach. In now appears to be more natural to formulate the factorization and modularity constraints in terms of boundary CFT data [fu02].

1.10 Minimal models and chiral algebras

The construction of the representation spaces V_i is similar to the representation theory of $su(2)$, where unitarity or finite dimension of representations imply the existence of a state of vanishing norm, on which a creation operator vanishes. Such states are called null vectors, and the vanishing norm conditions leads to the quantization of their eigenvalues. In CFT it can be shown that rationality implies the existence of such null vectors [be84, DI97], i.e. certain formal descendents of the primary fields that label the conformal families must have vanishing norm. These descendent states, as well as all their descendents, thus can be identified with the zero vector in their representation space. There are also important differences to the representation theory of $su(2)$ because the Virasoro algebra is infinite dimensional: Representations become inifinite dimensional, while the number of unitary representations becomes finite. In any case it turnes out that the conformal weights and the possible values of the central charge become rational in a rational CFT. Most importantly, the correlations functions decompose into finite linear combinations of products of holomorphic and antiholomorphic factors, called conformal blocks, and the condition that all correlations of null states have to vanish provide differential equations for these conformal block that have finite-dimensional solution spaces.

The first successful implementation of this bootstrap program of directly constructing a full quantum field theory from its symmetries and its consistency conditions is due to Belavin, Polyakov and Zamolodchikov (BPZ) [be84], who found that all rational theories, called Virasoro minimal models, are parametrized by a pair of relatively prime integers $p < p'$ with central charge and conformal weights given by

$$c = 1 - \tfrac{6(p-p')}{pp'} \qquad h_{r,s} = \tfrac{(rp-sp')^2-(p-p')^2}{4pp'} = h_{p'-r,p-s}, \; (0 < r < p',\; 0 < s < p,\; sp' < rp) \quad (1.168)$$

At about the same time a subset of the theories, the unitary minimal models with $p' = p + 1$ and $c = 1 - \frac{6}{p(p+1)}$, was constructed by Friedan, Qiu and Shenker [fr84]. In a unitary CFT all normes are positive (except for the zero vector in the Hilbert space), which implies that $c \ge 0$ because

$$c/2 = \langle 0|[L_2, L_{-2}]|0\rangle = \langle 0|L_2L_{-2}|0\rangle = ||L_{-2}|0\rangle||^2 \ge 0 \qquad (1.169)$$

where we used scale invariance $L_0|0\rangle = 0$ of the vacuum and the hightest-weight condition $L_2|0\rangle = 0$. Simlarly, for highest weight vectors with $L_n|h\rangle = 0$ for $n > 0$ and $L_0|h\rangle = h|h\rangle$

$$0 \le ||L_{-n}|h\rangle||^2 = \langle h|[L_n, L_{-n}]|h\rangle = \left(\tfrac{c}{12}(n^3 - n) + 2nh\right)\langle h|h\rangle \qquad (1.170)$$

implies that conformal weights are nonnegative in a unitary theory because the r.h.s. is dominated by the first term for $n \to \infty$. $h = 0$ is only possible for a translation invariant state, i.e. for the vacuum. Unitarity is important for the internal sector of a string model in order to guarantee positivity of physical states. It is, however, clearly violated for unphysical states with time-like polatizations and in the ghost sector. There are also applications of non-unitary models in solid state physics.

Since the repertoire of Virasoro rational theories is quite limited it is useful to look for additional symmetries that allow us to construct more interesting families of models. The relevant objects are the holomorphic and the antiholomorphic subalgebras of the operator algebra, which contain the energy momentum tensor and thus extend the Virasoro algebra to a larger chiral and anti-chiral algebra, often denoted by $\mathcal{A}_L$ and $\mathcal{A}_R$, respectively. These organize the fields into larger conformal families and rationality is defined as the finiteness of number of these larger families. In the rational case, Ward identities again are sufficiently to constrain the conformal blocks to finite dimensional solution spaces of differential equations. The holomorphic/chiral data are interesting by themselves. In mathematics their study is the subject of the theory of 'vertex algebras'. They also have direct applications in physics in the quantum Hall effect, where the external magnetic field breaks parity and leads to chiral currents.

Important examples of chiral algebras $\mathcal{A}$ include the *current algebras*, which are also called *affine Lie algebras* and are an subclass of the Kac–Moody algebras (i.e. infinite dimensional Lie algebras that can be defined in terms of generalized Cartan matrices that violate positivity). Current algebras are generated by currents $J^a(z)$ and their OPEs are of the form

$$\underbrace{J^a(z)J^b(w)}= \frac{\kappa^{ab}K}{(z-w)^2} - \frac{f^{ab}{}_c J^c(w)}{z-w} \tag{1.171}$$

with K a central operator, i.e. $K = k\mathbf{1}$ in an irreducible representation. In terms of the Fourier modes $T^a_n = \oint \frac{dz}{2\pi i} J(z) z^{n+1}$

$$[T^a_m, T^b_n] = f^{ab}{}_c T^c_{m+n} + m\delta_{m+n}\kappa^{ab}K, \qquad J^a(a) = \sum T^a_n z^{-n-1}. \tag{1.172}$$

Jacobi identities imply that $f^{ab}{}_c$ are the structure constants of a Lie algebra $\mathfrak{g}$, which is actually contained as the subalgebra of 0-modes $T^a = T^a_0$. In turn, affine Lie algebras $\hat{\mathfrak{g}}$ can be constructed as central extensions (quantizations) of loop algebras, i.e. gauge theories on the circle with gauge group $\mathfrak{g}$. κ^{ab} has to be ad_{T^a}-invariant and hence, in the compact case, equal to the Killing metric of $\mathfrak{g}$. Unitarity implies quantization of the *level* $k \in \mathbb{Z}$. An interesting property of affine Lie algebras is that the energy momentum tensor is bilinear in the currents

$$T(z) = \frac{\kappa_{ab}}{2(k+g^\vee)} : J^a(z)J^b(z) :, \qquad c = \frac{kd}{k+g^\vee} \tag{1.173}$$

because this composite operator has just the right OPEs with the currents, which can be used to compute the central charge c in terms of the level k, and the dimension d and the dual Coxeter number $g^\vee$ of the Lie algebra $\mathfrak{g}$. An extension $\mathcal{A}$ of the Virasoro algebra by fields of higher conformal weight $h \geq 3$ is called a W algebra [bo93].

The $N = 1$ superconformal or supervirosoro algebra, which emerges in the RNS formalism, can be regarded as an algebra extension by the supercurrent T_F of conformal weight $h = 3/2$,

$$\underbrace{T_F(z)T_F(w)} = \frac{2c/3}{(z-w)^3} + \frac{2T(w)}{(z-w)}, \qquad \{G_r, G_s\} = 2L_{r+s} + \frac{c}{3}(r^2 - \tfrac{1}{4})\delta_{r+s}, \tag{1.174}$$

with $T_F(z) = \sum_r \frac{G_r}{z^{r+3/2}}$. Strictly speaking this is not a chiral algebra because modular T-invariance implies that only fields of integral conformal weight can be chiral so that after the GSO projection only composites of the supercurrent with other fields can belong to $\mathcal{A}$. For superstring compactifications from 10 to 4 dimensions with a superconformal theory with $c = 9$ as 'internal sector' it has been shown that the supercurrents is actually split into a positive and a negative part $T_F(z) = G^+(z)+G^-(z)$ by a $U(1)$ current $J(z)$ that extends the superconformal algebra to an $N = 2$ algebra [ba88],

$$\{G^-_r, G^+_s\} = 2L_{r+s} - (r-s)J_{r+s} + \tfrac{c}{3}(r^2 - \tfrac{1}{4})\delta_{r+s}, \qquad [L_n, G^\pm_r] = (\tfrac{n}{2} - r)G^\pm_{n+r}, \tag{1.175}$$

$$[J_m, J_n] = \tfrac{c}{3}m\delta_{m+n}, \qquad [J_n, G^\pm_r] = \pm G^\pm_{n+r}, \qquad [L_n, J_m] = -mJ_{m+n} \tag{1.176}$$

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