DIPLOMARBEIT

Superfields and Supersymmetry

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Contents

Symbols

D_A	Super covariant spinorial/spacetime derivatives.
$\mathbb{D}_A$	Realization of spinorial/spacetime derivatives in Superspace.
$\mathcal{D}_A$	Gauge covariant spinorial/spacetime derivatives.
∇_A	Gauge covariant spinorial/spacetime derivatives in Superspace
L	Lagrangian
$\mathcal{L}$	Lagrangian density
H	Hamiltonian
$\mathcal{H}$	Hamiltonian density
δ_i	Generators of the gauge group
T_i	Hermitian representation matrices of δ_i
$A_\alpha, \overline{A}_{\dot{\alpha}}$	(Gauge) prepotential
A_a	(Gauge) potential
$\overline{Q}_{\dot{\alpha}}$, Q_α	Noether charges for SUSY transformation (SUSY generators)
$\overline{\mathbb{Q}}_{\dot{\alpha}}$, $\mathbb{Q}_\alpha$	Realization of the SUSY generators in Superspace
PI	Partial integration

Introduction

Elementary particle physics is based on symmetry principles that are realized in relativistic quantum field theory. The observable object in quantum field theory is the S-matrix. Thus it is important to study those symmetries of the S-matrix that are compatible with quantum field theory. In 1967 Coleman and Mandula stated in their famous "no-go"theorem that it is impossible to combine spacetime symmetries with internal symmetries nontrivially [co67]. This destroyed the hope for a fusion between the Poincare group and the internal symmetries. However, it was realized that this result can be circumvented by supersymmetry (SUSY), which introduces graded symmetry algebras. SUSY relates fermions with bosons. It unifies spacetime symmetries with internal symmetries and it was proven to be the only possible further symmetry beside those two [ha75] (again, under certain assumptions).

There are two ways of looking at SUSY. One is to consider the SUSY algebra that acts on ordinary fields (functions of spacetime). These fields are called component or tensor fields. The other possibility is to consider the Superspace approach, which is an extension of spacetime that comes when introducing additional anticommuting coordinates. Superfields are then functions on Superspace, and the SUSY algebra is realized as rotations and translations involving the Superspace coordinates. The advantage of the Superfield approach is that it introduces auxiliary fields that are needed in order to have off-shell closure of the SUSY algebra [vp03]. But this is also the disadvantage of the Superspace approach: in many cases it introduces too many unphysical fields.

The "symmetries" that describe the fundamental interactions are the Yang-Mills (YM) theories, which come from generalizations of the definition of a physical state in quantum mechanics. That's why YM symmetries are not real symmetries (i.e. transform physical states into physical states) but rather describe the redundancy of physical states [FU97]. It is interesting how the supersymmetric extensions of Yang-Mills theories look like. This will be the main focus of this work.

This thesis is organized as follows: in chapter 1 the basic tools are listed. These are the jet space approach, the structure group and symmetry algebras. Since the algebraic side will use graded Lie algebras, Bianchi identities have to be considered.

Chapter 2 introduces $\mathcal{N} = 1$ SUSY in field theories and the most general supersymmetric Lagrangian for matter fields is constructed. This will be used to make a consistency check of the algebra, by calculating the Dirac bracket of the Noether charges of a SUSY transformation (Appendix A). Then the Superspace formalism is introduced, and the connection between Superfields and component fields is established. In the next step a similar relation will be formulated between operators on component fields and operators on superfields.

Chapter 3 deals with supersymmetric Yang-Mills theories. First the Superspace approach is presented which introduces vectorfields, whose components are the gauge fields. In the next section the algebraic approach via covariant derivatives is formulated. The constraints that have to be imposed are then solved with the Bianchi Identities. Using the knowledge of chapter 2 the two approaches are related and it is shown that they are equivalent after some partial gauge fixing.

Chapter 4 provides an outlook to Supergravity (SUGRA). The symmetry algebra which will be more general than for YM theories is presented, and Dragon's Theorem is stated (Appendix B).

In Appendix C there is a summary of the used notations and conventions and a list of useful formulae.

Chapter 1

Tools

1.1 Jet space

The concept of jet spaces [OL93] provides a framework for the discussion of symmetries. Jet spaces are spaces whose coordinates are the ordinary coordinates x^m of a base space M (e.g. spacetime), and additional variables $\partial_{m_1 \cdots m_k}\phi^i$ with $k = 0, 1, 2 \cdots$. The fields and their formal partial derivatives are thus regarded as independent algebraic objects. These additional variables are also called *jet bundle coordinates*. They are denoted by $[\phi^i] = \{\partial_{m_1 \cdots m_k}\phi^i, k = 0, 1, 2 \cdots\} \quad [\phi^i] \in U^i$. The total space $M \times U$ is called *jet space* of the underlying space M. The fields as functions of the coordinate $x^m \in M$ can be viewed as a map from the jet space $M \times U$ to some number field F

$$\Delta : M \times U \to F \tag{1.1}$$

such that the formal derivatives agree with the partial derivatives.

1.2 Structure group

Gauge generators are denoted by δ_i. They act linearly on fields $\delta_i\phi$ and fullfill the Lie algebra

$$[\delta_i, \delta_j] = f_{ij}^k \delta_k \tag{1.2}$$

A finite dimensional representation of δ_i may be constructed with matrices T_i, such that $\delta_i\phi = -T_i\phi$ where ϕ is some field in the representation space [dr87]. Let the commutator act on some field $[\delta_i, \delta_j]\phi = \delta_i\delta_j\phi - \delta_j\delta_i\phi = -(\delta_i T_j\phi -$

$\delta_j T_i \phi) = -(T_j \delta_i \phi - T_i \delta_j \phi) = [T_j, T_i]$. On the other hand is $[\delta_i, \delta_j]\phi = f_{ij}^k \delta_k \phi = -f_{ij}^k T_k \phi$ Therefore the Lie algebra for the matrices T_i is of the form

$$[T_i, T_j] = f_{ij}^k T_k \tag{1.3}$$

1.3 Symmetry algebras and Bianchi identities

In this section the general structure of closed irreducible symmetry algebras (or gauge algebras) is analyzed for which the infinitesimal symmetry transformations are derivations. Concider a set of graded derivations Δ_M that form a closed graded commutator algebra

$$[\Delta_M, \Delta_N\} = \mathcal{F}_{MN}{}^P \Delta_P \tag{1.4}$$

with some graded structure functions $\mathcal{F}_{MN}{}^P = -(-)^{MN} \mathcal{F}_{NM}{}^P$. The grading of the structure functions is given by $|\mathcal{F}_{MN}{}^P| = |M| + |N| + |P| \mod 2$. The Δ_M have to fulfill the Jacobi identity because they are derivations and commutators of derivations are again derivations. Introducing the non-associative product $A \circ B := [A, B]$ introduces a derivation on the algebra of derivations : $A \circ (B \circ C) = (A \circ B) \circ C + (-)^{AB} B \circ (A \circ C)$. This is the Jacobi identity, using commutators $\sum_{ABC} (-)^{AC} [A, [B, C\}\} = 0$. Hence extra signs occur as compared to the non graded cyclic sum $\sum_{ABC} X_{ABC} := X_{ABC} + X_{BCA} + X_{CAB}$. Plugging (1.4) into the Jacobi identities yields the Bianchi Identities

$$\sum_{MNP} (-)^{MP} \left(\Delta_M \mathcal{F}_{NP}{}^Q - \mathcal{F}_{MN}{}^R \mathcal{F}_{RP}{}^Q \right) = 0. \tag{1.5}$$

These equations will be the crucial consistency conditions for the symmetry algebras that will be considered. The aim is to consider field theories with a set of elementary fields φ. Therefore it will always be assumed that the Δ_M are realized on functions $\phi^i(\varphi) =: \phi^i$ of the elementary fields and their derivatives. The fields ϕ^i are called *tensor fields* [`br91`]. If the realization of the Δ_M is linear in the $\frac{\partial}{\partial \phi^i}$, it is called a *linear realization.* Then $\Delta_M = f_M^i(\phi) \frac{\partial}{\partial \phi^i}$ with structure functions that are in general field-dependent $\mathcal{F}_{MN}{}^P = \mathcal{F}_{MN}{}^P(\phi)$.

The physical theories are obtained by specifying the fields $\mathcal{F}_{MN}{}^P$ in such a way that the requirements (1.5) are satisfied. For $D = 4$, $N = 1$ SUGRA, which contains $D = 4$, $N = 1$ super YM, the covariant symmetry transformations Δ_M are split into *space-time* symmetries $\{\mathcal{D}_A\} = \{\mathcal{D}_a, \mathcal{D}_{\underline{\alpha}}\}$ and *internal symmetries* $\{\delta_I\} = \{l_{ab}, \delta_i\}$, which generate Lorentz transformations and Yang–Mills group actions [1]. Thus $\{\Delta_M\} = \{\mathcal{D}_A\ , \delta_I\}$ and the graded commutator relations read

$$[\mathcal{D}_A, \mathcal{D}_B\} = -T_{AB}{}^C\mathcal{D}_C + F_{AB}{}^I\delta_I, \tag{1.6}$$
$$[\delta_I, \mathcal{D}_A\} = -g_{IA}{}^B\mathcal{D}_B, \tag{1.7}$$
$$[\delta_I, \delta_J\} = f_{IJ}{}^K\delta_K, \tag{1.8}$$

with torsions $T_{AB}{}^C$, field strengths $F_{AB}{}^I$, representation matrices $(g_I)_A{}^B$, and structure constants $f_{IJ}{}^K$. This is not the most general form a closed symmetry algebra of this type can have, because (1.7) contains no term with δ_I on the right hand side. It is imposed that the $(g_I)_A{}^B$ and the $f_{IJ}{}^K$ are constant, whereas the $T_{AB}{}^C$ and $F_{AB}{}^I$ are in general field dependent [br02]. The Bianchi identities (1.5) then read for the various index pictures ${}_{MNP}{}^Q$:

$${}_{IJK}{}^L: \quad f_{IJ}{}^M f_{MK}{}^L + f_{JK}{}^M f_{MI}{}^L + f_{KI}{}^M f_{MJ}{}^L = 0 \tag{1.9}$$
$${}_{IJK}{}^A: \quad 0 = 0 \tag{1.10}$$
$${}_{IJA}{}^K: \quad 0 = 0 \tag{1.11}$$
$${}_{IJA}{}^B: \quad g_{IA}{}^C g_{JC}{}^B - g_{JA}{}^C g_{IC}{}^B = f_{IJ}{}^K g_{KA}{}^B \tag{1.12}$$
$${}_{IAB}{}^C: \quad \delta_I T_{AB}{}^C = -g_{IA}{}^D T_{DB}{}^C + (-)^{AB} g_{IB}{}^D T_{DA}{}^C + T_{AB}{}^D g_{ID}{}^C \tag{1.13}$$
$${}_{IAB}{}^J: \quad \delta_I F_{AB}{}^J = -g_{IA}{}^C F_{CB}{}^J + (-)^{AB} g_{IB}{}^C F_{CA}{}^J + (-)^{IA+IB} F^K_{AB} f_{KI}{}^J \tag{1.14}$$
$$\text{BI 1:} \quad {}_{ABC}{}^D: \quad \sum_{ABC} (-)^{AC}(\mathcal{D}_A T_{BC}{}^D + T_{AB}{}^E T_{EC}{}^D - F_{AB}{}^I g_{IC}{}^D) = 0 \tag{1.15}$$
$$\text{BI 2:} \quad {}_{ABC}{}^I: \quad \sum_{ABC} (-)^{AC}(\mathcal{D}_A F_{BC}{}^I + T_{AB}{}^D F_{DC}{}^I) = 0 \tag{1.16}$$

Equation (1.9) is the Jacobi identity for the structure constants of the Lie algebra of the gauge generators. (1.13) and (1.14) state that the torsions and fieldstrengths transform like ordinary tensors with respect to their index picture. (1.15) is known as the *first Bianchi identity* and (1.16) is called the *second Bianchi identity.*

[1] More generally there can be more symmetry generators which belong to the internal symmetries: dilatons δ_W, Weyl symmetries δ_R and if there are more supersymmetries R symmetries δ_R

Chapter 2

Supersymmetric field theories

2.1 SUSY realization on fields

The aim of this section is to construct local quantum field theories whose symmetry algebras contain SUSY generators. What we actually want to construct is local actions depending on some set of elementary fields that transform into total derivatives under SUSY transformations. In this context it is useful to think in terms of jet bundles. Conceptually it is important to distinguish between the supercharge Q_α, the supersymmetry transformation D_α that should act linearly on the elementary fields, the implementation $\mathbb{Q}_\alpha$ of SUSY transformations in terms of a superspace differential operator acting on superfields, and the covariant derivative $\mathbb{D}_\alpha$ that also acts in superspace. Denoting the canonical coordinates by q^i and a symmetry transformation by $\delta_I q^i = f_I{}^i(q,\dot{q})$ the time derivative of the Noether charge is $\dot{Q}_I = \delta_I q^i\ (\delta L/\delta q^i)$. Using the Poisson brackets

$$\{A,B\}_{PB} := (-)^{iA}\left(\frac{\partial A}{\partial q^i}\frac{\partial B}{\partial p_i} - (-)^i \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q^i}\right), \qquad \{p_i, q^j\}_{PB} = -\delta_i{}^j \tag{2.1}$$

with the Noether charge we can, in turn, recover the symmetry transformation

$$\delta_I A := \{Q_I, A\}_{PB}, \qquad \{Q_I, H\}_{PB} = 0, \qquad \dot{Q}_I = \delta_I q^i\ \frac{\delta L}{\delta q^i}. \tag{2.2}$$

Upon quantization Poisson brackets are replaced by $-i/\hbar$ times commutators

$$i\hbar\{p_i, q^j\}_{PB} \ \rightarrow\ [P_i, Q^j] = -i\hbar\delta_i{}^j, \qquad \delta_I A = \tfrac{i}{\hbar}\,[Q_I, A]. \tag{2.3}$$

The Schrödinger equation $i\partial_t\psi = H\psi$ implies the time evolution $\dot{\mathcal{O}} = i[H,\mathcal{O}]$ of Heisenberg operators. This is consistent with

$$\{P_m,\phi\}_{PB} = -\partial_m\phi, \qquad [P_m,\phi] = -i\partial_m\phi \tag{2.4}$$
$$\{Q_\alpha,\phi\}_{PB} = -D_\alpha\phi, \qquad [Q_\alpha,\phi] = -iD_\alpha\phi \tag{2.5}$$
$$\{\overline{Q}_{\dot\alpha},\phi\}_{PB} = -\overline{D}_{\dot\alpha}\phi, \qquad [\overline{Q}_{\dot\alpha},\phi] = -i\overline{D}_{\dot\alpha}\phi \tag{2.6}$$

$$\{Q_\alpha,\overline{Q}_{\dot\beta}\} = 2\sigma^m_{\alpha\dot\beta}P_m \tag{2.7}$$
$$\{D_\alpha,\overline{D}_{\dot\alpha}\} = 2i\partial\!\!\!/_{\alpha\dot\alpha} := 2i\sigma^m_{\alpha\dot\alpha}\partial_m \tag{2.8}$$
$$[D_\alpha,\partial_m] = [\overline{D}_{\dot\alpha},\partial_m] = 0 \tag{2.9}$$

because $\{D_\alpha,\overline{D}_{\dot\alpha}\}\phi = i[Q_\alpha,i[\overline{Q}_{\dot\alpha},\phi\}\} + i[\overline{Q}_{\dot\alpha},i[Q_\alpha,\phi\}\} = -[\{Q_\alpha,\overline{Q}_{\dot\alpha}\},\phi] = 2i\sigma^m_{\alpha\dot\alpha}\partial_m$, where $[A,B\} := AB - (-)^{AB}BA$ denotes the graded commutator [WE83, dr87]. We also use the abbreviation $v_{\alpha\dot\alpha} := v\!\!\!/_{\alpha\dot\alpha} := \sigma^m_{\alpha\dot\alpha}v_m$ to write vectors in terms of spinor indices. A consistency check of the signs may be performed by calculating the Noether charges Q_α and $\overline{Q}_{\dot\alpha}$ of a SUSY action, which come from a SUSY transformation and then calculate the Dirac bracket between them.

In order to construct a field theory with a linear realization of supersymmetry we next have to find representations of the algebra $\{D_\alpha,\overline{D}_{\dot\alpha}\} = 2i\sigma^m_{\alpha\dot\alpha}\partial_m$ and then constructions of invariant actions depending on those fields. The most natural representation is obtained by declaring $\overline{D}_{\dot\alpha}$ to be 'annihilation operators' on some elementary scalar field ϕ. Since $D_\alpha D_\beta D_\gamma = 0$ the resulting (scalar) *chiral multiplet* consists of ϕ, the Weyl spinor $\chi_\alpha := D_\alpha\phi/\sqrt{2}$ and the auxiliary field $F := -D^2\phi/4$ (F is not dynamical in a renormalizable theory since it has mass dimension 2 if ϕ has its canonical dimension 1). Note that ϕ must be a complex field since reality of ϕ would imply that it is also antichiral $D_\alpha\phi = 0$ and thus, because of the SUSY algebra, constant. The action of $\overline{D}_{\dot\alpha}$ on χ_α and F is fixed by the SUSY algebra and the definition of these fields. Denoting the SUSY transformation with constant commuting parameters $\xi^{\underline{\alpha}}$ by $s = \xi^\alpha D_\alpha + \overline{\xi}^{\dot\alpha}\overline{D}_{\dot\alpha}$ we find

$$\overline{D}_{\dot\alpha}\phi = 0, \qquad \chi_\alpha = \tfrac{1}{\sqrt{2}}D_\alpha\phi, \qquad F = -\tfrac{1}{4}D^2\phi, \tag{2.10}$$
$$D_\alpha\phi s\phi = \sqrt{2}\xi\chi, \qquad s\chi_\alpha = \sqrt{2}(\xi_\alpha F + i\sigma^a_{\alpha\dot\alpha}\overline{\xi}^{\dot\alpha}\partial_a\phi), \qquad sF = \sqrt{2}i\sigma^a_{\alpha\dot\alpha}\overline{\xi}^{\dot\alpha}\partial_a\chi^\alpha. \tag{2.11}$$

as is easily checked using the identities

$$D^2 = -\varepsilon^{\alpha\beta}D_\alpha D_\beta, \qquad D_\alpha D_\beta = \tfrac{1}{2}\varepsilon_{\alpha\beta}D^2,$$
$$\left[D^2,\overline{D}_{\dot\alpha}\right] = 4iD^\alpha\partial\!\!\!/_{\alpha\dot\alpha}, \quad [D_\alpha,\overline{D}^2] = 4i\partial\!\!\!/_{\alpha\dot\alpha}\overline{D}^{\dot\alpha},$$

which follow from our conventions

$$D^\alpha = \varepsilon^{\alpha\beta} D_\beta, \qquad \varepsilon_{\alpha\beta}\varepsilon^{\beta\gamma} = \delta_\alpha^\gamma, \tag{2.12}$$

$$D^2 := D^\alpha D_\alpha, \qquad \overline{D}^2 := \overline{D}_{\dot\alpha}\overline{D}^{\dot\alpha}. \tag{2.13}$$

2.2 SUSY Lagrangians

Invariant Lagrangians can be constructed by observing that $D^2\overline{D}^2$ acting on any (composite) field and D^2 acting on a (composite) chiral field always give expressions that transform into total derivatives. It can be shown [br92] that the most general supersymmetric lagrangian $\mathcal{L}$ that depends on a set $\{\phi^i\}$ of chiral fields and the corresponding hermitian conjugate anti-chiral fields $\{\overline{\phi}^i\}$ is of the form[1]

$$\mathcal{L} = -\tfrac{1}{4}D^2 L + h.c., \qquad L = \tfrac{3}{8}\overline{D}^2 K([\phi, D\phi, D^2\phi], [\overline{\phi}, \overline{D\phi}, \overline{D}^2\overline{\phi}]) + g(\phi), \tag{2.14}$$

where g is called *superpotential* and K is called *Kähler potential.* Note that the superpotential can be chosen not to contain any derivatives (no ∂'s and no D's). A redefinition $K \to K + f(\phi) + f^*(\overline{\phi})$, which changes the action only by total derivatives, is called Kähler transformation. Such a transformation together with a suitable normalization of the chiral fields can be used to bring an analytic Kähler potential into the form $K = -\frac{1}{3}\sum_i \overline{\phi}^i\phi^i + \ldots$ if the kinetic energies are positive. The dots denote terms of dimension 3 or higher. If we demand renormalizability such terms are forbidden and the superpotential

[1] It is easy to see that all terms are of the form $D^2X + c.c$ and that all terms containing chiral and antichiral fields can be written as $D^2\overline{D}^2 Y$: We define the operator t^α by $t^\alpha D_\beta\phi = \delta^\alpha_\beta\phi^i$, $t^\alpha D^2\phi = -2D^\alpha\phi^i$ and $t^\alpha\phi^i = t^\alpha\overline{\phi}^i = \{t^\alpha, \overline{D}_{\dot\alpha}\} = [t^\alpha, \partial_a] = 0$ so that $\{t^\alpha, D_\beta\} = \delta^\alpha_\beta\mathcal{E}(\phi^i, \chi^i, F^i)$, where $\mathcal{E}$ is the Euler operator that counts the degree of homogeneity in the

component fields of chiral multiplets (formally one may write '$t^\alpha = \partial/\partial(D_\alpha)$' when acting on chiral fields). As t and D act linearly we may decompose the action into terms $\mathcal{L}_n$ of definite degree n in (ϕ^i, χ^i, F^i). Since $[D^2, t^\alpha] = 2\mathcal{E}D^\alpha$ and $[D^2, t^2] = 4\mathcal{E}(tD - \mathcal{E})$ a supersymmetric action with $D_\alpha\mathcal{L}_n = \partial_a X^a_\alpha$ can be written as $\mathcal{L}_n = -\frac{1}{4n^2}D^2(t^2\mathcal{L}_n) + \frac{1}{4n^2}\partial_a(t^2 D^\alpha X^a_\alpha + 4nt^\alpha X^a_\alpha)$ for $n > 0$, i.e. $\mathcal{L}_n$ can be written as D^2 acting on some local function up to total derivatives. Similarly it can be shown that terms depending on antichiral fields can be written as $\overline{D}^2(-\overline{t}^2\mathcal{L}_n/4n^2)$ and terms that depend on both, chiral and antichiral fields, are of the form $D^2\overline{D}^2 K$.

To show that X can be assumed to depend only on ϕ (without derivatives) is more complex and this result depends on the 'QDS-structure' of the SUSY representation on chiral fields [br92]; note that the linear SUSY representations on local fields are infinite dimensional because $\{D, \overline{D}\}$ contains the partial derivative.

must be cubic, so that

$$\mathcal{L} = -\frac{1}{4}D^2\left(-\frac{1}{8}\overline{D}^2\overline{\phi}\phi + g(\phi)\right) + h.c. \tag{2.15}$$
$$g = \gamma + \lambda_i\phi^i + \frac{1}{2}m_{ij}\phi^i\phi^j + \frac{1}{6}\kappa_{ijk}\phi^i\phi^j\phi^k.$$

In order to express the Lagrangian in terms of the chiral and antichiral multiplet one needs the relation

$$[D^2, \overline{D}^2] = 4i\partial\!\!\!/_{\alpha\dot{\alpha}}[D^\alpha, \overline{D}^{\dot{\alpha}}] = (8iD\partial\!\!\!/\overline{D} + 16\Box) \tag{2.16}$$

because of $\{D^\alpha, \overline{D}^{\dot{\alpha}}\} = 2i\overline{\partial\!\!\!/}^{\dot{\alpha}\alpha}$ and $\partial\!\!\!/\overline{\partial\!\!\!/} = \Box\mathbf{1}$ with $\overline{\partial\!\!\!/} := \overline{\sigma}^m\partial_m$ and $\mathrm{tr}\,\mathbf{1} = \delta^\alpha_\alpha = 2$. Using the Leibnitz rule evaluation of

$$\begin{aligned}\mathcal{L} = \quad & (\overline{D}^2\overline{\phi}D^2\phi + 2D\phi D\overline{D}^2\overline{\phi} + D^2\overline{D}^2\overline{\phi}\phi)/32- \\ & -(D^2\phi^i\partial_i g + D\phi^i D\phi^j\partial_i\partial_j g)/4 + h.c.\end{aligned} \tag{2.17}$$

thus yields

$$\begin{aligned}\mathcal{L} = & -\tfrac{1}{2}\Box\overline{\phi}^i\phi^i + i\chi^i\sigma^a\partial_a\overline{\chi}^i + F^i\overline{F}^i + \\ & + F^i\partial_i g - \tfrac{1}{2}\chi^i\chi^j\partial_i\partial_j g + \overline{F}^i\partial_i g^* - \tfrac{1}{2}\overline{\chi}^i\overline{\chi}^j\partial_i\partial_j g^*,\end{aligned} \tag{2.18}$$

where the kinetic terms and $F\overline{F}$ come from the Kähler potential. Integrating out the auxiliary fields by inserting their equations of motion $\overline{F}_i = -\partial_i g$ we find the potential

$$V(\phi,\phi^*) = \sum_i |\partial_i g|^2 = |F(\phi)|^2 \tag{2.19}$$

for the scalar fields. The terms $-\frac{1}{2}\chi^i\chi^j\partial_i\partial_j g$ and their hermitian conjugates are the Yukawa couplings. (2.16) implies that we can define projection operators

$$\Pi_+ = \frac{\overline{D}^2D^2}{16\Box}, \quad \Pi_- = \frac{D^2\overline{D}^2}{16\Box}, \quad \Pi_T = -\frac{D\overline{D}^2D}{8\Box} = -\frac{\overline{D}D^2\overline{D}}{8\Box}, \tag{2.20}$$

$$\Pi_+ + \Pi_- + \Pi_T = \mathbf{1}, \tag{2.21}$$

where Π_+ and Π_- project onto chiral and anti-chiral fields, respectively (to see this, evaluate $D\overline{D}^2D = D[\overline{D}^2, D] + D^2\overline{D}^2$ and $\overline{D}D^2\overline{D} = \overline{D}[D^2,\overline{D}] + \overline{D}^2D^2$). Π_T is called transversal projector.

2.3 Superspace

In the *superspace* approach SUSY transformations are interpreted as motions in a space with anticommuting coordinates θ_α and $\overline{\theta}_{\dot\alpha}$ in addition to the space-time coordinates x^m [BU98, GA83]. Complete SUSY multiplets like (ϕ, χ, F) are combined into a single *superfield* $\Phi(x, \theta, \overline{\theta})$.

The supersymmetry transformation acting on a superfield is then represented by a linear combination of an ordinary partial derivative and a derivative with respect to the anticommuting coordinates. With an appropriate ansatz we find the operators

$$\mathbb{Q}_\alpha = \frac{\partial}{\partial\theta^\alpha} + i\partial\!\!\!/_{\alpha\dot\beta}\overline{\theta}^{\dot\beta}, \qquad \overline{\mathbb{Q}}_{\dot\alpha} = -\frac{\partial}{\partial\overline{\theta}^{\dot\alpha}} - i\theta^\beta\partial\!\!\!/_{\beta\dot\alpha}, \tag{2.22}$$

$$\{\mathbb{Q}_\alpha, \overline{\mathbb{Q}}_{\dot\alpha}\} = -2i\sigma^a_{\alpha\dot\alpha}\partial_a \tag{2.23}$$

that obey the appropriate algebraic relations (since $\sigma^{m*} = \sigma^{mT}$ and $\partial/\partial\psi^* = (-)^{|\psi|}(\partial/\partial\psi)^*$ we have $\mathbb{Q}^* = \overline{\mathbb{Q}}$). A superfield Φ is then a function in superspace that satisfies

$$\mathbb{Q}_\alpha\Phi = D_\alpha\Phi, \qquad \overline{\mathbb{Q}}_{\dot\alpha}\Phi = \overline{D}_{\dot\alpha}\Phi, \tag{2.24}$$

where D and $\overline{D}$ act on the component fields.[2]

$\mathbb{Q}$ does not map superfields to superfields since $\{\mathbb{Q}, \overline{D}\} = 0$ but $\{\mathbb{Q}, \overline{\mathbb{Q}}\} \neq 0$. To impose the chirality condition on superfields we thus need another differential operator in superspace, the *covariant derivative*

$$\mathbb{D}_\alpha = \frac{\partial}{\partial\theta^\alpha} - i\partial\!\!\!/_{\alpha\dot\beta}\overline{\theta}^{\dot\beta}, \qquad \overline{\mathbb{D}}_{\dot\alpha} = -\frac{\partial}{\partial\overline{\theta}^{\dot\alpha}} + i\theta^\beta\partial\!\!\!/_{\beta\dot\alpha} \tag{2.25}$$

which satisfies

$$\begin{aligned} &\{\mathbb{D}_\alpha, D_\beta\} = \{\mathbb{D}_\alpha, \mathbb{Q}_\beta\} = \{\mathbb{D}_\alpha, \overline{D}_{\dot\alpha}\} = \{\mathbb{D}_\alpha, \overline{\mathbb{Q}}_{\dot\alpha}\} = 0, \\ &\{\mathbb{D}_\alpha, \overline{\mathbb{D}}_{\dot\alpha}\} = 2i\sigma^a_{\alpha\dot\alpha}\partial_a, \end{aligned} \tag{2.26}$$

so that it preserves the superfield property [WE83]. Indeed, the chirality condition $\overline{\mathbb{D}}_{\dot\alpha}\Phi = 0$ for a superfield is equivalent to the chirality of its θ-independent part since

$$\mathbb{D}_\alpha e^{\theta D + \overline{\theta D}} = e^{\theta D + \overline{\theta D}}(\mathbb{D}_\alpha + i\partial\!\!\!/_{\alpha\dot\alpha}\overline{\theta}^{\dot\alpha} + D_\alpha) = e^{\theta D + \overline{\theta D}}(\partial_{\theta^\alpha} + D_\alpha) \tag{2.27}$$

[2] $\{\mathbb{Q}, \overline{\mathbb{Q}}\} = -\{D, \overline{D}\}$ is consistent with this equation because $\mathbb{Q}\Phi = D\Phi$ is no superfield.

The components of a chiral superfield are easily evaluated using the formulas

$$e^{\theta D+\overline{\theta D}} = e^{-i\theta\partial\!\!\!/\overline{\theta}} e^{\theta D} e^{\overline{\theta D}} = e^{i\theta\partial\!\!\!/\overline{\theta}} e^{\overline{\theta D}} e^{\theta D}, \tag{2.28}$$

which follow from $[\theta D, \overline{\theta D}] = 2i\theta^\alpha \partial\!\!\!/_{\alpha\dot\alpha}\overline{\theta}^{\dot\alpha}$ and the Baker–Campbell–Hausdorff formula

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]+\frac{1}{12}([A,[A,B]]-[B,[A,B]])+\text{ multiple commutators}}. \tag{2.29}$$

We thus obtain

$$\Phi(x,\theta,\overline{\theta}) = e^{-i\theta\partial\!\!\!/\overline{\theta}} e^{\theta D}\phi(x) = \phi(y) + \theta D\phi(y) - \tfrac{1}{4}\,\theta^2 D^2\phi(y), \tag{2.30}$$

$$y^m = x^m - i\theta\sigma^m\overline{\theta} \tag{2.31}$$

and the analogous formula for anti-chiral fields by complex conjugation. To obtain the $\overline{\theta}$-dependent components explicitly we just have to formally Taylor-expand $\phi(y)$, $\chi(y)$ and $F(y)$ in $y-x$.

Lemma 1. *Every superfield $\mathcal{F}$ can be written in the form $\mathcal{F}(\Phi) = e^{(\theta D+\overline{\theta D})}f(\phi)$ where $f(\phi)$ is the $\theta_{\underline{\alpha}}$ independent part of Φ.*

Proof. In the first step it is shown that $\Phi = e^{(\theta D+\overline{\theta D})}f(\phi)$ is a superfield. Evaluating $Be^A = e^A(B - [A,B] + \frac{1}{2}[A,[A,B]] - ...)$ with $B := \mathbb{Q}_\alpha - D_\alpha$ and $A := \theta D + \overline{\theta D})$ gives $[A,B] = -D_\alpha + 2i\partial\!\!\!/_{\alpha\dot\alpha}\overline{\theta}^{\dot\alpha}$ and $[A,[A,B]] = 2i\partial\!\!\!/_{\alpha\dot\alpha}\overline{\theta}^{\dot\alpha}$. Putting everything together $\mathbb{Q}\mathcal{F} = D\mathcal{F}$ follows from $\mathbb{Q}_\alpha f = i\overline{\theta}^{\dot\alpha}\partial\!\!\!/_{\alpha\dot\alpha} f$ and $\overline{\mathbb{Q}} = \overline{D}$ by complex conjugation. The second step shows that any nonvanishing superfield must have a nonvanishing θ-independent part. Splitting the superfield according to the subspaces of Superpace $S = \bigoplus S_{mn}$: $\mathcal{F} = \sum \mathcal{F}_{mn}$. The $\mathcal{F}_{mn}$ term is then a term of degree m in θ and of degree n in $\overline{\theta}$. The superfield conditions imply recursion relations which allow to express all $\mathcal{F}_{mn}$ linearly in $f = \mathcal{F}_{00}$. Since the difference of the superfields $\mathcal{F}$ and $e^{\theta D+\overline{\theta D}}f(\phi)$ is again a superfield, this difference must vanish, which completes the proof of the lemma. □

The map $\exp(\theta D + \overline{\theta D}) : \mathcal{S}_{00} \to \mathcal{S}$ is a bijection. It is surjective because $\forall \Phi \in \mathcal{S}$ there exists with Lemma 1 $\phi \in \mathcal{S}_{00}$ such that $\Phi = \exp(\theta D + \overline{\theta D})\phi$. Let $\Phi_1, \Phi_2 \in \mathcal{S}$, with $\Phi_1 = \exp(\theta D + \overline{\theta D})\phi_1, \Phi_2 = \exp(\theta D + \overline{\theta D})\phi_2$. Then $0 = \Phi_1 - \Phi_2 = \exp(\theta D + \overline{\theta D})(\phi_1 - \phi_2) \Rightarrow \phi_1 = \phi_2$ which shows that the map is injective. Therefore the map $\exp(\theta D + \overline{\theta D})$ is invertible and the θ-independent part is arbitrary and determines a unique superfield.

Lemma 2. *An operator $\mathcal{O}$ maps Superfields into Superfields iff it is of the form $e^{(\theta D+\overline{\theta D})}\mathcal{A}e^{-(\theta D+\overline{\theta D})}$, where $\mathcal{A}$ acts on $\mathcal{S}_{00}$. $\mathcal{A}$ is unique.*

Proof. "$\Leftarrow$": let Φ be a Superfield and $\mathcal{A}$ an operator that acts on $\mathcal{S}_{00}$, then $\mathcal{O}\Phi = e^{(\theta D+\overline{\theta D})}\mathcal{A}e^{-(\theta D+\overline{\theta D})}e^{(\theta D+\overline{\theta D})}\phi = e^{(\theta D+\overline{\theta D})}\mathcal{A}\phi$. This is with the previous Lemma a Superfield. "$\Rightarrow$": Let $\mathcal{O}$ be an operator that maps superfields into superfields, and Φ a superfield with $\Phi = e^{(\theta D+\overline{\theta D})}\phi$. Define the map $\mathcal{A} : \mathcal{S}_{00} \to \mathcal{S}_{00} \quad \phi \mapsto e^{-(\theta D+\overline{\theta D})}\mathcal{O}e^{(\theta D+\overline{\theta D})}\phi$. Then the diagram

$$\begin{tikzcd}
\mathcal{S} \arrow[r, "\mathcal{O}"] & \mathcal{S} \arrow[d, "e^{(\theta D+\overline{\theta D})}"] \\
\mathcal{S}_{00} \arrow[u, "e^{-(\theta D+\overline{\theta D})}"] \arrow[r, "\mathcal{A}"'] & \mathcal{S}_{00}
\end{tikzcd}$$

is commutative because $e^{(\theta D+\overline{\theta D})}$ is invertible. This shows the remaining direction. In order to show that $\mathcal{A}$ is unique, suppose for a given $\mathcal{O}$ there exist $\mathcal{A}, \tilde{\mathcal{A}}$ such that $\mathcal{O} = e^{(\theta D+\overline{\theta D})}\mathcal{A}e^{-(\theta D+\overline{\theta D})} = e^{(\theta D+\overline{\theta D})}\tilde{\mathcal{A}}e^{-(\theta D+\overline{\theta D})}$. Then $0 = e^{(\theta D+\overline{\theta D})}(\mathcal{A} - \tilde{\mathcal{A}})e^{-(\theta D+\overline{\theta D})}$ and therefore $\mathcal{A} = \tilde{\mathcal{A}}$. $\square$

From now on operators that map superfields into superfields are called *superfieldoperators*.

Example. $\mathbb{D}_\alpha = e^{(\theta D+\overline{\theta D})}D_\alpha e^{-(\theta D+\overline{\theta D})}$. $\mathbb{D}_\alpha$ is a superfieldoperator because $\{D_\alpha, \mathbb{D}_\beta\} = \{\overline{D}_{\dot\alpha}, \mathbb{D}_\beta\} = \{\mathbb{Q}_\alpha, \mathbb{D}_\beta\} = \{\overline{\mathbb{Q}}_{\dot\alpha}, \mathbb{D}_\beta\} = 0$ and therefore Lemma 2 may be used.

Indeed one could have used Lemma 2 to define the $\mathbb{D}$ as a natural superfield extensions of the operators D which act on $\mathcal{S}_{00}$.

The space of all superfields forms a ring. The zerofunction is a superfield. This shows together with the linearity of the D and $\mathbb{Q}$ the vector space structure. The fact that the product of two superfields is again a superfield follows from the fact that D and $\mathbb{Q}$ are derivations. Take the superfields Ψ and Φ then $\mathbb{Q}_\alpha(\Phi\Psi) = (\mathbb{Q}_\alpha\Phi)\Psi + (-)^{|\Phi|}\Phi(\mathbb{Q}_\alpha\Psi) = (D_\alpha\Phi)\Psi + (-)^{|\Phi|}\Phi(D_\alpha\Psi) = D_\alpha(\Phi\Psi)$. The same for $\overline{\mathbb{Q}}_{\dot\alpha}$ and $\overline{D}_{\dot\alpha}$

In the same way the space of all superfieldoperators with $(\mathcal{O}_1 + \mathcal{O}_2)\Phi := \mathcal{O}_1\Phi + \mathcal{O}_2\Phi$ and the multiplication $(\mathcal{O}_1 \circ \mathcal{O}_2)\Phi := \mathcal{O}_1(\mathcal{O}_2\Phi)$ forms a ring.

The advantage of the superspace formulation is that we can rewrite the action as a superspace integral and extend the Feynman rules to a supergraph calculus [WE83, br96]. To this end we define superspace integration with $\{z^M\} = \{x^m, \theta^\alpha, \overline{\theta}_{\dot\alpha}\}$ and δ-functions by

$$\begin{aligned}
\int d\theta^{\underline{\alpha}} = \frac{\partial}{\partial\theta^{\underline{\alpha}}}, \qquad & \int d^2\theta = \int d\theta^2 d\theta^1, \\
\int d^2\overline{\theta} = \int d\overline{\theta}^{\dot 1} d\overline{\theta}^{\dot 2}, \qquad & \int d^4\theta = \int d^2\theta d^2\overline{\theta}, \\
\int d^6 z = \int d^4x d^2\theta, \qquad & \int d^6\overline{z} = \int d^4x d^2\overline{\theta}, \\
\delta^2(\theta-\theta') = -\tfrac{1}{2}(\theta-\theta')^2, \qquad & \delta^6(z-z') = \delta^2(\theta-\theta')\delta^4(x-x'), \\
\delta^2(\overline{\theta}-\overline{\theta}') = -\tfrac{1}{2}(\overline{\theta}-\overline{\theta}')^2, \qquad & \delta^6(\overline{z}-\overline{z}') = \delta^2(\overline{\theta}-\overline{\theta}')\delta^4(x-x').
\end{aligned} \tag{2.32}$$

2.4 Superspace actions

Up to total derivatives the action can then be rewritten in terms of θ-integrations. Expanding $e^{\theta D+\overline{\theta D}} = \frac{1}{2}(e^{-i\theta\partial\!\!\!/\overline{\theta}}e^{\theta D}e^{\overline{\theta D}} + c.c.)$ we find

$$\begin{aligned}
\tfrac{1}{2}(\theta D+\overline{\theta D})^2 &= \tfrac{1}{2}\theta^\alpha\overline{\theta}^{\dot\alpha}[D_\alpha,\overline{D}_{\dot\alpha}] - \tfrac{1}{4}(\theta^2D^2+\overline{\theta}^2\overline{D}^2), \\
\tfrac{1}{3!}(\theta D+\overline{\theta D})^3 &= \theta^2(-\tfrac{1}{4}D^2\overline{D}_{\dot\alpha} + \tfrac{i}{2}D^\alpha\partial\!\!\!/_{\alpha\dot\alpha})\overline{\theta}^{\dot\alpha} + \overline{\theta}^2\theta^\alpha(-\tfrac{1}{4}\overline{D}^2D_\alpha - \tfrac{i}{2}\partial\!\!\!/_{\alpha\dot\alpha}\overline{D}^{\dot\alpha}), \\
\tfrac{1}{4!}(\theta D+\overline{\theta D})^4 &= \tfrac{1}{4}\theta^2\overline{\theta}^2(\tfrac{1}{8}(D^2\overline{D}^2+\overline{D}^2D^2) - \Box) \quad .
\end{aligned} \tag{2.33}$$

We obtain the action including the surface terms, which don't contribute to the equations of motion.

$$\begin{aligned}
&\int d^2\theta \exp(\theta D+\overline{\theta D})f(\phi) = \\
&\qquad \tfrac{1}{2}D^2\exp(\overline{\theta D})\,f(\phi) + \partial_m((\frac{i}{2}D^\alpha\sigma^m_{\alpha\dot\alpha}\overline{\theta}^{\dot\alpha} + \overline{\theta}^2\frac{i}{4}D^\alpha\sigma^m_{\alpha\dot\alpha}\overline{D}^{\dot\alpha} + \frac{\overline{\theta}^2}{4}\partial^m)f(\phi)), \\
&\int d^4\theta \exp(\theta D+\overline{\theta D})f(\phi) = \\
&\qquad \frac{1}{16}D^2\overline{D}^2 f(\phi) + \partial_m((\frac{1}{4}\partial^m + \frac{i}{4}D^\alpha\sigma^m_{\alpha\dot\alpha}\overline{D}^{\dot\alpha})f(\phi))
\end{aligned} \tag{2.34}$$

As usual, propagators are most easily obtained by solving the equations of motion for the sources via evaluation of all possible projections.

Usually one is not interested in the surface terms of the action. In that case there is an alternative way to find the SUSY action [GA83] in superspace. From $\mathbb{D}_\alpha = \frac{\partial}{\partial\theta^\alpha} - i\partial_{\alpha\dot\beta}\overline{\theta}^{\dot\beta}$ follows immediately $\int d^4x\mathbb{D}_\alpha\Phi = \int d^4x\frac{\partial}{\partial\theta^\alpha}\Phi +$

$tot.deriv. = \int d^4x d\theta^\alpha \Phi + tot.deriv.$. Therefore under the integral $\mathbb{D}_{\underline{\alpha}}$ can be replaced by $\int d\theta_{\underline{\alpha}}$. With this the SUSY action can be immediately translated into the superspaceformalism. For the kinetic energy this gives

$$\int d^4x D^2\overline{D}^2\overline{\phi}\phi \;\; = \int d^4x \mathbb{D}^2\overline{\mathbb{D}}^2\overline{\Phi}\Phi| = \tag{2.35}$$

$$\int d^4x d^2\theta d^2\overline{\theta}\overline{\Phi}\Phi| = \int d^4x d^2\theta d^2\overline{\theta}\overline{\Phi}\Phi \tag{2.36}$$

Where $|$ means projection onto S_{00}, i.e. put $\theta_{\underline{\alpha}} = 0$. The first equality sign holds because of Lemma 2 and because of the fact that Superfields form a ring, which particularly implies with Lemma 1 $\overline{\Phi}\Phi = \exp(\theta D + \overline{\theta D})(\overline{\phi}\phi)$. The last equality sign holds because $\int d^2\theta d^2\overline{\theta}$ picks out the highest component. The equal signs have to be read modulo total derivatives. The same is true for the superpotential:

$$\int d^4x D^2 g(\phi) = \int d^4x \theta^2 g(\Phi) \tag{2.37}$$

Here the projections could be neglected because ϕ is chiral.

Chapter 3

Supersymmetric Yang–Mills theories

There are two apparently independent approaches to a supersymmetric generalization of Yang-Mills theory. One approach is the superspace formalism. The idea is that the gauge fields are contained in superfields as component fields. The other, more systematic approach is via the symmetry algebra. Here the idea is to impose constraints and to solve them via the Bianchi Identities. In this chapter these two different approaches are presented and it is shown that they are equivalent (after some partial gauge fixing).

3.1 Superspace approach

3.1.1 Abelian gauge theories

We look for a superfield that contains the gauge fields: We might think about a real superfield whose highest component is the gauge field. For such a field, however, we would have to impose complicated constraints to get rid of higher spin components. It is much easier to start from a superfield that is based on a real scalar field $C = C^*$ [WE83]. Using lemma 1 and (2.33) gives

$$V = V^\dagger = C + \theta \not{A} \overline{\theta} + \tfrac{1}{2}\theta^2 \overline{\theta}^2 (D - \tfrac{1}{2}\Box C) + \Big((\theta\chi + \theta^2 M + \overline{\theta}^2 \theta(\lambda - \tfrac{i}{2}\not{\partial}\overline{\chi})) + h.c. \Big), \tag{3.1}$$

which already contains a real vector field A as its $\theta\overline{\theta}$-component. The linear SUSY representation that comes with the *real scalar superfield* is therefore called *vector multiplet.* To find the multiplet structure of the component fields we use $\mathcal{F}(\Phi) = \exp(\theta D + \overline{\theta D}) f(\phi)$. The SUSY representation defined by a superfield V is then

$$\begin{aligned}
\not{A}_{\alpha\dot\alpha} &= \tfrac{1}{2}[D_\alpha, \overline{D}_{\dot\alpha}]C = (D_\alpha \overline{D}_{\dot\alpha} - i\not{\partial}_{\alpha\dot\alpha})C = (i\not{\partial}_{\alpha\dot\alpha} - \overline{D}_{\dot\alpha} D_\alpha)C, \qquad (3.2)\\
\chi_\alpha &= D_\alpha C, \qquad \lambda_\alpha = -\tfrac{1}{4}\overline{D}^2 D_\alpha C,\\
M &= -\tfrac{1}{4}D^2 C, \qquad D_F = \tfrac{1}{16}\{D^2\overline{D}^2 + \overline{D}^2 D^2\}C.
\end{aligned}$$

The component fields χ_α, M and λ_α are complex. The real fields D_F transforms into a total derivative under SUSY transformations (such terms are called *Fayet–Iliopoulos* or D-terms; they are gauge invariant and thus can contribute to the action only for abelian factors of the gauge group). The gauge invariant field strength $v_{mn} = \partial_m A_n - \partial_n A_m$ of the real gauge connection A_m is contained in $D_\alpha \lambda^\beta = \not{v}_\alpha^{\ \beta} + i\delta_\alpha^\beta D_F$.

Consider a chiral superfield Λ with lowest component L. Again, using superfield expansion:

$$\Lambda = L + \theta DL + i\theta\not{\partial}\overline{\theta}L - \frac{1}{4}\theta^2 D^2 L + \frac{i}{2}\theta^2 \not{\partial} DL\overline{\theta} + \frac{1}{4}\theta^2\overline{\theta}^2 \Box L \qquad (3.3)$$

Out of Λ we can construct a special real superfield by adding its complex conjugate. A comparison with the formula above suggests the following super symmetrization of gauge transformations

$$\begin{aligned}
\delta V &= i(\Lambda^\dagger - \Lambda), && \qquad (3.4)\\
\delta C &= 2\,\mathrm{Im}\, L, & \delta\chi &= DL,\\
\delta A_m &= 2\,\mathrm{Re}\,\partial_m L, & \delta M &= -\frac{1}{4}D^2 L,\\
\delta\lambda &= \delta D_F = \delta F_{mn} = 0.
\end{aligned}$$

This is plausible because the usual gauge transformation occurs in the term $\delta A_m = 2\,\mathrm{Re}\,\partial_m L$. Note that the transversal projector Π_T in (2.20) projects onto the gauge invariant content of the real superfield. For a chiral superfield of charge q the gauge transformation and a gauge invariant kinetic energy may thus be defined by

$$\begin{aligned}
&\Phi \to e^{iq\Lambda}\Phi, \qquad V \to V + iq(\Lambda^\dagger - \Lambda), \qquad (3.5)\\
&K(\Phi, \overline{\Phi}, V) = \overline{\Phi} e^{qV}\Phi \qquad \overline{\Phi} = \Phi^\dagger
\end{aligned}$$

From now on q will not be written explicitly, which amounts to $q\Lambda \to \Lambda$.

In the superspace version polynomials and exponentials in the superfields are rather tedious to evaluate and we can use supergauge transformations to set $C = \chi_\alpha = M = 0$. This is called the *Wess–Zumino gauge*. The remaining gauge freedom is the ordinary gauge transformation of the vector component field A_m: $\delta A_m = 2\,\mathrm{Re}\,\partial_m L$. The Wess–Zumino gauge breaks supersymmetry [GA83]. This can be seen by performing a SUSY transformation on a vanishing component field of V

$$\delta_{SUSY}\chi_\alpha = (\xi D + \overline{\xi D})\chi_\alpha = 2\xi_\alpha M + \overline{\xi}^{\dot\alpha}(A_{\alpha\dot\alpha} - i\partial\!\!\!/_{\alpha\dot\alpha} C) \tag{3.6}$$

Therefore the nonvanishing component A_m does not belong to a linear SUSY representation (except for the trivial case). Further more a V field in the Wess–Zumino gauge is no longer a superfield, because it has vanishing lowest component, but a non-vanishing $\theta\overline{\theta}$-component which is not compatible with Lemma 1. For the remaining gauge freedom the Λ-fields which are compatibel with the Wess –Zumino gauge have only $\theta\overline{\theta}$ components which are real. Similar as above such a restricted Λ is no longer a superfield and its non-vanishing component is no linear SUSY representation (except for the trivial case where it is constant). In the Wess–Zumino gauge the gauge interaction is manifestly renormalizable [WE83].

Concider the superfields

$$\begin{aligned} \mathbb{W}_\alpha &:= -\frac{1}{4}\overline{\mathbb{D}}\overline{\mathbb{D}}\mathbb{D}_\alpha V \\ \overline{\mathbb{W}}_{\dot\alpha} &:= -\frac{1}{4}\mathbb{D}\mathbb{D}\overline{\mathbb{D}}_{\dot\alpha} V \end{aligned} \tag{3.7}$$

These fields are called supersymmetric field strengths [WE83]. Using Lemma 2 on $\mathbb{W}_\alpha = -\frac{1}{4}\overline{\mathbb{D}}\overline{\mathbb{D}}\mathbb{D}_\alpha e^{(\theta D + \overline{\theta D})}C$ and on $\overline{\mathbb{W}}_{\dot\alpha}$ shows that the lowest components of the $\mathbb{W}_{\underline{\alpha}}$ are the $\lambda_{\underline{\alpha}}$. Right from the definition follow the chirality conditions $\overline{\mathbb{D}}_{\dot\alpha}\mathbb{W}_a = \mathbb{D}_a\overline{\mathbb{W}}_{\dot\alpha} = 0$. Furthermore the $\mathbb{W}_{\underline{a}}$ are gauge invariant because Λ is chiral:

$$\begin{aligned} \mathbb{W}_\alpha \to -\frac{1}{4}\overline{\mathbb{D}}\overline{\mathbb{D}}\mathbb{D}_\alpha(V + \Lambda + \Lambda^\dagger) &= \mathbb{W}_\alpha + \frac{1}{4}\overline{\mathbb{D}}^{\dot\alpha}\{\overline{\mathbb{D}}_{\dot\alpha}, \mathbb{D}_\alpha\}\Lambda \\ &= \mathbb{W}_\alpha + \frac{i}{2}\partial\!\!\!/_{\alpha\dot\alpha}\overline{\mathbb{D}}^{\dot\alpha}\Lambda = \mathbb{W}_\alpha \end{aligned} \tag{3.8}$$

Analogously for $\overline{\mathbb{W}}_{\dot\alpha}$. In order not to get confused with the bunch of fields which come with the superfield formalism, we summarize to

name	superfield	lowest component	physical content
vectorfield	V	C	A_m , $\lambda_{\underline{\alpha}}$
chiral field	Λ	L	$\partial_m L$
supersymmetric field strength	$\mathbb{W}_{\underline{\alpha}}$	$\lambda_{\underline{\alpha}}$ (gaugino)	F_{mn}, λ, D

3.1.2 Non-abelian gauge theories

The generalization to non-abelian gauge theories introduces for each element δ_i of the structure group a vector superfield V^i. Denote $V = V^i\delta_i$ and $\Lambda = \Lambda^i\delta_i$, which is consistent with the abelian case where the structure group consists only of one element. For a given element δ_i of the structure group there comes a whole Superfield V^i. The corresponding vector multiplet of the component fields is called *gauge multiplet of the δ_i factor*. It will be convenient to work with a unitary representation of the structure group $\delta_i\phi = -iT_i\phi$, with hermitian representation matrices T_i. The abbreviations from above read then $V = -V^iT_i$ and $\Lambda = -\Lambda^iT_i$. We let Φ become vectors that transform in some representation. of the gauge group. Then supergauge transformations are defined by [WE83]

$$\Phi' = e^{i\Lambda}\Phi, \qquad e^{V'} = e^{i\Lambda^\dagger}e^Ve^{-i\Lambda} \quad \Rightarrow \quad V' = V + i(\Lambda^\dagger - \Lambda) + O(\Lambda^2). \tag{3.9}$$

and supersymmetric gauge-covariant field strength can be defined by

$$\mathbb{W}_\alpha = -\tfrac{1}{4}\overline{\mathbb{D}}^2 e^{-V}\mathbb{D}_\alpha e^V \quad \Rightarrow \quad \mathbb{W}'_\alpha = e^{i\Lambda}W_\alpha e^{-i\Lambda}, \tag{3.10}$$

3.2 Supercovariant derivatives and Bianchi identities

In an alternative approach to super Yang–Mills we start with the covariant derivatives

$$\mathcal{D}_a = \partial_a + A_a^i\delta_i \quad \rightarrow \quad \mathcal{D}_\alpha = D_\alpha + A_\alpha^i\delta_i \tag{3.11}$$

and try to impose reasonable constraints on the covariant field strengths F_{AB} defined by

$$[\mathcal{D}_A, \mathcal{D}_B] = -T_{AB}{}^C\mathcal{D}_C + F^i_{AB}\delta_i, \tag{3.12}$$

$$[\delta_i, \delta_j] = f_{ij}{}^k\delta_k, \tag{3.13}$$

$$T_{\alpha\dot\beta}{}^c = -2i\sigma^c_{\alpha\dot\beta}, \tag{3.14}$$

with all other torsion components vanishing. The constraints must be consistent with the Bianchi identities (1.9)-(1.16).The first Bianchi identity, which arises as the coefficient of D_A in the Jacobi identity, is trivial in flat space with only internal symmetries. The second Bianchi identity reads

$$\sum_{ABC} (-)^{AC} \left(\mathcal{D}_A F^i_{BC} + T_{AB}{}^D F^i_{DC} \right) = 0 \tag{3.15}$$

In SUSY there are two types of constraints: The first type can be imposed by a mere redefinition of what we call the covariant derivative. Such *conventional constraints* are familiar from Riemannian geometry: There we can absorb the torsion $T_{ab}{}^c$ into a redefinition of the spin connection $\omega_m{}^{ab}$ that determines the covariant derivative and thus replace a general metric-compatible connection by the Christoffel connection. This is a mere change of basis of the covariant local coordinates of the jet bundle and the torsion then becomes a particular tensor field that may (or may not) be set to 0. Computing the field strengths in terms of the connections we find

$$F^i_{\alpha\dot\beta} = D_\alpha \overline{A}^i_{\dot\beta} + \overline{D}_{\dot\beta} A^i_\alpha + A^j_\alpha \overline{A}^k_{\dot\beta} f_{jk}{}^i - 2i A\!\!\!/^i_{\alpha\dot\beta}, \tag{3.16}$$

so that $F^i_{\alpha\dot\beta} = 0$ can be imposed as a conventional constraint.[1]

In order to construct gauge invariant interactions for matter fields we want to impose a covariant chirality condition $\overline{\mathcal{D}}_{\dot\alpha}\phi = 0$. Covariantly chiral fields can, however, be charged under the gauge group only if $\{\mathcal{D}_\alpha, \mathcal{D}_\beta\} = F^i_{\alpha\beta}\delta_i$ vanishes. We thus impose the *standard constraints*

$$F^i_{\alpha\beta} = F^i_{\dot\alpha\dot\beta} = 0, \qquad F^i_{\alpha\dot\beta} = 0. \tag{3.17}$$

The general form of the gauge algebra, with the non-vanishing commutation relations

$$[\mathcal{D}_a, \mathcal{D}_b] = F^i_{ab}\delta_i, \tag{3.18}$$

$$[\mathcal{D}_\alpha, \mathcal{D}_a] = i\sigma_{a\,\alpha\dot\beta}\overline{W}^{i\dot\beta}\delta_i, \tag{3.19}$$

$$\{\mathcal{D}_\alpha, \overline{\mathcal{D}}_{\dot\beta}\} = 2i\mathcal{D}\!\!\!/_{\alpha\dot\beta} \tag{3.20}$$

can then be obtained by solving the Bianchi identities, which also imply

$$\overline{\mathcal{D}}_{\dot\alpha}W = 0, \qquad D^i := \frac{1}{2}\mathcal{D}^\alpha W^i_\alpha, \qquad \mathcal{D}^2 W_\alpha = -4\mathcal{D}_\alpha D^i \tag{3.21}$$
$$\mathcal{D}_\alpha W^{i\beta} = \sigma^{ab\beta}_\alpha F^i_{ab} + i\delta^\beta_\alpha D^i.$$

[1] Then the gauge potential A^i_m can be written in terms of (covariant derivatives of) A^i_α and $\overline{A}^i_{\dot\alpha}$, which are therefore called *prepotentials*. This is similar to the fact that we can express the spin connection in terms of the vielbein if we impose $T_{ab}{}^c = 0$.

To derive this result we should analyze the identities with contributions from torsions:

$$\sum_{\alpha\beta\dot{\gamma}} (...) = \sigma^a_{\alpha\dot{\gamma}} F_{a\beta} + \sigma^a_{\beta\dot{\gamma}} F_{a\alpha} = 0 \tag{3.22}$$

$$\Rightarrow \quad \sigma^a_{\alpha\dot{\gamma}} F_{a\beta} = \varepsilon_{\alpha\beta} \overline{W}_{\dot{\gamma}}, \qquad F_{a\alpha} = \sigma_{a\alpha\dot{\beta}} \overline{W}^{\dot{\beta}}$$

i.e. $F_{a\alpha}$ contains no spin 3/2 component. Except for the complex conjugate of the above the only other BI with contributions from torsions is

$$\sum_{\alpha\dot{\beta}c} (...) = \mathcal{D}_\alpha F_{c\dot{\beta}} + \mathcal{D}_{\dot{\beta}} F_{c\alpha} + 2i\sigma^a_{\alpha\dot{\beta}} F_{ab} = 0 \tag{3.23}$$

$$\Rightarrow \quad F_{ab} = -\frac{1}{4}(\overline{\mathcal{D}}\overline{\sigma}_a \sigma_b \overline{W} - \mathcal{D}\sigma_a \overline{\sigma}_b W)$$

Antisymmetry of F_{ab} thus implies

$$\overline{\mathcal{D}W} = \mathcal{D}W, \qquad F_{ab} = \frac{1}{2}(\mathcal{D}\sigma_{ab}W - \overline{\mathcal{D}}\overline{\sigma}_{ab}\overline{W}). \tag{3.24}$$

The only remaining Bianchi identity that contains new information is

$$\sum_{\dot{\alpha}\dot{\beta}c} (...) = \overline{\mathcal{D}}_{\dot{\alpha}} F_{c\dot{\beta}} + \overline{\mathcal{D}}_{\dot{\beta}} F_{c\dot{\alpha}} = 0 \tag{3.25}$$

$$\Rightarrow \quad (\sigma^c_{\alpha\dot{\alpha}} \overline{\mathcal{D}}_{\dot{\beta}} + \sigma^c_{\alpha\dot{\beta}} \overline{\mathcal{D}}_{\dot{\alpha}}) W^\alpha = 0, \qquad \overline{\mathcal{D}}_{\dot{\alpha}} W_\alpha = 0,$$

i.e. W_α is covariantly chiral (use $\sigma^c_{\alpha\dot{\alpha}} \overline{\sigma}^{\dot{\beta}\beta}_c = 2\delta^\beta_\alpha \delta^{\dot{\beta}}_{\dot{\alpha}}$ after contraction with $\overline{\sigma}^{\gamma\dot{\gamma}}_c$).

The field W_α is also caled *gaugino field*. This name is justified because the covariant derivatives close on the multiplet $(A_a, W_\alpha, \overline{W}_{\dot{\alpha}}, D)^i$ [dr87]. It is important to distinguish between $W_{\underline{\alpha}}$ and the superfield $\mathbb{W}_{\underline{\alpha}}$ with lowest component $\lambda_{\underline{\alpha}}$. The relation between these fields is established in section 3.5.1.

3.3 From superspace to algebra

The aim of this section is to show how the superspace approach to YM theories leads to the algebra of covariant derivatives. The transformation property of the fields leads to the transformation property of covariant derivatives, which can then be written down explicitly and they can be shown to fullfill the gauge algebra together with the constraints. With the results about superspace a connection can be established between the potentials and prepotentials on the one side and the component fields on the other side. A gauge transformation on a scalar superfield is performed with a chiral gauge field $\Lambda = -\Lambda^i T_i$.

$$\begin{aligned} \Phi &\longrightarrow \Phi' = e^{i\Lambda}\Phi, \qquad\qquad \overline{\mathbb{D}}_{\dot\alpha}\Lambda \ = 0 \\ \overline{\Phi} = \Phi^\dagger &\longrightarrow \Phi'^\dagger = \Phi^\dagger e^{-i\Lambda^\dagger} \\ \Phi^* &\longrightarrow \Phi'^* = e^{-i\Lambda^*}\Phi^* \end{aligned} \tag{3.26}$$

Where in general $\Lambda^\dagger \neq \Lambda$ as we would expect it in quantum mechanics. Beside a modification of the derivatives this also causes a modification of the kinetic term $\bar\Phi\Phi$ by $\bar\Phi e^V \Phi$ which transforms as $e^V \longrightarrow e^{V'} = e^{V'} = e^{i\Lambda^\dagger} e^V e^{-i\Lambda}$. In order to have a real lagrangian the kinetic energy must be real. Therefore it is natural to use a hermitian representation of the structure group. From the reality of the Lagrangian we get a reality condition for $V = V^\dagger$. We want to find covariant derivatives $\nabla_{\underline{\alpha}}$ which transform covariantly under a gauge transformation:

$$\begin{aligned} {\nabla_\alpha}' &= e^{i\Lambda}\nabla_\alpha e^{-i\Lambda} \\ {\overline{\nabla}_{\dot\alpha}}' &= e^{i\Lambda^\dagger}\overline{\nabla}_{\dot\alpha} e^{-i\Lambda^\dagger} \end{aligned} \tag{3.27}$$

Lemma 3. *The solutions for the covariant derivatives are*

$$\begin{aligned} \nabla_\alpha &= e^{-V}\mathbb{D}_\alpha e^V \\ \overline{\nabla}_{\dot\alpha} &= e^V \overline{\mathbb{D}}_{\dot\alpha} e^{-V} \end{aligned} \tag{3.28}$$

and they fulfill the gauge algebra together with the constraints. [1]

Proof. We use the transformation property for the vectorfield which is *defined* by $e^{V'} = e^{i\Lambda^\dagger} e^V e^{-i\Lambda}$ and its inverse $e^{-V'} = e^{i\Lambda} e^{-V} e^{-i\Lambda^\dagger}$ and apply it in ${\nabla_\alpha}' = e^{-V'}{\mathbb{D}_\alpha}' e^{V'} = e^{-V'}\mathbb{D}_\alpha e^{V'} = e^{i\Lambda} e^{-V} e^{-i\Lambda^\dagger}\mathbb{D}_\alpha e^{i\Lambda^\dagger} e^V e^{-i\Lambda} \overset{(3.3)}{=} e^{i\Lambda} e^{-V}\mathbb{D}_\alpha e^V e^{-i\Lambda} = e^{i\Lambda}\nabla_\alpha e^{-i\Lambda}$ which is the desired transformation property. Similarly for ${\overline{\nabla}_{\dot\alpha}}'$,

[1] In principle we must consider all possible solutions of equation (11). But the solutions are unique in the following (physical) sense: the variation of some derivative, say $\mathbf{D}$ under a perturbation by some representation of the gauge group $X = X^i\delta_i$ is given by $\delta\mathbf{D} = \delta_X\mathbf{D} = -[X, \mathbf{D}]$. If we demand $\nabla \to \mathbf{D}$ for the gauge charge going to zero, we can write $\nabla = \mathbf{D} + g\delta\mathbf{D} + \frac{g^2}{2}\delta\delta\mathbf{D}... = \mathbf{D} - g[X, \mathbf{D}] + \frac{g^2}{2}[X, [X, \mathbf{D}]]... \overset{B.C.H}{=} e^{-X}\mathbf{D}e^X$. So it is enough to consider solutions of the above type and the factors have to be chosen such that $\nabla \to \mathbf{D}$ holds.

by hermitian conjugation. Next we have to show that the algebra is fulfilled. $\{\nabla_\alpha, \nabla_\beta\} = e^{-V}\{\mathbb{D}_\alpha, \mathbb{D}_\beta\}e^V = 0$, and analogously $\{\overline{\nabla}_{\dot\alpha}, \overline{\nabla}_{\dot\beta}\} = 0$ from which follows $\tilde{T}_{\alpha\beta} = \tilde{T}_{\dot\alpha\dot\beta} = \tilde{F}_{\alpha\beta} = \tilde{F}_{\dot\alpha\dot\beta} = 0$. We define the covariant spacetime derivative by $2i\sigma^a_{\alpha\dot\beta}\nabla_a := \{\nabla_\alpha, \overline{\nabla}_{\dot\beta}\} = \{e^{-V}\mathbb{D}_\alpha e^V, e^V\overline{\mathbb{D}}_{\dot\alpha}e^{-V}\}$, which yields the commutator and the constraints $\mathbb{F}_{\alpha\dot\alpha} = 0$, $\mathbb{T}_{\alpha\dot\alpha} = -2i\sigma^a_{\alpha\dot\alpha}$. Use the Baker Campbell Hausdorff formula and $-iT_i = \delta_i$ to split off the first contribution of $\nabla_a = \partial_a + g^j_a\delta_j$. Then $[\nabla_a, \nabla_b] = [\partial_a, \partial_b] + \partial_a g^k_b\delta_k - \partial_b g^k_a\delta_k + [g^i_a\delta_i, g^j_b\delta_j] = \mathbb{F}^k_{ab}\delta_k$. Therefore $\mathbb{T}_{ab}{}^C = 0$. In the same way $\nabla_{\underline{\alpha}} = \mathbb{D}_{\underline{\alpha}} + g^j_{\underline{\alpha}}\delta_j$. Because of $[\mathbb{D}_{\underline{\alpha}}, \partial_a] = 0$ it follows that $\{\nabla_a, \nabla_{\underline{\alpha}}\} = \mathbb{F}^k_{a\underline{\alpha}}\delta_k$, therefore $\mathbb{T}_{a\underline{\alpha}}{}^C = 0$. □

Summarizing the results gives the super YM algebra for the covariant derivatives.

$$\begin{aligned}
&[\nabla_A, \nabla_B\} = -\mathbb{T}_{AB}{}^C\nabla_C + \mathbb{F}^i_{AB}\delta_i, \quad [\delta_i, \delta_j] = f_{ij}{}^k\delta_k \\
&\tilde{T}_{\alpha\dot\beta}{}^c = -2i\sigma^c_{\alpha\dot\beta}, \quad \mathbb{T}_{\alpha\beta}{}^c = \mathbb{T}_{\dot\alpha\dot\beta}{}^c = \mathbb{T}_{ab}{}^c = \mathbb{T}_{a\dot\beta}{}^c = \mathbb{T}_{\alpha\beta}{}^c = 0 \\
&\mathbb{F}^i_{\alpha\dot\beta} = \mathbb{F}^i_{\alpha\beta} = \mathbb{F}^i_{\dot\alpha\dot\beta} = 0
\end{aligned} \tag{3.29}$$

Where the structure functions $\mathbb{F}^i, \mathbb{T}^C$are superfields. In order to have Bianchi identities, we have to show that the covariant derivatives are derivations. $V = V^i\delta_i$ is a derivation because the gauge generators are derivations and linear combinations of derivations are derivations (with multiplication from left). Then also $\nabla_\alpha = e^{-V}\mathbb{D}_\alpha e^V = \mathbb{D}_\alpha + [-V, \mathbb{D}_\alpha] + \frac{1}{2}[-V, [-V, \mathbb{D}_\alpha]] + ...$ is a derivation because $\mathbb{D}_\alpha$ is a derivation and commutators of derivations are again derivations. In a similar way $\overline{\nabla}_{\dot\alpha}$ and ∇_a are derivations.

Because of $[D_{\underline{\alpha}}, \delta_i] = [\mathbb{Q}_{\underline{\alpha}}, \delta_i] = 0$, e^V is a superfield operator. Since $\mathbb{D}_{\underline{\alpha}}$ are superfield operators and the space of all superfield operators forms a ring statement 2 establishes then superfieldoperators ∇_A. With Lemma 2 we may write $e^{-(\theta D+\overline{\theta D})}\nabla_A e^{(\theta D+\overline{\theta D})} = \mathcal{D}_A$ with a unique operator $\mathcal{D}_A$ which acts on the θ, $\overline{\theta}$ independent sector $\mathcal{S}_{00}$. Obviously the $\mathcal{D}_A$ fulfill the algebra too.

$$\begin{aligned}
&[\mathcal{D}_A, \mathcal{D}_B\} = -T_{AB}{}^C\mathcal{D}_C + F^i_{AB}\delta_i, \quad [\delta_i, \delta_j] = f_{ij}{}^k\delta_k \\
&T_{\alpha\dot\beta}{}^c = -2i\sigma^c_{\alpha\dot\beta}, \quad T_{\alpha\beta}{}^c = T_{\dot\alpha\dot\beta}{}^c = T_{ab}{}^c = T_{a\dot\beta}{}^c = T_{\alpha\beta}{}^c = 0 \\
&F^i_{\alpha\dot\beta} = F^i_{\alpha\beta} = F^i_{\dot\alpha\dot\beta} = 0
\end{aligned} \tag{3.30}$$

The F^i, T^C are the structure functions on $\mathcal{S}_{00}$. They are related via $F^i = e^{-(\theta D+\overline{\theta D})}\mathbb{F}^i e^{(\theta D+\overline{\theta D})} = e^{-(\theta D+\overline{\theta D})}\mathbb{F}^i$ and $T^C = e^{-(\theta D+\overline{\theta D})}\mathbb{T}^C e^{(\theta D+\overline{\theta D})} = e^{-(\theta D+\overline{\theta D})}\mathbb{T}^C$. By the same argument as above the $\mathcal{D}_A$ are derivations. Therefore we arrived at our goal to show that the superspace formulation leads to the algebra of covariant derivatives and the Bianchi identities hold.

Now we want to see how the potentials are encoded in the vector field. We compute the covariant derivatives (12) on $\mathcal{S}_{00}$ and compare the result with the usual ansatz for the covariant derivatives [2] $D_A - iA^i_A T_i = \mathcal{D}_A = \nabla_A|_{\theta=\overline{\theta}=0}$. This yields:

$$
\begin{aligned}
-iA^i_\alpha T_i &= e^{-C}[D_\alpha, e^C] = -D_\alpha C^i T_i - \frac{1}{2} C^j D_\alpha C^i f^k_{ji} T_k \cdots \\
-i\overline{A}^i_{\dot\alpha} T_i &= e^{C}[\overline{D}_{\dot\alpha}, e^{-C}] = \quad \overline{D}_{\dot\alpha} C^i T_i - \frac{1}{2} C^j \overline{D}_{\dot\alpha} C^i f^k_{ji} T_k \cdots
\end{aligned}
\tag{3.31}
$$

Because if we apply ∇_α on a superfield $\Phi = e^{(\theta D + \overline{\theta D})}\phi$ we get with lemmas 1 and 2, $\nabla_\alpha \Phi = e^{-V} e^{(\theta D+\overline{\theta D})} D_\alpha e^{-(\theta D+\overline{\theta D})} e^V \Phi = e^{-V} e^{(\theta D+\overline{\theta D})} D_\alpha (e^{-(\theta D+\overline{\theta D})} e^{(\theta D+\overline{\theta D})}) e^C \Phi$. This yields in the $\mathcal{S}_{00}$ sector $(e^{-C} D_\alpha e^C)\phi$ and compared with the ansatz shows $e^{-C}[D_\alpha, e^C] = A_\alpha$. In particular in the **abelian case** where $D_\alpha e^C = e^C D_\alpha C$ holds,

$$
\begin{aligned}
A_\alpha &= -iD_\alpha C, \\
\overline{A}_{\dot\alpha} &= \quad i\overline{D}_{\dot\alpha} C.
\end{aligned}
\tag{3.32}
$$

As a check one may compute in the linearized case the potential A_m out of the algebra, which should be the same as the relevant component of the superfield. $A\!\!\!/_{\alpha\dot\beta} = \frac{1}{2i}(D_\alpha \overline{A}_{\dot\beta} + \overline{D}_{\dot\beta} A_\alpha) = \frac{1}{2}[D_\alpha, \overline{D}_{\dot\beta}]C$. A comparison with (3.1.1) shows that this is indeed the case.

3.4 From algebra to superspace

In this section the other direction is done. It will be shown that the algebra of gauge covariant derivatives together with the constraints can be parameterized by a real field in Superspace. Whereas the second Bianchi identities have been used to solve the constraints and determine the field content in the algebraic approach, the remaining non trivial ones (1.14) will be used to grant integrability conditions which yields to the desired real field parameterization. Gauge covariant derivatives are obtained by adding connection fields, which are 1-forms, to the derivatives

$$
\mathcal{D}_A = D_A + A^i_A \delta_i. \tag{3.33}
$$

[2] The covariant derivatives have to act unitarily [dr87] with respect to their matrix structure. Using hermitian matrices T_i, gives an additional imaginary unit for the covariant derivatives.

The algebra of the (gauge)covariant derivatives is given by

$$[\mathcal{D}_A, \mathcal{D}_B] = -T_{AB}^{C}\mathcal{D}_C + F_{AB}^{i}\delta_i, \tag{3.34}$$

$$[\delta_i, \delta_j] = f_{ij}^{k}\delta_k, \tag{3.35}$$

$$[\delta_i, D_A] = 0 \tag{3.36}$$

$$T_{\alpha\dot{\beta}}^{c} = -2i\sigma_{\alpha\dot{\beta}}^{c}, \qquad T_{ab} = T_{\alpha\beta} = T_{\dot{\alpha}\dot{\beta}}, \tag{3.37}$$

$$F_{\alpha\dot{\beta}}^{i} = F_{\dot{\alpha}\dot{\beta}}^{i} = F_{\alpha\beta}^{i} = 0. \tag{3.38}$$

We start with this algebra together with the constraints and show that they imply a vector field. We assume a Lie Algebra of the gauge group with real structure constants. The constraints on the torsions are not changed by the gauging, whereas the constraints on the fieldstrengths lead to a real vectorfield. From the algebra follows

$$D_{(\beta}A_{\alpha)} = -\{A_\beta, A_\alpha\} \tag{3.39}$$

$$A_a^i = +\frac{1}{4i}\overline{\sigma}_a^{\dot{\alpha}\alpha}(D_\alpha\overline{A}_{\dot{\alpha}}^i + \overline{D}_{\dot{\alpha}}A_\alpha^i + f_{ck}^{i}A_\alpha^c\overline{A}_{\dot{\alpha}}^k) \tag{3.40}$$

The first equation follows from $0 = \{\mathcal{D}_\alpha, \mathcal{D}_\beta\} = \{A_\beta, A_\alpha\} + D_\beta A_\alpha + D_\alpha A_\beta + \{D_\alpha, D_\beta\}$, where the last term vanishes due to the SUSY algebra. The second equation uses the constraint $0 = F_{\alpha\dot{\beta}}^i = F_{\alpha\dot{\beta}}^i\delta_i = \{\mathcal{D}_\alpha, \overline{\mathcal{D}}_{\dot{\beta}}\} + T_{\alpha\dot{\beta}}^c\mathcal{D}_c = \{D_\alpha + A_\alpha, \overline{D}_{\dot{\beta}} + \overline{A}_{\dot{\beta}}\} - 2i\sigma_{\alpha\dot{\beta}}^a(D_a + A_a) = \{A_\alpha, \overline{A}_{\dot{\beta}}\} + \{D_\alpha, \overline{A}_{\dot{\beta}}\} + \{A_\alpha, \overline{D}_{\dot{\beta}}\} - 2i\sigma_{\alpha\dot{\beta}}^a A_a = \{A_\alpha, \overline{A}_{\dot{\beta}}\} - 2i\sigma_{\alpha\dot{\beta}}^a A_a + (D_\alpha\overline{A}_{\dot{\beta}}^i)\delta_i - \overline{A}_{\dot{\beta}}^i[D_\alpha, \delta_i] + (\overline{D}_{\dot{\beta}}A_\alpha^i)\delta_i - A_\alpha^i[\overline{D}_{\dot{\beta}}, \delta_i]$. With $\{A_\alpha, \overline{A}_{\dot{\beta}}\} = A_\alpha^c\overline{A}_{\dot{\beta}}^k[\delta_c, \delta_k] = A_\alpha^c\overline{A}_{\dot{\beta}}^k f_{ck}^i\delta_i$ and $[\delta_i, D_{\underline{\alpha}}] = 0$ follows the equation. The second equation expresses the potential in terms of the prepotential. Under the assumption $\overline{A}_{\dot{\alpha}} = A_\alpha^*$ it follows that A_a is always real. This is necessary because an imaginary part of the Potential would lead to two Potentials (in ordinary gauge theory the gauge parameter is real. This naturally leads to a real potential). The Bianchi identities give equations for the prepotentials, i.e every *given* prepotential has to fulfill it. Therefore one can make conclusions on their structure.

3.4.1 Linearized case

First consider the linearized constraint $D_{(\beta}A_{\alpha)} = 0$, which is equivalent to looking at an abelian gauge theory. The aim is the construction of an explicit solution. Start with the ansatz $-iD_\alpha C = A_\alpha$ and $i\overline{D}_{\dot{\alpha}}C = \overline{A}_{\dot{\alpha}}$ with some real field C. If such a C exists then the A solve the constraint equation. With this ansatz it follows that $2i\partial\!\!\!/_{\alpha\dot{\beta}}C = \overline{D}_{\dot{\beta}}D_\alpha C + D_\alpha\overline{D}_{\dot{\beta}}C = i(\overline{D}_{\dot{\beta}}A_\alpha - D_\alpha\overline{A}_{\dot{\beta}})$.

Multiplying both sides with $\overline{\partial\!\!\!/}^{\dot\beta\alpha}$ and using the greensfunction $\frac{1}{\Box}$ gives $C = \frac{1}{4}\frac{1}{\Box}\overline{\partial\!\!\!/}^{\dot\beta\alpha}(\overline{D}_{\dot\beta}A_\alpha - D_\alpha\overline{A}_{\dot\beta})$.

Lemma 4. *Let A, B be (graded) commuting operators $[A,B]=0$ and B invertible, then also $[A,B^{-1}]=0$*

Proof. Multiply $AB = BA$ with B^{-1} from the right and the left $B^{-1}ABB^{-1} = B^{-1}BAB^{-1}$. Therefore $[A,B^{-1}]=0$. $\Box$

Conjecture 1 (homogeneous case)**.** *Let A_α and $\overline{A}_{\dot\alpha}$ be given and be members of a realization of the SUSY multiplet such that $D_{(\beta}A_{\alpha)}=0$, then the field $C := \frac{1}{4}\frac{1}{\Box}\overline{\partial\!\!\!/}^{\dot\beta\alpha}(\overline{D}_{\dot\beta}A_\alpha - D_\alpha\overline{A}_{\dot\beta})$ is real and it holds:*

$$A_\alpha = -iD_\alpha C \tag{3.41}$$
$$\overline{A}_{\dot\alpha} = \quad i\overline{D}_{\dot\alpha}C \tag{3.42}$$

C is well defined because the $A_{\underline{\alpha}}$ are members of a SUSY multiplet. Obviously C is real. It remains to show that C solves the constraint equations. Use Lemma 4 and compute

$$\begin{aligned}
D_\gamma C &= \frac{1}{4}\frac{1}{\Box}\overline{\partial\!\!\!/}^{\dot\beta\alpha}D_\gamma(\overline{D}_{\dot\beta}A_\alpha - D_\alpha\overline{A}_{\dot\beta})\\
&= \frac{1}{4}\frac{1}{\Box}\overline{\partial\!\!\!/}^{\dot\beta\alpha}(-\overline{D}_{\dot\beta}\underbrace{D_\gamma A_\alpha}_{=-D_\alpha A_\gamma} + 2i\partial\!\!\!/_{\gamma\dot\beta}A_\alpha - D_\gamma D_\alpha\overline{A}_{\dot\beta})\\
&= \frac{1}{4}\frac{1}{\Box}\overline{\partial\!\!\!/}^{\dot\beta\alpha}(2i\partial\!\!\!/_{\alpha\dot\beta}A_\gamma - D_\alpha\overline{D}_{\dot\beta}A_\gamma + D_\alpha D_\gamma\overline{A}_{\dot\beta} + 2i\partial\!\!\!/_{\gamma\dot\beta}A_\alpha)\\
&= iA_\gamma + \frac{1}{\Box}\underbrace{\overline{\partial\!\!\!/}^{\dot\beta\alpha}(-D_\alpha\overline{D}_{\dot\beta}A_\gamma + D_\alpha D_\gamma\overline{A}_{\dot\beta} + 2i\partial\!\!\!/_{\gamma\dot\beta}A_\alpha)}_{=X_\gamma}
\end{aligned}$$

In order that the conjecture holds X_γ has to vanish. The data are the imposed constraints, which will be called *primary constraints*. They imply via the algebra

$$F_{\alpha\beta} = T_{\alpha\beta} = 0 \qquad \Rightarrow \qquad D_{(\alpha}A_{\beta)} = 0 \tag{3.43}$$
$$F_{\dot\alpha\dot\beta} = T_{\dot\alpha\dot\beta} = 0 \qquad \Rightarrow \qquad \overline{D}_{(\dot\alpha}\overline{A}_{\dot\beta)} = 0 \tag{3.44}$$
$$F_{\alpha\dot\beta} = 0 \qquad \Rightarrow \qquad A^i_a = \frac{1}{4i}\overline{\sigma}^{\dot\alpha\alpha}_a(D_\alpha\overline{A}^i_{\dot\alpha} + \overline{D}_{\dot\alpha}A^i_\alpha) \tag{3.45}$$
$$T_{a\alpha} = T_{a\dot\alpha} = T_{ab} = 0 \tag{3.46}$$

The primary constraints can be solved via the Bianchi identities. This gives new constraints which will be called *secondary constraints*. They read ($\delta_I F^J_{AB} = 0$ in the abelian case):

$$\begin{array}{lcll} \mathcal{D}_\beta \overline{W}^{\dot\alpha} = 0 & & D_\beta \overline{W}^{\dot\alpha} = 0 & (3.47) \\ \overline{\mathcal{D}}_{\dot\beta} W^\alpha = 0 & \xrightarrow{\text{abelian}} & \overline{D}_{\dot\beta} W^\alpha = 0 & (3.48) \\ \overline{\mathcal{D}W} = \mathcal{D}W & & \overline{DW} = DW & (3.49) \end{array}$$

A systematic approach to dealing with systems of equations which are linear in some fields is in the Jet space formalism, where the fields and their derivatives are considered as independent algebraic objects. Solving systems of equations which are linear in the fields corresponds then to performing linear algebra with the independent algebraic objects as basis elements. Due to the constraints, however, not all possible objects are independent. In order to find a basis in Jet space start with A_α and apply D_β and $\overline{D}_{\dot\beta}$ on it. This gives from $\overline{D}_{\dot\beta} A_\alpha$ four independent objects. Since $D_{(\alpha} A_{\beta)} = 0$, it is possible to write $D_\beta A_\alpha = 1/2 \varepsilon_{\beta\alpha} DA$. Therefore $D_\beta A_\alpha$ gives only one independent object, which is chosen to be DA. For the higher derivatives we chose the order that the $\overline{D}$ stand to the left of the D. Applying D_γ and $\overline{D}_{\dot\gamma}$ on $\overline{D}_{\dot\beta} A_\alpha$ gives the 12 independent objects $\{\overline{D}_{\dot\alpha} DA, \partial_n A_\alpha, \overline{D} A_\alpha\}$. Because of the constraints the object $D_\gamma D_\beta A_\alpha$ is antisymmetric in all three indices and therefore has to vanish (since the spinorial indices may only take the values $1, 2$). Therefore $D_\alpha DA = 0$. The only nonvanishing objects that come from DA are then $\overline{D}_{\dot\alpha} DA$ but these are already taken into account. Therefore from DA there are no extra contributions to the second order. Analogously for higher derivative orders. The same scheme applies to $\overline{A}_{\dot\alpha}$ with the opposite ordering rule. Summarizing gives

$$\begin{array}{lll} & A_\alpha & \overline{A}_{\dot\alpha} \\ \text{1. order} & \overline{D}_{\dot\beta} A_\alpha & D_\beta \overline{A}_{\dot\alpha} \\ & DA & \overline{DA} \\ \text{2. order} & \overline{D}_{\dot\alpha} DA & D_\alpha \overline{DA} \\ & \overline{D} A_\alpha & D \overline{A}_{\dot\alpha} \\ & \partial_n A_\alpha & \partial_n \overline{A}_{\dot\alpha} \\ \text{3. order} & \partial_n \overline{D}_{\dot\beta} A_\alpha & \partial_n D_\beta \overline{A}_{\dot\alpha} \\ & \partial_n DA & \partial_n \overline{DA} \\ & \overline{D} DA & D \overline{DA} \\ \vdots & \vdots & \vdots \\ n.\ \text{order} & \partial_m (n-2)\text{order} & \text{for } n \geq 4 \end{array}$$

With this it can be shown that $D_\mu X_\gamma = 0$. Indeed, using $[D^2, \overline{D}_{\dot\alpha}] = 4i\partial\!\!\!/_{\alpha\dot\alpha} D^\alpha$

$$\begin{aligned} D_\mu X_\gamma &= \overline{\partial\!\!\!/}^{\dot\beta\alpha}(-D_\mu D_\alpha \overline{D}_{\dot\beta} A_\gamma + 2i\partial\!\!\!/_{\gamma\dot\beta} D_\mu A_\alpha) \\ &= \overline{\partial\!\!\!/}^{\dot\beta\alpha}(-\frac{1}{2}\varepsilon_{\mu\alpha} D^2 \overline{D}_{\dot\beta} A_\gamma + i\varepsilon_{\mu\alpha}\partial\!\!\!/_{\gamma\dot\beta} DA) \\ &= \overline{\partial\!\!\!/}^{\dot\beta\alpha}(-i\varepsilon_{\mu\alpha}\partial\!\!\!/_{\gamma\dot\beta} DA + i\varepsilon_{\mu\alpha}\partial\!\!\!/_{\gamma\dot\beta} DA) = 0 \end{aligned} \tag{3.50}$$

In the chosen basis X_γ reads

$$\partial_a \overline{\sigma}^{a\dot\beta\alpha}\left(\frac{1}{2}\varepsilon_{\alpha\gamma}\overline{D}_{\dot\beta} DA - 2i\sigma^n_{\alpha\dot\beta}\partial_n A_\gamma + \frac{1}{2}\varepsilon_{\alpha\gamma} D^2 \overline{A}_{\dot\beta} + 2i\sigma^n_{\gamma\dot\beta}\partial_n A_\alpha\right) = X_\gamma \tag{3.51}$$

Since we have only used primary constraints to find the independent algebraic objects, we may use them as a basis to show the following

Lemma 5. *If the primary constraints are fullfilled, then the secondary constraints follow from the primary constraints.*

Proof. First it is shown that $D_\beta \overline{W}^{\dot\alpha} = 0$.

$$\begin{aligned} D_\beta \overline{W}^{\dot\alpha} &= \overline{\sigma}^{a\dot\alpha\alpha} D_\beta F_{a\alpha} \\ &= \overline{\sigma}^{a\dot\alpha\alpha} D_\beta \left(\partial_a A_\alpha - D_\alpha \frac{1}{4i}\overline{\sigma}_a^{\dot\gamma\gamma}(D_\gamma \overline{A}^i_{\dot\gamma} + \overline{D}_{\dot\gamma} A^i_\gamma)\right) \\ &= \frac{1}{2}\varepsilon_{\beta\alpha}\overline{\sigma}^{a\dot\alpha\alpha}\partial_a DA - \frac{1}{4i}\varepsilon^{\dot\alpha\dot\gamma}\delta^\gamma_\beta D^2 \overline{D}_{\dot\gamma} A_\gamma \\ &= \frac{1}{2}\varepsilon_{\beta\alpha}\overline{\sigma}^{a\dot\alpha\alpha}\partial_a DA - \varepsilon^{\dot\alpha\dot\gamma}\partial\!\!\!/_{\mu\dot\gamma} D^\mu A_\beta = 0 \end{aligned} \tag{3.52}$$

where $D^2 A_\gamma = 0$ and $[D^2, \overline{D}_{\dot\gamma}] = 4i\partial\!\!\!/_{\mu\dot\gamma} D^\mu$ was used. $\overline{D}_{\dot\beta} W^\alpha = 0$ follows by complex conjugation. In the next step it is shown that $\overline{DW} = DW$ holds. Calculate

$$\begin{aligned} \overline{D}_{\dot\beta}\overline{W}^{\dot\beta} &= \overline{D}_{\dot\beta}\overline{\sigma}^{a\dot\beta\beta} F_{a\beta} \\ &= \overline{\sigma}^{a\dot\beta\beta}\partial_a \overline{D}_{\dot\beta} A_\beta - \frac{1}{2i}\varepsilon^{\dot\gamma\dot\beta}\varepsilon^{\gamma\beta}\left(\overline{D}_{\dot\beta} D_\beta(\overline{D}_{\dot\gamma} A_\gamma + D_\gamma \overline{A}_{\dot\gamma})\right) \\ &= 2\overline{\sigma}^{a\dot\beta\beta}(\partial_a \overline{D}_{\dot\beta} A_\beta - \partial_a D_\beta \overline{A}_{\dot\beta}) + \frac{1}{2i} D\overline{DA} - \frac{1}{2i}\overline{D}DA \end{aligned} \tag{3.53}$$

which is the same as its complex conjugate and therefore $\overline{DW} = DW$. $\square$

The statement says that the secondary constraints are redundant and therefore don't yield new information. This means that the Bianchi Identities and the constraints alone are not enough to show the conjecture.

A further restriction that has yet not been taken into account is that we only want to consider chiral gauge transformations in superspace. The observation

$$\Pi_+ X_\gamma = \frac{\overline{D}^2 D^2}{16\Box} X_\gamma = 0 \tag{3.54}$$

shows that X_γ has no chiral part. This leads to the assumption that X_γ can be gauged away by a partial gauge fixing with a transversal field $g = \Pi_T g$. In the abelian case gauge transformations on the connections are of the form

$$A_A \to A_A + D_A g \tag{3.55}$$

In terms of algebras such transformations are also called *similarity transformations*, because they don't change the structure of the algebra [br91]. In the following a suitable g will be constructed which transforms away X_γ and it is shown that it is transversal. This means that the equation

$$\begin{aligned} X_\gamma(A_{\underline{\alpha}} + \delta A_{\underline{\alpha}}) &= \quad 0 \qquad \Leftrightarrow \\ X_\gamma(A_{\underline{\alpha}} + D_{\underline{\alpha}} g) &= \quad 0 \qquad \Leftrightarrow \\ \overline{\partial\!\!\!/}^{\dot\beta\alpha}(-D_\alpha \overline{D}_{\dot\beta} D_\gamma g + D_\alpha D_\gamma \overline{D}_{\dot\beta} g + 2i\partial\!\!\!/_{\gamma\dot\beta} D_\alpha g) &= -X_\gamma \qquad \Leftrightarrow \\ \varepsilon_{\alpha\gamma}\overline{\partial\!\!\!/}^{\dot\beta\alpha} D^2 \overline{D}_{\dot\beta} g &= -X_\gamma \end{aligned} \tag{3.56}$$

has to be solved for g. It is convenient to split g according to its chiral, antichiral and transversal part

$$g = \Pi_+ g + \Pi_- g + \Pi_T g = g_+ + g_- + g_T. \tag{3.57}$$

With this it immediately follows that any chiral part g_+ of g does not contribute to (3.56) because $\overline{D}_{\dot\beta}(g_+ + g_- + g_T) = \overline{D}_{\dot\beta}(g_- + g_T)$. Using the formula $[D^2, \overline{D}_{\dot\alpha}] = 4i\partial\!\!\!/_{\alpha\dot\alpha} D^\alpha$ shows the same for any antichiral g_-. The g we are looking for is therefore transversal (if it exists). The existence is shown by direct calculation of

$$\begin{aligned} & \varepsilon_{\alpha\gamma}\overline{\partial\!\!\!/}^{\dot\beta\alpha} D^2 \overline{D}_{\dot\beta} g = -X_\gamma \\ \Rightarrow \quad & \overline{\partial\!\!\!/}^{\dot\beta\delta} \partial\!\!\!/_{\delta\dot\mu} D^2 \overline{D}_{\dot\beta} g = -\varepsilon^{\gamma\delta} \partial\!\!\!/_{\delta\dot\mu} X_\gamma \\ \Rightarrow \quad & g_T = \Pi_T g = -\frac{\overline{D} D^2 \overline{D}}{8\Box} g = \frac{\overline{D}\,\overline{\partial\!\!\!/} X}{8\Box^2} \end{aligned} \tag{3.58}$$

where $\overline{\partial\!\!\!/}^{\dot\beta\delta}\partial\!\!\!/_{\delta\dot\mu} = \Box\delta^{\dot\beta}_{\dot\mu}$ was used. Summarizing, gives

Lemma 6. *Let A_α and $\overline{A}_{\dot\alpha}$ be given and be members of a realization of the SUSY multiplet such that $D_{(\beta}A_{\alpha)} = 0$, then there exists a similarity transformation $A_\alpha \to A'_\alpha = A_\alpha + D_\alpha g$ such that $A'_\alpha = -iD_\alpha C'$. g is of the form $g := \frac{\overline{D}\overline{\partial\!\!\!/}X}{8\Box^2}$ and is transversal.*

Proof. g is well defined because the $A_{\underline{\alpha}}$ are members of the SUSY multiplet. Transversality has already been shown above, or simply observe that $\Pi_- g = \Pi_+ g = 0$ by using $D_\alpha X_\beta = 0$ and $[D^2, \overline{D}_{\dot\alpha}] = 4i\partial\!\!\!/_{\alpha\dot\alpha}D^\alpha$. Therefore $\Pi_T g = g$ because of $\Pi_+ + \Pi_- + \Pi_T = \mathbf{1}$. In the next step it is shown that $X_\gamma(A'_{\underline{\alpha}}) = 0$. Calculate $\varepsilon_{\alpha\gamma}\overline{\partial\!\!\!/}^{\dot\beta\alpha}D^2\overline{D}_{\dot\beta}\frac{\overline{D}\overline{\partial\!\!\!/}X}{8\Box^2} = -\frac{1}{2}\partial\!\!\!/_{\gamma\dot\mu}\overline{\partial\!\!\!/}^{\dot\mu\mu}\frac{D^2\overline{D}^2X_\mu}{8\Box^2} = -\frac{D^2\overline{D}^2X_\gamma}{16\Box} = -X_\gamma$. The last step was achieved by using $[D^2, \overline{D}^2] = 8i\overline{D}\overline{\partial\!\!\!/}D + 16\Box$ together with $D_\alpha X_\beta = 0$. With (3.56) follows therefore $X_\gamma(A'_{\underline{\alpha}}) = 0$, which also shows $A'_\alpha = -iD_\alpha C'$. ∎

Obviously g is not unique. There can be always added arbitrary chiral and antichiral fields g_+ and g_- which is the remaining gauge freedom in superspace.

Subsequently the prime will be dropped, and it will be assumed that the transversal gauge is fixed such that $A_\alpha = -iD_\alpha C$ holds. The next step is to calculate the transformation property of C. However since the calculation is the same for the nonabelian case, the framework of the nonabelian case will be used. The constant unit-function, which is the unit element of the gauge group is chiral $\overline{D}_{\dot\alpha}1 = 0$. g^{-1} must be chiral too, because $0 = \overline{D}_{\dot\alpha}1 = \overline{D}_{\dot\alpha}(gg^{-1})$. The transformation property of e^C follows from the transformation property of the prepotentials $A'_\alpha = gD_\alpha g^{-1} + gA_\alpha g^{-1} = gD_\alpha g^{-1} + g(e^{-C}[D_\alpha, e^C])g^{-1} = ge^{-C}D_\alpha e^C g^{-1} = ge^{-C}g^\dagger g^{\dagger-1}D_\alpha(e^C g^{-1}) = (ge^{-C}g^\dagger)D_\alpha(g^{\dagger-1}e^C g^{-1})$. Where the unitoperator $g^\dagger g^{\dagger-1}$ was inserted. On the other hand $A'_\alpha = (e^{-C}D_\alpha e^C)' = e^{-C'}D_\alpha e^{C'}$. These two expressions compared shows the disired transformation property

$$\begin{aligned} e^{C'} &= g^{\dagger-1}e^C g^{-1} \\ e^{-C'} &= ge^{-C}g^\dagger \end{aligned} \tag{3.59}$$

Now we are done. The only remaining thing is to translate the results into the whole Superspace formalism which is done with lemma 1 and 2 and $[D_A, \delta_i] = 0$. The gauge algebra can be parameterized by a real field $V = V^i\delta_i = e^{(\theta D + \overline{\theta}\overline{D})}Ce^{-(\theta D + \overline{\theta}\overline{D})}$. The chirality of the gauge group element shows that in Superspace it must be of the form $g = e^{i\Lambda}$ with $\Lambda = \Lambda^i \delta_i$ and $\overline{\mathbb{D}}_{\dot\alpha}\Lambda^i = 0$. The transformation property of the vectorfield follows from (3.59) $e^{V'} = e^{i\Lambda^\dagger} e^V e^{-i\Lambda}$.

3.5 Super YM Lagrangians

There are two ways to look at the Super YM action. One is to start with a gauge chiral multiplet and try to find an analogous way to construct the action like for the matter action [dr87]. The problem is that the F term does not transform into a total derivative because of the covariantization of the derivative $\mathcal{D}_\alpha$. It will be therefore convenient to start from the SUSY Lagrangian and adapt it to YM theories [WE83].

3.5.1 Field strengths

The field strengths for the superspace and the algebraic approach are compared. Start with the lowest dimensional non-vanishing field strength in the algebraic approach $F_{a\dot\alpha}$. The commutator relation for the spinorial field strength (3.22) reads $-4iW^\alpha = 2i\overline{\sigma}^{a\dot\alpha\alpha}F_{a\dot\alpha} = \left[\{\overline{\mathcal{D}}^{\dot\alpha}, \mathcal{D}^\alpha\}, \overline{\mathcal{D}}_{\dot\alpha}\right]$. This expression is rather lengthy to evaluate for $\mathcal{D}_\alpha = e^{-C}D_\alpha e^C, \overline{\mathcal{D}}_{\dot\alpha} = e^C\overline{D}_{\dot\alpha}e^{-C}$, the gauge covariant derivatives which were found in section 3.3. It is more economical to perform first a similarity transformation

$$\mathcal{D}_A \longrightarrow \tilde{\mathcal{D}}_A = e^Y \mathcal{D}_A e^{-Y}, \tag{3.60}$$

$$[\tilde{\mathcal{D}}_A, \tilde{\mathcal{D}}_B] = -\tilde{T}_{AB}{}^C\tilde{\mathcal{D}}_C + \tilde{F}^i_{AB}\tilde{\delta}_i, \tag{3.61}$$

$$\tilde{T}_{AB} = e^Y T_{AB} e^{-Y}, \quad \tilde{F}^i_{AB} = e^Y F^i_{AB} e^{-Y}, \quad \tilde{\delta}_i = e^Y \delta_i e^{-Y}. \tag{3.62}$$

In particular the trace of the contracted field strengths remains invariant $\operatorname{tr} \tilde{F}_{AB}\tilde{F}^{BA} = \operatorname{tr} F_{AB}F^{BA}$. Especially the choice $Y = -C$ gives $\tilde{\mathcal{D}}_{\underline{\alpha}} = \{\overline{D}_{\dot\alpha}, e^{-2C}D_\alpha e^{2C}\}$, for which we immediately find $2i\overline{\sigma}^{a\dot\alpha\alpha}\tilde{F}_{a\dot\alpha} = \left[\tilde{\overline{\mathcal{D}}}^{\dot\alpha}, \{\tilde{\overline{\mathcal{D}}}_{\dot\alpha}, \tilde{\mathcal{D}}^\alpha\}\right] = \overline{D}^2 e^{-2C}D_\alpha e^{2C}$, where the operators $D, \overline{D}$ act on C. From $0 = \tilde{\overline{\mathcal{D}}}_{\dot\beta}\tilde{F}_{a\dot\alpha} = \overline{\mathcal{D}}_{\dot\beta}F_{a\dot\alpha}$ it follows that $\overline{\mathcal{D}}_{\dot\beta}W_\beta = 0$. The inverse similarity transformation gives the fieldstrength explicitly

$$-4iW^\alpha = 2i\overline{\sigma}^{a\dot\alpha\alpha}F_{a\dot\alpha} = e^C\left(\overline{D}^2 e^{-2C}D_\alpha e^{2C}\right)e^{-C} \tag{3.63}$$

The problem that comes with this fieldstrength is that it does not transform covariantly under gauge transformations because of $e^{2C'} = e^{iL^\dagger}e^C e^{-iL}e^{iL^\dagger}e^C e^{-iL}$

(remember L is the lowest component of the chiral gauge field Λ). Just like the kinetic energy also the field strength has to be covariantized, by introducing $e^{\pm C}$. The modified gauge covariant field strength $\hat{W}^\alpha$ must be

$$i\hat{W}^\alpha = -\frac{1}{4}e^C\left(\overline{D}^2 e^{-C}D_\alpha e^C\right)e^{-C} = e^C\lambda^\alpha e^{-C} \tag{3.64}$$

Where λ^α is the lowest component of the field strengths in the superspace approach. In particular $\operatorname{tr}\lambda^2 = -\operatorname{tr}\hat{W}^2$. Therefore the relevant quantity for constructing the action will be the field strength from the superspace approach, which has already shown to be gauge covariant and chiral. In the **abelian case** the fieldstrengths are equal up to a factor of $2i$.

$\overline{W}_{\dot\alpha}, \hat{\overline{W}}_\alpha$ can be calculated analogously, or by complex conjugation.

3.5.2 Construction of the Lagrangian

The SUSY Lagrangian is of the form

$$\mathcal{L} = \frac{1}{32}D^2\overline{D}^2\overline{\phi}\phi - \frac{1}{4}D^2 g(\phi) + h.c. \tag{3.65}$$

where the superpotential is chiral and the Kähler potential real. In order to get the Lagrangian gauge invariant, use the knowledge of the previous subsection [WE83, GA83]:

$$\begin{aligned}\mathcal{L} =& \frac{1}{4k}\operatorname{tr}\left(D^2(\lambda^\alpha\lambda_\alpha) + h.c.\right) + \frac{1}{32}D^2\overline{D}^2(\overline{\phi}e^C\phi + \frac{1}{k}\operatorname{tr}\sum_{\delta_a\in U(1)}\mu'^2_a C^a\delta_a) + \\ &+ \left(D^2 g_{inv}(\phi) + h.c.\right)\end{aligned} \tag{3.66}$$

where g_{inv} indicates that the superpotential has to be gauge invariant and k comes from the normalization of the trace. The additional contributions $\tilde{\mu}_a^2 C^a\delta_a$ are for abelian factors δ_a of the gauge group. Only for them the expression is gauge invariant because of $[\delta_i, \delta_a] = 0$. Using $\operatorname{tr}\delta_a = k$ the lagrangian can be recast into

$$\mathcal{L} = \frac{1}{4k}\operatorname{tr}\left(D^2(\lambda^\alpha\lambda_\alpha) + h.c.\right) + \frac{1}{32}D^2\overline{D}^2(\overline{\phi}e^C\phi) + \left(D^2 g_{inv}(\phi) + h.c.\right) + \mu_a^2 D_F^a \tag{3.67}$$

where the sum of the D-terms is only over abelian factors.

3.5.3 Component action

In the next step the Lagrangian is evaluated in terms of the component fields. The idea is to switch into a convenient representation of the fields, which become then covariantly chiral by a similarity transformation [GA83]. The **kinetic energy** $D^2\overline{D}^2(\overline{\phi}e^C\phi)$ is evaluated in two steps:

- Show that $D^2\overline{D}^2(\overline{\phi}e^C\phi) = \tilde{\mathcal{D}}^2\tilde{\overline{\mathcal{D}}}^2(\tilde{\overline{\phi}}\tilde{\phi})$ with some gauge covariant derivatives $\tilde{\mathcal{D}}_A$ and covariantly chiral fields $\tilde{\phi}$.

- Calculate $\tilde{\mathcal{D}}^2\tilde{\overline{\mathcal{D}}}^2(\tilde{\overline{\phi}}\tilde{\phi})$, using the algebra of gauge covariant derivatives.

First observe that on gauge invariant quantities the similarity transformation $\overline{D}_{\dot{\alpha}} \to \tilde{\overline{\mathcal{D}}}_{\dot{\alpha}} = e^X\overline{D}_{\dot{\alpha}}e^{-X}$ with $X = X^i\delta_i$ may be performed such that $D^2\overline{D}^2(inv) = D^2\tilde{\overline{\mathcal{D}}}^2(inv)$ [GA83]. This can be seen by using $[\delta_i, D_A] = 0$ and the Baker Campell Hausdorff formula, which shows $\tilde{\overline{\mathcal{D}}}^2 = \overline{D}^2 + Z^i\delta_i$ and therefore $D^2\tilde{\overline{\mathcal{D}}}^2(inv) = D^2\overline{D}^2(inv) + Z^i\delta_i(inv)$. $\delta_i(inv)$ vanishes because of gauge invariance. By the very same argument an additional similarity transformation could have been performed on D_α. We choose $\tilde{\phi} := e^{\frac{C}{2}}\phi$ and $\tilde{\overline{\mathcal{D}}}_{\dot{\alpha}} := e^{\frac{C}{2}}\overline{D}_{\dot{\alpha}}e^{-\frac{C}{2}} = \overline{D}_{\dot{\alpha}} - i\tilde{\overline{A}}{}^i_{\dot{\alpha}}T_i$. Therefore a new set of operators and fields can be defined such that the kinetic energy can be written as

$$D^2\overline{D}^2(\overline{\phi}e^C\phi) = \tilde{\mathcal{D}}^2\tilde{\overline{\mathcal{D}}}^2(\tilde{\overline{\phi}}\tilde{\phi}) \tag{3.68}$$

with the *covariantly chiral multiplet*

$$\tilde{\overline{\mathcal{D}}}_{\dot{\alpha}}\tilde{\phi} = 0, \qquad \tilde{\chi}_\alpha = \tfrac{1}{\sqrt{2}}\tilde{\mathcal{D}}_\alpha\tilde{\phi}, \qquad \tilde{F} = -\tfrac{1}{4}\tilde{\mathcal{D}}^2\tilde{\phi}, \tag{3.69}$$

$$\tilde{\mathcal{D}}_\alpha\tilde{\overline{\phi}} = 0, \qquad \tilde{\overline{\chi}}_{\dot{\alpha}} = \tfrac{1}{\sqrt{2}}\tilde{\overline{\mathcal{D}}}_{\dot{\alpha}}\tilde{\overline{\phi}}, \qquad \tilde{\overline{F}} = -\tfrac{1}{4}\tilde{\overline{\mathcal{D}}}^2\tilde{\overline{\phi}}, \tag{3.70}$$

With our choice this we have $\tilde{\overline{\phi}} = \tilde{\phi}^\dagger$ and $\tilde{\overline{\mathcal{D}}}_{\dot{\alpha}} = \tilde{\mathcal{D}}^\dagger_\alpha$. In Lemma 3 it was shown that covariant derivatives of the form $e^{-C}D_\alpha e^C$, $e^C\overline{D}_{\dot{\alpha}}e^{-C}$ fulfill the gauge algebra. By the same argument any covariant derivative which is a similarity transformation of the flat spinor derivatives fulfills the gauge algebra too.

Therefore we can use the gauge algebra

$$[\tilde{\mathcal{D}}_a, \tilde{\mathcal{D}}_b] = \tilde{F}^i_{ab}\delta_i, \tag{3.71}$$

$$[\tilde{\mathcal{D}}_\alpha, \tilde{\mathcal{D}}_a] = i\sigma_{a\,\alpha\dot{\beta}}\overline{\tilde{W}}^{i\dot{\beta}}\delta_i, \tag{3.72}$$

$$\{\tilde{\mathcal{D}}_\alpha, \overline{\tilde{\mathcal{D}}}_{\dot{\beta}}\} = 2i\tilde{\not{\mathcal{D}}}_{\alpha\dot{\beta}}, \tag{3.73}$$

$$\tilde{D}^i := \frac{1}{2}\tilde{\mathcal{D}}^\alpha\tilde{W}^i_\alpha, \qquad \tilde{\mathcal{D}}^2\tilde{W}_\alpha = -4\tilde{\mathcal{D}}_\alpha\tilde{D}^i, \tag{3.74}$$

$$\tilde{\mathcal{D}}_\alpha\tilde{W}^{i\beta} = \sigma^{ab\beta}_\alpha\tilde{F}^i_{ab} + i\delta^\beta_\alpha\tilde{D}^i. \tag{3.75}$$

and evaluate

$$\begin{aligned}\tilde{\mathcal{D}}^2\overline{\tilde{\mathcal{D}}}^2(\overline{\tilde{\phi}}\tilde{\phi}) = &\left(\tilde{\mathcal{D}}^2\overline{\tilde{\mathcal{D}}}^2\overline{\tilde{\phi}}\right)\tilde{\phi} - 2\left(\tilde{\mathcal{D}}_\alpha\overline{\tilde{\mathcal{D}}}^2\overline{\tilde{\phi}}\right)\tilde{\mathcal{D}}^\alpha\tilde{\phi} + \overline{\tilde{\mathcal{D}}}^2\overline{\tilde{\phi}}\tilde{\mathcal{D}}^2\tilde{\phi} = \\ =&8i\sqrt{2}\overline{\tilde{W}}^{\dot{\beta}i}\overline{\tilde{\chi}}_{\dot{\beta}}T_i\tilde{\phi} - 16\tilde{\mathcal{D}}_a\tilde{\mathcal{D}}^a\overline{\tilde{\phi}}\tilde{\phi} + 8i\tilde{D}^i\overline{\tilde{\phi}}T_i\tilde{\phi} \\ &- 16i\sigma^m_{\alpha\dot{\alpha}}\tilde{\mathcal{D}}_m\overline{\tilde{\chi}}^{\dot{\alpha}}\tilde{\chi}^\alpha - 8i\sqrt{2}\tilde{W}^i_\alpha\overline{\tilde{\phi}}T_i\tilde{\chi}^\alpha \\ &+ 16\overline{\tilde{F}}\tilde{F}\end{aligned} \tag{3.76}$$

because of

$$\tilde{\mathcal{D}}_\alpha\overline{\tilde{\mathcal{D}}}^2\overline{\tilde{\phi}} = 4\sqrt{2}i\sigma^m_{\alpha\dot{\alpha}}\tilde{\mathcal{D}}_m\overline{\tilde{\chi}}^{\dot{\alpha}} + 4i\tilde{W}^i_\alpha(\overline{\tilde{\phi}}T_i), \tag{3.77}$$

$$\tilde{\mathcal{D}}^2\overline{\tilde{\mathcal{D}}}^2\overline{\tilde{\phi}} = 8i\sqrt{2}\overline{\tilde{W}}^{\dot{\beta}i}\overline{\tilde{\chi}}_{\dot{\beta}}T_i - 16\tilde{\mathcal{D}}_a\tilde{\mathcal{D}}^a\overline{\tilde{\phi}} + 8i\tilde{D}^i\overline{\tilde{\phi}}T_i. \tag{3.78}$$

Now we consider the **field strength** $\mathrm{tr}(D^2\lambda^2)$. Using the property of the trace and the special chosen operator $\tilde{\mathcal{D}}_\alpha := e^{-\frac{C}{2}}D_\alpha e^{\frac{C}{2}}$, we find

$$\mathrm{tr}(D^2\lambda^2) = -\,\mathrm{tr}(\tilde{\mathcal{D}}^2\tilde{W}^2) \tag{3.79}$$

Using the invariance of the trace under (graded) cyclic permutations and (3.21) this can be evaluated to

$$\begin{aligned}\mathrm{tr}(\tilde{\mathcal{D}}^2\tilde{W}^2) &= 2\,\mathrm{tr}((\tilde{\mathcal{D}}^2\tilde{W})\tilde{W} - \tilde{\mathcal{D}}^\alpha\tilde{W}^\beta\tilde{\mathcal{D}}_\alpha\tilde{W}_\beta) = \\ &= 2\,\mathrm{tr}(-4i\tilde{W}\sigma^m\tilde{\mathcal{D}}_m\overline{\tilde{W}} - \tilde{F}_{ab}\tilde{F}^{ab} + 2\tilde{D}).\end{aligned} \tag{3.80}$$

It was used that

$$\tilde{\mathcal{D}}^2\tilde{W}^\beta = -2\tilde{\mathcal{D}}^\beta\tilde{\mathcal{D}}\tilde{W} = -2\tilde{\mathcal{D}}^\beta\overline{\tilde{\mathcal{D}}}\overline{\tilde{W}} = -4i\varepsilon^{\beta\gamma}\sigma^m_{\gamma\dot{\alpha}}\tilde{\mathcal{D}}_m\overline{\tilde{W}}^{\dot{\alpha}} \tag{3.81}$$

because of $\tilde{\mathcal{D}}\tilde{W} = \overline{\tilde{\mathcal{D}}}\overline{\tilde{W}}$ and $\tilde{\mathcal{D}}_\alpha\overline{\tilde{W}}_{\dot{\alpha}} = 0$.

For the **abelian factors** we find for the chosen $\tilde{\mathcal{D}}_{\underline{\alpha}}$ that $\tilde{D}^a = -\frac{i}{8} D^\alpha \overline{D}^2 D_\alpha C$ and therefore we can rewrite in the Lagragian the Fayet-Iliopoulis terms as $\tilde{\mu}_a^2 \tilde{D}^a$ (up to a total derivative). $\tilde{D}^a$ stands for the abelian factors of the D-terms (3.74).

Summarizing, we started with the Lagragian (3.67) which contains flat derivatives, chiral fields, and a real field (This is the approach in [WE83]). By a similarity transformation we could find covariant derivatives and covariant chiral fields. With these we can rewrite the Lagragian (the form of [dr87]) into

$$\mathcal{L} = \tilde{\mathcal{D}}^2 \overline{\tilde{\mathcal{D}}}^2 (\overline{\tilde{\phi}} \tilde{\phi}) - \frac{1}{4k} \operatorname{tr} \left(\tilde{\mathcal{D}}^2 \tilde{W}^2 + h.c. \right) + \tilde{\mu}_a^2 \tilde{D}^a + \left(\mathcal{D}^2 g_{inv}(\tilde{\phi}) + h.c. \right). \tag{3.82}$$

Where the last term comes from the superpotential, for which $g(\phi) = g(\tilde{\phi})$ was used because of the assumed gauge invariance of g. Since the covariant derivatives fulfill the gauge algebra we can find the component action, in terms of the covariant chiral multiplet, and in terms of field strengths, the gaugino fields, and the D-terms from the algebra. Putting the things from above together gives the component action

$$\begin{aligned} \mathcal{L} = & -\frac{1}{2} \tilde{\mathcal{D}}_a \tilde{\mathcal{D}}^a \overline{\tilde{\phi}} \tilde{\phi} - \frac{i}{2} \sigma^m_{\alpha\dot{\alpha}} \tilde{\mathcal{D}}_m \overline{\tilde{\chi}}^{\dot{\alpha}} \tilde{\chi}^\alpha + \frac{1}{2} \overline{\tilde{F}} \tilde{F} \\ & + \frac{i\sqrt{2}}{4} \left(\overline{\tilde{W}}^{\dot{\beta} i} \overline{\tilde{\chi}}_{\dot{\beta}} T_i \tilde{\phi} - \tilde{W}^i_\alpha \overline{\tilde{\phi}} T_i \tilde{\chi}^\alpha \right) + \frac{i}{4} \tilde{D}^i \overline{\tilde{\phi}} T_i \tilde{\phi} \\ & + \frac{1}{2k} \operatorname{tr} \left(4i \tilde{W} \sigma^m \tilde{\mathcal{D}}_m \overline{\tilde{W}} + \tilde{F}_{ab} \tilde{F}^{ab} - 2\tilde{D} + h.c. \right) \\ & + \tilde{\mu}_a^2 \tilde{D}^a + \left(\partial_i g \tilde{F}^i - \frac{1}{2} \partial_i \partial_j g \tilde{\chi}^i \tilde{\chi}^j + h.c. \right) \end{aligned} \tag{3.83}$$

where $\tilde{\mu}_a^2 \tilde{D}^a$ is the sum over the abelian factors of the gauge group. This Lagrangian contains the kinetic energies of the gauge field, the fermions $\tilde{\chi}$ and $\tilde{W}$ and the scalars $\tilde{\phi}$. The gauge couplings are via the covariant derivatives $\tilde{\mathcal{D}}_A = D_A - i\tilde{A}^i_A T_i$ The gaugino field $W_{\underline{\alpha}}$ has Yukawa couplings to the chiral multiplet which are determined by the gauge coupling.

Conversely, if one starts with a Lagrangian of the form (3.82) which is formulated in terms of covariant derivatives and covariantly chiral fields (this is the approach in [dr87]), we can cast it into the form of (3.67). Lemma 6 assures the existence of some real field C' such that the covariant derivatives are of the form $\tilde{\mathcal{D}}_\alpha = e^{-C'} D_\alpha e^{C'}$, $\overline{\tilde{\mathcal{D}}}_{\dot{\alpha}} = e^{C'} \overline{D}_{\dot{\alpha}} e^{-C'}$. The covariantly chiral fields are related to the chiral ones by $\tilde{\phi} = e^{C'}\phi$, $\overline{D}_{\dot\alpha}\phi = 0$. With this we can go the way back and arrive at (3.67), after specifying $C' \to C/2$.

3.6 Algebraic Structure

A *representation* R of a group (algebra) G is a map $R : G \to \mathrm{End}(V)$ from G into the group of linear transformations on a vector space V over a field $\mathbb{K}$ that is consistent with the group (algebra) structures. An *intertwiner* between two representations R_1 and R_2 of some group (algebra) G is a map $A : V_1 \to V_2$ that is compatible with the representations, i.e. $AR_1(g) = R_2(g)A \ \ \forall g \in G$. The representations R_1 and R_2 are called *equivalent* if there exists an invertible intertwiner.

Example. The SUSY algebra may be represented on the chiral multiplet $\mathcal{C}$ [WE83]. Similarly one can represent the SUSY algebra on the antichiral multiplet $\overline{\mathcal{C}}$. A priori no (analytical) relation between the operators or fields with and without bars can be made. The usual identification $\overline{\phi} = \phi^*$ [dr79] is equivalent to the commutative diagram (intertwiner):

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\ *\ } & \overline{\mathcal{C}} \\ {\scriptstyle\mathcal{A}}\big\downarrow & & \big\downarrow{\scriptstyle\mathcal{A}} \\ \mathcal{C} & \xrightarrow[\ *\ ]{} & \overline{\mathcal{C}} \end{array} \tag{3.84}$$

where the same symbol $\mathcal{A}$ was used for the representations of the SUSY algebra on $\mathcal{C}$ and $\overline{\mathcal{C}}$. This allows the identification $\overline{D}_{\dot\alpha} = D^*_\alpha$ (because of $0 = D_\alpha\overline{\phi} = (D_\alpha\overline{\phi})^* = D^*_\alpha\phi$). It remains to find the representation of the SUSY algebra on the (tensor) product spaces of some given representation spaces. This is done in the standart fashion [FU97]. Let there be given two representations R_1, R_2 of a Lie algebra on representation spaces V_1 and V_2, then the Lie algebra can be represented on $V_1 \otimes V_2$ by declaring $(R_1 \otimes R_2)(x \otimes y) := (R_1(x)) \otimes y + (-)^{|x||R_1|} x \otimes (R_2(y))$. This is consistent with the properties of a derivation.

When looking at gauge covariant derivatives it will be interesting to consider a special class of intertwiners, namely those who can be written as an exponential map (Lie group element with some underlying Lie algebra).

Definition. Two Operators $\mathcal{O}_1$ and $\mathcal{O}_2$ are called *gauge adjoint* if there exists a map e^X such that the following diagram is commutative

$$\begin{array}{ccc} V & \xrightarrow{e^X} & W \\ {\scriptstyle\mathcal{O}_1}\Big\downarrow & & \Big\downarrow{\scriptstyle\mathcal{O}_2} \\ V & \xrightarrow[e^X]{} & W \end{array} \tag{3.85}$$

This map can be also used to construct gauge adjoint operators. Obviously if in some representation of the D_α, $\{D_\alpha, D_\beta\} = 0$ holds, then it also holds for the gauge adjoints $\mathcal{D}_\alpha := e^X D_\alpha e^{-X}$ with $X = X^i \delta_i$. This will also be the main purpose of this construction, i.e. to carry over representations of a given algebra from one representation space to another. e^X acts then as an intertwiner.

Example. The commutative diagram which was used in the proof of lemma 2 has this structure. In this sense, $\exp(\theta D + \overline{\theta D})$ intertwines a representation of the SUSY algebra on fields in $\mathcal{S}_{00}$ with a representation on superfields in $\mathcal{S}$.

In the next step we want to make contact to gauge transformations, which is done by the following commutative diagram:

$$\begin{array}{ccccc} V & \xrightarrow{e^X} & W & \xrightarrow{e^Y} & W \\ \Big\downarrow{\scriptstyle\mathcal{O}_1} & & \Big\downarrow{\scriptstyle\mathcal{O}_2} & & \Big\downarrow{\scriptstyle\mathcal{O}_2'} \\ V & \xrightarrow[e^X]{} & W & \xrightarrow[e^Y]{} & W \end{array} \tag{3.86}$$

The operator $\mathcal{O}_2$ is then said to transform *gauge covariant* under a gauge group element e^Y with $Y = Y^i \delta_i$. In the above terminology gauge covariant operators transform gauge adjoint.

Example. This structure occurred in lemma 3, where we were given an operator $\mathcal{O}_1$ ($\mathbb{D}$) and a gauge transformation e^Y ($e^{i\Lambda}$) and we had to find a X (V) such that the gauge adjoint operator $\mathcal{O}_2$ ($\mathcal{D}$) transforms covariantly under e^Y ($e^{i\Lambda}$).

For a given gauge transformation e^Y and a given Operator $\mathcal{O}_1$ the gauge adjoint operators are not unique. There can always be a Lie group element added which lies in the kernel of $\mathcal{O}_1$, because of $\mathcal{O}_2 = e^{-X}\mathcal{O}_1 e^X = e^{-X}e^{-N}\mathcal{O}_1 e^N e^X$, with $O_1 e^N = 0$. This gives the transformation property of $e^{X'} = e^N e^X e^Y$.

In the next step two such structures are combined. Consider an invertible intertwiner f , such that the following diagram is commutative.

$$\begin{tikzcd}
\tilde{W} \arrow[d, "\tilde{\mathcal{O}}_2{}'"] & \tilde{W} \arrow[l, "e^{Y'}"'] \arrow[d, "\tilde{\mathcal{O}}_2"] & \tilde{V} \arrow[l, "e^{X'}"'] \arrow[r, "f"] \arrow[d, "\tilde{\mathcal{O}}_1"] & V \arrow[r, "e^X"] \arrow[d, "\mathcal{O}_1"] & W \arrow[r, "e^Y"] \arrow[d, "\mathcal{O}_2"] & W \arrow[d, "\mathcal{O}_2'"] \\
\tilde{W} & \tilde{W} \arrow[l, "e^{Y'}"] & \tilde{V} \arrow[l, "e^{X'}"] \arrow[r, "f"'] & V \arrow[r, "e^X"'] & W \arrow[r, "e^Y"'] & W
\end{tikzcd} \tag{3.87}$$

Since f is an intertwiner and because of the Lie group structure of e^X, $e^{-X'} \circ f \circ e^X =: \tilde{f}$ is again an intertwiner

$$\begin{tikzcd}
\tilde{W} \arrow[d, "\tilde{\mathcal{O}}_2{}'"] & \tilde{W} \arrow[l, "e^{Y'}"'] \arrow[r, "\tilde{f}"] \arrow[d, "\tilde{\mathcal{O}}_2"] & W \arrow[r, "e^Y"] \arrow[d, "\mathcal{O}_2"] & W \arrow[d, "\mathcal{O}_2'"] \\
\tilde{W} & \tilde{W} \arrow[l, "e^{Y'}"] \arrow[r, "\tilde{f}"'] & W \arrow[r, "e^Y"'] & W
\end{tikzcd} \tag{3.88}$$

So $\tilde{f}$ is an intertwiner between the representations on W and $\tilde{W}$.

Example. In superspace the gauge covariant derivatives are of the form $\nabla_\alpha = e^{-V}\mathbb{D}_\alpha e^V$ and $\overline{\nabla}_{\dot{\alpha}} = e^V\overline{\mathbb{D}}_{\dot{\alpha}}e^{-V}$. The intertwiner is hermitian conjugation, and the transformation property of e^V is $e^{V'} = e^{i\Lambda^\dagger}e^V e^{-i\Lambda}$.

3.7 Covariant derivatives and the Poincaré lemma

The general form of the spinorial covariant derivatives are $\mathcal{D}_\alpha = D_\alpha + A_\alpha$. Because of the constraints on the fieldstrength it should furthermore hold that $\{\mathcal{D}_{\dot{\beta}}, \mathcal{D}_{\dot{\alpha}}\} = 0$. From (3.86) it follows that covariant derivatives which are obtained from gauge adjoint operators are of this form, because of the Baker Campbell Hausdorff formula. Conversely, given a covariant derivative $\mathcal{D}_A$ one could ask whether it is some gauge adjoint of the flat derivative D_A. If $\mathcal{D}_A$ is considered to emerge from a deformation then this is the case (following the argument on page 24). In general the question is: given $\mathcal{D}_A = D_A + A_A$, exists then a X, such that $\mathcal{D}_A = e^X D_A e^{-X}$. In the abelian case this means that the equation $A_a = D_a X$ has to be solved for X. If such a X exists, then also holds $D_{(\alpha}A_{\beta)} = 0$. The structure of solutions of

$D_{(\alpha}A_{\beta)} = 0$ can be regarded as the spinorial version of the Poincaré lemma. In [br91] it has been shown that equations of this type cannot be solved independently of the representation of the SUSY algebra. This could have been also observed in the calculations of section 3.4, where the solution was of the form $A_\alpha = -iD_\alpha C + transversal$. The extra transversal terms were gauged away. Therefore one would expect that the representation of the SUSY algebra enters also in the gauge freedom.

Chapter 4

Supergravity

In super YM theories it was shown that $V = V^i \delta_i$ parameterizes the most general solution to the constraints, so that the real scalar superfield saves us all the work with the Bianchi identities. In SUGRA, however, no such nice magic is known and we have to do it the hard way by solving the Bianchi identities with constraints. We can either work in superspace with the super-vielbein and super-spin connection, and eventually use a superspace coordinate transformation to go to a Wess–Zumino gauge when life becomes too tedious, or we may avoid to introduce the redundant fields that are eliminated by that gauge from scratch and work with the structure of the gauge algebra. This is the approach that we will follow.

4.1 Symmetry algebras

A comparison with the symmetry algebra shows that there are more non-trivial relations than in super YM. The Bianchi identities read

$$\text{BI 1:} \qquad \sum_{ABC} (-)^{AC} \left(\mathcal{D}_A T_{BC}{}^D + T_{AB}{}^E T_{EC}{}^D - F_{AB}{}^I g_{IC}{}^D \right) = 0, \tag{4.1}$$

$$\text{BI 2:} \qquad \sum_{ABC} (-)^{AC} \left(\mathcal{D}_A F_{BC}{}^I + T_{AB}{}^D F_{DC}{}^I \right) = 0. \tag{4.2}$$

and the meaning of the remaining identities is that $T_{AB}{}^C$ and $F_{AB}{}^K$ transform as representations under δ_I according to their indices,

$$\delta_I F_{AB}{}^K = -g_{IA}{}^D F_{DB}{}^K + (-)^{AB} g_{IB}{}^D F_{DA}{}^K + (-)^{IA+IB} F_{AB}{}^J f_{JI}{}^K, \tag{4.3}$$

$$\delta_I T_{AB}{}^C = -g_{IA}{}^D T_{DB}{}^C + (-)^{AB} g_{IB}{}^D T_{DA}{}^C + T_{AB}{}^D g_{ID}{}^C, \tag{4.4}$$

and that the representations matrices g_I and the structure constants $f_{IJ}{}^K$ are invariant tensors $\delta_I g_{JA}{}^B = 0$ (the representation property of g) and

$\delta_I f_{JK}{}^L = 0$ (the Jacobi identity for f). On the torsions the algebra acts then according to

$$
\begin{aligned}
[\mathcal{D}_A, \mathcal{D}_B\} T_{CD}{}^E \quad &= (-T_{AB}{}^K \mathcal{D}_K + F_{AB}{}^I \delta_I) T_{CD}{}^E \qquad (4.5) \\
&= -T_{AB}{}^K \mathcal{D}_K T_{CD}{}^E + \\
&\quad + F_{AB}{}^I (-g_{IC}{}^K T_{KD}{}^E + (-)^{CD} g_{ID}{}^K T_{KC}{}^E + T_{CD}{}^K g_{IK}{}^E) \\
&= -T_{AB}{}^K \mathcal{D}_K T_{CD}{}^E - F_{ABC}{}^K T_{KD}{}^E + (-)^{CD} F_{ABD}{}^K T_{KC}{}^E + \\
&\quad (-)^{(I+K+E)(C+D+K)} F_{ABK}{}^E T_{CD}{}^K
\end{aligned}
$$

where $F_{ABC}{}^E$ stands for $F_{AB}{}^I g_{IC}{}^F$.

We assume that the δ_I are linearly represented on tensor fields and that ∂_m can be written as a linear combination of the covariant derivatives

$$\partial_m \phi = -\mathcal{A}_m{}^N(\varphi) \nabla_N \phi. \qquad (4.6)$$

To specify the field content we assume that the connection one forms $\mathcal{A}^N = dx^m \mathcal{A}_m{}^N$ and their (symmetrized) derivatives can be chosen to be the only non-covariant variables of the jet bundle. (The formalism can be extended to the case of p-form gauge fields and reducible gauge algebras, as well as to algebras that only close off-shell [br96₁]). With $e_m{}^a := -\mathcal{A}_m{}^a$ and $e_m{}^a E_a{}^n = \delta_m^n$ we define

$$
\begin{aligned}
\{\mathcal{A}_m{}^M\} &= \{-e_m{}^a, \mathcal{A}_m{}^\mu\} = \{-e_m{}^a, \psi_m{}^{\underline{\alpha}}, \mathcal{A}_m{}^I\} = \qquad (4.7) \\
&= \{-e_m{}^a, \psi_m{}^{\underline{\alpha}}, \omega_m{}^{ab}, \mathcal{A}_m{}^i + \ldots\}, \\
\mathcal{D}_a &= E_a{}^m (\partial_m + \mathcal{A}_m{}^\mu \nabla_\mu) = \qquad (4.8) \\
&= E_a{}^m (\partial_m + \psi_m{}^{\underline{\alpha}} \mathcal{D}_{\underline{\alpha}} + \tfrac{1}{2} \omega_m{}^{ab} l_{ab} + A_m{}^i \delta_i + \ldots).
\end{aligned}
$$

In these equations the vielbein $e_m{}^a$ is assumed to be invertible and vielbein and gravitino (Rarita–Schwinger field) are interpreted as connections for translations and SUSY transformations. Commutation of the partial derivatives $[\partial_m, \partial_n] = 0$ and independence of $\nabla_N \phi$ then imply

$$\partial_m \mathcal{A}_n{}^P - \partial_n \mathcal{A}_m{}^P - \mathcal{A}_m{}^M \mathcal{A}_n{}^N \mathcal{F}_{NM}{}^P = 0, \qquad (4.9)$$

which can be solved for the field strengths with bosonic indices

$$e_m{}^a e_n{}^b \mathcal{F}_{ab}{}^N = \partial_m \mathcal{A}_n{}^N - \partial_n \mathcal{A}_m{}^N - e_m{}^c \mathcal{A}_n{}^\mu \mathcal{F}_{\mu c}{}^N + e_n{}^c \mathcal{A}_m{}^\mu \mathcal{F}_{\mu c}{}^N + \mathcal{A}_n{}^\nu \mathcal{A}_m{}^\mu \mathcal{F}_{\mu\nu}{}^N. \qquad (4.10)$$

This equation could again be split into equations for field strengths and torsions in terms of the various connections to obtain the usual lengthy formulas (the last term with $\nu = j, \mu = i, N = k$, for example, gives the A^2-term in YM).

It is straightforward to set up the BRST formalism for symmetry algebras of this type. The BRST transformations of the matter fields is defined by replacing the gauge parameters by ghost fields of opposite grading $|C^I| \equiv |\nabla_I| + 1 \bmod 2$, i.e. $s\phi^i = C^N \nabla_N \phi^i$. For any closed and irreducible gauge algebra one may check that $s^2\phi^i = 0$ uniquely fixes the BRST transformations of the ghost fields.

$$s\phi^i = C^N \nabla_N \phi^i \quad \Rightarrow \quad sC^P = \tfrac{(-)^M}{2} C^M C^N \mathcal{F}_{NM}{}^P. \tag{4.11}$$

$s^2 C^P = 0$ is then equivalent to the Bianchi identity (1.5).

Anti-commutativity of s and d, which follow from $[s, \partial_m] = \{s, dx^m\} = 0$, may then be used to define a new nilpotent operator $\tilde{s} := s + d$ and $\tilde{C}^N = C^N + \mathcal{A}^N$ so that $s + d = \tilde{C}^N \nabla_N$ on tensor fields. (4.11) implies because of formal identity of the algebras that

$$(s + d)\, \tilde{C}^P = \tfrac{1}{2}(-)^N \tilde{C}^N \tilde{C}^M \mathcal{F}_{MN}{}^P \tag{4.12}$$

whose split into parts with ghost number 0, 1 and 2 yields

$$\begin{aligned} sC^P &= \tfrac{1}{2}(-)^N C^N C^M \mathcal{F}_{MN}{}^P, && (4.13)\\ s\mathcal{A}^P + dC^P &= C^M \mathcal{A}^N \mathcal{F}_{NM}{}^P, && (4.14)\\ d\mathcal{A}^P &= \tfrac{1}{2}\mathcal{A}^M \mathcal{A}^N \mathcal{F}_{NM}{}^P. && (4.15) \end{aligned}$$

The first two equations define the BRST transformations of connections and ghost fields. Consistency of the last equation with the tensor transformation law of the field strengths can be checked by a straightforward computation.

To obtain the more conventional form of this transformation law we use the reparameterization

$$\xi^a := C^m e_m{}^a, \qquad \xi^\mu := C^\mu + C^m \mathcal{A}_m{}^\mu = C^\mu + i_C \mathcal{A}^\mu. \tag{4.16}$$

ξ^m corresponds to the vector field entering the Lie derivative and we thus obtain

$$\begin{aligned} s\,\phi &= (\xi^m \partial_m + \xi^\mu \nabla_\mu)\, \phi, && (4.17)\\ s\, e_m{}^a &= \xi^n \partial_n e_m{}^a + (\partial_m \xi^n)\, e_n{}^a + \xi^\mu \mathcal{A}_m{}^N \mathcal{F}_{N\mu}{}^a, && (4.18)\\ s\, \mathcal{A}_m{}^\mu &= \xi^n \partial_n \mathcal{A}_m{}^\mu + (\partial_m \xi^n)\, \mathcal{A}_n{}^\mu + \partial_m \xi^\mu + \xi^\nu \mathcal{A}_m{}^N \mathcal{F}_{N\nu}{}^\mu, && (4.19)\\ s\, \xi^m &= \xi^n \partial_n \xi^m + \tfrac{1}{2}(-)^\mu \xi^\mu \xi^\nu \mathcal{F}_{\nu\mu}{}^a E_a{}^m, && (4.20)\\ s\, \xi^\mu &= \xi^n \partial_n \xi^\mu + \tfrac{1}{2}(-)^\nu \xi^\nu \xi^\rho (\mathcal{F}_{\rho\nu}{}^\mu - \mathcal{F}_{\rho\nu}{}^a E_a{}^m \mathcal{A}_m{}^\mu). && (4.21) \end{aligned}$$

The C^N are called covariant ghosts: The necessity of a redefinition of ghost variables in covariant equations can already be observed in Riemannian geometry: Since the Lie derivative maps tensors into tensors it should be possible to write it in terms of covariant derivatives. But this works out only if we combine it with a Lorentz transformation and redefine the parameter Λ:

$$\mathcal{L}_\xi+\tfrac{1}{2}\Lambda_{ab}l^{ab}=\xi^l D_l-(D_i\xi^k+\xi^l T_{li}{}^k)\Delta_k{}^i+\tfrac{1}{2}\hat{\Lambda}_{ab}l^{ab},\qquad \hat{\Lambda}_{ab}=\Lambda_{ab}-\xi^l\omega_{lab}. \quad (4.22)$$

($\Delta_i{}^j$ and l^{ab} are the GL_n and Lorentz generators; for simplicity we avoid any world indices on tensors by contraction with the vielbein, which is a connection in the present context, or with differentials in case of field strengths). Using $\hat{\Lambda}$ we also find $s\omega_{na}{}^b=-D_n\hat{\Lambda}_a{}^b-\xi^l R_{lna}{}^b$, in analogy with the tensorial property of the variation of the connection coefficients $s\Gamma_{nl}{}^m=D_nD_l\xi^m+D_n(\xi^k T_{kl}{}^m)+\xi^k R_{knl}{}^m$. Of course these results are contained in their above extension to more general algebras of covariant derivatives if world indices are avoided.

Returning to the construction of supergravity theories, the next step is to impose constraints since the connections we introduced so far yield highly reducible theories that, furthermore, usually do not allow for matter fields obeying equation of motion of the type that we expect. First one ones redefinitions $\nabla_M\to X_M{}^N\nabla_N$ with $X_M{}^N=\delta_M^N+H_M{}^N(\mathcal{F})$ of the covariant derivatives to bring the gauge algebra into a standard form, where we have the conventional constaints [br91]

$$T_{\alpha\dot\beta}{}^a=2i\sigma^a_{\alpha\dot\beta},\qquad T_{ab}{}^c=T_{\alpha\beta}{}^\gamma=T_{\dot\alpha\dot\beta}{}^{\dot\gamma}=T_{a\dot\beta}{}^{\underline{\gamma}}=0,\qquad F^i_{\alpha\dot\beta}=0. \quad (4.23)$$

To allow for chiral matter multiplets one extends this to the following collection of standard constraints:

$$T_{ab}{}^c=0,\qquad T_{\underline{\alpha\beta}}{}^a=2i\gamma^a_{\underline{\alpha\beta}},\qquad F^i_{\underline{\alpha\beta}}=0,\qquad T_{\alpha\beta}{}^{\dot\gamma}=0. \quad (4.24)$$

(which of these constraints are conventional slightly depends on whether we gauge R and Weyl symmetries).

Consistency of the constraints requires that the Bianchi identities are fulfilled, the check of which is the crucial (and most tedious) step in the construction of a SUGRA theory. These identities usually imply additional constraints and the general parameterization of the allowed curvatures and torsions requires the introduction of auxiliary fields that, together with the vielbein $e_m{}^a$ and the gravitino $\psi_m{}^{\underline{\alpha}}$, constitute the (off-shell) graviton multiplet. In some complicated cases, like 10-dimensional SUGRA and $N=4$–extended

SUGRA in 4 dimension, it has be shown that our approach cannot lead to a satisfactory theory. In these cases on must extend our framework and admit open and reducible gauge algebras.

The standard constraints are usually not sufficient and finding a useful complete set of constraints (i.e. obtaining an irreducible SUGRA theory) requires some experience (informed guesses and tedious evaluation of the consequences). In four dimensions, for example, there are 3 known sets of solutions, called old minimal, new minimal and non-minimal SUGRA. Non-minimal SUGRA has some ugly features as far as allowed matter couplings are concerned and new minimal SUGRA is the one that automatically comes out of superstring theory.

It turns out that not all of the BIs are independent. This is the content of the following
Theorem (Dragon): The second BI follows from equation (4.1) and the first set of BIs [`dr79`,`MU89`].

Proof. The idea is to exploit the properties of the generators. Define $M_{ABCD}{}^E := \sum_{ABC}(-)^{AC}(\mathcal{D}_A F_{BCD}{}^E + T_{AB}{}^F F_{FCD}{}^E)$. We have to show that $M_{ABCD}{}^E = 0$, where A,B,C take all possible indices for some D,E (the D,E indices correspond to the Lorentz generators). In the first step we show that $M_{ABCD}{}^E - (-)^{AB+AC+AD} M_{BCDA}{}^E + (-)^{AC+AD+BC+BD} M_{CDAB}{}^E - (-)^{AD+BD+CD} M_{DABC}{}^E$ holds. This is done by grouping together terms which involve derivatives of the $\sum_{ABC} R_{ABC}{}^D$, and replacing $\sum_{ABC} R_{ABC}{}^D$ via BI1 by terms of the torsion. The occurring commutators which act on the torsion are then evaluated by (4.1). All terms are canceled because of BI1 (see Appendix B). Now we show that every $M_{ABCD}{}^E$ has to vanish. The structure group consists of the Lorentz generators

$$(G_{ab})_C{}^D = \begin{pmatrix} (G_{ab})_c{}^d & 0 & 0 \\ 0 & (\sigma_{ab})_\alpha{}^\beta & 0 \\ 0 & 0 & (\overline{\sigma}_{ab})^{\dot\alpha}{}_{\dot\beta} \end{pmatrix} \tag{4.25}$$

with $(G_{ab})_c{}^d = -(\eta_{ac}\delta_b{}^d - \eta_{bc}\delta_a{}^d)$. Observe that $M_{ABCD}{}^E$ has the structure of a generator, therefore it vanishes by definition if it does not have the index picture of a generator. Take A,B,C spinorial and D,E spacetime like shows $M_{\underline{\alpha\beta\gamma}d}{}^e = 0$. Now take for A,B,C one vector index and two spinor indices. Choosing vector indices for D,E shows $M_{a\underline{\beta\gamma}d}{}^e - M_{\underline{\beta\gamma}da}{}^e = 0 \Rightarrow M_{\underline{\beta\gamma}ad}{}^e - M_{\underline{\beta\gamma}da}{}^e = 0$. The object $M_{\underline{\beta\gamma}dae}$ is antisymmetric in a,e and symmetric in a,d and therefore has to vanish. Now take for A,B vector indices, for C a spinor index and for D,E spinor indices of the opposite type:

$M_{ab\alpha\dot{\beta}}{}^{\dot{\gamma}} = 0$ and $M_{ab\dot{\alpha}\beta}{}^{\gamma} = 0$. It remains the index picture where A, B, C are vectorial. Chose D, E spinorial, then $M_{abc\alpha}{}^{\beta} = 0$. Putting everything together and using the linear independence of the Lorentz generators G_{ab} shows $M_{ABC}{}^{ab} = 0$, which are the second Bianchi Identities. $\square$

The statement also holds with additional generators for internal symmetries. The theorem decreases the number of consistency checks one has to do if equations on the torsions are imposed. Using the first Bianchi identities it can be shown that the curvature can be expressed in terms of the torsions and the covariant derivatives [dr79]. Therefore in supergravity the curvature is a redundant object and the equations of motion and the constraints should be formulated as conditions for the torsion.

To find the most general local action that is invariant under a given gauge algebra the BRST formalism can be used to derive the descent equations, which reduce the problem to the computation of cohomologies of (super) Lie algebra [br92].

Appendix A

Noether charges for SUSY

Task

Take a supersymmetric lagrangian and calculate the Noether charges Q_α,$\overline{Q}_{\dot{\alpha}}$ for a susy transformation. Then calculate the Dirac bracket, which is the constrained pendant to the Poisson bracket and perform canonical quantization (alternatively to the Dirac procedure one could go into the first order formalism). Prove that the sign for the graded commutator $\{Q_\alpha, \overline{Q}_{\dot{\alpha}}\}$ is correct (i.e. it has to imply positive energy). Otherwise one has to change a sign in the SUSY algebra such that it gets correct.

Signs

The anti-commutator relation for the Noether charges $\{Q_\alpha, \overline{Q}_{\dot{\alpha}}\} = 2\sigma^m_{\alpha\dot{\alpha}} P_m$ implies positive energy. In the rest frame it has with the chosen conventions the form $\{Q_\alpha, \overline{Q}_{\dot{\alpha}}\} = 2\delta_{\alpha\dot{\alpha}} P_0$, with the energy $P_0 = H$ which is assumed to be positive. In quantum mechanics this means a positive expectation value. On the other hand $\langle\psi, \{Q_\alpha, \overline{Q}_{\dot{\alpha}}\}\psi\rangle = \langle\psi, Q_\alpha \overline{Q}_{\dot{\alpha}}\psi\rangle + \langle\psi, \overline{Q}_{\dot{\alpha}} Q_\alpha\psi\rangle = \parallel Q_\alpha\psi \parallel + \parallel Q_{\dot{\alpha}}\psi \parallel \geq 0$ in a positive definit Hilbert space. Therefore the sign of the anticommutator of the charges is fixed, if positive energy is assumed. Positive energy and the choice of η together with the Legendre transformation also fixes the global sign of the supersymmetric Lagrangian density. It must have the form $\mathcal{L} = +\partial_n\phi\partial^n\overline{\phi}$.... This also fixes the sign for the Noether current. $H = P_0 = \int T^0_0$ therefore the energy momentum tensor T^0_a must have the sign $T^0_a = \partial_a\Omega^i \frac{\partial\mathcal{L}}{\partial(\partial_0\Omega^i)} - \delta^0_a\mathcal{L}$ to assure positive energy (then $T^0_0 = 2\partial_0\phi\partial_0\overline{\phi} - \partial_n\phi\partial^n\overline{\phi} = +\sum_{n=0}^3 \partial_n\phi\partial_n\overline{\phi} = \sum_{n=0}^3 |\partial_n\phi|^2 \geq 0$). Since T^0_a is the Noether current which is obtained by translations, a general Noether current should have the sign $J^m_I = \delta_I\Omega^i \frac{\partial\mathcal{L}}{\partial(\partial_m\Omega^i)} - K^m_I$

SUSY transformation

Take the supersymmetric Lagrangian density

$$\mathcal{L} = +\partial_n\phi\partial^n\overline{\phi} + \frac{i}{2}(\chi^\alpha\sigma^m_{\alpha\dot{\alpha}}\partial_m\overline{\chi}^{\dot{\alpha}} + \overline{\chi}_{\dot{\alpha}}\overline{\sigma}^{m\dot{\alpha}\alpha}\partial_m\chi_\alpha) + F\overline{F} \tag{A.1}$$

Starting with a chiral field ϕ and an antichiral one $\overline{\phi}$, the field content is then

$$\begin{aligned} &\overline{D}_{\dot{\alpha}}\phi = 0 \qquad \chi_\alpha := \frac{1}{\sqrt{2}}D_\alpha\phi \qquad F := -\frac{1}{4}D^2\phi \\ &D_\alpha\overline{\phi} = 0 \qquad \overline{\chi}_{\dot{\alpha}} := \frac{1}{\sqrt{2}}\overline{D}_{\dot{\alpha}}\overline{\phi} \qquad \overline{F} := -\frac{1}{4}\overline{D}^2\overline{\phi} \end{aligned} \tag{A.2}$$

The SUSY transformation is then of the form

$$\begin{aligned} s &= \xi^\alpha D_\alpha + \overline{\xi}^{\dot{\alpha}}\overline{D}_{\dot{\alpha}} \\ [s, \partial_m] &= 0 \end{aligned} \tag{A.3}$$

where the parameters ξ $\overline{\xi}$ commute. Therefore the SUSY transformation has an odd grading, although the index picture is even. Assuming that the commutator of the spinorial derivatives has the sign $\{D_\alpha, \overline{D}_{\dot{\alpha}}\} = 2i\sigma^m_{\alpha\dot{\alpha}}\partial_m$, the SUSY transformation of the fields is then

$$\begin{array}{ll} s\phi = \sqrt{2}\xi\chi & s\overline{\phi} = -\sqrt{2}\overline{\xi}\overline{\chi} \\ s\chi_\alpha = \sqrt{2}(\xi_\alpha F + i\sigma^m_{\alpha\dot{\alpha}}\overline{\xi}^{\dot{\alpha}}\partial_m\phi) & s\overline{\chi}^{\dot{\alpha}} = -\sqrt{2}(\overline{\xi}^{\dot{\alpha}}\overline{F} + i\overline{\sigma}^{m\dot{\alpha}\alpha}\xi_\alpha\partial_m\overline{\phi}) \\ sF = \sqrt{2}i\sigma^m_{\alpha\dot{\alpha}}\overline{\xi}^{\dot{\alpha}}\partial_m\chi^\alpha & \overline{F} = -\sqrt{2}i\sigma^m_{\alpha\dot{\alpha}}\xi^\alpha\partial_m\overline{\chi}^{\dot{\alpha}} \end{array}$$

Using the Leibnitz rule the variation of the Lagrangian density is given by

$$\begin{aligned} s\mathcal{L} = &(s\partial_n\phi)(\partial^n\overline{\phi}) + (s\partial^n\overline{\phi})(\partial_n\phi) + \\ &+ \frac{i}{2}[(s\chi^\alpha)\sigma^m_{\alpha\dot{\alpha}}\partial_m\overline{\chi}^{\dot{\alpha}} - (s\partial_m\overline{\chi}^{\dot{\alpha}})\sigma^m_{\alpha\dot{\alpha}}\chi^\alpha + \\ &\quad + (s\overline{\chi}_{\dot{\alpha}})\overline{\sigma}^{m\dot{\alpha}\alpha}\partial_m\chi_\alpha - (s\partial_m\chi_\alpha)\overline{\sigma}^{m\dot{\alpha}\alpha}\overline{\chi}_{\dot{\alpha}}] + \\ &+ (sF)\overline{F} + (s\overline{F})F \end{aligned} \tag{A.4}$$

Plugging in the expressions for the field transformations

$$
\begin{aligned}
s\mathcal{L} = {} & \partial_n(\sqrt{2}\xi\chi)(\partial^n\overline{\phi}) + \partial^n(-\sqrt{2}\overline{\xi}\overline{\chi})(\partial_n\phi) + \\
& + \frac{i}{2}[\varepsilon^{\alpha\beta}\sqrt{2}(\xi_\beta F + i\sigma^m_{\beta\dot{\beta}}\overline{\xi}^{\dot{\beta}}\partial_m\phi)\sigma^n_{\alpha\dot{\alpha}}\partial_n\overline{\chi}^{\dot{\alpha}} - \\
& \quad - \sqrt{2}\partial_n(-)(\overline{\xi}^{\dot{\alpha}}\overline{F} + i\overline{\sigma}^{m\dot{\alpha}\beta}\xi_\beta\partial_m\overline{\phi})\sigma^n_{\alpha\dot{\alpha}}\chi^\alpha + \\
& \quad + \varepsilon_{\dot{\alpha}\dot{\beta}}(-)\sqrt{2}(\overline{\xi}^{\dot{\beta}}\overline{F} + i\overline{\sigma}^{m\dot{\beta}\beta}\xi_\beta\partial_m\overline{\phi})\overline{\sigma}^{n\dot{\alpha}\alpha}\partial_n\chi_\alpha - \\
& \quad - \sqrt{2}\partial_n(\xi_\alpha F + i\sigma^m_{\alpha\dot{\beta}}\overline{\xi}^{\dot{\beta}}\partial_m\phi)\overline{\sigma}^{n\dot{\alpha}\alpha}\overline{\chi}_{\dot{\alpha}}] + \\
& + \sqrt{2}i\sigma^m_{\alpha\dot{\alpha}}\overline{\xi}^{\dot{\alpha}}(\partial_m\chi^\alpha)\overline{F} - \sqrt{2}i\sigma^m_{\alpha\dot{\alpha}}\xi^\alpha(\partial_m\overline{\chi}^{\dot{\alpha}})F
\end{aligned}
\tag{A.5}
$$

collecting terms where F and $\overline{F}$ occur gives a total derivative

$$
\begin{aligned}
& \frac{i}{2}[\sqrt{2}\xi^\alpha F\sigma^n_{\alpha\dot{\alpha}}\partial_n\overline{\chi}^{\dot{\alpha}} + \sqrt{2}\partial_n(\overline{\xi}^{\dot{\alpha}}\overline{F})\sigma^n_{\alpha\dot{\alpha}}\chi^\alpha - \\
& - \sqrt{2}\overline{\xi}_{\dot{\alpha}}\overline{F}\overline{\sigma}^{n\dot{\alpha}\alpha}\partial_n\chi_\alpha - \sqrt{2}\xi_\alpha\partial_n F\overline{\sigma}^{n\dot{\alpha}\alpha}\overline{\chi}_{\dot{\alpha}}] + \\
& + \sqrt{2}i\sigma^m_{\alpha\dot{\alpha}}\overline{\xi}^{\dot{\alpha}}(\partial_m\chi^\alpha)\overline{F} - \sqrt{2}i\sigma^m_{\alpha\dot{\alpha}}\xi^\alpha(\partial_m\overline{\chi}^{\dot{\alpha}})F \\
= {} & \frac{i}{\sqrt{2}}[\overline{\xi}_{\dot{\alpha}}\overline{F}\overline{\sigma}^{n\dot{\alpha}\alpha}\partial_n\chi_\alpha + \partial_n(\overline{\xi}^{\dot{\alpha}}\overline{F})\sigma^n_{\alpha\dot{\alpha}}\chi^\alpha - \xi^\alpha F\sigma^n_{\alpha\dot{\alpha}}\partial_n\overline{\chi}^{\dot{\alpha}} - \sigma^n_{\alpha\dot{\alpha}}\xi^\alpha(\partial_n\overline{\chi}^{\dot{\alpha}})F] \\
= {} & \partial_n(\frac{i}{\sqrt{2}}\overline{\xi}\overline{\sigma}^n\chi\overline{F} - \frac{i}{\sqrt{2}}\xi\sigma^n\overline{\chi}F)
\end{aligned}
\tag{A.6}
$$

The remaining terms are then

$$
\begin{aligned}
& \sqrt{2}\partial_n(\xi\chi)(\partial^n\overline{\phi}) - \sqrt{2}\partial^n(\overline{\xi}\overline{\chi})(\partial_n\phi) - \\
& - \frac{1}{\sqrt{2}}[\underbrace{\varepsilon^{\alpha\beta}\sigma^m_{\beta\dot{\beta}}\overline{\xi}^{\dot{\beta}}(\partial_m\phi)\sigma^n_{\alpha\dot{\alpha}}(\partial_n\overline{\chi}^{\dot{\alpha}})}_{\overline{\xi}^{\dot{\beta}}(\partial_m\phi)(\overline{\sigma}^n\sigma^m)^{\dot{\alpha}}_{\dot{\beta}}(\partial_n\overline{\chi}_{\dot{\alpha}})} + \underbrace{\partial_n(\overline{\sigma}^{m\dot{\alpha}\beta}\xi_\beta\partial_m\overline{\phi})\sigma^n_{\alpha\dot{\alpha}}\chi^\alpha}_{(1)} - \\
& \qquad - \underbrace{\varepsilon_{\dot{\alpha}\dot{\beta}}\overline{\sigma}^{m\dot{\beta}\beta}\xi_\beta(\partial_m\overline{\phi})\overline{\sigma}^{n\dot{\alpha}\alpha}(\partial_n\chi_\alpha)}_{\xi_\beta(\sigma^n\overline{\sigma}^m)^\beta_\alpha(\partial_m\overline{\phi})(\partial_n\chi^\alpha)} - \underbrace{\partial_n(\sigma^m_{\alpha\dot{\beta}}\overline{\xi}^{\dot{\beta}}\partial_m\phi)\overline{\sigma}^{n\dot{\alpha}\alpha}\overline{\chi}_{\dot{\alpha}}]}_{(2)}
\end{aligned}
\tag{A.7}
$$

Use σ matrix relations

$$
\begin{aligned}
(1) \qquad & \partial_n(\overline{\sigma}^{m\dot{\alpha}\beta}\xi_\beta\partial_m\overline{\phi})\sigma^n_{\alpha\dot{\alpha}}\chi^\alpha = \xi_\beta(\partial_n\partial_m\overline{\phi})(\sigma^n\overline{\sigma}^m)^\beta_\alpha\chi^\alpha \\
& = \xi_\beta(\partial_n\partial_m\overline{\phi})[-(\sigma^m\overline{\sigma}^n)^\beta_\alpha + 2\eta^{nm}\delta^\beta_\alpha]\chi^\alpha \\
& = -2\xi\chi(\partial^m\partial_m\overline{\phi}) - \xi_\beta(\sigma^m\overline{\sigma}^n)^\beta_\alpha(\partial_n\partial_m\overline{\phi})\chi^\alpha \\
& = -2\xi\chi(\partial^n\partial_n\overline{\phi}) - \xi_\beta(\sigma^n\overline{\sigma}^m)^\beta_\alpha(\partial_m\partial_n\overline{\phi})\chi^\alpha \\
& = -2\xi\chi(\partial^n\partial_n\overline{\phi}) - \xi_\beta(\sigma^n\overline{\sigma}^m)^\beta_\alpha(\partial_n\partial_m\overline{\phi})\chi^\alpha
\end{aligned}
\tag{A.8}
$$

$$(2) \qquad \partial_n(\sigma^m_{\alpha\dot\beta}\overline{\xi}^{\dot\beta}\partial_m\phi)\overline{\sigma}^{n\dot\alpha\alpha}\overline{\chi}_{\dot\alpha} = \overline{\xi}^{\dot\beta}(\partial_n\partial_m\phi)(\overline{\sigma}^n\sigma^m)^{\dot\alpha}_{\dot\beta}\overline{\chi}_{\dot\alpha} \tag{A.9}$$
$$= \overline{\xi}^{\dot\beta}(\partial_n\partial_m\phi)[-(\overline{\sigma}^m\sigma^n)^{\dot\alpha}_{\dot\beta} + 2\eta^{nm}\delta^{\dot\alpha}_{\dot\beta}]\overline{\chi}_{\dot\alpha}$$
$$= -2\overline{\xi}\overline{\chi}(\partial^m\partial_m\phi) - \overline{\xi}^{\dot\beta}(\partial_n\partial_m\phi)(\overline{\sigma}^m\sigma^n)^{\dot\alpha}_{\dot\beta}\overline{\chi}_{\dot\alpha}$$
$$= -2\overline{\xi}\overline{\chi}(\partial^n\partial_n\phi) - \overline{\xi}^{\dot\beta}(\partial_m\partial_n\phi)(\overline{\sigma}^n\sigma^m)^{\dot\alpha}_{\dot\beta}\overline{\chi}_{\dot\alpha}$$
$$= -2\overline{\xi}\overline{\chi}(\partial^n\partial_n\phi) - \overline{\xi}^{\dot\beta}(\partial_n\partial_m\phi)(\overline{\sigma}^n\sigma^m)^{\dot\alpha}_{\dot\beta}\overline{\chi}_{\dot\alpha}$$

In the last two lines a change of the summation index was performed $m \leftrightarrow n$ and the partial derivatives interchanged. The splits (1) and (2) show that the remaining terms of $s\mathcal{L}$ add up to total derivatives.

$$\sqrt{2}\partial_n(\xi\chi\partial^n\overline{\phi}) - \sqrt{2}\partial^n(\overline{\xi}\overline{\chi}\partial_n\phi) + \tag{A.10}$$
$$+\frac{1}{\sqrt{2}}\partial_n[\xi_\beta(\sigma^n\overline{\sigma}^m)^\beta_\alpha(\partial_m\overline{\phi})\chi^\alpha] - \frac{1}{\sqrt{2}}\partial_n[\overline{\xi}^{\dot\beta}(\partial_m\phi)(\overline{\sigma}^n\sigma^m)^{\dot\alpha}_{\dot\beta}\overline{\chi}_{\dot\alpha}]$$

Putting together all terms, the SUSY transformation of the given Lagrangian density is then of the form.

$$s\mathcal{L} = \partial_n K^n \tag{A.11}$$

$$K^n = \sqrt{2}\xi\chi\partial^n\overline{\phi} - \sqrt{2}\overline{\xi}\overline{\chi}\partial^n\phi + \tag{A.12}$$
$$+\frac{1}{\sqrt{2}}\chi\sigma^n\overline{\sigma}^m\xi(\partial_m\overline{\phi}) - \frac{1}{\sqrt{2}}\overline{\chi}\overline{\sigma}^n\sigma^m\overline{\xi}(\partial_m\phi) +$$
$$+\frac{i}{\sqrt{2}}\overline{\xi}\overline{\sigma}^n\chi\overline{F} - \frac{i}{\sqrt{2}}\xi\sigma^n\overline{\chi}F$$

This shows that the SUSY transformation is indeed a symmetry transformation. The explicit form of K^n is necessary to calculate the Noether current and Noether charges.

Noether currents

The Noether current is given by the expression $J^n_{susy} = s\Omega^i\frac{\partial\mathcal{L}}{\partial(\partial_n\Omega^i)} - K^n_{susy}$ where $\Omega^i \in \{\phi, \overline{\phi}, \chi, \overline{\chi}, F, \overline{F}\}$. For the given Lagrangian density the second term is already calculated (13), whereas the first term is given by

$$s\Omega^i \frac{\partial \mathcal{L}}{\partial(\partial_n \Omega^i)} = (s\phi)(\partial^n \overline{\phi}) + (s\overline{\phi})(\partial^n \phi) + \tag{A.13}$$

$$\frac{i}{2}[-(s\overline{\chi}^{\dot{\alpha}})\chi^\alpha \sigma^n_{\alpha\dot{\alpha}} - (s\chi_\alpha)\overline{\chi}_{\dot{\alpha}}\overline{\sigma}^{n\dot{\alpha}\alpha}]$$

$$= \sqrt{2}\xi\chi(\partial^n\overline{\phi}) - \sqrt{2}\overline{\xi}\overline{\chi}(\partial^n\phi) +$$

$$+ \frac{i}{2}[\sqrt{2}(\overline{\xi}^{\dot{\alpha}}\overline{F} + i\overline{\sigma}^{m\dot{\alpha}\beta}\xi_\beta\partial_m\overline{\phi})\chi^\alpha\sigma^n_{\alpha\dot{\alpha}} - \sqrt{2}(\xi_\alpha F + i\sigma^m_{\alpha\dot{\beta}}\overline{\xi}^{\dot{\beta}}\partial_m\phi)\overline{\chi}_{\dot{\alpha}}\overline{\sigma}^{n\dot{\alpha}\alpha}]$$

$$= \sqrt{2}\xi\chi(\partial^n\overline{\phi}) - \sqrt{2}\overline{\xi}\overline{\chi}(\partial^n\phi) + \frac{i}{\sqrt{2}}[\overline{\xi}\overline{\sigma}^n\chi\overline{F} - \xi\sigma^n\overline{\chi}F]$$

$$+ \frac{1}{\sqrt{2}}[\overline{\chi}\overline{\sigma}^n\sigma^m\overline{\xi}(\partial_m\phi) - \chi\sigma^n\overline{\sigma}^m\xi(\partial_m\overline{\phi})]$$

Therefore the Noether current can be written as

$$J^n_{susy} = \sqrt{2}[\overline{\chi}\overline{\sigma}^n\sigma^m\overline{\xi}(\partial_m\overline{\phi}) - \chi\sigma^n\overline{\sigma}^m\xi(\partial_m\phi)] \tag{A.14}$$

Now split the current according to the transformation parameters ξ^α and $\overline{\xi}^{\dot{\alpha}}$ in order to get the chiral and anitchiral currents:

$$\begin{aligned} J^n_{susy} &= \xi^\alpha J^n_\alpha + \overline{\xi}^{\dot{\alpha}}\overline{J}^n_{\dot{\alpha}} \\ J^n_\alpha &= \sqrt{2}\varepsilon_{\alpha\beta}(\chi\sigma^n\overline{\sigma}^m)^\beta(\partial_m\overline{\phi}) \\ &= \sqrt{2}(\chi\sigma^n\overline{\sigma}^m)_\alpha(\partial_m\overline{\phi}) \\ \overline{J}^n_{\dot{\alpha}} &= \sqrt{2}(\overline{\chi}\overline{\sigma}^n\sigma^m)_{\dot{\alpha}}(\partial_m\phi) \end{aligned} \tag{A.15}$$

Noether charges

The Noether charges are obtained by integrating over the spacial variables of the zero component of the currents.

$$\begin{aligned} Q_\alpha &= \int d^3x J^0_\alpha = \int d^3x \sqrt{2}(\chi\sigma^0\overline{\sigma}^m)_\alpha(\partial_m\overline{\phi}) \\ \overline{Q}_{\dot{\alpha}} &= \int d^3x J^0_{\dot{\alpha}} = \int d^3x \sqrt{2}(\overline{\chi}\overline{\sigma}^0\sigma^m)_{\dot{\alpha}}(\partial_m\phi) \end{aligned} \tag{A.16}$$

Remark: One could have used the bosonic SUSY transformation $s = \xi^\alpha D_\alpha + \overline{\xi}^{\dot{\alpha}}\overline{D}_{\dot{\alpha}}$ with anticommuting parameters $\xi, \overline{\xi}$. Then the result would be the same.

Dirac procedure

Since in the lagrangian density occur fermions and auxiliary fields, the conjugated momenta are not all free. One has to deal with a constrained system. In this case the constraints are second class. So the Dirac bracket has to be used, when the system is quantized [HE92]. Denote the constraints by ϕ_A. Then the Dirac bracket is defined by

$$\{F, G\}_D = \{F, G\}_{PB} - \{F, \phi_A\}_{PB} C^{-1AB} \{\phi_B, G\}_{PB} \tag{A.17}$$

where the matrix C^{-1AB} is the inverse of the matrix $C_{BD} := \{\phi_B, \phi_D\}_{PB}$, $C^{-1AB} C_{BD} = \delta^A_D$. The conjugated momenta $\pi_{\Omega^i} = \frac{\partial \mathcal{L}}{\partial(\partial_0 \Omega^i)}$ are

$$\begin{aligned} \pi_\phi &= \partial_0 \overline{\phi} \qquad & \pi^\alpha := \pi_{\chi_\alpha} &= -\frac{i}{2} \overline{\chi}_{\dot\alpha} \overline{\sigma}^{0\dot\alpha\alpha} \qquad & \pi_F &= 0 \\ \pi_{\overline{\phi}} &= \partial_0 \phi \qquad & \overline{\pi}_{\dot\alpha} := \pi_{\overline{\chi}^{\dot\alpha}} &= -\frac{i}{2} \chi^\alpha \sigma^0_{\alpha\dot\alpha} \qquad & \pi_{\overline{F}} &= 0 \end{aligned} \tag{A.18}$$

The constraints $\pi_F = \pi_{\overline{F}} = 0$ give the secondary constraints $F = \overline{F} = 0$ (Take e.g. π_F the Poisson bracket with all constraints vanishes. Therefore the extraconstraint $0 = \{\pi_F, H\}_{PB} = \overline{F}$ has to be imposed). Denoting the constraints by ϕ_A, they can be summarized to

$$\begin{aligned} \phi^\alpha &= \pi^\alpha + \frac{i}{2} \overline{\chi}_{\dot\alpha} \overline{\sigma}^{0\dot\alpha\alpha} \qquad & \phi_{\pi_F} &= \pi_F \qquad & \phi_F &= F \\ \phi_{\dot\alpha} &= \overline{\pi}_{\dot\alpha} + \frac{i}{2} \chi^\alpha \sigma^0_{\alpha\dot\alpha} \qquad & \phi_{\pi_{\overline{F}}} &= \pi_{\overline{F}} \qquad & \phi_{\overline{F}} &= \overline{F} \end{aligned} \tag{A.19}$$

The Noether charges are then in the Hamiltonian picture of the form of the form

$$Q_\alpha = \int d^3x 2\sqrt{2} i \overline{\pi} (\overline{\sigma}_{0\alpha} \pi_\phi - \vec{\overline{\sigma}}_\alpha \vec{\nabla} \overline{\phi}) \tag{A.20}$$

$$\overline{Q}_{\dot\alpha} = \int d^3x 2\sqrt{2} i \pi (\sigma_{0\dot\alpha} \pi_{\overline{\phi}} - \vec{\sigma}_{\dot\alpha} \vec{\nabla} \phi) \tag{A.21}$$

In order to compute $\{Q_\alpha, \overline{Q}_{\dot\alpha}\}_D$ one splits the bracket into contributions from the bosonic fields ϕ, $\overline{\phi}$ where it just reduces to the ordinary Poisson bracket because there are no constraints on them and contributions from the fermionic fields χ, $\overline{\chi}$, where the Dirac bracket has to be evaluated. There are no contributions from the auxiliary fields because they don't occur in the charges. In the following the convention $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ will be used. It will always be assumed that the fields vanish at infinity rapidly enough such that integration by parts yields vanishing boundary terms. Thus, integrating by parts, introducing the charge densities $\mathcal{Q}$ $\overline{\mathcal{Q}}$ and calculating the $\overline{\phi}$,$\pi_{\overline{\phi}}$ part gives

$$\int (\frac{\mathcal{Q}_\alpha}{\partial \overline{\phi}} \frac{\overline{\mathcal{Q}}_{\dot{\alpha}}}{\partial \pi_{\overline{\phi}}} - \frac{\mathcal{Q}_\alpha}{\partial \pi_{\overline{\phi}}} \frac{\overline{\mathcal{Q}}_{\dot{\alpha}}}{\partial \overline{\phi}}) = -8 \int (\vec{\nabla} \overline{\pi}_{\dot{\beta}}) \vec{\overline{\sigma}}^{\dot{\beta}\beta} \varepsilon_{\alpha\beta} \pi^\gamma \sigma_{0\gamma\dot{\alpha}} \tag{A.22}$$

Analogously for the ϕ, π_ϕ part.

$$\int (\frac{\partial \mathcal{Q}_\alpha}{\partial \phi} \frac{\partial \overline{\mathcal{Q}}_{\dot{\alpha}}}{\partial \pi_\phi} - \frac{\partial \mathcal{Q}_\alpha}{\partial \pi_\phi} \frac{\partial \overline{\mathcal{Q}}_{\dot{\alpha}}}{\partial \phi}) = 8 \int \overline{\pi}_{\dot{\beta}} \overline{\sigma}_0^{\dot{\beta}\beta} \varepsilon_{\alpha\beta} \vec{\nabla} \pi^\gamma \vec{\sigma}_{\gamma\dot{\alpha}} \tag{A.23}$$

In order to be able to compare the results with the energy and momentum of the given Lagrangian later on, the above results will be spaned in the basis of the $\sigma^n_{\alpha\dot{\alpha}}$ matrices (for the given index structure and in 2 dimensions the only basiselements are the $\sigma^n_{\alpha\dot{\alpha}}$ matrices). Especially for (A.22)

$$-8 \int (\vec{\nabla} \overline{\pi}_{\dot{\beta}}) \vec{\overline{\sigma}}^{\dot{\beta}\beta} \varepsilon_{\alpha\beta} \pi^\gamma \sigma_{0\gamma\dot{\alpha}} = \sigma_{n\alpha\dot{\alpha}} Z^n = \sigma_{0\alpha\dot{\alpha}} Z_0 - \vec{\sigma}_{\alpha\dot{\alpha}} \vec{Z} \tag{A.24}$$

The components can be calculated by contracting with $\overline{\sigma}_{n\dot{\alpha}\alpha}$ using the relation $\mathrm{tr}\, \sigma_m \overline{\sigma}_n = 2\eta_{mn}$. For Z_0 contract with $\overline{\sigma}_{0\dot{\alpha}\alpha}$ and for $Z_k \quad k \neq 0$ contract with $\overline{\sigma}_{k\dot{\alpha}\alpha}$.

$$\begin{aligned} 2Z_0 &= -8 \int (\vec{\nabla} \overline{\pi}_{\dot{\beta}}) \vec{\overline{\sigma}}^{\dot{\beta}\beta} \varepsilon_{\alpha\beta} \pi^\gamma \sigma_{0\gamma\dot{\alpha}} \overline{\sigma}_{0\dot{\alpha}\alpha} &\quad \text{(A.25)} \\ &= 8 \int \pi^\gamma (\vec{\nabla} \overline{\pi}_{\dot{\beta}}) \varepsilon_{\alpha\beta} \delta^\alpha_\gamma = -8 \int \pi \vec{\sigma} \vec{\nabla} \overline{\pi} & \\ 2Z_k &= -8 \int \overline{\pi \sigma_0} \sigma_k \vec{\overline{\sigma}} \vec{\nabla} \pi \qquad k \neq 0 &\quad \text{(A.26)} \end{aligned}$$

The same applies to (A.23)

$$8 \int \overline{\pi}_{\dot{\beta}} \overline{\sigma}_0^{\dot{\beta}\beta} \varepsilon_{\alpha\beta} \vec{\nabla} \pi^\gamma \vec{\sigma}_{\gamma\dot{\alpha}} = \sigma_{0\alpha\dot{\alpha}} Y_0 - \vec{\sigma}_{\alpha\dot{\alpha}} \vec{Y} \tag{A.27}$$

$$Y_0 = -4 \int \pi \vec{\sigma} \vec{\nabla} \overline{\pi} \tag{A.28}$$

$$Y_k = -4 \int \overline{\pi \vec{\overline{\sigma}}} \sigma_k \overline{\sigma_0} \vec{\nabla} \pi \qquad k \neq 0 \tag{A.29}$$

Putting (A.22) and (A.23) together the bosonic part of the Poisson bracket reads

$$\begin{aligned}\{Q_\alpha,\overline{Q}_{\dot\alpha}\}_D^{bos} &= \{Q_\alpha,\overline{Q}_{\dot\alpha}\}_{PB}^{bos} \\ &= -8\sigma_{0\alpha\dot\alpha}\int \pi\vec{\sigma}\vec{\nabla}\overline{\pi} \\ &\quad+4\sum_{n,k=1}^{3}\sigma_{k\alpha\dot\alpha}\int\overline{\pi}\left(\overline{\sigma}_n\sigma_k\overline{\sigma_0}+\overline{\sigma_0}\sigma_k\overline{\sigma}_n\right)\partial_n\pi \qquad k\neq 0 \\ &= -8\sigma_{0\alpha\dot\alpha}\int\pi\vec{\sigma}\vec{\nabla}\overline{\pi}-8\vec{\sigma}_{\alpha\dot\alpha}\int\overline{\pi\sigma_0}\vec{\nabla}\pi\end{aligned} \tag{A.30}$$

Where use of the formula $\overline{\sigma}_n\sigma_k\overline{\sigma_0}+\overline{\sigma_0}\sigma_k\overline{\sigma}_n = 2\eta_{kn}\sigma_0 = -2\delta_{kn}\sigma_0$ (Appendix C) was made.
In order to get the fermionic contributions one has to calculate the functional matrix C_{AB}. Since the auxiliary fields F,$\overline{F}$ don't occur in the charges it's enough to consider $C_{\underline{\alpha\beta}}$.

$$\begin{aligned}C_{\underline{\beta\gamma}}(\vec{x}-\vec{y}) &= \begin{pmatrix}\{\phi_\beta(\vec{x}),\phi_\gamma(\vec{y})\}_{PB} & \{\phi_\beta(\vec{x}),\phi_{\dot\gamma}(\vec{y})\}_{PB} \\ \{\phi_{\dot\beta}(\vec{x}),\phi_\gamma(\vec{y})\}_{PB} & \{\phi_{\dot\beta}(\vec{x}),\phi_{\dot\gamma}(\vec{y})\}_{PB}\end{pmatrix} = \\ &= \begin{pmatrix}0 & i\sigma^0_{\beta\dot\gamma}\delta(\vec{x}-\vec{y}) \\ i\sigma^0_{\gamma\dot\beta}\delta(\vec{x}-\vec{y}) & 0\end{pmatrix}\end{aligned} \tag{A.31}$$

The inverse matrix is given by

$$C^{-1\underline{\gamma\delta}}(\vec{y}-\vec{z}) = \begin{pmatrix}0 & -i\overline{\sigma}^{0\dot\delta\gamma}\delta(\vec{y}-\vec{z}) \\ -i\overline{\sigma}^{0\dot\gamma\delta}\delta(\vec{y}-\vec{z}) & 0\end{pmatrix} \tag{A.32}$$

The Poisson brackets $\{Q_\alpha,\phi_{\dot\gamma}\}$ and $\{\phi_\delta,\overline{Q}_{\dot\alpha}\}$ vanish. Therefore the only contribution to the Dirac bracket from $C^{-1\underline{\gamma\delta}}$ is of the index picture $C^{-1\gamma\dot\delta}$. The remaining Poisson brackets are

$$\begin{aligned}\{Q_\alpha,\phi_\gamma(\vec{y})\}_{PB} &= \sqrt{2}\partial_m\overline{\phi}(\vec{y})\sigma^m_{\alpha\dot\mu}\overline{\sigma}_0^{\dot\mu\mu}\varepsilon_{\gamma\mu} \\ &= \sqrt{2}(\pi_\phi(\vec{y})\varepsilon_{\gamma\alpha}-\vec{\nabla}\overline{\phi}(\vec{y})\vec{\sigma}_{\alpha\dot\mu}\overline{\sigma}_0^{\dot\mu\mu}\varepsilon_{\gamma\mu}) \\ \{\phi_{\dot\delta}(\vec{z}),\overline{Q}_{\dot\alpha}\}_{PB} &= \sqrt{2}\partial_n\phi(\vec{z})\overline{\sigma}^{n\dot\nu\nu}\sigma_{0\nu\dot\delta}\varepsilon_{\dot\alpha\dot\nu} \\ &= \sqrt{2}(\pi_{\overline{\phi}}(\vec{z})\varepsilon_{\dot\alpha\dot\delta}-\vec{\nabla}\phi(\vec{z})\vec{\overline{\sigma}}^{\dot\nu\nu}\sigma_{0\nu\dot\delta}\varepsilon_{\dot\alpha\dot\nu})\end{aligned} \tag{A.33}$$

Putting everything together the Dirac bracket for the fermionic part is then

$$\{Q_\alpha, \overline{Q}_{\dot\alpha}\}_D^{ferm} = \underbrace{\{Q_\alpha, \overline{Q}_{\dot\alpha}\}_{PB}^{ferm}}_{=0} - \tag{A.34}$$

$$- \int d^3y \int d^3z \{Q_\alpha, \phi_\gamma\}_{PB}(\vec{y}) C^{-1\gamma\dot\delta} \delta(\vec{y}-\vec{z}) \{\phi_{\dot\delta}, \overline{Q}_{\dot\alpha}\}_{PB}(\vec{z})$$

$$= 2i \int d^3y \; (\pi_\phi(\vec{y})\varepsilon_{\gamma\alpha} - \vec{\nabla}\overline{\phi}(\vec{y})\vec{\sigma}_{\alpha\dot\mu}\overline{\sigma}_0^{\dot\mu\mu}\varepsilon_{\gamma\mu})\overline{\sigma}^{0\dot\delta\gamma}$$

$$(\pi_{\overline{\phi}}(\vec{y})\varepsilon_{\dot\alpha\dot\delta} - \vec{\nabla}\phi(\vec{y})\vec{\overline{\sigma}}^{\dot\nu\nu}\sigma_{0\nu\dot\delta}\varepsilon_{\dot\alpha\dot\nu})$$

$$= -2i \int \{\sigma_{0\alpha\dot\alpha}(\pi_\phi\pi_{\overline{\phi}} + \vec{\nabla}\overline{\phi}\vec{\nabla}\phi) - \vec{\sigma}_{\alpha\dot\alpha}(\pi_\phi\vec{\nabla}\phi + \pi_{\overline{\phi}}\vec{\nabla}\overline{\phi})\}$$

The term with $\vec{\nabla}\overline{\phi}\vec{\nabla}\phi$ is obtained by symmetrizing, integrating twice by parts and the use of the σ matrix relation $\sigma_a\overline{\sigma}_b\sigma_c + \sigma_c\overline{\sigma}_b\sigma_a = 2(-\eta_{ac}\sigma_b + \eta_{bc}\sigma_a + \eta_{ab}\sigma_c)$

$$\int (\vec{\nabla}\overline{\phi}\vec{\sigma}_{\alpha\dot\mu})\overline{\sigma}_0^{\dot\mu\mu}\varepsilon_{\gamma\mu}\overline{\sigma}^{0\dot\delta\gamma}(\vec{\nabla}\phi\vec{\overline{\sigma}}^{\dot\nu\nu})\sigma_{0\nu\dot\delta}\varepsilon_{\dot\alpha\dot\nu} \tag{A.35}$$

$$= -\sum_{m,n=1}^{3} \int \partial_n\overline{\phi}\partial_m\phi(\sigma_n\overline{\sigma}_0\sigma_m)_{\alpha\dot\alpha}$$

$$= -\frac{1}{2}\sum_{m,n=1}^{3} \{\int \partial_n\overline{\phi}\partial_m\phi(\sigma_n\overline{\sigma}_0\sigma_m)_{\alpha\dot\alpha} + \underbrace{\int \partial_m\overline{\phi}\partial_n\phi}_{\overset{PI}{=}\int \partial_n\overline{\phi}\partial_m\phi}(\sigma_m\overline{\sigma}_0\sigma_n)_{\alpha\dot\alpha}\}$$

$$= -\frac{1}{2}\sum_{m,n=1}^{3} \int \partial_n\overline{\phi}\partial_m\phi \underbrace{(\sigma_n\overline{\sigma}_0\sigma_m + \sigma_m\overline{\sigma}_0\sigma_n)_{\alpha\dot\alpha}}_{=2\delta_{nm}\sigma_{0\alpha\dot\alpha}}$$

$$= -\sigma_{0\alpha\dot\alpha}\sum_{n=1}^{3} \int \partial_n\overline{\phi}\partial_n\phi$$

$$= -\sigma_{0\alpha\dot\alpha}\vec{\nabla}\overline{\phi}\vec{\nabla}\phi$$

Putting all the terms together gives the Dirac bracket for the charges

$$\{Q_\alpha, \overline{Q}_{\dot\alpha}\}_D = -2i\sigma_{0\alpha\dot\alpha}\int(\pi_\phi\pi_{\overline{\phi}} + \vec{\nabla}\overline{\phi}\vec{\nabla}\phi - 4i\pi\vec{\sigma}\vec{\nabla}\overline{\pi}) \tag{A.36}$$

$$+ 2i\vec{\sigma}_{\alpha\dot\alpha}\int(\pi_\phi\vec{\nabla}\phi + \pi_{\overline{\phi}}\vec{\nabla}\overline{\phi} + 4i\overline{\pi\sigma_0}\vec{\nabla}\pi)$$

Energy-momentum-tensor

Energy and momentum are the Noether charges of the symmetry transformations along the four spacetime directions. For a given action one starts with the corresponding currents which are often called energy-momentum-tensor

$$T_a^b = \partial_a \Omega^i \frac{\partial \mathcal{L}}{\partial(\partial_b \Omega^i)} - \delta_a^b \mathcal{L}. \tag{A.37}$$

Spacial integration over the zero component leads to the energy and momentum of the system $P_a = \int d^3y T_a^0$ where the $a = 0$ component is the energy. For the given Lagrangian density this gives for the energy

$$\begin{aligned} P_0 &= P_0^{bos} + P_0^{ferm} - \int F\overline{F} \qquad\qquad (A.38)\\ P_0^{bos} &= \int (\partial_0\phi\partial_0\overline{\phi} + \vec{\nabla}\phi\vec{\nabla}\overline{\phi}) \\ P_0^{ferm} &= \int \frac{i}{2}(\overline{\chi}\vec{\sigma}\vec{\nabla}\chi + \chi\vec{\sigma}\vec{\nabla}\overline{\chi}) \overset{PI}{=} i\int \chi\vec{\sigma}\vec{\nabla}\overline{\chi} \end{aligned}$$

and for the momenta

$$P_n = P_n^{bos} + P_n^{ferm} \qquad n \neq 0 \tag{A.39}$$

$$P_n^{bos} = \int (\partial_n\phi\partial_0\overline{\phi} + \partial_n\overline{\phi}\partial_0\phi) \tag{A.40}$$

$$P_n^{ferm} = \int \frac{i}{2}(\overline{\chi\sigma}^0\partial_n\chi + \chi\sigma^0\partial_n\overline{\chi}) \overset{PI}{=} i\int \chi\sigma^0\partial_n\overline{\chi}$$

These quantities are written in terms of the Lagrangian variables. In order to compare them with the Dirac bracket they have to be translated into the phase space coordinates.

$$P_0^{bos} = \int (\pi_{\overline{\phi}}\pi_\phi + \vec{\nabla}\phi\vec{\nabla}\overline{\phi}) \qquad P_0^{ferm} = -4i\int \pi\vec{\sigma}\vec{\nabla}\overline{\pi} \tag{A.41}$$

$$n \neq 0 \qquad P_n^{bos} = \int (\partial_n\phi\pi_\phi + \partial_n\overline{\phi}\pi_{\overline{\phi}}) \qquad P_n^{ferm} = 4i\int \overline{\pi\sigma_0}\partial_n\pi \tag{A.42}$$

Quantization

With the expressions for energy and momentum (A.36) reads

$$\{Q_\alpha, \overline{Q}_{\dot{\alpha}}\}_D = -2i\sigma^n_{\alpha\dot{\alpha}}P_n \tag{A.43}$$

In order to quantize this relation the canonical quantization rule $i\{A, B\}_D \to [A, B]$ has to be applied. This gives

$$\{Q_\alpha, \overline{Q}_{\dot{\alpha}}\} = 2\sigma^n_{\alpha\dot{\alpha}}P_n \tag{A.44}$$

which has the same sign as in the algebra. Therefore the choice of the signs is consistent.

Appendix B

Relation for Dragon's Theorem

We have to check

$$M_{ABCD}{}^E - (-)^{AB+AC+AD} M_{BCDA}{}^E + (-)^{AC+AD+BC+BD} M_{CDAB}{}^E - \tag{B.1}$$
$$(-)^{AD+BD+CD} M_{DABC}{}^E (-)^{AB+AC+AD+BC+BD+BA} M_{CDAB}{}^E$$
$$- (-)^{AB+AC+AD+BC+BD+BA+CD+CA+CB} M_{DABC}{}^E$$
$$= 0$$

using the Bianchi identities and formula (4.1). One could think of dealing with (graded) cyclic sums whose elements are the cycles C_r of the permutation group $\mathcal{P}_n$ $\sum_{i_1 \cdots i_n} X_{i_1 \cdots i_n} = \sum_{C_n \in \mathcal{P}_n} X_{i_1 \cdots i_n}$. The sum has n elements. Since $\sum_{C_{n+1} \in \mathcal{P}_{n+1}} X_{i_{n+1}} \sum_{C_n \in \mathcal{P}_n} X_{i_1 \cdots i_n} \neq \sum_{C_{n+1} \in \mathcal{P}_{n+1}} X_{i_1 \cdots i_{n+1}}$ (just count the number of elements. left: $(n+1)n$, right: $n+1$) it is better to think of totally (graded) antisymmetrized objects, where all elements $\pi \in \mathcal{P}_n$ of the permutation group act. Here one has $X_{[i_{n+1}} X_{[i_1 \cdots i_n]]} = X_{[i_1 \cdots i_{n+1}]}$.

Start with an object with n indices which is anti-symmetric in $n-1$ indices: $X_{i_1 \cdots i_n} = X_{i_1 [i_2 \cdots i_n]}$. In order to get the whole object anti-symmetrized one has to antisymmetrize i_1 with every $i_j \in \{i_2 \cdots i_n\}$. These are alltogether n terms. Using again anti-symmetry one gets sort of a cyclic sum, with possible signs. These signs depend on the order of $\mathcal{P}_n$ i.e. on the number of transpositions which occur in the cycle. A cycle of n elements is generated by $n-1$ transpositions. Therefore for each cycle there will be a factor of $(-)^{n-1}$. Hence $X_{[i_1 \cdots i_n]} = X_{i_1 \cdots i_n} + (-)^{n-1} X_{i_2 \cdots i_1} + (-)^{2(n-1)} X_{i_3 \cdots i_2} + cyclic$ when the last $n-1$ indices are anti-symmetrized. The same formula holds for graded anti-symmetry with additional signs depending on the grading of the indices. As example consider the graded anti-symmetrization of X_{ABC} which is already graded anti-symmetric in the last two indices. Since $n = 3$ the only signs come from the grading of the indices $X_{[ABC]} = X_{ABC} + (-)^{AB+AC} X_{BCA} +$

$(-)^{AB+AC+CB+AB}X_{CAB}$. If $X_{[ABC]} = 0$ then this can be cast into the form $(-)^{AC}X_{ABC}+(-)^{BA}X_{BCA}+(-)^{CB}X_{CAB} = 0$. A comparison with the (first) bianchi identity, which is graded anti-symmetric in two indices, shows that it can be written as a total graded anti-symmetrization. Repeating this then with $n = 4$ (B.1) is a total graded antisymmetrization. Summarizing, we have to show

$$M_{[ABCD]}{}^{E} = 0 \tag{B.2}$$

where $[ABCD]$ stands for total graded anti-symmetrization with respect to $ABCD$, which will be sometimes denoted by $[\cdots]_{ABCD}$ too. The data are the first Bianchi Identity, which reads

$$\mathcal{D}_{[A}T_{BC]}{}^{E} + T_{[AB}{}^{K}T_{KC]}{}^{E} - F_{[ABC]}{}^{E} = 0 \tag{B.3}$$

and the transformation property of the torsion,

$$[\mathcal{D}_A, \mathcal{D}_B]T_{CD}{}^{E} = -T_{AB}{}^{K}\mathcal{D}_K T_{CD}{}^{E} - F_{ABC}{}^{K}T_{KD}{}^{E} + F_{ABD}{}^{K}T_{KC}{}^{E} + F_{ABK}{}^{E}T_{CD}{}^{K} \tag{B.4}$$

The first observation is for the field strength F_{ABC}, which is anti-symmetric in the first indices

$$\begin{aligned} F_{[ABC]} &= F_{[A[BC]]} = F_{A[BC]} + F_{B[CA]} + F_{C[AB]} \\ &= F_{A[BC]} + F_{BCA} - F_{BAC} + F_{CAB} - F_{CBA} \\ &= 2F_{A[BC]} + F_{[BC]A} \end{aligned} \tag{B.5}$$

now calculate

$$
\begin{aligned}
M_{[ABCD]}{}^{E} &= \mathcal{D}_{[A}F_{BCD]}{}^{E} + T_{[AB}{}^{K}F_{KCD]}{}^{E} \qquad \text{(B.6)}\\
&\overset{(B.3)}{=} \mathcal{D}_{[A}(\mathcal{D}_{[B}T_{CD]}{}^{E} + T_{[BC|}{}^{K}T_{K|D]]}{}^{E}) + T_{[AB|}{}^{K}F_{K|CD]}{}^{E}\\
&= \frac{1}{2}[\mathcal{D}_{[A},\mathcal{D}_{B}]T_{CD]}{}^{E} + \mathcal{D}_{[A}(T_{BC|}{}^{K}T_{K|D]}{}^{E}) + T_{[AB|}{}^{K}F_{K|CD]}{}^{E}\\
&\overset{(B.4)}{=} \frac{1}{2}(-T_{[AB}{}^{K}\mathcal{D}_{K}T_{CD]}{}^{E} \underbrace{-F_{[ABC|}{}^{K}T_{K|D]}{}^{E} + F_{[ABD|}{}^{K}T_{K|C]}{}^{E}}_{=2F_{[ABD|}{}^{K}T_{K|C]}{}^{E}} + F_{[AB|K}{}^{E}T_{|CD]}{}^{K})\\
&\quad + \mathcal{D}_{[A}(T_{BC|}{}^{K}T_{K|D]}{}^{E}) + T_{[AB|}{}^{K}F_{K|CD]}{}^{E}\\
&= \left[\frac{1}{2}T_{AB}{}^{K}(\underbrace{2\mathcal{D}_{C}T_{KD}{}^{E} - \mathcal{D}_{K}T_{CD}{}^{E}}_{=\mathcal{D}_{[C}T_{KD]}{}^{E}} + \underbrace{2F_{KCD}{}^{E} + F_{CDK}{}^{E}}_{=F_{[KCD]}{}^{E}})\right]_{ABCD} +\\
&\quad \left[(F_{ABD}{}^{K} - \mathcal{D}_{A}T_{BD}{}^{K})T_{KC}{}^{E}\right]_{ABCD}\\
&\overset{(B.3)}{=} \left[\frac{1}{2}T_{AB}{}^{K}(\underbrace{\mathcal{D}_{[C}T_{KD]}{}^{E} + \mathcal{D}_{[K}T_{CD]}{}^{E}}_{=0} + T_{[KC|}{}^{N}T_{N|D]}{}^{E})\right]_{ABCD} +\\
&\quad \left[(\underbrace{\mathcal{D}_{A}T_{BD}{}^{K} - \mathcal{D}_{A}T_{BD}{}^{K})T_{KC}{}^{E}}_{=0} + T_{AB}{}^{N}T_{ND}{}^{K}T_{KC}{}^{E}\right]_{ABCD}\\
&= \left[T_{AB}{}^{N}\underbrace{(-\frac{1}{2}T_{[CD}{}^{K}T_{N]K}{}^{E} - T_{N[D}{}^{K}T_{C]K}{}^{E})}_{=\frac{1}{2}T_{[DC}{}^{K}T_{N]K}{}^{E}}\right]_{ABCD}\\
&= -\frac{1}{2}T_{[AB}{}^{N}T_{CD]}{}^{K}T_{NK}^{E} = 0
\end{aligned}
$$

Appendix C

Notation, Convention, Formulas

Conventions

- Metric $\eta = diag(+,-,-,-)$
- Use leftderivatives
- Graded commutator $[A,B] = AB - (-)^{|A||B|}BA$
- Anticommutator $\{A,B\} = AB + BA$
- Legendre transformation $H(q,p) = \dot{q}^i p_i - L(q,\dot{q}) \Rightarrow p_i = \frac{\partial L}{\partial \dot{q}^i}$
- Graded poisson bracket $\{A,B\}_{PB} = (-)^{iA}(\frac{\partial A}{\partial q^i}\frac{\partial B}{\partial p_i} - (-)^i \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q^i})$
- Quantization rule $i\{A,B\}_{PB} \rightarrow [A,B]$. For second class constrained systems: Diracbracket $i\{A,B\}_D \rightarrow [A,B]$
- $\frac{\delta}{\delta\phi(x)}$ functional derivative
  $\frac{\partial}{\partial\phi}$ partial derivative with respect to the (jet space) variable
- Van der Waerden Notation
- $\varepsilon^{\alpha\beta}\varepsilon_{\beta\gamma} = \delta^\alpha_\gamma$ (Wess-Bagger)
- Overall index $A \in \{a,\alpha,\dot{\alpha}\}$, for spinorial indices $\underline{\alpha} \in \{\alpha,\dot{\alpha}\}$, where $\eta_{\dot{\alpha}} := \overline{\eta}_{\dot{\alpha}}$.
- Summation conventions. Undotted North-West: $\Psi^\alpha\Phi_\alpha$. Dotted North-East: $\overline{\Psi}_{\dot{\alpha}}\overline{\Phi}^{\dot{\alpha}}$

σ matrices

$$\sigma_0 = \mathbf{1}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{C.1}$$

complex conjugation by transposition $\sigma^* = \sigma^T$ in components $\sigma^{m*}_{\alpha\dot\beta} = \sigma^m_{\beta\dot\alpha}$
ε intertwines $\vec\sigma$ and $-\vec\sigma^{\,*}$, hence also σ^a and $(\overline{\sigma}^a)^T$.

$$\overline{\sigma}^{m\alpha\dot\alpha} = \varepsilon^{\alpha\beta}\varepsilon^{\dot\alpha\dot\beta}\sigma^m_{\beta\dot\beta} \tag{C.2}$$

Conjugation

We define complex conjugation by $(\phi\chi)^* = (-)^{|\phi||\chi|}\phi^*\chi^*$, whereas we define hermitian conjugation as $\dagger = * \circ t$, which means for objects with matrix indices only (not in the functional analytical sense). For the covariant derivatives one usually sets: $D_{\dot\alpha} = D^*_\alpha$ in accordance with $\{D_\alpha, D_{\dot\beta}\} = 2i\sigma^m_{\alpha\dot\beta}\partial_m$. In Superspace the coordinates are related by $\theta^{\alpha *} = \overline{\theta}^{\dot\alpha}$. Because of $\mathbb{R} \ni \frac{\partial}{\partial\theta^\alpha}\theta^\alpha = -\frac{\partial}{\partial\overline{\theta}^{\dot\alpha}}\overline{\theta}^{\dot\alpha}$ the complex conjugate of the partial derivative is $\frac{\partial}{\partial\theta^\alpha}^* = -\frac{\partial}{\partial\overline{\theta}^{\dot\alpha}}$.

Formulas

$$\sigma^a\overline{\sigma}^b\sigma^c + \sigma^c\overline{\sigma}^b\sigma^a = 2(-\eta^{ac}\sigma^b + \eta^{bc}\sigma^a + \eta^{ab}\sigma^c) \qquad \sigma^n_{\alpha\dot\alpha}\overline{\sigma}^{\dot\beta\beta}_n = 2\delta^\beta_\alpha\delta^{\dot\beta}_{\dot\alpha} \tag{C.3}$$

$$(\sigma^m\overline{\sigma}^n + \sigma^n\overline{\sigma}^m)_\alpha^{\ \beta} = 2\eta^{mn}\delta^\beta_\alpha \qquad Tr\sigma_m\overline{\sigma}_n = 2\eta_{mn} \tag{C.4}$$

$$D_\alpha D_\beta D_\gamma = 0 \qquad \overline{D}_{\dot\alpha}\overline{D}_{\dot\beta}\overline{D}_{\dot\gamma} = 0 \tag{C.5}$$

$$D_\alpha D_\beta = \frac{1}{2}\varepsilon_{\alpha\beta}D^2 \qquad D^\alpha D^\beta = -\frac{1}{2}\varepsilon^{\alpha\beta}D^2 \tag{C.6}$$

$$\overline{D}_{\dot\alpha}D_{\dot\beta} = -\frac{1}{2}\varepsilon_{\dot\alpha\dot\beta}\overline{D}^2 \qquad \overline{D}^{\dot\alpha}\overline{D}^{\dot\beta} = \frac{1}{2}\varepsilon^{\dot\alpha\dot\beta}\overline{D}^2 \tag{C.7}$$

$$[D^2, \overline{D}_{\dot\alpha}] = 4i\partial\!\!\!/_{\alpha\dot\alpha}D^\alpha \qquad [\overline{D}^2, D_\alpha] = -4i\partial\!\!\!/_{\alpha\dot\alpha}\overline{D}^{\dot\alpha} \tag{C.8}$$

$$D^2\overline{D}^2 + \overline{D}^2D^2 = 2D^\alpha\overline{D}^2D_\alpha - 16\Box \tag{C.9}$$

$$[D^2, \overline{D}^2] = 8iD\partial\!\!\!/\overline{D} + 16\Box \qquad D^2\overline{D}_{\dot\alpha}D^2 = 0 \tag{C.10}$$

$$(\partial\!\!\!/\overline{\partial\!\!\!/})_\alpha^{\ \beta} = \Box\delta^\beta_\alpha \qquad \partial\!\!\!/\overline{\partial\!\!\!/} = 2\Box \tag{C.11}$$

Poisson brackets in field theory

The graded Poisson bracket is defined by $\{A, B\}_{PB} = (-)^{iA}(\frac{\partial A}{\partial q^i}\frac{\partial B}{\partial p_i} - (-)^i \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q^i})$. This is the definition for finite degrees of freedom. There are two ways to adapt the Poisson bracket to field theory: changing the reading rules or changing the writing rules. A change of the reading rules lead to the de Wit notation [De84]. $\{A, B\}_{PB} = (-)^{iA}(\frac{\partial A}{\partial \phi^i}\frac{\partial B}{\partial \pi_i} - (-)^i \frac{\partial A}{\partial \pi_i}\frac{\partial B}{\partial \phi^i})$. The index i has the double duty to label the fields and the space points. Summation is then a combination of summing of the field indices and integrating over space at equal times. In a less condensed notation the Poisson bracket reads then (changing the writing rules):

$$\{A, B\}_{PB} = (-)^{iA} \sum_i \int d^3x (\frac{\delta A}{\delta \phi^i(x)} \frac{\delta B}{\delta \pi_i(x)} - (-)^i \frac{\delta A}{\delta \pi_i(x)} \frac{\delta B}{\delta \phi^i(x)}) \quad \text{(C.12)}$$

$$\{\phi(\vec{x}, t), \pi(\vec{y}, t)\}_{PB} = \delta(\vec{x} - \vec{y}) \quad \text{(C.13)}$$

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