DIPLOMARBEIT

Open String Theory and Non-commutative Geometry

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Chapter 1

Introduction

First considerations dealing with a non-commutative (or quantized) space-time are due to Snyder [1, 2] and date back to the year 1947. The idea was to overcome the ultra-violet divergences in quantum electrodynamics by the introduction of an effective short distance cut-off in the field theory. In contrast to earlier attempts, which replaced the space-time continuum by a lattice structure, the non-commutative structure maintains the translational invariance. At the same time, however, the renormalization program succeeded in predicting numbers from the theory of quantum electrodynamics and the ideas of Snyder were for the most part ignored. Some time later von Neumann introduced the term "non-commutative geometry" to refer in general to a geometry in which an algebra of functions is replaced by a non-commutative algebra. As in the quantization of the classical phasespace, coordinates are replaced by generators of the algebra. Since these do not commute they cannot be diagonalized simultaneously and the space disappears. Similarly to the uncertainty principle of quantum mechanics,

$$[\hat{x}^\mu, \hat{p}_\nu] = i\hbar\delta^\mu{}_\nu \ , \tag{1.1}$$

one may replace the Minkowski coordinates x^μ by generators $\hat{x}^\mu$ of a non-commutative algebra which satisfy commutation relations of the form

$$[\hat{x}^\mu, \hat{x}^\nu] = i\alpha\Theta^{\mu\nu} \ , \tag{1.2}$$

where the parameter α is a fundamental area scale. If the right-hand side does not vanish some of the coordinates $\hat{x}^\mu$ do not commute and thus cannot simultaneously be measured with arbitrary accuracy. Non-commutative effects could take place on mikroskopic scales and from dimensional considerations of the fundamental constants we suppose the value of α to be of the order of the Planck area,

$$\alpha \simeq m_P^{-2} = G\hbar \; . \tag{1.3}$$

However, the experimental bounds would be much larger. On makroskopic scales we cannot see the algebraic structure (1.2), since $\lim_{\alpha\to 0} \hat{x}^\mu = x^\mu$ so that the coordinates commute.

In a sense, string theory introduces a concept that is very similar to non-commutative geometry. Point particles are replaced by strings, open and closed ones. The structure of these extended objects gets relevant at the Planck scale and it provides, just as non-commutative geometry, an effective short distance cut-off. So it is not astonishing that string theory and non-commutative geometry are somehow related, even more if one takes into account that the space-time coordinates, being fields on a two dimensional world sheet, become operators upon quantization. However, things are not so simple and an algebra like (1.2) was first found by Schomerus [4] no more than two years ago. The non-commutative structure originates from a two form background field B and appears only on so-called Dp-branes that are $(p+1)$-dimensional dynamical objects on which the ends of open strings are fixed. So in a theory of only closed strings there does not arise non-commutativity (at least not by the same mechanism). Before this discovery, both fields of research, the non-commutative geometry as well as the open string theory, were intensively investigated seperately. Thereafter a lot of progress was achieved in the relation of the two and interests went in several directions, such as D-brane physics, the differential structure, or the relation between commutative and non-commutative geometry, provided by the Seiberg-Witten map.

In the subsequent sections we present recent developements in open string theory and non-commutative geometry as far as they are relevant for our considerations.

1.1 Non-commutative Geometry

The mathematical description of non-commutative coordinates can be implemented in two different ways. In the introduction we used the operator notation. The order of the operators $\hat{x}^\mu$ plays an essential role and if the non-commutative parameter $\Theta^{\mu\nu}$ is not constant we have to specify in what order it depends on the coordinates $\hat{x}^\mu$. The second description is known as deformation quantization and uses ordinary c-numbers x^μ, i.e., x^μ are the coordinates of a point P on a differentiable manifold $\mathcal{M}$. Non-commutativity is realized by a bilinear, associative product of functions, which is parametrized by a tensor field $\Theta^{\mu\nu}$ on $\mathcal{M}$. Henceforth, we will make use of the second kind of description.

A constant field $\Theta^{\mu\nu}$ defines for instance the Moyal-Weyl star product

$$f(x) * g(x) = e^{\frac{i\alpha}{2}\Theta^{\mu\nu}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial y^\mu}} f(x)g(y)\big|_{x=y}. \tag{1.4}$$

Taking the functions to be the coordinates themselves one obtains immediately

$$[x^\mu, x^\nu]_* = x^\mu * x^\nu - x^\mu * x^\nu = i\alpha\Theta^{\mu\nu} \ , \tag{1.5}$$

which is similar to equation (1.2). The Moyal-Weyl product (1.4) has, apart from its associativity, the property that under an integral the product of two functions simplifies to an ordinary product. Accordingly, the integration acts on the product of an arbitrary number of functions as a trace, i.e.,

$$\int_M \mathrm{d}^D x\, f_1 * \ldots * f_{N-1} * f_N = \int_M \mathrm{d}^D x\, f_N * f_1 * \ldots * f_{N-1} \ , \tag{1.6}$$

and it is allowed to omit one of the stars.

Recently, a lot of success was achieved investigating non-commutative Yang-Mills theories. We give a short description of the underlying model. The Moyal-Weyl product is a very simple example for a non-commutative space, so that it was used for most considerations. If we take, in addition, a flat Minkowski metric, the action for a non-commutative $U(N)$ Yang-Mills theory is

$$S = -\frac{1}{4g^2}\int_M \mathrm{d}^D x \mathrm{Tr}(F_{\mu\nu} * F^{\mu\nu}) \ , \tag{1.7}$$

where g is a coupling constant. The field strength $F_{\mu\nu}$ corresponding to the gauge field A_μ is defined as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]_* \ . \tag{1.8}$$

Both A and F are $N \times N$ hermitian matrices and comply with the infinitesimal gauge transformations

$$\delta_\lambda A_\mu = \partial_\mu \lambda + i[\lambda, A_\mu]_* \ , \quad \delta_\lambda F_{\mu\nu} = i[\lambda, F_{\mu\nu}]_* \ . \tag{1.9}$$

Even in the $U(1)$ case equations (1.9) keep the structure of a non-abelian gauge transformation. In the limit $\Theta^{\mu\nu} \to 0$ the theory reduces to an ordinary $U(N)$ gauge theory.

So far we considered the very special case of the Moyal-Weyl product. The generalization to a non-constant field $\Theta^{\mu\nu}(x)$ was investigated in the context of deformation quantization of Poisson manifolds [8]. A manifold with a Poisson structure $\Theta^{\mu\nu}$ is endowed with a bilinear, associative product given by[1]

$$\begin{aligned} f \circ g \ = \ & fg + \frac{i\alpha}{2}\Theta^{\mu\nu}\partial_\mu f \ \partial_\nu g - \frac{\alpha^2}{8}\Theta^{\mu\nu}\Theta^{\rho\sigma}\partial_\mu\partial_\rho f \ \partial_\nu\partial_\sigma g - \\ & -\frac{\alpha^2}{12}\Theta^{\mu\rho}\partial_\rho\Theta^{\nu\sigma}\left(\partial_\mu\partial_\nu f \ \partial_\sigma g - \partial_\nu f \ \partial_\mu\partial_\sigma g\right) + \mathcal{O}(\alpha^3) \ . \end{aligned} \tag{1.10}$$

If the manifold is, moreover, symplectic, the Poisson condition simplifies to the condition that the inverse of $\Theta^{\mu\nu}$ is closed, i.e., $d(\Theta^{-1}) = 0$.

1.2 Open String Theory

In analogy to the action of a point particle, the Nambu-Goto action of a (bosonic) string is the area of a surface, the world sheet, that is embedded in a D-dimensional target space:

$$S_{NG} = \frac{1}{2\pi\alpha'}\int_\Sigma \mathrm{d}^2\sigma \sqrt{-\det g^*(X(\sigma))} \ , \tag{1.11}$$

[1] The exact definition of the product containing all orders of α can be found in the original paper [8]. It would go beyond the scope to introduce the notation in order to give the full definition. However, throughout this work we only need approximation (1.10).

where g^*_{ab} is the induced target space metric on the world sheet Σ. σ^1 and σ^2 are the local coordinates on Σ. In terms of the action principle the minimalization of the area gives the path and the oszillation mode of the string.

There are two possible types of strings. The closed string is a loop, so that its world sheet has no boundary. The open string has two ends, which means that it gives rise to a world sheet with boundaries. The oscillation modes of a string correspond to the spectrum of various particles. The closed string, for instance, gives rise to a graviton, a 2-form gauge field and a dilaton. The open string model includes in addition gauge bosons.

When deriving the equations of motion for the coordinate fields X^μ in terms of the action principle, in the open string case we have to impose boundary conditions, either Dirichlet, $X^\mu\big|_{\partial\Sigma} = a^\mu$ (or equivalent $\partial_\tau X^\mu\big|_{\partial\Sigma} = 0$), where a^μ is constant, or von Neumann, $\partial_n X^\mu\big|_{\partial\Sigma} = 0$. It is also possible to use different types for different directions, for instance, the time and p spatial directions satisfying von Neumann and the remaining $(D{-}p{-}1)$ directions satisfying Dirichlet conditions. In such a case the string ends are fixed on a $(p{+}1)$-dimensional hypersurface, which is called a Dp-brane and is itself a dynamical object and interacts with a string through its ends.

The Nambu-Goto action (1.11) can also be reformulated in terms of a sigma model, the Polyakov action,

$$S_P = \frac{1}{4\pi\alpha'} \int_\Sigma \mathrm{d}^2\sigma \sqrt{h} h^{ab} \partial_a X^\mu \partial_b X^\nu g_{\mu\nu}(X) \ , \tag{1.12}$$

where $g_{\mu\nu}$ and X^μ denote the metric and the coordinates in the target space and h_{ab} the metric on the world sheet. The Nambu-Goto action can be retrieved by solving the algebraic equation of motion of h_{ab}. But doing so, the world sheet metric can be determined only up to an arbitrary function $\rho(\sigma)$, i.e., $h_{ab} = \rho g^*_{ab}$. This means that the Polyakov action (1.12) has, different from the Nambu-Goto action (1.11), an additional symmetry, the Weyl symmetry, i.e., it is invariant under the transformation $h_{ab} \to \rho h_{ab}$. This symmetry is very important for the sigma model of strings. Without, one could never get back to the Nambu-Goto action and the interpretation of minimalizing an area would break down. The Weyl symmetry occurs only in two dimensions.

The Weyl symmetry plays a key role in the quantization of a string theory with interactions of the target space coordinates X^μ, for instance (1.12). The quantized theory is no longer invariant and one has to require the condition that the Weyl anomalies vanish, which leads to equations of motion for the background fields in the target space, such as the Einstein equation for the metric $g_{\mu\nu}$, i.e., $R_{\mu\nu} = 0$.

Another important symmetry on the world sheet is the diffeomorphism invariance, so that it is locally always possible to transform h_{ab} into the flat metric δ_{ab}. However, for several issues it is more useful to take the conformal gauge $h_{ab} = e^{2\omega}\delta_{ab}$. Then, choosing in addition a flat target space $g_{\mu\nu} = \eta_{\mu\nu}$, the action reads

$$S_P = \frac{1}{4\pi\alpha'} \int_\Sigma \mathrm{d}^2\sigma \partial_a X^\mu \partial^a X_\mu \ . \tag{1.13}$$

Although we have fixed the gauge, action (1.13) is still invariant under conformal (angle preserving) transformation, i.e., all world sheets that are connected by conformal transformations describe equivalent theories. The simplest interaction of open strings without any holes in the world sheet can thus be formulated as a theory on the disk or on the complex upper half plane, where the in- and outgoing open strings shrink to points on the boundary. All our considerations will be restricted to this tree level interaction.

In a quantized conformal field theory each state of the Hilbert space is associated to an operator, i.e., there exists a state-operator isomorphism (see, e.g., [3]). In terms of string theory this means that all particles of the string spectrum correspond to an operator. For instance, the two form gauge field is represented by $\mathcal{V}_b(k) = i\epsilon^{ab}\partial_a X^\mu \partial_b X^\nu e^{ikX} b_{\mu\nu}$, where $b_{\mu\nu}$ is the polarization of the particle. We could now consider strings in the presence of a background of such a field. This can be done by introducing a coherent superposition of particle operators in the action. As an example, a background of antisymmetric states $\mathcal{V}_b(k)$ is represented by

$$\frac{1}{4\pi\alpha'} \int_\Sigma \mathrm{d}^2\sigma \sqrt{h} i\epsilon^{ab}\partial_a X^\mu \partial_b X^\nu B_{\mu\nu}(X) \ . \tag{1.14}$$

1.3 The Connection

A very interesting open string model is the simple case of a constant antisymmetric background field $B_{\mu\nu}$. Expression (1.14) becomes a surface term and the whole action reads

$$S = \frac{1}{4\pi\alpha'} \int_{\Sigma} \mathrm{d}^2\sigma \partial_a X^\mu \partial^a X^\nu \eta_{\mu\nu} + \frac{i}{4\pi\alpha'} \oint_{\partial\Sigma} \mathrm{d}\tau\, X^\mu \partial_\tau X^\nu B_{\mu\nu} \ , \qquad (1.15)$$

again in conformal gauge. In [4] it was shown that on the disk this theory leads to a non-commutative product of functions, in fact, the Moyal-Weyl product (1.4). Without any calculation this is plausible from the structure of (1.15). While the first term gives rise to a "propagation" of the coordinate fields X^μ, the second describes an interaction of different coordinate directions and thus originates a non-commutative geometry. Furthermore, the antisymmetric part in model (1.15) is a pure boundary term and thus non-commutativity arises only on D-branes.

In [7] a limit was introduced in order to decouple the metric $\eta_{\mu\nu}$ from the gauge field $B_{\mu\nu}$, i.e., to switch off gravitational effects and to keep only non-commutative effects. We use the slightly different limit $\eta_{\mu\nu} \sim \epsilon \to 0$ and $B_{\mu\nu} \sim$ const., which has in fact the same consequence. In this decoupling limit the non-commuative product appears in a very clear way through the correlator of N functions inserted at the boundary of the disc with the order $(\tau_1, \ldots, \tau_N)$, i.e.,

$$\langle f_1[X(\tau_1)] \ldots f_N[X(\tau_N]\rangle = \int_x f_1(x) * \ldots * f_N(x) \ . \qquad (1.16)$$

'$*$' indicates the Moyal-Weyl product with the non-commutative parameter $\Theta^{\mu\nu} = (B^{-1})^{\mu\nu}$. Because of the simple structure within the limit, it was used in almost all considerations of non-commutative geometry within string theory, even in the case of non-trivial B-field backgrounds.

From the point of view of string theory the trace property of the Moyal-Weyl product is not an accident but a consequence of the conformal invariance of the theory. On the upper half plane (the disk) the boundary conditions restrict the possible conformal transformations, so that only the $SL(2,\mathbb{R})$ group remains. The correlation functions must be invariant under such transformations and it is exactly the inversion part that is responsible for the cyclic permutation. Since the conformal invariance is a consistency requirement of string theory, in the decoupling limit the trace property must be satisfied even for non-trivial backgrounds.

A generalization to non-constant B-fields was first considered in ref. [9] but not directly in the context of string theory. In fact, a topological Poisson sigma model was found to be the field theory behind Kontsevich's product (1.10). The connection to open string theory is based on the special case of a symplectic model. Then the Poisson sigma model coincides with the open string model in the decoupling limit and with a non-constant, but closed 2-form field B.

A generalization to a completely arbitrary non-commutative parameter $\Theta^{\mu\nu}$ in the framework of open string theory was first tried by the authors of ref. [11]. They showed that the product retains the Kontsevich form but is of course non-associative, where the field strength $H = dB$ controls non-associativity. We already pointed out that the trace property is important for a product originating from string theory. Althought treated in [11] there are still open questions concerning the trace property.

The final goal of considering non-commutativity within string theory is to reproduce the full low energy field theory arising from open string theory away from the decoupling limit. In [7] a calculation of the correlation function of three photons in the "constant" model (1.15) was taken to reproduce the $U(1)$ Yang-Mills theory (1.7)

$$S = -\frac{1}{4g^2}\int_M \mathrm{d}^D x\sqrt{G}G^{\mu\rho}G^{\nu\sigma}F_{\mu\nu} * F^{\rho\sigma} \ , \tag{1.17}$$

where the metric $G^{\mu\nu}$ and the field $\Theta^{\mu\nu}$ are defined in terms of the open string quantities by $(G-\Theta)^{\mu\nu} = (\eta - B)^{-1\mu\nu}$. The field strength and the gauge transformations are as in equations (1.8) and (1.9). Mind that the choice of the integration measure $\sqrt{G}$ in (1.17) is in fact arbitrary, since $G^{\mu\nu}$ is constant. So, the form of the actual measure still needs to be clarified. This can only be done if one considers non-trivial backgrounds.

1.4 Methods and Summary

In this work we tie in with ref. [11] and consider the problem of open strings in general backgrounds, in particular B-field backgrounds with non-vanishing field strength. We address the issue of the measure for the integration as it appears in equations (1.17) as well as in (1.16) where it was silently suppressed since it plays no role in the case of constant fields. The main goal will be to derive the non-commutative product of functions to first order in derivatives of the background fields and investigate its properties. The feature of our approach will be the use of the on-shell condition for the background fields. In previous work the decoupling limit $g_{\mu\nu} \to 0$ disguised the importance of the equations of motion. Therefore, we omit it except for comparative purposes.

Following a similar strategy as the authors in [11] we will work with a derivative expansion of the background fields to extract the star product from correlation functions computed on the disk. Furthermore, we do not choose any gauge conditions for the background gauge fields. Here our setting deviates vitally from the one used in [11], where radial gauge was imposed on the two form gauge potential B. With this choice of gauge and neglecting the field strength F of the boundary interaction only the field strength $H = dB$ contributes in the derivative expansion of the background fields. Due to H being totally antisymmetric this obscures the underlying structure of the product. Instead we prefer to work with the gauge invariant quantity $B + F$ and keep the full dynamics of F. Furthermore, we only perform a perturbation expansion around the constant zero modes, but do not use the approximation of slowly varying background fields as done in [11]. This keeps the full zero mode dependence of the background fields and even simplifies the calculations.

Our main concern will be to discuss the properties of the product obtained by the procedure described above. Although this product is non-commutative and even non-associative we will show that associativity of the product of three functions and the trace property of the integrated product for an arbitrary number of functions is guaranteed up to first order in the derivative expansion and up to surface terms. This is achieved by including the full Born-Infeld measure and the equations of motion of the space-time background fields. However, no on-shell condition is needed for the functions inserted in the product! Because of the prominente role of the Weyl anomalies we present a detailed derivation thereof.

Finally, we comment on the relation to the recent work of Cornalba and Schiappa [11]. Using the limit $g_{\mu\nu} \to 0$ they found that with the choice of radial gauge it is possible to adjust the integration measure in such a way that the integral still acts as a trace. However, we will show that this works only in radial gauge. Moreover, the consistency of the topological limit of [11] severely constrains the background fields through the equations of motion. In the second order of the derivative expansion the Einstein equation already implies the vanishing of the field strength H [12] and hence one is restricted to the symplectic case.

The organization of this work is as follows.

In chapter 2 we introduce the setup for the models under consideration. We give the derivative expansions of the background fields in terms of Riemannian normal coordinates and introduce the additional interaction vertices. The split of the constant zero mode and the quantum fluctuations in the path integral is explained in detail [20]. Moreover, we cite several relations in Riemannian normal coordinates.

In chapter 3 we review the calculations of [4] for the free field theory defined by the constant parts of the background fields and identify the effective open string parameters G and Θ. Furthermore, we compute the vacuum amplitude of the free theory on the disk. It contributes the "Born-Infeld" measure to the integration over the zero modes in the path integral.

Chapter 4 contains a calculation of the Weyl anomalies of the open string theory. Dimensional regularization gives rise to anomalies on the bulk, such as the Einstein equation and the equation of motion for the B-field. The boundary anomaly, the non-linear Maxwell equation, is computed by the use of a displacement regularization.

Then in chapter 5 the disk correlators are computed in order to extract the non-commutative and non-associative Kontsevich-type product.

The properties are discussed in chapter 6. In particular we show that the trace property of the two point function holds due to the equations of motion of the background fields. The "Born-Infeld" measure exactly cancels the additional contributions arising from partial integration. By the same mechanism the product of three functions does not depend on the way one introduces brackets, i.e. the non-associativity is a surface term. This, in turn, implies the trace property for an arbitrary number of functions. We finish this chapter with some comments on the relations of our approach to the recent work of Cornalba and Schiappa. In particular we examine the implications of the radial gauge and the consistency of the topological limit used in [11].

In the last chapter we conclude with comments on some open questions.

Appendix A finally presents the dilogarithm function and some relations thereof. Appendix B contains the detailed calculations of Greens functions for chapter 5.

Chapter 2

Open String Sigma Model

The starting point of our considerations is the non-linear sigma model of the bosonic open string [20, 22, 23]

$$\begin{aligned} S &= \frac{1}{4\pi\alpha'}\int_\Sigma \mathrm{d}^2\sigma\sqrt{h}\Big(h^{ab}\partial_a X^\mu \partial_b X^\nu g_{\mu\nu}(X) + i\epsilon^{ab}\partial_a X^\mu \partial_b X^\nu B_{\mu\nu}(X)\Big) \\ &+ i\int_{\partial\Sigma}\mathrm{d}s\Big(\partial_s X^\mu A_\mu(X)\Big), \end{aligned} \tag{2.1}$$

which includes the space-time metric $g_{\mu\nu}(X)$, the 2-form gauge potential $B_{\mu\nu}(X)$ and the 1-form gauge potential $A_\mu(X)$. h_{ab} denotes the Euclidean metric on the world sheet Σ and $\mathrm{d}s$ is the induced line element on the boundary.

In (2.1) the boundary term of the 1-form gauge potential A can be written as a bulk term

$$\int_\Sigma \mathrm{d}^2\sigma\sqrt{h}\, i\epsilon^{ab}\partial_a X^\mu \partial_b X^\nu F_{\mu\nu}(X), \tag{2.2}$$

where $F = \mathrm{d}A$ is the corresponding 2-form field strength.

2.1 Space-Time Gauge Fields

Both, the 1-form potential A and the 2-form potential B, are associated with space-time gauge invariances. For the former the gauge transformation

$$\delta A = \mathrm{d}\lambda \tag{2.3}$$

leaves the action (2.1) invariant. In open string theory there does not exist a gauge transformation for the 2-form potential B alone, because surface terms require a combined transformation

$$\begin{aligned} \delta B &= \mathrm{d}\Lambda, \\ \delta A &= -\frac{\Lambda}{2\pi\alpha'} \end{aligned} \tag{2.4}$$

that does not change the action (2.1). From (2.3) and (2.4) one can see that the combination $\mathcal{F} = B + 2\pi\alpha' F = B + 2\pi\alpha' \mathrm{d}A$ is invariant under both gauge symmetries. Therefore, gauge invariant expressions contain the 2-form $\mathcal{F}$ and the 3-form field strength $H = \mathrm{d}\mathcal{F} = \mathrm{d}B$.

If one considers a brane that is not space-time filling, the gauge field A and hence $\mathcal{F}$ are only defined along the brane. For simplicity we will restrict our considerations to the special case of a space-time filling D25-brane. Furthermore, in topologically non-trivial backgrounds the gauge potentials A and B may not be globally well defined. Such considerations are, however, irrelevant in the present context.

In the classical approximation of open string theory the world sheet Σ is a disk. Taking advantage of the conformal invariance of the theory, we map the disk to the upper half plane $\mathbb{H}$ and perform our calculations there. Furthermore, we choose the conformal gauge and use complex coordinates $z = \sigma^1 + i\sigma^2$. Thus the world sheet metric becomes $h_{z\bar{z}} = e^{2\omega(z,\bar{z})}\delta_{z\bar{z}}$ and the invariant line element at the boundary is $\mathrm{d}s = e^{\omega}\mathrm{d}\tau$. The derivatives tangential and normal to the boundary are $\partial_\tau = (\partial + \bar{\partial})$ and $\partial_n = i(\bar{\partial} - \partial)$, respectively. In this parametrization the action (2.1) is given by

$$S = \frac{1}{2\pi\alpha'} \int_{\mathbb{H}} \mathrm{d}^2 z\, \partial X^\mu \bar{\partial} X^\nu \Big(g_{\mu\nu}(X) + \mathcal{F}_{\mu\nu}(X) \Big), \tag{2.5}$$

and the corresponding mixed boundary condition along the brane is

$$g_{\mu\nu}(X)(\partial - \bar{\partial})X^\nu - \mathcal{F}_{\mu\nu}(X)(\partial + \bar{\partial})X^\nu \Big|_{\bar{z}=z} = 0. \tag{2.6}$$

2.2 Constant Zero Mode and Derivative Expansion

Following the procedure explained in [20] we expand the field $X^\mu(z,\bar{z})$ around the constant zero mode contribution x,

$$X^\mu(z,\bar{z}) = x^\mu + \zeta^\mu(z,\bar{z}), \tag{2.7}$$

so that the path integral over the field $X^\mu(z,\bar{z})$ splits into an ordinary integral over the constant zero modes x^μ and a path integral over the quantum fluctuations $\zeta^\mu(z,\bar{z})$. This separation is, in fact, not unique. A constant part, c^μ, can always be exchanged between the two parts, i.e.,

$$\begin{aligned} X^\mu(z,\bar{z}) &= x^\mu + \zeta^\mu(z,\bar{z}) = x'^\mu + \zeta'^\mu(z,\bar{z}) \\ x'^\mu &= x^\mu - c^\mu \\ \zeta'^\mu &= \zeta^\mu + c^\mu \end{aligned}$$

So, we have to impose a "gauge" condition in order to fix c^μ. A unique way to perform the split (2.7) is to insert the following "unity" into the path integral:

$$1 = \int \mathrm{d}^D x \int [\mathrm{d}\zeta]\; \delta^D\big(X(z,\bar{z}) - (x + \zeta(z,\bar{z})\big)\; \delta^D(P^\mu[x,\zeta])\; \Delta[x,\zeta]\;, \tag{2.8}$$

with

$$\Delta[x,\zeta] = \det\Big(\frac{\partial P^\mu[x-c,\zeta+c]}{\partial c}\Big)_{c=0}\;.$$

$P^\mu[x,\zeta] = 0$ is the "gauge" condition and $\Delta[x,\zeta]$ is the corresponding "ghost" determinant. Subsequently, we will use the condition

$$P^\mu = \int \mathrm{d}s\, \zeta^\mu(s) = 0\;, \quad \Delta = L^D\;, \quad L = \int \mathrm{d}s\;. \tag{2.9}$$

Since Δ is only a constant factor it does not play an essential role and we can incorporate it in the normalization of the path integral. The delta functional for $P^\mu = 0$ will not be written explicitely, but we impose the condition by hand. Whereever there appears an integral of the quantum fluctuations over the boundary of the world sheet, we set it to zero.

Therefore, we get

$$\begin{aligned}\langle :f_1[X(z_1)]: \ \ldots \ :f_N[X(z_N)]:\rangle &= \\ &= \frac{1}{L^D}\int[\mathrm{d}X]\, e^{-S[X]} f_1[X_1]\ldots f_N[X_N] = \\ &= \int \mathrm{d}^D x \int [\mathrm{d}\zeta]\, e^{-S[x+\zeta]} f_1[x+\zeta_1]\ldots f_N[x+\zeta_N]\,, \end{aligned} \tag{2.10}$$

where the functions $f_i[X(z_i)]$ denote arbitrary insertions in the path integral. For the expansion of the action $S[X] = S[x+\zeta]$ around the zero modes we simplify our computation by making use of Riemannian normal coordinates [18, 24],

$$g_{\mu\nu}(x+\zeta) = \eta_{\mu\nu} - \frac{1}{3}R_{\mu\rho\nu\sigma}(x)\zeta^\rho\zeta^\sigma + \mathcal{O}(\zeta^3), \tag{2.11}$$

$$\mathcal{F}_{\mu\nu}(x+\zeta) = \mathcal{F}_{\mu\nu}(x) + \partial_\rho\mathcal{F}_{\mu\nu}(x)\zeta^\rho + \frac{1}{2}\partial_\rho\partial_\sigma\mathcal{F}_{\mu\nu}(x)\zeta^\rho\zeta^\sigma + \mathcal{O}(\zeta^3). \tag{2.12}$$

In contrast to [11] we do not choose radial gauge for $\mathcal{F}_{\mu\nu}(X)$. In that case (2.12) would split into two separate expansions for B and F, where the non-constant part of the B expansion contains only the field strength H. The radial gauge fixes the combined transformation (2.4), whereas transformation (2.3) remains unaffected. With (2.11) and (2.12) we are able to write the action (2.5) as

$$\begin{aligned} S = \frac{1}{2\pi\alpha'}\int_{\mathbb{H}} \mathrm{d}^2 z \ \Big\{ &\partial\zeta^\mu\bar\partial\zeta^\nu(\eta_{\mu\nu}+\mathcal{F}_{\mu\nu}) + \partial\zeta^\mu\bar\partial\zeta^\nu\zeta^\rho\,\partial_\rho\mathcal{F}_{\mu\nu} + \\ + \ &\partial\zeta^\mu\bar\partial\zeta^\nu\zeta^\rho\zeta^\sigma\,(\frac{1}{2}\partial_\rho\partial_\sigma\mathcal{F}_{\mu\nu} - \frac{1}{3}R_{\mu\rho\nu\sigma}) + \mathcal{O}(\partial_\rho^3)\Big\}. \end{aligned} \tag{2.13}$$

In the following we will restrict our considerations to terms of at most first order in derivatives of the space-time background fields.

2.3 More About Riemannian Normal Coordinates

For later reference we explain some properties of Riemannian normal coordinates in this section.

The basic idea behind Riemannian normal coordinates is to use the geodesics through a given point to define the coordinates for nearby points [16,17]. Take a point P with coordinates x^μ and a nearby point Q. If Q is close enough to P then there exists a unique geodesic joining P to Q. Let a^μ be the components of the unit tangent vector to this geodesic at P and let s be the geodesic arc length measured from P to Q. Then the Riemannian normal coordinates of Q are defined to be $X^\mu = x^\mu + sa^\mu$.

An equivalent but for our purposes more useful definition of Riemannian normal coordinates at a point P is that they are a set of coordinates for which

$$\Gamma^\mu{}_{\rho\sigma}(x) = 0 \tag{2.14}$$

$$\Gamma^\mu{}_{\rho\sigma,\lambda}(x) + \Gamma^\mu{}_{\lambda\rho,\sigma}(x) + \Gamma^\mu{}_{\sigma\lambda,\rho}(x) = 0\ . \tag{2.15}$$

As a consequence, one obtains equation (2.11) by a Taylor series expansion around P. The flat metric $\eta_{\mu\nu}$ at P requires the additional property that the tangent vectors of the geodesics which build our coordinate system are chosen to be orthogonal at P.

Finally, we present some useful relations between the metric, the Christoffel symbols and the curvature:

$$\Gamma^\mu{}_{\rho\sigma,\lambda}(x) = -\frac{1}{3}\big(R^\mu{}_{\rho\sigma\lambda}(x) + R^\mu{}_{\sigma\rho\lambda}(x)\big) \tag{2.16}$$

$$g_{\mu\nu,\rho\sigma}(x) = -\frac{1}{3}\big(R_{\mu\rho\nu\sigma}(x) + R_{\mu\sigma\nu\rho}(x)\big) \tag{2.17}$$

$$R_{\mu\nu\rho\sigma}(x) = g_{\rho\nu,\mu\sigma}(x) - g_{\rho\mu,\nu\sigma}(x) \tag{2.18}$$

These equations will intensively be used in section 4.7.

Chapter 3

The Free Theory

3.1 The Propagator

As a warm up for later calculations and to set up the relevant techniques of our approach let us first calculate the propagator for the free field theory defined by the Gaussian part of (2.13) in the path integral,

$$S_{\text{free}} = \frac{1}{2\pi\alpha'} \int_{\mathbb{H}} \mathrm{d}^2 z\, \partial\zeta^\mu \bar{\partial}\zeta^\nu \eta_{\mu\nu} + \frac{i}{4\pi\alpha'} \oint_{\partial\mathbb{H}} \mathrm{d}\tau\, \zeta^\mu \partial_\tau \zeta^\nu \mathcal{F}_{\mu\nu}. \tag{3.1}$$

Here, $\partial\mathbb{H}$ denotes the boundary of the upper half plane, i.e., the real line.[1] The second term contributes to the boundary condition which takes the same form as (2.6) with $\eta_{\mu\nu}$ and $\mathcal{F}_{\mu\nu}(x)$ replacing the full metric $g_{\mu\nu}(X)$ and $\mathcal{F}_{\mu\nu}(X)$, respectively. The boundary term can be regarded as a perturbative correction [4] to the free propagator

$$\langle \zeta^\mu(u,\bar{u})\, \zeta^\nu(w,\bar{w}) \rangle = -\frac{\alpha'}{2}\eta^{\mu\nu} \ln|u-w|^2 - \frac{\alpha'}{2}\eta^{\mu\nu} \ln|u-\bar{w}|^2. \tag{3.2}$$

The homogeneous (image charge) part accounts for the Neumann boundary condition $\partial_n \zeta^\mu|_{\partial\mathbb{H}} = 0$ of the theory without perturbation. The propagator of the perturbed theory is then given by

$$\langle \zeta^\mu(u,\bar{u})\, \zeta^\nu(w,\bar{w}) \rangle_{\mathcal{F}} = \langle \zeta^\mu(u,\bar{u})\, \zeta^\nu(w,\bar{w}) \mathrm{e}^{-\frac{i}{4\pi\alpha'} \oint_{\partial\mathbb{H}} \mathrm{d}\tau\, \zeta^\rho \partial_\tau \zeta^\sigma \mathcal{F}_{\rho\sigma}} \rangle. \tag{3.3}$$

[1] We have used the divergence theorem for complex coordinates, which reads $\int_\Sigma \mathrm{d}^2 z\,(\partial_z v^z \pm \partial_{\bar{z}} v^{\bar{z}}) = i \oint_{\partial\Sigma} (\mathrm{d}\bar{z} v^z \mp \mathrm{d}z v^{\bar{z}})$.

For the calculation of the propagator only tree contributions are relevant. We will consider loops separately in the next section. Expanding in a perturbation series the term of order n

$$\langle\zeta^\mu(u,\bar u)\zeta^\nu(w,\bar w)\frac{1}{n!}\Big\{\frac{i}{4\pi\alpha'}\Big[\oint_{\partial\mathbb{H}}\mathrm{d}z\,\partial\zeta^\rho\zeta^\sigma\mathcal{F}_{\rho\sigma}+\oint_{\partial\mathbb{H}}\mathrm{d}\bar z\,\bar\partial\zeta^\rho\zeta^\sigma\mathcal{F}_{\rho\sigma}\Big]\Big\}^n\rangle \quad (3.4)$$

gives two slightly different contributions, depending on whether n is even or odd. By using the derivative of the propagator (3.2) it is straightforward to obtain the result[2]

$$\frac{i}{2\pi}(\mathcal{F}^n)_{\lambda\kappa}\Big\{(-1)^{n-1}\oint_{\partial\mathbb{H}}\mathrm{d}z\,\eta^{\mu\lambda}\frac{1}{\bar u-z}\langle\zeta^\kappa(z,\bar z)\zeta^\nu(w,\bar w)\rangle + \oint_{\partial\mathbb{H}}\mathrm{d}\bar z\,\eta^{\mu\lambda}\frac{1}{u-\bar z}\langle\zeta^\kappa(z,\bar z)\zeta^\nu(w,\bar w)\rangle\Big\} \quad . \quad (3.5)$$

The remaining divergent integrals are regularized by differentiating with respect to w and $\bar w$, respectively. This yields a finite result plus an infinite additive constant $C^{\mu\nu}_{(\infty)}$,

$$\alpha'(\mathcal{F}^n)^{\mu\nu}\Big\{(-1)^{n-1}\ln(\bar u-w)-\ln(u-\bar w)\Big\}+C^{\mu\nu}_{(\infty)}. \quad (3.6)$$

Now, it is possible to sum up all orders in a geometric series, which finally gives the desired propagator [22, 23]

$$\langle\zeta^\mu(u,\bar u)\,\zeta^\nu(w,\bar w)\rangle_{\mathcal{F}}=-\alpha'\Big\{\eta^{\mu\nu}(\ln|u-w|-\ln|u-\bar w|) + G^{\mu\nu}\ln|u-\bar w|^2-\Theta^{\mu\nu}\ln\Big(\frac{\bar w-u}{\bar u-w}\Big)\Big\}+C^{\mu\nu}_{(\infty)}, \quad (3.7)$$

where we have introduced the quantities[3]

$$G^{\mu\nu}:=\Big(\frac{1}{g-\mathcal{F}}\;g\;\frac{1}{g+\mathcal{F}}\Big)^{\mu\nu} \quad \text{and} \quad \Theta^{\mu\nu}:=-\Big(\frac{1}{g-\mathcal{F}}\;\mathcal{F}\;\frac{1}{g+\mathcal{F}}\Big)^{\mu\nu}. \quad (3.8)$$

[2] In this calculation there appear integrals of the form $\oint_{\partial\mathbb{H}}\mathrm{d}z\frac{1}{\bar u-z}\frac{1}{\bar z-w}$. The part along the real axis $\mathbb{R}$ is $\int_{\mathbb{R}}\mathrm{d}r\frac{1}{\bar u-r}\frac{1}{r-w}$, whereas the integral along the semicircle in the upper half plane with infinite radius is zero. Therefore, the original integral can be written as

$$\oint_{\partial\mathbb{H}}\mathrm{d}z\frac{1}{\bar u-z}\frac{1}{\bar z-w}=\oint_{\partial\mathbb{H}}\mathrm{d}z\frac{1}{\bar u-z}\frac{1}{z-w},$$

which can be evaluated using the residue theorem.

[3] For later reference we have expressed $G^{\mu\nu}$ and $\Theta^{\mu\nu}$ by the full bulk metric $g_{\mu\nu}$, whereas the correct terms in (3.7) contain of course the Minkowski metric $\eta_{\mu\nu}$ because of the Riemannian normal coordinates.

The integration constant $C^{\mu\nu}_{(\infty)}$ plays no essential role and can be set to a convenient value, e.g. $C^{\mu\nu}_{(\infty)} = 0$ [7]. Restricted to the boundary ($u = \bar{u} = \tau$ and $w = \bar{w} = \tau'$) the propagator has the simple form

$$\begin{aligned} \alpha' i\pi \Delta^{\mu\nu}(\tau, \tau') &:= \langle \zeta^\mu(\tau)\, \zeta^\nu(\tau') \rangle_{\mathcal{F}} \\ &= -\alpha' G^{\mu\nu} \ln(\tau - \tau')^2 - \alpha' i\pi \Theta^{\mu\nu} \epsilon(\tau - \tau'). \end{aligned} \tag{3.9}$$

As discussed in [7] the boundary propagator (3.9) suggests to interpret $G_{\mu\nu}$ as an effective metric seen by the open strings, in contrast to $g_{\mu\nu}$, which is to be viewed as the closed string metric in the bulk.

For later purposes we elaborate on the distinction between the open string quantities $G^{\mu\nu}$ and $\Theta^{\mu\nu}$ and the closed string quantities $g_{\mu\nu}$ and $B_{\mu\nu}$. In order to make a clear distinction between the bulk and the boundary quantities, we mark all expressions that refer to boundary quantities with bars. To this end we define

$$\bar{G}_{\mu\nu} := (g - \mathcal{F}^2)_{\mu\nu} \quad \text{and} \quad \bar{\Theta}^\mu{}_\nu := -\mathcal{F}^\mu{}_\nu. \tag{3.10}$$

The first of the above definitions is equivalent to setting $\bar{G}^{\mu\nu} = G^{\mu\nu}$ and requiring $\bar{G}_{\mu\nu}$ to be its inverse. The second definition follows from setting $\bar{\Theta}^{\mu\nu} = \Theta^{\mu\nu}$ and pulling indices with $\bar{G}_{\mu\nu}$. In an analogous way we label all expressions that are built out of these quantities with bars, e.g. the Christoffel symbol $\bar{\Gamma}_{\mu\nu}{}^\rho$ and the covariant derivative $\bar{D}_\mu$ compatible with the open string metric $\bar{G}_{\mu\nu}$.

3.2 Vacuum Amplitude and Integration Measure

Let us now consider loop contributions arising from an even number of insertions of the boundary perturbation of (3.1)[4]. In this calculation there appear divergences when the insertion points approach the boundary. We regularize these terms by keeping a fixed distance d_0 with respect to the metric in conformal gauge to the boundary $\partial\mathbb{H}$, i.e., we impose $|z - \bar{z}| \geq 2\mathrm{Im}(z) \geq e^{-\omega} d_0$.

[4] Odd powers vanish because of the antisymmetry of $\mathcal{F}_{\mu\nu}$

To make this more explicit let us consider the one loop contribution of the $\mathcal{F}^2$ term,

$$\frac{1}{2}\langle\Big(\frac{-i}{4\pi\alpha'}\Big)^2 \oint_{\partial\mathbb{H}} \mathrm{d}\tau\, \zeta^\mu\partial_\tau\zeta^\nu\mathcal{F}_{\mu\nu} \times \oint_{\partial\mathbb{H}} \mathrm{d}\tau'\, \zeta^\rho\partial'_\tau\zeta^\sigma\mathcal{F}_{\rho\sigma}\rangle_{1-\mathrm{loop}}. \tag{3.11}$$

Using the same techniques as for the chains (3.4) gives the divergent contribution

$$\Big(\frac{1}{4\pi d_0}\int \mathrm{d}s\Big)\frac{1}{2}\mathcal{F}_\mu{}^\nu\mathcal{F}_\nu{}^\mu, \tag{3.12}$$

where $\mathrm{d}s = \mathrm{d}\tau e^\omega$ is the invariant line element in conformal gauge. Summing up all powers of $\mathcal{F}$ in the 1-loop contribution yields

$$\Big(\frac{1}{4\pi d_0}\int \mathrm{d}s\Big)\sum_{n=1}^{\infty}\frac{1}{2n}\mathrm{Tr}(\mathcal{F}^{2n}) = -\Big(\frac{1}{4\pi d_0}\int \mathrm{d}s\Big)\frac{1}{2}\ln(\det(\delta-\mathcal{F}^2)_\mu{}^\nu). \tag{3.13}$$

As observed in [19, 25] this linear divergence is in fact regularization scheme dependent and can be absorbed into the tachyon by a field redefinition. But a finite constant part

$$b_0\ln(\det(\delta-\mathcal{F}^2)_\mu{}^\nu) \tag{3.14}$$

may remain after subtraction of appropriate counterterms. The analysis given in [20, 25] determined the constant b_0 to be $\frac{1}{4}$ in order to yield the Born-Infeld action for a vanishing tachyon field.

In (3.14) we have added up all powers of $\mathcal{F}$ contributing to the connected vacuum graphs. Taking into account all disconnected one loop graphs to all orders of the interaction leads in fact to the Born-Infeld Lagrangian

$$\sum_{n=0}^{\infty}\frac{1}{n!}\big(\ln(\det(\delta-\mathcal{F}^2)_\mu{}^\nu)^{\frac{1}{4}}\big)^n = \sqrt[4]{\det(\delta-\mathcal{F}^2)_\mu{}^\nu} = \sqrt{\det(\delta-\mathcal{F})_\mu{}^\nu}\;. \tag{3.15}$$

Here we used the antisymmetry of $\mathcal{F}_{\mu\nu}$ to change the sign in the determinant. Expression (3.15) can also be interpreted as a contribution to the measure of the integration over the zero modes in the path integral. Although we make use of Riemannian normal coordinates for the perturbation expansion, we can write the measure in a covariant way by including the term $\sqrt{\det g_{\mu\nu}}$.

Therefore, if there are no operator insertions in the path integral (2.10), we obtain the Born-Infeld action

$$\begin{aligned}\int \mathrm{d}^D x \sqrt{\det g_{\mu\nu}} \sqrt[4]{\det(\delta - \mathcal{F}^2)_\mu{}^\nu} &= \int \mathrm{d}^D x \sqrt[4]{\det g_{\mu\nu}} \sqrt[4]{\det \bar{G}_{\mu\nu}} = \\ &= \int \mathrm{d}^D x \sqrt{\det(g - \mathcal{F})_{\mu\nu}}, \qquad (3.16)\end{aligned}$$

where $\bar{G}_{\mu\nu}$ is the boundary metric as defined in (3.10).

So far we have regarded all possible diagrams of the boundary insertion of (3.1). Therefore, we can now work with the full propagator (3.7) for all higher order interaction terms.

For the remainder we make use of the abbreviations $g = \det g_{\mu\nu}$ and $\int_x = \int \mathrm{d}^D x \sqrt{g - \mathcal{F}} = \int \mathrm{d}^D x \sqrt[4]{g} \sqrt[4]{\bar{G}}$. Furthermore, we set $2\pi\alpha' = 1$.

Chapter 4

Weyl Anomalies

In section 1.2 we pointed out that the Weyl invariance of the sigma model is important to maintain the connection to the Nambu-Goto action (1.11). In this chapter we are interested in the scaling behaviour of the quantized open string sigma model. The regularization of the divergent diagrams entails the introduction of a scale dependent parameter and, therefore, the quantum corrections cause a breaking of the symmetry, i.e., the theory is Weyl anomalous.

The renormalization group theory provides a quantity that extracts the Weyl anomalous parts from the regularized diagrams. It is called the β-function and is associated with the renormalization of the coupling constant. In the case of the sigma model (2.1) we have, in fact, infinitely many coupling constants. Expanding the functional $\mathcal{F}_{\mu\nu}[X(z,\bar{z})]$ in a Taylor series (2.12) we get a sequence of coupling constants $\mathcal{F}_{\mu\nu}(x)$, $\partial_\rho\mathcal{F}_{\mu\nu}(x)$, $\frac{1}{2}\partial_\rho\partial_\sigma\mathcal{F}_{\mu\nu}(x)$, ... for the interaction vertices $\partial\zeta^\mu\bar{\partial}\zeta^\nu$, $\partial\zeta^\mu\bar{\partial}\zeta^\nu\zeta^\rho$, $\partial\zeta^\mu\bar{\partial}\zeta^\nu\zeta^\rho\zeta^\sigma$, ... , respectively. In order to maintain the Weyl invariance of the quantized theory one has to require that the corresponding β-functions vanish.

In string theory the functions $g_{\mu\nu}$ and $\mathcal{F}_{\mu\nu}$ have two different meanings. From the world sheet point of view they represent a series of couplings, as stated above. But in the target space they mean various particle fields. The condition $\beta = 0$ corresponds to equations of motion for the particle fields (in the target space).

In practice, the computation of the β-function requires a separation of the counterterms into contributions to the wave function and the coupling constant renormalization. Therefore, it is necessary to work out the counterterms for two different vertices. In order to avoid this separation it is more convenient to compute the Weyl anomaly more directly by choosing conformal gauge. The anomalous terms then contain the conformal function ω and appear in the finite part of the regularized diagrams.[1] The divergent part must be compensated by counterterms, which can be written in a general coordinate invariant way.

In the simpler model of closed strings only the inhomogeneous part of the propagator contributes to divergent loop diagrams which is due to the absence of a boundary. So the problem can be treated on the complex plane with natural boundary conditions and the divergences appear all over the world sheet. Since the open string theory is defined on a world sheet with boundary the homogeneous part of the propagator gives also rise to additional divergences that appear only at the boundary. For both kinds of divergence we obtain of course corresponding Weyl anomalies.

4.1 The Action in Continuous Dimensions

We start with the calculation of the bulk anomalies using the inhomogeneous part of the propagator (3.7). We choose the dimensional regularization scheme and thus start rewriting action (2.1) in continuous dimensions $n = 2 - \epsilon$ with $\epsilon > 0$. Applying conformal gauge $h_{ab} = e^{2\omega}\delta_{ab}$ the integration measure on the world sheet becomes $\sqrt{h} = e^{n\omega}$ and in combination with the inverse metric we get $\sqrt{h}h^{ab} = e^{(n-2)\omega}\delta^{ab}$. The treatment of the antisymmetric tensor ϵ^{ab} is a bit more subtle. It is defined as $\epsilon^{ab} = \frac{\tilde{\epsilon}^{ab}}{\sqrt{h}}$ with the ϵ-symbol $\tilde{\epsilon}^{ab}$,[2] and in conformal gauge we have $\epsilon^{ab} = e^{-2\omega}\tilde{\epsilon}^{ab}$. If one generalizes to arbitrary dimensions the entries in ϵ^{ab} do not change, but the continuous dimension emerges again from the measure, so that we obtain

[1] The ω independent finite terms are no matter of interest in our discussion and will be neglected throughout this chapter.

[2] We use the convention that $\tilde{\epsilon}^{12} = 1$.

$\sqrt{h}\epsilon^{ab} = e^{(n-2)\omega}\tilde{\epsilon}^{ab}$. All in all the action (2.1) is given by

$$S = \frac{1}{4\pi\alpha'}\int_{\mathbb{H}} \mathrm{d}^n\sigma\, e^{-\epsilon\omega}\Big\{\partial_a\zeta^\mu\partial^a\zeta^\nu(\eta_{\mu\nu} - \zeta^\rho\zeta^\sigma\frac{1}{3}R_{\mu\rho\nu\sigma}) \tag{4.1}$$
$$+ \quad i\tilde{\epsilon}^{ab}\partial_a\zeta^\mu\partial_b\zeta^\nu(\mathcal{F}_{\mu\nu} + \zeta^\rho\partial_\rho\mathcal{F}_{\mu\nu} + \zeta^\rho\zeta^\sigma\frac{1}{2}\partial_\rho\partial_\sigma\mathcal{F}_{\mu\nu})\Big\} \quad .$$

Because of the factor $e^{-\epsilon\omega}$ the first term in (4.1) is not a free part of the theory. In order to seperate a free part we perform a field redefinition $\zeta^\mu = e^{\frac{\epsilon}{2}\omega}\tilde{\zeta}^\mu$, so that

$$S_\eta = \frac{1}{4\pi\alpha'}\int_{\mathbb{H}} \mathrm{d}^n\sigma\,\Big\{\partial_a\tilde{\zeta}^\mu\partial^a\tilde{\zeta}_\mu + \epsilon\partial_a\omega\tilde{\zeta}^\mu\partial^a\tilde{\zeta}_\mu + O(\epsilon^2)\Big\} \quad . \tag{4.2}$$

The first term yields the propagator and the second can be treated perturbatively as "soft mass insertion" [21].

4.2 The Inhomogeneous Part of the Propagator

In order to account for the continuous dimensions the inhomogeneous part of the propagator should be represented through its Fourier transformation

$$< \tilde{\zeta}^\mu(\sigma)\tilde{\zeta}^\nu(\sigma') >_{\text{free}} = 2\pi\alpha'\eta^{\mu\nu}\int \frac{\mathrm{d}^n k}{(2\pi)^n}\frac{e^{ik(\sigma-\sigma')}}{k^2+m^2} \quad . \tag{4.3}$$

Since the propagator is also infrared divergent, we introduced an appropriate regulator mass m. The ultraviolet behaviour of the free propagator (4.3) is

$$< \tilde{\zeta}^\mu(\sigma)\tilde{\zeta}^\nu(\sigma) >_{\text{free}} \sim \alpha'\eta^{\mu\nu}\frac{1}{\epsilon} \quad . \tag{4.4}$$

Within dimensional regularization there appear only logarithmic divergences in a massless theory, so that

$$\begin{aligned} < \partial_a\tilde{\zeta}^\mu(\sigma)\tilde{\zeta}^\nu(\sigma) >_{\text{free}} &\sim 0\ , \\ < \partial_a\tilde{\zeta}^\mu(\sigma)\partial_b\tilde{\zeta}^\nu(\sigma) >_{\text{free}} &\sim 0\ , \\ < \partial_a\partial_b\tilde{\zeta}^\mu(\sigma)\tilde{\zeta}^\nu(\sigma) >_{\text{free}} &\sim 0\ . \end{aligned} \tag{4.5}$$

However, the "soft mass" term can change the order of divergence and thus generate non-vanishing results. Since it is linear in ϵ we expect a finite result that depends on the conformal factor ω. To see how this works we recognize first that the insertion can be rewritten as

$$\begin{aligned} &\epsilon\partial_a\omega\tilde{\zeta}^\mu\partial^a\tilde{\zeta}_\mu \\ &= \epsilon\partial_a(\omega\tilde{\zeta}^\mu\partial^a\tilde{\zeta}_\mu) - \epsilon\omega\partial_a\tilde{\zeta}^\mu\partial^a\tilde{\zeta}_\mu \qquad (4.6) \\ &= \frac{1}{2}\epsilon\partial^a(\partial_a\omega\tilde{\zeta}^\mu\tilde{\zeta}_\mu) - \frac{1}{2}\epsilon\partial^2\omega\tilde{\zeta}^\mu\tilde{\zeta}_\mu \qquad (4.7) \end{aligned}$$

Here we used the fact that in the dimensional regularization scheme the free classical equation of motion $\partial^2\tilde{\zeta}^\mu = 0$ is satisfied as operator equation as can be seen from (4.5). This can easily be seen from (4.5). The boundary terms in expressions (4.6) and (4.7) do not generate divergencies and therefore vanish in the limit $\epsilon \to 0$. The interesting contributions arise from the bulk insertions. They give rise to $\frac{1}{\epsilon}$ poles which are cancelled by the ϵ in the "soft mass" term. Take, for instance, insertion (4.6). Using the propagator (4.3) we obtain

$$< \tilde{\zeta}^\mu(\sigma)\tilde{\zeta}^\nu(\sigma)\frac{1}{4\pi\alpha'}\int_{\mathbb{H}} \mathrm{d}^n\sigma'\,\epsilon\omega\partial_a\tilde{\zeta}^\rho\partial^a\tilde{\zeta}_\rho >_{\text{free}} \sim \alpha'\omega\eta^{\mu\nu} \quad . \qquad (4.8)$$

If we repeat the procedure with the differentiated field $\tilde{\zeta}^\mu$ in a similar way we end up with the contractions

$$\begin{aligned} < \tilde{\zeta}^\mu(\sigma)\tilde{\zeta}^\nu(\sigma) > &\sim \alpha'\eta^{\mu\nu}(\frac{1}{\epsilon} + \omega) \quad , \\ < \partial_a\tilde{\zeta}^\mu(\sigma)\tilde{\zeta}^\nu(\sigma) > &\sim 0 \quad , \\ < \partial_a\partial_b\tilde{\zeta}^\mu(\sigma)\tilde{\zeta}^\nu(\sigma) > &\sim -\frac{1}{4}\alpha'\eta^{\mu\nu}\delta_{ab}\partial^2\omega \quad \text{and} \\ < \partial_a\tilde{\zeta}^\mu(\sigma)\partial_b\tilde{\zeta}^\nu(\sigma) > &\sim \frac{1}{4}\alpha'\eta^{\mu\nu}\delta_{ab}\partial^2\omega \quad . \qquad (4.9) \end{aligned}$$

Finally, returning to the original field ζ^μ, we find the contractions

$$\begin{aligned} < \zeta^\mu(\sigma)\zeta^\nu(\sigma) > &\sim e^{\epsilon\omega}\alpha'\eta^{\mu\nu}(\frac{1}{\epsilon} + \omega) \quad , \\ < \partial_a\zeta^\mu(\sigma)\zeta^\nu(\sigma) > &\sim e^{\epsilon\omega}\frac{\alpha'}{2}\eta^{\mu\nu}\partial_a\omega \quad , \\ < \partial_a\zeta^\mu(\sigma)\partial_b\zeta^\nu(\sigma) > &\sim e^{\epsilon\omega}\frac{\alpha'}{4}\eta^{\mu\nu}\delta_{ab}\partial^2\omega \qquad \text{and} \qquad (4.10) \\ < \partial^2\zeta^\mu(\sigma)\zeta^\nu(\sigma) > &\sim 0 \quad . \end{aligned}$$

Figure 4.1: The one loop diagrams under consideration.

4.3 Counterterms and Weyl Anomaly I

Now we are prepared to calculate divergent and ω-dependent contributions to the effective action arising from diagrams as shown in figure 4.1. In every interaction vertex of (4.1) two fields must be contracted.

First, we consider the curvature 4-point vertex

$$-\frac{1}{12\pi\alpha'}\int_{\mathbb{H}} \mathrm{d}^n\sigma\, e^{-\epsilon\omega}\partial_a\zeta^\mu\partial^a\zeta^\nu\zeta^\rho\zeta^\sigma R_{\mu\rho\nu\sigma}\ . \tag{4.11}$$

The coincidence limits (4.10) yield the result

$$-\frac{1}{12\pi}\int_{\mathbb{H}} \mathrm{d}^n\sigma \left\{ (\frac{1}{\epsilon}+\omega)\partial_a\zeta^\mu\partial^a\zeta^\nu R_{\mu\nu} - \partial_a\omega\partial^a\zeta^\mu\zeta^\nu R_{\mu\nu} + \frac{1}{2}\partial^2\omega\zeta^\mu\zeta^\nu R_{\mu\nu}\right\}\ .$$

The $\frac{1}{\epsilon}$-divergence must be compensated by an appropriate counterterm in the action. We are not further interested in it. In more detail we look into the Weyl anomalous part. In order to compare the result with later ones we return to complex coordinates (cf. chapter 2). After partial integration the ω-dependent terms can be written as

$$\begin{aligned}\Gamma^{(\omega)}_{R,1} &= -\frac{1}{2\pi}\int_{\mathbb{H}} \mathrm{d}^2z\,\omega\,\partial\zeta^\mu\bar\partial\zeta^\nu R_{\mu\nu} - \\ &\quad - \frac{1}{12\pi}\int_{\partial\mathbb{H}} \mathrm{d}\tau \left\{\frac{1}{2}\partial_n\omega\,\zeta^\mu\zeta^\nu R_{\mu\nu} - 2\omega\,\partial_n\zeta^\mu\zeta^\nu R_{\mu\nu}\right\}\ .\end{aligned} \tag{4.12}$$

We postpone the interpretation of (4.12) to a later section when all contributions to the Weyl anomaly are known.

For the remaining 2-form vertices

$$\frac{1}{4\pi\alpha'}\int_{\mathbb{H}}\mathrm{d}^n\sigma\, e^{-\epsilon\omega} i\tilde{\epsilon}^{ab}\partial_a\zeta^\mu\partial_b\zeta^\nu\Big\{\zeta^\rho\partial_\rho\mathcal{F}_{\mu\nu}+\zeta^\rho\zeta^\sigma\frac{1}{2}\partial_\rho\partial_\sigma\mathcal{F}_{\mu\nu})\Big\} \qquad (4.13)$$

we apply the same procedure and obtain the Weyl symmetry breaking terms

$$\begin{aligned}\Gamma^{(\omega)}_{\mathcal{F},1} &= \frac{1}{4\pi}\int_{\mathbb{H}}\mathrm{d}^2z\,\omega\,\partial\zeta^\mu\bar{\partial}\zeta^\nu\big(\eta^{\rho\sigma}\partial_\rho H_{\sigma\mu\nu}\big)+ \qquad (4.14)\\ &+\frac{i}{4\pi}\int_{\partial\mathbb{H}}\mathrm{d}\tau\,\omega\,\partial_\tau\zeta^\mu\big(\eta^{\rho\sigma}\partial_\rho\mathcal{F}_{\sigma\mu}+\zeta^\nu\partial_\nu\eta^{\rho\sigma}\partial_\rho\mathcal{F}_{\sigma\mu}\big)\ .\end{aligned}$$

4.4 The Homogeneous Part of the Propagator

As mentioned in the introduction of this chapter the main new feature of the open string model is the appearence of divergences and anomalies that are located at the boundary. These infinities are related to the homogeneous part of the propagator (3.7). The coincidence limits thereof, i.e.,

$$\begin{aligned}<\zeta^\mu(z,\bar z)\,\zeta^\nu(z,\bar z)>_{\text{hom}} &= -\alpha'\tilde{G}^{\mu\nu}\ln|z-\bar z|^2 \quad ,\\ <\zeta^\mu(z,\bar z)\,\partial\zeta^\nu(z,\bar z)>_{\text{hom}} &= -\alpha'\tilde{G}^{\mu\nu}\frac{1}{z-\bar z}-\alpha'\Theta^{\mu\nu}\frac{1}{z-\bar z} \quad , \qquad (4.15)\\ <\zeta^\mu(z,\bar z)\,\bar\partial\zeta^\nu(z,\bar z)>_{\text{hom}} &= +\alpha'\tilde{G}^{\mu\nu}\frac{1}{z-\bar z}-\alpha'\Theta^{\mu\nu}\frac{1}{z-\bar z} \quad \text{and}\\ <\partial\zeta^\mu(z,\bar z)\,\bar\partial\zeta^\nu(z,\bar z)>_{\text{hom}} &= -\alpha'\tilde{G}^{\mu\nu}\frac{1}{(z-\bar z)^2}+\alpha'\Theta^{\mu\nu}\frac{1}{(z-\bar z)^2}\ ,\end{aligned}$$

are finite inside the world sheet but divergent at the boundary and require therefore an appropriate regularization. For convenience, we introduced the appreviation $\tilde{G}^{\mu\nu}=G^{\mu\nu}-\frac{1}{2}\eta^{\mu\nu}$.

4.5 Once More the Regularization of Boundary Divergences

We reapply the same method as in section 3.2 and restrict z to the domain

$$\mathbb{H}_{(d_0)} := \big\{z\in\mathbb{H}\ \big|\ 2\mathrm{Im}(z)\geq e^{-\omega}d_0\big\}\ , \qquad (4.16)$$

where d_0 is a small displacement. The weight $e^{-\omega}$ appears because we have chosen the metric in conformal gauge. This regularization scheme will be illustrated for the diagrams arising from the curvature 4-point vertex

$$-\frac{1}{6\pi\alpha'}\int_{\mathbb{H}_{(d)}} \mathrm{d}^2 z \partial\zeta^\mu \bar{\partial}\zeta^\nu \zeta^\rho \zeta^\sigma R_{\mu\rho\nu\sigma} \ . \tag{4.17}$$

Using the contractions (4.15) we obtain

$$\begin{aligned}\Gamma_{R,2} = \frac{1}{6\pi}\int_{\mathbb{H}_{(d)}} \mathrm{d}^2 z \Big\{ & \frac{1}{(z-\bar{z})^2}\,\zeta^\mu\zeta^\nu(\tilde{G}R)_{\mu\nu} \\ & -\frac{1}{z-\bar{z}}\,\zeta^\mu\bar{\partial}\zeta^\nu(\tilde{G}R)_{\mu\nu} \\ & +\frac{1}{z-\bar{z}}\,\partial\zeta^\mu\zeta^\nu(\tilde{G}R)_{\mu\nu} \\ & +\ln|z-\bar{z}|^2\,\partial\zeta^\mu\bar{\partial}\zeta^\nu(\tilde{G}R)_{\mu\nu} \\ & +\frac{2}{z-\bar{z}}\,\bar{\partial}\zeta^\nu\zeta^{(\rho}\Theta^{\sigma)\mu}R_{\mu\rho\nu\sigma} \\ & +\frac{2}{z-\bar{z}}\,\partial\zeta^\mu\zeta^{(\rho}\Theta^{\sigma)\nu}R_{\mu\rho\nu\sigma}\Big\} \quad ,\end{aligned} \tag{4.18}$$

where $(\tilde{G}R)_{\mu\nu} = \tilde{G}^{\rho\sigma}R_{\mu\rho\nu\sigma}$. If we consider (4.18) we recognize the appearance of different boundary divergences in the integrand: a logarithmic one as well as a linear and a quadratic one. However, the former is not that bad. Althought divergent, it is integrable if one performs two further contractions including the remaining fields $\partial\zeta^\mu$ and $\bar{\partial}\zeta^\nu$. In fact, it leads to a finite result. For the later two types we observe the following behaviour. The integral as a whole is divergent, whereas the integrand is singular only at the boundary. So one can try to split the integral into a finite intergral over the bulk $\mathbb{H}_{(d_0)}$ and a integral over the boundary $\partial\mathbb{H}_{(d_0)}$ with a divergent argument as $d_0 \to 0$. Take, for instance, the quadratic term in (4.18). By partial integration we get

$$\begin{aligned} & \frac{1}{3\pi}\int_{\mathbb{H}} \mathrm{d}^2 z \ln|z-\bar{z}|\,\partial\zeta^\mu\bar{\partial}\zeta^\nu(\tilde{G}R)_{\mu\nu} + \\ + \ & \frac{i}{6\pi}\Big\{\int_{\partial\mathbb{H}_{(d_0)}} \mathrm{d}z \ln|z-\bar{z}|\,\partial\zeta^\mu\zeta^\nu - \int_{\partial\mathbb{H}_{(d_0)}} \mathrm{d}\bar{z}\ln|z-\bar{z}|\,\bar{\partial}\zeta^\mu\zeta^\nu\Big\}(\tilde{G}R)_{\mu\nu} - \\ - \ & \frac{i}{12\pi}\Big\{\int_{\partial\mathbb{H}_{(d_0)}} \mathrm{d}z\frac{1}{z-\bar{z}}\zeta^\mu\zeta^\nu + \int_{\partial\mathbb{H}_{(d_0)}} \mathrm{d}\bar{z}\frac{1}{z-\bar{z}}\zeta^\mu\zeta^\nu\Big\}(\tilde{G}R)_{\mu\nu} \ .\end{aligned} \tag{4.19}$$

So, with this method we are able to isolate the boundary divergences from an integrable bulk contribution in the first line.

Since we are interested in the Weyl anomaly, we keep only the divergent and ω-dependent part in the integrals over the boundary and therefore substitute $(z-\bar{z})\big|_{\partial\mathbb{H}} = ie^{-\omega}d_0$. Lines two and three in (4.19) give

$$\begin{aligned} &- \frac{1}{6\pi}\int_{\partial\mathbb{H}} \mathrm{d}\tau(\ln d_0 - \omega)\,\partial_n\zeta^\mu\zeta^\nu(\tilde{G}R)_{\mu\nu} - \\ &- \frac{1}{6\pi}\frac{1}{d_0}\int_{\partial\mathbb{H}} \mathrm{d}\tau e^{\omega}\zeta^\mu\zeta^\nu(\tilde{G}R)_{\mu\nu}\ . \end{aligned} \tag{4.20}$$

The other terms in (4.18) can be treated in the same way and one obtains a finite, a divergent and a ω-dependent contribution. In order to get managable parts we split $\Gamma_{R,2}$ into

$$\Gamma_{R,2} = \Gamma^{(fin)}_{R,2} + \Gamma^{(div)}_{R,2} + \Gamma^{(\omega)}_{R,2}\ . \tag{4.21}$$

Similarly, we consider the vertices

$$\frac{1}{2\pi\alpha'}\int_{\mathbb{H}} \mathrm{d}^2z\,\partial\zeta^\mu\bar{\partial}\zeta^\nu\Big\{\zeta^\rho\,\partial_\rho\mathcal{F}_{\mu\nu} + \zeta^\rho\zeta^\sigma\,\frac{1}{2}\partial_\rho\partial_\sigma\mathcal{F}_{\mu\nu}\Big\}, \tag{4.22}$$

and, accordingly, use the split

$$\Gamma_{\mathcal{F},2} = \Gamma^{(fin)}_{\mathcal{F},2} + \Gamma^{(div)}_{\mathcal{F},2} + \Gamma^{(\omega)}_{\mathcal{F},2}\ . \tag{4.23}$$

For the sake of completeness we present the finite terms, althought not interesting for our purposes,

$$\begin{aligned} \Gamma^{(fin)} &= \Gamma^{(fin)}_{R,2} + \Gamma^{(fin)}_{\mathcal{F},2} = \\ &= \frac{1}{\pi}\int_{\mathbb{H}} \mathrm{d}^2z\ln|z-\bar{z}|\,\partial\zeta^\mu\bar{\partial}\zeta^\nu\Big\{(\tilde{G}R)_{\mu\nu} - \frac{1}{3}(\Theta R)_{\mu\nu}\Big\} + \\ &+ \frac{1}{2\pi}\int_{\mathbb{H}} \mathrm{d}^2z\ln|z-\bar{z}|\,\partial\zeta^\mu\bar{\partial}\zeta^\nu\Big\{\Theta^{\rho\sigma}\partial_{(\mu}H_{\nu)\rho\sigma} - \tilde{G}^{\rho\sigma}\partial_\rho H_{\sigma\mu\nu}\Big\}\ , \end{aligned} \tag{4.24}$$

where $(\Theta R)_{\mu\nu} = \Theta^{\rho\sigma}(R_{\mu\nu\rho\sigma} + R_{\mu\rho\nu\sigma})$.

4.6 Counterterms and Weyl Anomaly II

As already stated in section 4.3 the divergent contributions $\Gamma_2^{(div)} = \Gamma_{\mathcal{F},2}^{(div)} + \Gamma_{R,2}^{(div)}$ must be compensated by appropriate counterterms. Applying the methods of the previous section gives

$$
\begin{aligned}
\Gamma_2^{(div)} \;=\; & -\frac{1}{3\pi}\ln d_0 \int_{\partial\mathbb{H}} \mathrm{d}\tau\, \partial_n\zeta^\mu\zeta^\nu(\tilde{G}R)_{\mu\nu} \\
& -\frac{i}{6\pi}\ln d_0 \int_{\partial\mathbb{H}} \mathrm{d}\tau\, \partial_\tau\zeta^\mu\zeta^\nu(\Theta R)_{\mu\nu} \\
& -\frac{i}{2\pi}\ln d_0 \int_{\partial\mathbb{H}} \mathrm{d}\tau\, \partial_\tau\zeta^\mu(\partial_\rho\mathcal{F}_{\sigma\mu}\tilde{G}^{\rho\sigma} + \zeta^\nu\partial_\nu\partial_\rho\mathcal{F}_{\sigma\mu}\tilde{G}^{\rho\sigma}) \\
& -\frac{1}{4\pi}\ln d_0 \int_{\partial\mathbb{H}} \mathrm{d}\tau\, \partial_n\zeta^\mu(H_{\mu\rho\sigma}\Theta^{\rho\sigma} + \zeta^\nu\partial_\nu H_{\mu\rho\sigma}\Theta^{\rho\sigma})\ .
\end{aligned}
\tag{4.25}
$$

In the following Weyl anomalous parts we also included the divergent tachyon like contributions since the compensation of the divergence by a counterterm may leave a finite ω-dependence. Thus we have

$$
\begin{aligned}
\Gamma_{R,2}^{(\omega)} = \; & -\frac{1}{6\pi}\frac{1}{d_0}\int_{\partial\mathbb{H}} \mathrm{d}\tau\, e^\omega\zeta^\mu\zeta^\nu(\tilde{G}R)_{\mu\nu} \\
& +\frac{1}{3\pi}\int_{\partial\mathbb{H}} \mathrm{d}\tau\,\omega\,\partial_n\zeta^\mu\zeta^\nu(\tilde{G}R)_{\mu\nu} \\
& +\frac{i}{6\pi}\int_{\partial\mathbb{H}} \mathrm{d}\tau\,\omega\,\partial_\tau\zeta^\mu\zeta^\nu(\Theta R)_{\mu\nu}
\end{aligned}
\tag{4.26}
$$

and

$$
\begin{aligned}
\Gamma_{\mathcal{F},2}^{(\omega)} \;=\; & -\frac{1}{2\pi}\frac{1}{d_0}\int_{\partial\mathbb{H}} \mathrm{d}\tau\, e^\omega(\zeta^\mu\partial_\mu\mathcal{F}_{\rho\sigma}\Theta^{\rho\sigma} + \frac{1}{2}\zeta^\mu\zeta^\nu\partial_\mu\partial_\nu\mathcal{F}_{\rho\sigma}\Theta^{\rho\sigma}) + \\
& +\frac{i}{2\pi}\int_{\partial\mathbb{H}} \mathrm{d}\tau\,\omega\,\partial_\tau\zeta^\mu(\partial_\rho\mathcal{F}_{\sigma\mu}\tilde{G}^{\rho\sigma} + \zeta^\nu\partial_\nu\partial_\rho\mathcal{F}_{\sigma\mu}\tilde{G}^{\rho\sigma}) \\
& +\frac{1}{4\pi}\int_{\partial\mathbb{H}} \mathrm{d}\tau\,\omega\,\partial_n\zeta^\mu(H_{\mu\rho\sigma}\Theta^{\rho\sigma} + \zeta^\nu\partial_\nu H_{\mu\rho\sigma}\Theta^{\rho\sigma})\ .
\end{aligned}
\tag{4.27}
$$

4.7 The Space-Time Equations of Motion

Now we have calculated all Weyl anomalies arising from the diagrams in figure 4.1 and are ready to interpret the results. First, we recognize that the homogeneous part of the propagator gave only rise to Weyl anomalies on the boundary, as we have already mentioned earlier. Whereas the inhomogeneous part generated both, boundary and bulk anomalies. For easier reference we summarize the results (4.12,4.14,4.26,4.27),

$$\begin{aligned}
\Gamma^{(\omega)} &= -\frac{1}{2\pi}\int_{\mathbb{H}} \mathrm{d}^2 z\, \omega\, \partial\zeta^\mu \bar{\partial}\zeta^\nu \big(R_{\mu\nu} - \frac{1}{2}\eta^{\rho\sigma}\partial_\rho H_{\sigma\mu\nu}\big) \\
&- \frac{1}{24\pi}\int_{\partial\mathbb{H}} \mathrm{d}\tau\, \partial_n\omega\, \zeta^\mu\zeta^\nu R_{\mu\nu} + \frac{1}{12\pi}\frac{1}{d_0}\int_{\partial\mathbb{H}} \mathrm{d}\tau\, e^\omega \zeta^\mu\zeta^\nu R_{\mu\nu} \\
&+ \frac{1}{3\pi}\int_{\partial\mathbb{H}} \mathrm{d}\tau\, \omega\, \partial_n\zeta^\mu\zeta^\nu (GR)_{\mu\nu} \\
&+ \frac{i}{6\pi}\int_{\partial\mathbb{H}} \mathrm{d}\tau\, \omega\, \partial_\tau\zeta^\mu\zeta^\nu (\Theta R)_{\mu\nu} \\
&- \frac{1}{2\pi}\frac{1}{d_0}\int_{\partial\mathbb{H}} \mathrm{d}\tau\, e^\omega \big(\zeta^\mu\partial_\mu\mathcal{F}_{\rho\sigma}\Theta^{\rho\sigma} + \zeta^\mu\zeta^\nu\frac{1}{2}(\partial_\mu\partial_\nu\mathcal{F}_{\rho\sigma}\Theta^{\rho\sigma} + \frac{2}{3}(GR)_{\mu\nu})\big) \\
&+ \frac{i}{2\pi}\int_{\partial\mathbb{H}} \mathrm{d}\tau\, \omega\, \partial_\tau\zeta^\mu (\partial_\rho\mathcal{F}_{\sigma\mu}G^{\rho\sigma} + \zeta^\nu\partial_\nu\partial_\rho\mathcal{F}_{\sigma\mu}G^{\rho\sigma}) \\
&+ \frac{1}{4\pi}\int_{\partial\mathbb{H}} \mathrm{d}\tau\, \omega\, \partial_n\zeta^\mu (H_{\mu\rho\sigma}\Theta^{\rho\sigma} + \zeta^\nu\partial_\nu H_{\mu\rho\sigma}\Theta^{\rho\sigma})\ . \qquad (4.28)
\end{aligned}$$

At first sight their are some terms with no obvious meaning. For instance, there appear anomalies which contain a derivative of the field $\partial_n\zeta$, but we did not introduce a vertex operator like

$$\int_{\partial\mathbb{H}} \mathrm{d}\tau\, \partial_n X^\mu V_\mu(X)\ . \qquad (4.29)$$

However, we can use the boundary conditions (2.6) for the quantum fields, i.e.,

$$\partial_n\zeta^\mu\big|_{\bar{z}=z} = -i\mathcal{F}^\mu{}_\nu\partial_\tau\zeta^\nu\big|_{\bar{z}=z} + \dots\ , \qquad (4.30)$$

and rewrite the normal derivative as tangential derivative.

Furthermore, we have to take into consideration that in section 2.2 we have introduced Riemannian normal coordinates. This means that in equation (4.28) the lines three and four and the last expression in line five could

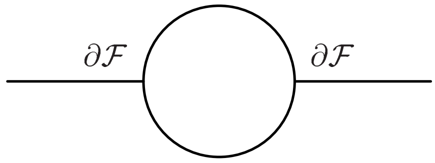

Figure 4.2: The $(\partial\mathcal{F})^2$-diagram was not taken into consideration.

contribute to covariant derivatives. Indeed using formulas (2.14 - 2.18) one can show

$$\begin{aligned} G^{\rho\sigma} D_\nu D_\rho \mathcal{F}_{\sigma\mu} &= G^{\rho\sigma}(\partial_\nu\partial_\rho\mathcal{F}_{\sigma\mu} - \partial_\nu\Gamma^\lambda{}_{\rho\sigma}\mathcal{F}_{\lambda\mu} + \partial_\nu\Gamma^\lambda{}_{\rho\mu}\mathcal{F}_{\lambda\sigma}) = \\ &= G^{\rho\sigma}\partial_\nu\partial_\rho\mathcal{F}_{\sigma\mu} - \frac{2}{3}(GR)_{\lambda\nu}\mathcal{F}^\lambda{}_\mu + \frac{1}{3}(\Theta R)_{\mu\nu} \end{aligned} \tag{4.31}$$

and

$$\begin{aligned} \Theta^{\rho\sigma} D_\mu D_\nu \mathcal{F}_{\rho\sigma} &= \Theta^{\rho\sigma}(\partial_\mu\partial_\nu\mathcal{F}_{\rho\sigma} - 2\partial_\mu\Gamma^\lambda{}_{\nu\rho}\mathcal{F}_{\lambda\sigma}) = \\ &= \Theta^{\rho\sigma}\partial_\mu\partial_\nu\mathcal{F}_{\rho\sigma} + \frac{2}{3}(GR)_{\mu\nu} - \frac{2}{3}R_{\mu\nu}\,. \end{aligned} \tag{4.32}$$

If we apply the boundary condition and take advantage of the Riemannian normal coordinates, we see the covariant structure of (4.28),

$$\begin{aligned} \Gamma^{(\omega)} &= -\frac{1}{2\pi}\int_{\mathbb{H}} \mathrm{d}^2 z\,\omega\,\partial\zeta^\mu\bar\partial\zeta^\nu(R_{\mu\nu} - \frac{1}{2}D^\rho H_{\rho\mu\nu}) + \qquad (4.33) \\ &- \frac{1}{24\pi}\int_{\partial\mathbb{H}} \mathrm{d}\tau\,\partial_n\omega\,\zeta^\mu\zeta^\nu R_{\mu\nu} - \frac{1}{12\pi}\frac{1}{d_0}\int_{\partial\mathbb{H}} \mathrm{d}\tau\, e^\omega\zeta^\mu\zeta^\nu R_{\mu\nu} \\ &- \frac{1}{2\pi}\frac{1}{d_0}\int_{\partial\mathbb{H}} \mathrm{d}\tau\, e^\omega\big(\zeta^\mu D_\mu\mathcal{F}_{\rho\sigma}\Theta^{\rho\sigma} + \zeta^\mu\zeta^\nu\frac{1}{2}D_\mu D_\nu\mathcal{F}_{\rho\sigma}\Theta^{\rho\sigma}\big) \\ &+ \frac{i}{2\pi}\int_{\partial\mathbb{H}} \mathrm{d}\tau\,\omega\,\partial_\tau\zeta^\mu\Big\{(G^{\rho\sigma}D_\rho\mathcal{F}_{\sigma\mu} - \frac{1}{2}\Theta^{\rho\sigma}H_{\lambda\rho\sigma}\mathcal{F}^\lambda{}_\mu) \\ &\qquad + \zeta^\nu(G^{\rho\sigma}D_\nu D_\rho\mathcal{F}_{\sigma\mu} - \frac{1}{2}\Theta^{\rho\sigma}D_\nu H_{\lambda\rho\sigma}\mathcal{F}^\lambda{}_\mu)\Big\}\,. \end{aligned}$$

Mind that we have not taken into account all possible 1-loop diagrams. The diagram shown in figure 4.2 which is built of two 3-point vertices was not treated. On the bulk it gives rise to the well known H^2-term. At the boundary the missing terms of second order in derivatives combine with terms available in the last line of (4.33) to give $D_\nu(G^{\rho\sigma}D_\rho\mathcal{F}_{\sigma\mu} - \frac{1}{2}\Theta^{\rho\sigma}H_{\lambda\rho\sigma}\mathcal{F}^\lambda{}_\mu)$, the second term in a Taylor series expansion of the $\partial_\tau\zeta^\mu$ anomaly [22, 23].

The same holds for the third line of (4.33). This contribution reminds of a tachyon vertex and can indeed be absorbed by the tachyon in terms of a field redefinition (cf. the discussion of section 3.2) [19]. Note that the first term in this line vanishes because of condition (2.9) which was introduced to ensure a unique separation of the constant zero mode and the quantum fluctuation.

The remaining contributions cannot be removed by appropriate field redefinitions and must be set to zero in order to maintain the conformal invariance of the theory, i.e.,

$$\begin{aligned} R_{\mu\nu} - \frac{1}{4}H^2_{\mu\nu} &= 0 \, , \\ D^\rho H_{\rho\mu\nu} &= 0 \, , \\ G^{\rho\sigma}D_\rho\mathcal{F}_{\sigma\mu} - \frac{1}{2}\mathcal{F}^\lambda{}_\mu\Theta^{\rho\sigma}H_{\lambda\rho\sigma} &= 0 \, , \end{aligned} \tag{4.34}$$

where we have added the H^2-term arising from the diagram in figure 4.2. These are the equations of motion for the background fields: the Einstein equation for $g_{\mu\nu}$, the equation of motion for $B_{\mu\nu}$, and the non-linear Maxwell equation for A_μ. In chapter 6 the latter will turn out to be very important for the properties of the non-commutative product defined in section 5.3.

Chapter 5

Correlation Functions

In string theory interactions of different particles of the string spectrum are calculated by inserting the corresponding vertex operators in the path integral. Our goal is to extract a non-commutative product of functions out of the open string theory correlation functions [11]. To this end we do not restrict ourself to vertex operators, but investigate the correlator of two general functions $f[X(\tau)]$ and $g[X(\tau')]$ allowed to be off-shell. To simplify the calculations we take the order of insertions to be $\tau < \tau'$. Since the functions are composite operators, one has to introduce an appropriate normal ordering. As shown in the appendix 7 it is given by

$$\begin{aligned} :\zeta^\mu(\tau)\,\zeta^\nu(\tau'): \;=\; & \zeta^\mu(\tau)\,\zeta^\nu(\tau') \\ & + \frac{1}{2\pi} G^{\mu\nu} \ln(\tau - \tau')^2 + \frac{1}{2\pi} \partial_\rho G^{\mu\nu} \ln(\tau - \tau')^2 \; \zeta^\rho\!\left(\frac{\tau + \tau'}{2}\right). \end{aligned} \tag{5.1}$$

5.1 Moyal-Weyl Contribution

Taking into account the subtractions (5.1) the free propagator (3.9) yields [4]

$$\begin{aligned}
&\langle :f[X(\tau)]: :g[X(\tau')]: \rangle_{\text{Moyal}} = \\
&= \int_x \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{i}{2}\right)^n \Delta^{\mu_1\nu_1}\dots\Delta^{\mu_n\nu_n}\, \partial_{\mu_1}\dots\partial_{\mu_n} f(x)\, \partial_{\nu_1}\dots\partial_{\nu_n} g(x) \qquad (5.2)\\
&= \int_x \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{-1}{2\pi}\right)^n G^{\mu_1\nu_1}..\, G^{\mu_n\nu_n} \ln^n(\tau-\tau')^2\, \partial_{\mu_1}..\,\partial_{\mu_n} f(x) \,*\, \partial_{\nu_1}..\,\partial_{\nu_n} g(x).
\end{aligned}$$

In the last line we have summarized all $\Theta^{\mu\nu}$-dependent contributions in the product

$$\begin{aligned}
f * g &= \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{i}{2}\right)^n \Theta^{\mu_1\nu_1}(x)\dots\Theta^{\mu_n\nu_n}(x)\, \partial_{\mu_1}\dots\partial_{\mu_n} f(x)\, \partial_{\nu_1}\dots\partial_{\nu_n} g(x) \\
&= e^{\frac{i}{2}\Theta^{\mu\nu}(z)\partial_{x^\mu}\partial_{y^\nu}} f(x)g(y)\Big|_{x=y=z}, \qquad (5.3)
\end{aligned}$$

which we will refer to as “Moyal like” part of the final non-commutative product. It has the well known structure of the Moyal product and reduces to it if $\Theta^{\mu\nu}$ is constant. In this case (5.3) is clearly associative and satisfies the trace property. This is, however, no longer true, if $\Theta^{\mu\nu}$ is a generic field.

5.2 First Derivative Contribution

Going one step further in the derivative expansion we have to take into account the contribution to the non-commutative product arising from the interaction term

$$\frac{1}{2\pi\alpha'} \int_{\mathbb{H}} \mathrm{d}^2 z\, \partial\zeta^\mu \bar{\partial}\zeta^\nu \zeta^\rho\, \partial_\rho \mathcal{F}_{\mu\nu}. \qquad (5.4)$$

The rather cumbersome calculations are explained in appendix 7. Using (B.16) and (B.17) we obtain

$$\begin{aligned}
\langle :f[X(\tau)]::g[X(\tau')]: \rangle_{\partial\mathcal{F}} \quad = \\
&-\frac{1}{12}\int_x \Theta^{\mu\rho}\partial_\rho\Theta^{\nu\sigma}\left(\partial_\mu\partial_\nu f * \partial_\sigma g + \partial_\sigma f * \partial_\mu\partial_\nu g\right) \qquad (5.5)\\
&-\frac{i}{8\pi}\int_x \Theta^{\mu\rho}\partial_\rho G^{\nu\sigma}\left(\partial_\nu\partial_\sigma f * \partial_\mu g - \partial_\mu f * \partial_\nu\partial_\sigma g\right)\ln(\tau-\tau')^2\\
&-\frac{i}{4\pi}\int_x G^{\nu\sigma}\partial_\sigma\Theta^{\rho\mu}\left(\partial_\nu\partial_\rho f * \partial_\mu g - \partial_\mu f * \partial_\nu\partial_\rho g\right)\ln(\tau-\tau')^2\\
&-\frac{1}{16\pi^2}\int_x G^{\mu\rho}\partial_\rho G^{\nu\sigma}\left(\partial_\nu\partial_\sigma f * \partial_\mu g + \partial_\mu f * \partial_\nu\partial_\sigma g\right)\ln^2(\tau-\tau')^2\\
&+\frac{1}{8\pi^2}\int_x G^{\nu\rho}\partial_\rho G^{\sigma\mu}\left(\partial_\nu\partial_\sigma f * \partial_\mu g + \partial_\mu f * \partial_\nu\partial_\sigma g\right)\ln^2(\tau-\tau')^2 + \ldots,
\end{aligned}$$

where we only kept the $\Theta^{\mu\nu}$ terms from the contributions of the free propagator (3.7), since the $G^{\mu\nu}$ parts are irrelevant for our further discussion as we shall see shortly. Again, the first line of (5.5) contributes to our non-commutative product. The partial derivatives of the fields imply that the whole expression (5.5) vanishes for constant fields.

5.3 Definition of the Non-commutative Product

We define now the non-commutative product as

$$\sqrt{g-\mathcal{F}}\; f(x) \;\circ\; g(x) \;\; := \;\; \int [\mathrm{d}\zeta]\, e^{-S[x+\zeta]} f[X(0)]\; g[X(1)]\,. \qquad (5.6)$$

The choice of the distance $\tau' - \tau = 1$ is such that the scale dependent contributions of (5.2) and (5.5) are removed.[1] The resulting non-commutative product is the scale and translation invariant part of the 2-point correlation. This product is independent of $G^{\mu\nu}$, and we will see that only this part of the correlation has appropriate off-shell properties (as long as the background

[1]The value 1 is due to our choice of the infrared cut-off, i.e., the constant $C^{\mu\nu}_{(\infty)}$ in (3.7).

fields are on-shell). The full off-shell correlations will, of course, also have $G^{\mu\nu}$-dependent contributions.

From (5.2) and (5.5) we see that, up to first order in derivatives of $\Theta^{\mu\nu}$, the product reads[2]

$$\begin{aligned} f(x)\ \circ\ g(x) \ &= \ f * g - \frac{1}{12}\Theta^{\mu\rho} D_\rho \Theta^{\nu\sigma} \Big(D_\mu D_\nu f \ * D_\sigma g + D_\sigma f \ * D_\mu D_\nu g \Big) + \\ &+ \ \mathcal{O}\big((D\Theta)^2, DD\Theta\big). \end{aligned} \tag{5.7}$$

Here we have reintroduced the covariant notation. This is justified because in Riemannian normal coordinates the Christoffel symbol vanishes and (5.7) contains no derivatives thereof. The same is true for the $G^{\mu\nu}$-dependent parts.

A comparison of (5.7) with the formula given in [8] shows that, apart from the covariant derivatives, the non-commutative product (5.6) coincides with the Kontsevich formula. We do not require, however, that the field $\Theta^{\mu\nu}$ defines a Poisson structure. Note that in [11] the $\Theta\partial\Theta$ terms contains only Kontsevich contributions. This is due to the choice of radial gauge.

[2]Subsequently we abbreviate $\mathcal{O}\big((D\Theta)^2, DD\Theta\big)$ by $\mathcal{O}(D^2)$.

Chapter 6

Properties of the Non-commutative Product

In the limit $\alpha' \to 0$ the correlator of an arbitrary number of functions in the presence of a closed B-field background can be evaluated by an integration over the non-commutative product of these functions. On the disk the $SL(2, \mathbb{R})$ invariance of the correlators requires the product to satisfy the trace property.

The non-commutative product (5.6) defined without the use of the limit, however, does not describe the full correlation functions, because the $G^{\mu\nu}$-dependent contractions give additional contributions. Even so, we will show in this chapter that the trace property can be maintained for the product (5.7) if one imposes the equations of motion for the background fields, whereas the inserted functions are allowed to stay completely generic.

6.1 On-shell Condition for the Background Fields

In string theory the background field equations of motion are related to the renormalization group β functions, which probe the breaking of Weyl invariance (and hence the conformal invariance) of the theory. Since we perform our calculations up to first order in derivatives of the background fields, we expect that we have to account for the generalization of the Maxwell equation [23, 26],

$$G^{\rho\sigma} D_\rho \mathcal{F}_{\sigma\mu} - \frac{1}{2}\Theta^{\rho\sigma} H_{\rho\sigma\lambda}\mathcal{F}^\lambda{}_\mu = 0. \tag{6.1}$$

To show our proposition we rewrite (6.1) in a more appropriate way,

$$\begin{aligned} \partial_\mu\Big(\sqrt{g-\mathcal{F}}\ \Theta^{\mu\nu}\Big) &= \sqrt{g} D_\mu\Big(\frac{\sqrt[4]{G}}{\sqrt[4]{g}}\ \Theta^{\mu\nu}\Big) = \\ &= -\sqrt{g-\mathcal{F}}\Big(G^{\rho\sigma} D_\rho \mathcal{F}_{\sigma\mu} - \frac{1}{2}\Theta^{\rho\sigma} H_{\rho\sigma\lambda}\mathcal{F}^\lambda{}_\mu\Big) G^{\mu\nu} = 0\ , \end{aligned} \tag{6.2}$$

where we have used the relation

$$\Theta^{\rho\sigma} D_\mu \mathcal{F}_{\rho\sigma} = -\frac{1}{2}\bar{G}^{\rho\sigma} D_\mu \bar{G}_{\rho\sigma} = \Gamma_{\mu\lambda}{}^\lambda - \bar{\Gamma}_{\mu\lambda}{}^\lambda, \tag{6.3}$$

and the fact that the quotient $\frac{\sqrt[4]{G}}{\sqrt[4]{g}}$ is a scalar. We introduce the usual notation $\approx$ for equivalence up to equations of motion. Note, furthermore, that in the following all relations are valid only up to first order in derivatives of $\Theta^{\mu\nu}$.

6.2 Trace Property I

We start with the product of two functions and show that (5.7) is symmetric under the integral

$$\int \mathrm{d}^D x \sqrt{g-\mathcal{F}}\ f\ \circ\ g\ \approx\ \int \mathrm{d}^D x \sqrt{g-\mathcal{F}}\ g\ \circ\ f. \tag{6.4}$$

This relation holds due to (6.1) and (6.2), because then the first order term in $\Theta^{\mu\nu}$ of (5.7) transforms into a total divergence,

$$\int \mathrm{d}^D x \sqrt{g-\mathcal{F}}\ \Theta^{\mu\nu}\ D_\mu f\ D_\nu g \approx \int \mathrm{d}^D x\ \partial_\mu\Big(\sqrt{g-\mathcal{F}}\ \Theta^{\mu\nu}\ f\ D_\nu g\Big) = 0, \tag{6.5}$$

and the remaining antisymmetric parts can be written as contributions of second order in derivatives. Notice that here and in the subsequent relations it is essential that the constant b_0 in the integration measure takes the value $\frac{1}{4}$ in order to produce the total divergence.

6.3 Associativity up to Surface Terms

For a general field $\Theta^{\mu\nu}$ the product (5.7) is not associative. But again applying (6.1,6.2) associativity, except for a surface term, is ensured for the product of three functions. To see this we calculate $(f \circ g) \circ h - f \circ (g \circ h)$. Using the formula

$$\begin{aligned} \partial_\rho\big(f * g\big)(x) &= (\partial_{x^\rho} + \partial_{y^\rho} + \partial_{z^\rho}) e^{\frac{i}{2}\Theta^{\mu\nu}(z)\partial_{x^\mu}\partial_{y^\nu}} f(x) g(y)\Big|_{x=y=z} \\ &= \partial_\rho f * g + f * \partial_\rho g + \frac{i}{2}\partial_\rho \Theta^{\mu\nu}\partial_\mu f * \partial_\nu g. \end{aligned} \tag{6.6}$$

for the product (5.3) we obtain

$$\begin{aligned} (f * g) * h &= [f * g * h] + \frac{1}{4}\Theta^{\mu\sigma} D_\sigma \Theta^{\nu\rho} D_\nu f * D_\rho g * D_\mu h + \mathcal{O}(D^2), \\ f * (g * h) &= [f * g * h] - \frac{1}{4}\Theta^{\mu\sigma} D_\sigma \Theta^{\nu\rho} D_\mu f * D_\nu g * D_\rho h + \mathcal{O}(D^2), \end{aligned} \tag{6.7}$$

where $[f * g * h]$ denotes the part with no derivatives acting on $\Theta^{\mu\nu}$. So the non-associativity reads

$$\begin{aligned} (f \circ g) \circ h - f \circ (g \circ h) &= \\ &= \frac{1}{6}\big(\Theta^{\mu\sigma} D_\sigma \Theta^{\nu\rho} + (\text{cycl.}^{\mu\nu\rho})\big)\,\frac{1}{6} D_\mu f * D_\nu g * D_\rho h + \mathcal{O}(D^2) \\ &= \frac{1}{6}\Theta^{\mu\sigma}\Theta^{\nu\lambda}\Theta^{\rho\kappa}\bar{H}_{\sigma\lambda\kappa}\, D_\mu f * D_\nu g * D_\rho h + \mathcal{O}(D^2). \end{aligned} \tag{6.8}$$

In the last line we have introduced the 3-form field $\bar{H} = \mathrm{d}(\Theta^{-1})$ that is associated with the inverse of $\Theta^{\mu\nu}$,

$$(\Theta^{-1})_{\mu\nu} = -(g - \mathcal{F})_{\mu\rho}(\mathcal{F}^{-1})^{\rho\sigma}(g + \mathcal{F})_{\sigma\nu} = (\mathcal{F} - g\mathcal{F}^{-1}g)_{\mu\nu}. \tag{6.9}$$

Therefore, associativity is obtained (even off-shell) if

$$\bar{H}_{\mu\nu\rho} = 0. \tag{6.10}$$

At this point we want to stress that we nowhere have employed the limit $\alpha' \to 0$ in our considerations, so that the "full" $\Theta^{\mu\nu}$ occurs in all the relations. This means that (6.10) is a generalization of the well known property that in the limit $\alpha' \to 0$ the product becomes associative if $H = 0$.

However, open string theory does not require such a restriction and we investigate again the effects of the equation of motion (6.1). From (6.8) we obtain immediately that

$$\begin{aligned} & \int \mathrm{d}^D x \sqrt{g-\mathcal{F}}\, \big((f \circ g) \circ h - f \circ (g \circ h)\big) = \qquad (6.11) \\ = \;& \frac{1}{6} \int \mathrm{d}^D x \sqrt{g-\mathcal{F}}\, \big(\Theta^{\mu\sigma}\Theta^{\nu\lambda}\Theta^{\rho\kappa} \bar{H}_{\sigma\lambda\kappa}\, D_\mu f * D_\nu g * D_\rho h\big) + \mathcal{O}(D^2) \approx \\ \approx \;& \frac{1}{6} \int \mathrm{d}^D x \partial_\mu \big(\sqrt{g-\mathcal{F}} \ldots\big)^\mu + \mathcal{O}(D^2) \;=\; 0, \end{aligned}$$

so that we are allowed to omit the brackets.

6.4 Trace Property II

For more than three functions we are allowed to leave out the outermost lying bracket. In the case of four functions we obtain, for instance, the relation

$$\int \mathrm{d}^D x \sqrt{g-\mathcal{F}}\, (f \circ g) \circ h \circ l = \int \mathrm{d}^D x \sqrt{g-\mathcal{F}}\, f \circ g \circ (h \circ l). \qquad (6.12)$$

Finally, taking into account (6.4) and (6.11) we see immediately that the trace property holds for an arbitrary number of functions,

$$\begin{aligned} & \int \mathrm{d}^D x \sqrt{g-\mathcal{F}}\, \Big((\ldots(f_1 \circ \ldots)) \circ f_{N-1}\Big) \circ f_N \approx \\ \approx \;& \int \mathrm{d}^D x \sqrt{g-\mathcal{F}}\, f_N \circ \Big((\ldots(f_1 \circ \ldots)) \circ f_{N-1}\Big) \approx \\ \approx \;& \int \mathrm{d}^D x \sqrt{g-\mathcal{F}}\, \Big(f_N \circ (\ldots(f_1 \circ \ldots))\Big) \circ f_{N-1} \approx \;\ldots \quad . \qquad (6.13) \end{aligned}$$

6.5 Comparison with Recent Work

We close this chapter with a remark on the relation to the recent work of Cornalba and Schiappa [11]. They considered the special case of a slowly varying background field B in radial gauge, i.e., $B_{\mu\nu}(x) = B_{\mu\nu} + \frac{1}{3} H_{\mu\nu\rho} x^\rho + \mathcal{O}(x^2)$, and a vanishing field strength F for their path integral analysis. Taking the topological limit, $g_{\mu\nu} \sim \epsilon \to 0$,[1] the above properties of the product were achieved by adjusting a constant $\mathcal{N}$ in the integration measure $\sqrt{B}\,(1+\mathcal{N}(B^{-1})^{\mu\nu}H_{\mu\nu\rho}x^\rho)$. Using consistency arguments they determined the appropriate value of the constant to be $\mathcal{N}=\frac{1}{3}$.

However, dropping the radial gauge and repeating the calculations[2] of [11] for the trace property one obtains

$$-i\int\sqrt{B}[\mathcal{N}(B^{-1})^{\rho\sigma}(B^{-1})^{\mu\nu}-(B^{-1})^{\mu\sigma}(B^{-1})^{\rho\nu}]\partial_\mu B_{\sigma\rho}\,f\partial_\nu g. \qquad (6.14)$$

This expression does in general not vanish for any $\mathcal{N}$. Thus the trace property can not be restored by an appropriate choice for the constant $\mathcal{N}$ as it is possible for radial gauge, i.e., when $\partial_\mu B_{\sigma\rho}$ is replaced by $H_{\mu\sigma\rho}$!

On the other hand, expanding $B_{\mu\nu}(x)$ around its constant value and taking the topological limit in our setting, the Born-Infeld measure reduces to $\sqrt{B(x)}=\sqrt{B}\,(1+\frac{1}{2}(B^{-1})^{\mu\nu}\partial_\rho B_{\nu\mu}x^\rho)$. Then (6.14) can be recast into

$$\begin{aligned} - \quad & i\int\sqrt{B}\Big\{\frac{1}{2}(B^{-1})^{\rho\sigma}(B^{-1})^{\mu\nu}\partial_\mu B_{\sigma\rho} - \\ & \qquad -\frac{1}{2}(B^{-1})^{\mu\sigma}(B^{-1})^{\nu\rho}\partial_\rho B_{\mu\sigma}+\frac{1}{2}(B^{-1})^{\mu\sigma}(B^{-1})^{\nu\rho}H_{\mu\sigma\rho}\Big\}f\partial_\nu g \\ = \quad & \frac{i}{2}\int\sqrt{B}(B^{-1})^{\mu\sigma}H_{\mu\sigma\rho}(B^{-1})^{\rho\nu}f\partial_\nu g\ . \qquad (6.15) \end{aligned}$$

The last expression in square brackets in the second line is exactly what remains from the generalized Maxwell equation (6.1) in the topological limit, namely the constraint $(B^{-1})^{\rho\sigma}H_{\rho\sigma\lambda}=0$. So again the trace property holds, when the background fields are on-shell! Nevertheless, taking the topological limit mutilates the on-shell conditions in the sense that no dynamics is left and only a highly restrictive non-linear constraint remains. In dimensions up to four this constraint already implies the vanishing of the field strength H. Moreover, in the next order one has to take into account the beta function for the background metric, namely the Einstein equation, which imposes the

[1] Note that this limit is similar to the limit $\alpha'\to 0$ of Seiberg and Witten [7].

[2] Note that in this paragraph $B_{\sigma\rho}$ denotes the constant part of $B_{\sigma\rho}(x)$ and all dependencies on the zero modes are explicitly written.

even stronger restriction

$$R_{\mu\nu} - \frac{1}{4} H_{\mu\rho\sigma} H^{\rho\sigma}{}_{\nu} \sim -\frac{1}{4} H_{\mu\rho\sigma} H^{\rho\sigma}{}_{\nu} + \mathcal{O}(\epsilon^0) = 0, \qquad (6.16)$$

which enforces $H_{\mu\rho\sigma} = 0$ for any dimension (cf. [12]).[3] Hence the topological limit only seems to make sense considering the symplectic case.

[3]This can be seen by first setting $\mu = \nu = 0$ so that $H^2_{00} = 0$ and using the antisymmetry of H. This yields $H_{0ij} = 0$ for $i, j \neq 0$. The condition for purely spatial components follows immediately.

Chapter 7

Conclusion

On the world volume of a D-brane the product of functions (5.7) represents a non-associative deformation of a star product.[1] Nevertheless, it enjoys the properties that the integral acts as a trace and the product of three functions is associative up to total derivatives. This is accomplished by the equations of motion of the background fields (6.2) and the Born-Infeld measure. No on-shell conditions have to be imposed on the inserted functions! Note, however, that the product of four or more functions inserted in an integral is ambiguous if the brackets are omitted. This is due to the fact that associativity for three functions is valid only up to total derivatives. Only the outermost lying bracket may be omitted, but this suffices to ensure the trace property for an arbitrary number of functions.

Our results are correct up to first order in the derivative expansion of the background fields. In this approximation the influence of gravity amounts to the use of covariant derivatives in the generalized product (5.7) but the structure is still that of the formula given by Kontsevich. It would be interesting to investigate whether gravity induces a deviation from this structure at higher orders of the derivative expansion. One might also expect that higher order terms of the generalized Maxwell equation have to be used and even additional equations of motion must be imposed to maintain the properties of the product.

[1] Note that in the limit of vanishing gauge fields, $\mathcal{F}_{\mu\nu} \sim \epsilon \to 0$, the product reduces to the "ordinary" product of functions and the measure reduces to $\sqrt{g}$.

It would be furthermore interesting to address the question of how to use the open string non-commutative product and a perturbative operator product expansion in order to calculate correlation functions in general backgrounds. The property that the product of four or more functions is not unique without brackets seems related to the fact that these products are not independent of the moduli of the insertion points. For instance, in the case of four functions there are two distinct possibilities where to put the brackets, which coincides with the number of conformal blocks. This suggests that for higher n-point correlation functions one has to use linear combinations of the various orderings of the brackets weighted with coefficients depending on the moduli [13].

Since the correlation functions provide the S-matrix elements for scattering processes, the issue of finding the effective low energy field theory on D-branes is closely related and provides another motivation to calculate the correlators. Already the Born-Infeld measure denotes an important contribution for this issue.

Appendix A

The Dilogarithm

In appendix B we will encounter the dilogarithm [28]. As pointed out in ref. [11] it will play a prominent role in the calculation of correlation functions in open string theory. Therfore, we give definitions of the dilogarithm and related functions and cite some relations that will be used in appendix B.

A.1 Definition

Consider the series

$$\sum_{k=1}^{\infty} \frac{m^k}{k^2} \, , \tag{A.1}$$

which is only convergent for $|m| \leq 1$. One can find an integral representation of (A.1)

$$\mathrm{Li}_2(m) := -\int_0^m \mathrm{d}x \, \frac{\ln(1-x)}{x} \, . \tag{A.2}$$

$\mathrm{Li}_2(m)$ is called the dilogarithm. Although the series has a radius of convergence of 1 the integral (A.2) is not restricted to this limit. When m is real, $\mathrm{Li}_2(m)$ is well defined for $-\infty < m \leq 1$. For $m > 1$ the argument of the logarithm is negative and one has to use complex arguments and an appropriate position for the branch cut of the logarithm in order to assign a unique value to the dilogarithm. We choose the cut to be the negative real axis, so that we have

$$\ln(-x \pm i\epsilon) = \ln(x) \pm i\pi \quad x < 0 \ , \epsilon \to 0 \ . \tag{A.3}$$

Then the function[1]

$$\mathrm{Li}_2^{\pm}(m) := -\int_0^m \mathrm{d}x \, \frac{\ln(1 - x \pm i\epsilon)}{x} \tag{A.4}$$

is well defined for $m \in \mathbb{R}$.

A.2 Dilogarithm Relations

The dilogarithm contains a lot of symmetry relations. For instance, one can obtain the values of the function on the whole real axis just from the values in $0 < m < 1$,

$$\mathrm{Li}_2\Big(\frac{m}{m-1}\Big) = -\mathrm{Li}_2(m) - \frac{1}{2}\ln^2(1-m) \qquad \frac{m}{m-1} < 0 \ , \tag{A.5}$$

$$\mathrm{Li}_2^{\pm}\Big(\frac{1}{m}\Big) = \frac{\pi^2}{3} - \mathrm{Li}_2(m) - \frac{1}{2}\ln^2(m) \pm i\pi \ln(m) \qquad \frac{1}{m} > 1 \ . \tag{A.6}$$

These relations follow from simple substitutions in the integral representation (A.4). Equation (A.6) suggests a continuation of the dilogarithm to the whole real axis using the values of the function in $0 < m < 1$ by

$$\mathrm{Li}_2\Big(\frac{1}{m}\Big) := \frac{\pi^2}{3} - \mathrm{Li}_2(m) - \frac{1}{2}\ln^2(m) \ . \tag{A.7}$$

In the remainder of the appendix $\mathrm{Li}_2(m)$ for $m \in \mathbb{R}$ denotes the analytic continuation in terms of definition (A.7).

In appendix B it will turn out that the sum and the difference of $\mathrm{Li}_2(m)$ and $\mathrm{Li}_2(1-m)$ appear in the 3-point Greens function. The sum is just given by an expression of logarithms

$$\mathrm{Li}_2(m) + \mathrm{Li}_2(1-m) = \frac{\pi^2}{6} - \ln(m)\ln(1-m) \quad 0 < m < 1 \ . \tag{A.8}$$

[1] We leave the sign for the imaginary part of (A.4) open since later on we will encounter both conventions. We write the $\pm$ in $\mathrm{Li}_2^{\pm}(m)$ only if decisive, i.e., if $m > 1$.

And we abbreviate the difference by the function[2]

$$L^{(\text{-})}(m) := \mathrm{Li}_2(m) - \mathrm{Li}_2(1-m)\ . \tag{A.9}$$

Since we used the continuation (A.7) for $\mathrm{Li}_2(m)$, $L^{(\text{-})}(m)$ is well defined for $m \in \mathbb{R}$. Using equations (A.5), (A.8) and (A.7) one finds

$$\begin{aligned} L^{(\text{-})}\Big(\frac{m}{m-1}\Big) &= -L^{(\text{-})}(m) - \frac{\pi^2}{3}\ , &\text{(A.10)}\\ L^{(\text{-})}\Big(\frac{1}{m}\Big) &= -L^{(\text{-})}(m) + \frac{\pi^2}{3}\ , &\text{(A.11)}\end{aligned}$$

for $0 < m < 1$. It is by definition antisymmetric about $\frac{1}{2}$ and for $m \geq \frac{1}{2}$ it has the following special values:

$$\begin{aligned} L^{(\text{-})}\Big(\frac{1}{2}\Big) = 0 \quad &, \quad L^{(\text{-})}(1) = +\frac{\pi^2}{6}\ ,\\ L^{(\text{-})}(2) = +\frac{\pi^2}{3} \quad &, \quad L^{(\text{-})}(+\infty) = +\frac{\pi^2}{2}\ . \end{aligned} \tag{A.12}$$

[2]The function $L^{(\text{-})}(m)$ is closely related to Rogers dilogarithm $L(m)$, i.e., $L(m) = \frac{\pi^2}{12} + \frac{1}{2}L^{(\text{-})}(m)$.

Appendix B

The Contribution of the 3-Point Vertex

In the following we give an explicit calculation of the tree level contribution of the interaction term (5.4), i.e., the 3-point function $\langle\!\langle \zeta^\mu(\tau_i)\zeta^\nu(\tau_j)\zeta^\rho(\tau_k)\rangle\!\rangle$, which is needed in chapter 5. There we derive the correlator of two functions f and g. These functions contain an arbitrary power of quantum fluctuations ζ^μ. Therefore, the correlator has also contributions from 3-point Greens functions with two coinciding quantum fields ζ^μ: $\lim_{\tau_j \to \tau_i}\langle \zeta^\mu(\tau_i)\zeta^\nu(\tau_j)\zeta^\rho(\tau_k)\rangle$ and $\lim_{\tau_j \to \tau_k}\langle \zeta^\mu(\tau_i)\zeta^\nu(\tau_j)\zeta^\rho(\tau_k)\rangle$. The coincidence limits consist of both finite and divergent terms, which need different treatments. The divergent ones must be compensated by appropriate subtractions, which are accounted for in the normal ordering of the inserted functions, whereas the finite ones contribute explicitly to the correlator.

We will start with the introduction of convenient notations following [11]. Thereafter, we derive the Greens function $\langle \zeta^\mu(\tau_i)\zeta^\nu(\tau_j)\zeta^\rho(\tau_k)\rangle$, which needs a regularization similar to the propagator (3.3). We will see that the result is a generalization of the one in [11], because we do not use the limit $\alpha' \to 0$ and the radial gauge. Finally, we perform the coincidence limits to obtain the correct normal ordering and the finite contributions to the correlator.

B.1 Convenient Notations and Useful Relations

The free propagator (3.7) with one side connected to the boundary is

$$\langle \zeta^\mu(\tau_i)\, \zeta^\nu(z,\bar{z})\rangle = -\frac{1}{2\pi}\big(G^{\mu\nu}\mathcal{S}(\tau_i,z) - \Theta^{\mu\nu}\mathcal{A}(\tau_i,z)\big), \tag{B.1}$$

where $\mathcal{A}_i$ and $\mathcal{S}_i$ are defined as

$$\mathcal{A}_i = \mathcal{A}(\tau_i,z) = \ln\Big(\frac{\bar{z}-\tau_i}{\bar{\tau}_i - z}\Big) \quad \text{and} \quad \mathcal{S}_i = \mathcal{S}(\tau_i,z) = \ln|\tau_i - z|^2. \tag{B.2}$$

Note that $\mathcal{A}_i$ is an antisymmetric function in τ_i and z, whereas $\mathcal{S}_i$ is symmetric, i.e., $\mathcal{A}(\tau_i,z) = -\mathcal{A}(z,\tau_i)$ and $\mathcal{S}(\tau_i,z) = \mathcal{S}(z,\tau_i)$. From (B.2) we see that $\mathcal{A}_i$ and $\mathcal{S}_i$ satisfy the relations $\partial\mathcal{S}_i = -\partial\mathcal{A}_i$ and $\bar{\partial}\mathcal{S}_i = \bar{\partial}\mathcal{A}_i$. Therefore, we get

$$\begin{aligned}\langle \zeta^\mu(\tau_i)\, \partial\zeta^\nu(z,\bar{z})\rangle &= \frac{1}{2\pi}\big(\Theta^{\mu\nu} + G^{\mu\nu}\big)\, \partial\mathcal{A}_i \\ \langle \zeta^\mu(\tau_i)\, \bar{\partial}\zeta^\nu(z,\bar{z})\rangle &= \frac{1}{2\pi}\big(\Theta^{\mu\nu} - G^{\mu\nu}\big)\, \bar{\partial}\mathcal{A}_i.\end{aligned} \tag{B.3}$$

Furthermore we introduce the functions

$$f_A(\tau_a,\tau_b,\tau_c) = \int_{\mathbb{H}} \mathrm{d}^2 z\; \partial\mathcal{A}_a\bar{\partial}\mathcal{A}_b\mathcal{A}_c \tag{B.4}$$

$$f_S(\tau_a,\tau_b,\tau_c) = \int_{\mathbb{H}} \mathrm{d}^2 z\; \partial\mathcal{S}_a\bar{\partial}\mathcal{S}_b\mathcal{S}_c = -\int \mathrm{d}^2 z\; \partial\mathcal{A}_a\bar{\partial}\mathcal{A}_b\mathcal{S}_c, \tag{B.5}$$

which are finite except for an infinite constant. So the computation of (B.4) and (B.5) will need a regularization. With the above abbreviations and the relations

$$\begin{aligned}D_\rho G^{\mu\nu} &= -G^{\mu\lambda}D_\rho\mathcal{F}_{\lambda\sigma}\Theta^{\sigma\nu} - \Theta^{\mu\lambda}D_\rho\mathcal{F}_{\lambda\sigma}G^{\sigma\nu} \\ D_\rho\Theta^{\mu\nu} &= -\Theta^{\mu\lambda}D_\rho\mathcal{F}_{\lambda\sigma}\Theta^{\sigma\nu} - G^{\mu\lambda}D_\rho\mathcal{F}_{\lambda\sigma}G^{\sigma\nu} \quad ,\end{aligned} \tag{B.6}$$

the tree level amplitude of (5.4) reads

$$
\begin{aligned}
&\langle \zeta^{\kappa_i}(\tau_i)\, \zeta^{\kappa_j}(\tau_j) \zeta^{\kappa_k}(\tau_k) \Big\{ - \int_{\mathbb{H}} \mathrm{d}^2 z\, \partial\zeta^{\mu} \bar{\partial}\zeta^{\nu} \zeta^{\rho}\, \partial_{\rho}\mathcal{F}_{\mu\nu} \Big\} \rangle_{\text{tree}} \; = \\
= \; -\frac{1}{(2\pi)^3} \Big\{ &+\Theta^{\kappa_k\rho}\partial_{\rho}\Theta^{\kappa_i\kappa_j} \; \big(f_A(\tau_i,\tau_j,\tau_k) - f_A(\tau_j,\tau_i,\tau_k) \big) \\
&+\Theta^{\kappa_k\rho}\partial_{\rho}G^{\kappa_i\kappa_j} \; \big(f_A(\tau_i,\tau_j,\tau_k) + f_A(\tau_j,\tau_i,\tau_k) \big) \\
&+G^{\kappa_k\rho}\partial_{\rho}\Theta^{\kappa_i\kappa_j} \; \big(f_S(\tau_i,\tau_j,\tau_k) - f_S(\tau_j,\tau_i,\tau_k) \big) \\
&+G^{\kappa_k\rho}\partial_{\rho}G^{\kappa_i\kappa_j} \; \big(f_S(\tau_i,\tau_j,\tau_k) + f_S(\tau_j,\tau_i,\tau_k) \big) \\
&+\big(\text{cycl. perm. (ijk)}\big) \Big\} . \qquad \text{(B.7)}
\end{aligned}
$$

For the following computation of (B.7) we take the order $\tau_i < \tau_j < \tau_k$ on the real axis.

B.2 Regularization of f_A and f_S

To regularize f_A and f_S we differentiate the integral representations (B.4) and (B.5) with respect to τ_a, τ_b and τ_c, respectively. Then we can perform the integration over the upper half plane $\mathbb{H}$. This can be done by the well known method of a transformation into a contour integral and using the residue theorem. The pole prescriptions on the real axis are obtained by introducing a small imaginary shift $\pm i\epsilon$, so that

$$
\mathcal{A}_i = \ln\Big(\frac{\bar{z} - \tau_i - i\epsilon}{\tau_i - i\epsilon - z} \Big) \quad \text{and} \quad \mathcal{S}_i = \ln\big((\tau_i - i\epsilon - z)(\tau_i + i\epsilon - \bar{z}) \big) \; . \qquad \text{(B.8)}
$$

The prescription is chosen so that it is consistent with bulk insertions that emerge the boundary, i.e., we consider in fact insertions in the bulk which are very close to the boundary. The appearance of the logarithm needs a selection of a cut and it turns out that the negative real axis is a convenient choice. Finally, we determine the antiderivative with respect to τ_a, τ_b and τ_c. Now, the infinity is contained in the integration constant.

In such a way we get

$$\begin{aligned} f_A(\tau_a,\tau_b,\tau_c) &= 2\pi\int_0^t \mathrm{d}x\Big(\frac{\ln(x\pm i\epsilon')}{1-x}+\frac{\ln(1-x\pm i\epsilon')}{x}\Big)+C^A_{(\infty)}, \\ f_S(\tau_a,\tau_b,\tau_c) &= 2\pi\int_0^t \mathrm{d}x\Big(-\frac{\ln(x\pm i\epsilon')}{1-x}+\frac{\ln(1-x\pm i\epsilon')}{x}\Big) \\ &- \frac{\pi}{2}\ln^2(\tau_b-\tau_a)^2+i\pi^2\epsilon(\tau_b-\tau_a)\ln(\tau_b-\tau_a)^2+C^S_{(\infty)}, \end{aligned} \tag{B.9}$$

where the $\pm$ in the logarithm abbreviates in fact the sign function $+\epsilon(\tau_b-\tau_a)$. In (B.9) we have introduced the parameter t which is defined as the combination $t=\frac{\tau_c-\tau_a}{\tau_b-\tau_a}$. The shift ϵ' is needed to integrate along the correct side of the cut for negative arguments of the logarithm. This selection is determined by the pole prescription explained above.

The integrals (B.9) remind us of the dilogarithm (A.4) introduced in appendix A. Indeed, taking into account the branch cut of the logarithm, one can express equation (B.7) in terms of the analytically continued dilogarithm that was defined by (A.2) and (A.7). Because of the order $\tau_i<\tau_j<\tau_k$ the so-called modulus

$$m=\frac{\tau_j-\tau_i}{\tau_k-\tau_i}, \tag{B.10}$$

is restricted to $0<m<1$, a region that we met several times in appendix A.

B.3 The Tree Level Amplitude

What is left is to use (B.9) to bring together all combinations of the functions f_A and f_S in (B.7). This leads to the rather lengthy result

$$
\begin{aligned}
&-2\pi^2\langle\zeta^{\kappa_i}(\tau_i)\,\zeta^{\kappa_j}(\tau_j)\,\zeta^{\kappa_k}(\tau_k)\int_{\mathbb{H}} \mathrm{d}^2z\,\partial\zeta^\mu\bar\partial\zeta^\nu\zeta^\rho\,\partial_\rho\mathcal{F}_{\mu\nu}\rangle_{\text{tree}} \;=\\
&\Big\{+\Theta^{\kappa_k\rho}\partial_\rho\Theta^{\kappa_i\kappa_j}\big(\mathrm{Li}_2(\frac{\tau_{ik}}{\tau_{ij}})-\mathrm{Li}_2(\frac{\tau_{kj}}{\tau_{ij}})\big)\\
&\quad+\Theta^{\kappa_i\rho}\partial_\rho\Theta^{\kappa_j\kappa_k}\big(\mathrm{Li}_2(\frac{\tau_{ji}}{\tau_{jk}})-\mathrm{Li}_2(\frac{\tau_{ik}}{\tau_{jk}})\big)\\
&\quad+\Theta^{\kappa_j\rho}\partial_\rho\Theta^{\kappa_k\kappa_i}\big(\mathrm{Li}_2(\frac{\tau_{kj}}{\tau_{ki}})-\mathrm{Li}_2(\frac{\tau_{ji}}{\tau_{ki}})\big)\\
&+i\pi\Theta^{\kappa_k\rho}\partial_\rho G^{\kappa_i\kappa_j}\big(\ln\tau_{ji}-\ln\tau_{ki}\big)\\
&-i\pi\Theta^{\kappa_i\rho}\partial_\rho G^{\kappa_j\kappa_k}\big(\ln\tau_{kj}-\ln\tau_{ki}\big)\\
&-i\pi G^{\kappa_k\rho}\partial_\rho\Theta^{\kappa_i\kappa_j}\big(\ln\tau_{ki}\big)\\
&-i\pi G^{\kappa_i\rho}\partial_\rho\Theta^{\kappa_j\kappa_k}\big(\ln\tau_{ki}\big)\\
&+i\pi G^{\kappa_j\rho}\partial_\rho\Theta^{\kappa_k\kappa_i}\big(\ln\tau_{ki}\big)\\
&\quad+G^{\kappa_k\rho}\partial_\rho G^{\kappa_i\kappa_j}\big(+\ln\tau_{ji}\ln\tau_{kj}-\ln\tau_{kj}\ln\tau_{ki}+\ln\tau_{ji}\ln\tau_{ki}\big)\\
&\quad+G^{\kappa_i\rho}\partial_\rho G^{\kappa_j\kappa_k}\big(+\ln\tau_{ji}\ln\tau_{kj}+\ln\tau_{kj}\ln\tau_{ki}-\ln\tau_{ji}\ln\tau_{ki}\big)\\
&\quad+G^{\kappa_j\rho}\partial_\rho G^{\kappa_k\kappa_i}\big(-\ln\tau_{ji}\ln\tau_{kj}+\ln\tau_{kj}\ln\tau_{ki}+\ln\tau_{ji}\ln\tau_{ki}\big)\quad\Big\},
\end{aligned}
\tag{B.11}
$$

where we have set the integration constants of (B.9) to a convenient value, which can be done since they play no essential role (cf. equation (3.7)).

In the limit $g_{\mu\nu}\to 0$ all terms containing the boundary metric $G^{\mu\nu}$ vanish and we obtain

$$
\begin{aligned}
&-2\pi^2\langle\zeta^{\kappa_i}(\tau_i)\,\zeta^{\kappa_j}(\tau_j)\,\zeta^{\kappa_k}(\tau_k)\int_{\mathbb{H}} \mathrm{d}^2z\,\partial\zeta^\mu\bar\partial\zeta^\nu\zeta^\rho\,\partial_\rho\mathcal{F}_{\mu\nu}\rangle_{\text{tree},\alpha'\to 0} \;=\\
&\Big\{+\Theta^{\kappa_k\rho}\partial_\rho\Theta^{\kappa_i\kappa_j}\big(\mathrm{Li}_2(1-m)-\mathrm{Li}_2(m)+\frac{\pi^2}{3}\big)\\
&\quad+\Theta^{\kappa_i\rho}\partial_\rho\Theta^{\kappa_j\kappa_k}\big(\mathrm{Li}_2(1-m)-\mathrm{Li}_2(m)-\frac{\pi^2}{3}\big)\\
&\quad+\Theta^{\kappa_j\rho}\partial_\rho\Theta^{\kappa_k\kappa_i}\big(\mathrm{Li}_2(1-m)-\mathrm{Li}_2(m)\big)\quad\Big\}\,,
\end{aligned}
\tag{B.12}
$$

where we expressed (B.12) in terms of the modulus m instead of the explicit quotients in (B.11). Furthermore, we took advantage of relations (A.10) and (A.11).

When using, in addition, radial gauge and taking a vanishing gauge field A, the terms $\pm\frac{\pi^2}{3}$ in the first two lines disappear and we recover the result of [9]. This is due to the relation to Rogers dilogarithm $L(m)$, $\mathrm{Li}_2(1-m) - \mathrm{Li}_2(m) = \frac{\pi^2}{6} - 2L(m)$.

B.4 Coincidence Limits

In chapter 5 we calculate the correlator of two functions. For that purpose we have to consider the coincidence limits $\tau_j \to \tau_i$ and $\tau_j \to \tau_k$ of (B.11). In these limits there appear logarithmic singularities which can be regularized by a cut-off parameter Λ, i.e., $\lim_{\tau_j\to\tau_i} \ln(\tau_j - \tau_i) \to \ln\Lambda$. In terms of Λ we get

$$\begin{aligned}
-\langle\zeta^{\kappa_i}(\tau_i)\,\zeta^{\kappa_j}(\tau_i)\quad \zeta^{\kappa_k}(\tau_k)\ &\int_{\mathbb{H}} \mathrm{d}^2z\,\partial\zeta^\mu\bar\partial\zeta^\nu\zeta^\rho\,\partial_\rho\mathcal{F}_{\mu\nu}\rangle_{\text{tree,sing}} = \\
&= +\frac{i}{2\pi}\Theta^{\kappa_k\rho}\partial_\rho G^{\kappa_i\kappa_j}\ln\Lambda + \frac{1}{\pi^2}G^{\kappa_k\rho}\partial_\rho G^{\kappa_i\kappa_j}\ln\Lambda\ln(\tau_k-\tau_i) \\
&= -\frac{1}{\pi}\partial_\rho G^{\kappa_i\kappa_j}\ln\Lambda\langle\zeta^\rho(\tau_i)\,\zeta^{\kappa_k}(\tau_k)\rangle
\end{aligned} \tag{B.13}$$

for $\tau_j \to \tau_i$ and

$$\begin{aligned}
-\langle\zeta^{\kappa_i}(\tau_i)\quad \zeta^{\kappa_j}(\tau_k)\,\zeta^{\kappa_k}(\tau_k) &\int_{\mathbb{H}} \mathrm{d}^2z\,\partial\zeta^\mu\bar\partial\zeta^\nu\zeta^\rho\,\partial_\rho\mathcal{F}_{\mu\nu}\rangle_{\text{tree,sing}} = \\
&= -\frac{i}{2\pi}\Theta^{\kappa_i\rho}\partial_\rho G^{\kappa_j\kappa_k}\ln\Lambda + \frac{1}{\pi^2}G^{\kappa_i\rho}\partial_\rho G^{\kappa_j\kappa_k}\ln\Lambda\ln(\tau_k-\tau_i) \\
&= -\frac{1}{\pi}\partial_\rho G^{\kappa_j\kappa_k}\ln\Lambda\langle\zeta^{\kappa_i}(\tau_i)\,\zeta^\rho(\tau_k)\rangle
\end{aligned} \tag{B.14}$$

for $\tau_j \to \tau_k$. The singularities (B.13) and (B.14) must be compensated by appropriate subtractions, i.e., one has to introduce a normal ordering for the interacting theory (2.1). The correct subtractions can easily be read off from (B.13,B.14). Together with the singular part of the propagator (3.9) we get

$$\begin{aligned}
\zeta^\mu(\tau)\,\zeta^\nu(\tau') &= -\frac{1}{2\pi}G^{\mu\nu}\ln(\tau-\tau')^2 - \frac{1}{2\pi}\partial_\rho G^{\mu\nu}\ln(\tau-\tau')^2\ \zeta^\rho(\frac{\tau+\tau'}{2}) \\
&+\ (\text{ regular terms }).
\end{aligned} \tag{B.15}$$

In order to obtain the finite part of equation (B.11) in our limits we have to take into account the special values (A.12). So we get

$$\begin{aligned} &-\langle\, \zeta^{(\kappa_i}(\tau_i)\,\zeta^{\kappa_j)}(\tau_i)\ \ \zeta^{\kappa_k}(\tau_k)\ \ \int_{\mathbb{H}} \mathrm{d}^2 z\, \partial\zeta^\mu \bar{\partial}\zeta^\nu \zeta^\rho\, \partial_\rho \mathcal{F}_{\mu\nu}\rangle_{\text{tree,fin}} \ = \qquad (\text{B.16})\\ &= \ -\frac{1}{12}\big(\Theta^{\kappa_i\rho}\partial_\rho\Theta^{\kappa_j\kappa_k} - \Theta^{\kappa_j\rho}\partial_\rho\Theta^{\kappa_k\kappa_i}\big)\\ &\quad -\frac{i}{2\pi}\big(\Theta^{\kappa_k\rho}\partial_\rho G^{\kappa_i\kappa_j} + G^{\kappa_j\rho}\partial_\rho\Theta^{\kappa_i\kappa_k} + G^{\kappa_i\rho}\partial_\rho\Theta^{\kappa_j\kappa_k}\big)\ln(\tau_k-\tau_i)\\ &\quad -\frac{1}{2\pi^2}\big(G^{\kappa_k\rho}\partial_\rho G^{\kappa_i\kappa_j} - G^{\kappa_i\rho}\partial_\rho G^{\kappa_j\kappa_k} - G^{\kappa_j\rho}\partial_\rho G^{\kappa_k\kappa_i}\big)\ln^2(\tau_k-\tau_i) \end{aligned}$$

for $\tau_j \to \tau_i$ and

$$\begin{aligned} &-\langle\, \zeta^{\kappa_i}(\tau_i)\ \ \zeta^{(\kappa_j}(\tau_k)\,\zeta^{\kappa_k)}(\tau_k)\ \ \int_{\mathbb{H}} \mathrm{d}^2 z\, \partial\zeta^\mu \bar{\partial}\zeta^\nu \zeta^\rho\, \partial_\rho \mathcal{F}_{\mu\nu}\rangle_{\text{tree,fin}} \ = \qquad (\text{B.17})\\ &= \ +\frac{1}{12}\big(\Theta^{\kappa_k\rho}\partial_\rho\Theta^{\kappa_i\kappa_j} - \Theta^{\kappa_j\rho}\partial_\rho\Theta^{\kappa_k\kappa_i}\big)\\ &\quad +\frac{i}{2\pi}\big(\Theta^{\kappa_i\rho}\partial_\rho G^{\kappa_j\kappa_k} + G^{\kappa_j\rho}\partial_\rho\Theta^{\kappa_k\kappa_i} + G^{\kappa_k\rho}\partial_\rho\Theta^{\kappa_j\kappa_i}\big)\ln(\tau_k-\tau_i)\\ &\quad -\frac{1}{2\pi^2}\big(G^{\kappa_i\rho}\partial_\rho G^{\kappa_j\kappa_k} - G^{\kappa_j\rho}\partial_\rho G^{\kappa_k\kappa_i} - G^{\kappa_k\rho}\partial_\rho G^{\kappa_i\kappa_j}\big)\ln^2(\tau_k-\tau_i) \end{aligned}$$

for $\tau_j \to \tau_k$. We considered only the symmetric part of the limit, since the antisymmetric one does not contribute in chapter 5.

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