Exercises (Supersymmetry, LVA Nr. 136.030)

Exercise 1: Lorentz algebra and γ_* identitites

a) Derive the Lorentz algebra (1,2) from its vector representation.

b) Derive the identities (1,5) for γ_*.

Exercise 2: Recursion relations of Clifford algebras and construction of irreps

a) Derive the recursion relation (1.7).

b) Show that the fermionic creation/annihilation algebra $\{a_i, a_j^\dagger\} = \delta_{ij}$ with $i, j = 1, \ldots N$ is equivalent to a Clifford algebra in $D = 2N$ dimensions and that the Fock space representation provides an irreducible prepresentation with $\gamma_{2i-1} = a_i + a_i^\dagger$ and $\gamma_{2i} = i(a_i - a_i^\dagger)$. (Note that Lorentz transformations transform bosonic/fermionic states into themselves so that even/odd subspaces corresond to Weyl spinors.)

Exercise 3: Symmetries of charge conjugation C and $B = CA^T$

a) Derive the (anti)symmetry $C^T = (-1)^{\binom{D/2-1}{2}} C$ of charge conjugation (cf. eqs. (1.12) ff).

b) Derive the sign (1.22) of the coefficient b in $B = bB^T$.

Exercise 4: Symmetries of charge conjugation C and $B = CA^T$

The SYM action (cf. Green, Schwarz Witten, Superstring theory I, p. 244ff)

$$S_{SYM} = \int -\tfrac{1}{4}F^2 + \tfrac{i}{2}\bar{\lambda}\gamma^a D_a \lambda$$

is invariant under the SUSY transformations

$$\delta_e A_m^a = \tfrac{i}{2}\bar{\varepsilon}\gamma_m \lambda^a, \qquad \delta_\varepsilon \lambda^a = -\tfrac{1}{4}F_{mn}^a \Gamma^{mn}\varepsilon$$

for Weyl spinors in $d = 4, 6$ and for Majorana Weyl spinors in $d = 10$ dimensions. The Γ-matrices are defined as

$$\Gamma^{m_1 m_2} = \tfrac{1}{2!}\left(\gamma^{m_1}\gamma^{m_2} - \gamma^{m_2}\gamma^{m_1}\right),$$

and accordingly in cases with more indices.
Compute the variation of the action for the case of ten dimensions and show that all terms linear in the gauginos vanish up to total derivatives.

Useful relations:

$$\bar{\lambda}\gamma^l \Gamma^{mn}\varepsilon = -\bar{\varepsilon}\Gamma^{mn}\gamma^l \lambda$$

$$\Gamma^{mn}\gamma^l = \Gamma^{mnl} + \gamma^m \eta^{nl} - \gamma^n \eta^{ml}$$

$$D_{[m}F_{np]} = 0$$

[..] denotes antisymmetrization of the indices.

Remarks: The on-shell counting of degrees of freedom counts transversal bosonic degrees as $2(d-2)$ because of their 2nd order equations of motion, which matches $2^(d-1)$ for Weyl spinors and an extra factor 1/2 for Majorana in 10d.
At the quantum level gauge anomalies in 10d require a coupling to SUGRA, which is also anomalous. A cancellation mechanism between gauge and gravitational anomalies, which works exactly for the gauge groups $SO(32)$ and $E_8 \times E_8$ was discovered by Green and Schwarz in 1984 from the low energy limit of open string theory.

Exercise 5: Fierz identities and Super Yang-Mills theory in ten dimensions.
All spinors in this exercise are Majorana Weyl spinors in ten dimensions with Minkowskian signature.

a) Prove the following identities:

$$\begin{aligned} \gamma_m \gamma^n \gamma^m &= (2-D)\gamma^n \\ \gamma_m \Gamma^{nrs} \gamma^m &= (6-D)\Gamma^{nrs} \\ \gamma_m \Gamma^{nrslp} \gamma^m &= (10-D)\Gamma^{nrslp} \end{aligned}$$

b) Given two Weyl spinors λ^a, λ^b of the same chirality show that

$$\overline{\lambda}^a \Gamma^{m_1 \dots m_n} \lambda^b = 0$$

for all even n. Furthermore, show that

$$\overline{\lambda}^a \Gamma^{mnp} \lambda^b = \overline{\lambda}^b \Gamma^{mnp} \lambda^a .$$

c) Show that

$$c\epsilon_{m_1 \dots m_D} \gamma^{m_{n+1}} \dots \gamma^{m_D} \tilde{\gamma} = \gamma_{m_1} \cdots \gamma_{m_n}$$

holds if none of the indices m_i are equal and compute the coefficient c. You can also restrict yourself to the cases which are actually needed for exercise d).

d) As you have seen in Exercise 4, there is only one term which is trilinear in the gauginos λ^a that occurs when varying the action of 10D SYM Theory:

$$\Delta = f_{abc} \bar{\varepsilon} \gamma_m \lambda^a \bar{\lambda}^b \gamma^m \lambda^c \ .$$

Here f_{abc} are the totally antisymmetric structure constants of the Lie algebra of the gauge group and ε is the supersymmetry parameter.
Using the Fierz identity (eq. (1.32) in the script) and the relations a) to c), show that $\Delta = 0$.

Hints: Use the Fierz identity 1.32 by inserting unity into Δ in such a way that the γ-matrices align in a way that lets you exploit the relations a). Relation b) makes half of the remaining terms vanish, so that only four terms remain. Two of these vanish on their own and the remaining ones are proportional to Δ. The proof is complete by noting that this gives $\Delta = n\Delta$, $n \neq 1$, so that $\Delta = 0$ must hold.