DIPLOMARBEIT

BRST Cohomology of Dirichlet-Superstrings

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To my parents

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Chapter 1

Introduction

1.1 Motivation

String theory was discovered in the late 60s as a model for hadron resonances. While hadron resonances turned out to be more efficiently described by QCD, Scherk and Schwarz [sc74] showed that strings provide a promissing theory for quantum gravity. The introduction of ferminons by Ramond [ra71] and by Neveu and Schwarz [ns71] cured the inconsistencies of the bosonic string theory. It led to a reduction of the critical dimension from $D = 26$ to $D = 10$ and removed the tachyonic degree of freedom. Since then many developments in supersymmetry and in superstrings have been made including torus and lattice constructions, Calabi-Yau manifolds, WZW and Landau-Ginzburg models, topological field theories, duality and mirror symmetries.

D-Branes [po96] play a crucial role in advances towards an understanding of nonperturbative properties of string theories and supersymmetric quantum field theories. They provide a simple description of various nonperturbative objects required by string duality, since D-branes have the correct properties to fill out duality multiplets. Furthermore they give new insight into the quantum mechanics of black holes and into the nature of spacetime at the shortest distances. The Dirichlet-superstring of Neveu-Schwarz-Ramond can be formulated as $D = 2$ supergravity [ho79] coupled to bosonic and fermionic string coordinates with an additional $U(1)$-symmetry on the world sheet. The main purpose of this paper is to provide an appropriate basis for the cohomological analysis of Dirichlet superstrings.

Gauge invariance is a basic principle in models of fundamental interactions. The BRST formalism first established by Becchi, Rouet and Stora [brs74] provides a useful tool for dealing with gauge symmetries, since it encodes the gauge symmetry and its properties in a single antiderivative, which is strictly nilpotent on all the fields. This antiderivative is called BRST operator. The nilpotency of the BRST operator establishes the BRST cohomology in the space of local functionals of the fields. This cohomology is physically meaningful, since it determines gauge invariant actions, dynamical conservation laws and possible anomalies. Furthermore it is a useful tool in the renormalization of quantum field theories. A suitable general framework in which the cohomology of the BRST operator can be computed has been suggested by F. Brandt [br96]. The main part of this paper is heavily based on this framework, which relates the BRST cohomology to an underlying gauge covariant algebra. It includes a definition of tensor fields on which this algebra is realized and of generalized connections. It reduces the computation of the cohomology to a problem involving only these quantities. This involves the use of contracting homotopies in jet space, which allows to eliminate certain local jet coordinates from the cohomological analysis. These local jet coordinates are called trivial pairs.

Beltrami differentials parametrize conformal classes of two dimensional metrics [be58]. Thus it is natural to use these quantities as basic variables whenever a two dimensional field theory is coupled to gravity in a conformally invariant way [dg90,gd96,gr90,ta95,ta96, sc93]. A detailed understanding of this parametrization is essential for the computations in this work. Therefore the Beltrami and super-Beltrami parametrization is investigated in quite a detail. Thereby the properties of the Beltrami differential and its fermionic superpartner, the Beltramino are discussed and the factorization of the BRST algebra is established. This is comletely done in the component field formalism.

1.2 Outline of the paper

The paper is organized as follows. The algebraic approach to the BRST cohomology [Br97, br96] is sketched in chapter (2). It introduces some terminology and notation. In chapter (3) well known properties of clifford algebras and spinors are revisited and definitions and conventions of spinor space are given. Chapter (4) analyses the general structure of gauge covariant algebras. It is based on the formalism developed by F. Brandt [br96,br93,br91]. Eventually the BRST formalism is set up [Br93]. In chapter (5) the Bianchi identities for D=2 supergravity coupled to U(1) super-Yang-Mills theory are studied. Thereby the consistency of the imposed constraints is checked and the general parametrization of the allowed torsions and field strenghts is determined and the correponding auxiliary fields are introduced. The field content of the theory and the BRST transformations of the elementary fields are given. Chapter (7) introduces the Beltrami parametrization in the Riemannian manifold approach as the parametrization of conformal classes of metrics and in the Riemannian surface approach dealing with complex structures. Before turning to supersymmetric extensions of the Beltrami parametrization the theory of the bosonic D-sting is reviewed. The main part of the analysis is carried out in chapter (8). The BRST transformations of the the fields are supplemented by super-Weyl transformations and the corresponding ghost fields are introduced and their BRST transformations given. Then the super-Beltrami parametrization is introduced and the factorization of the BRST algebra is established. In order to constuct a suitable set of local jet coordinates the cohomological problem is mapped from the space of local functionals to the space of local total forms. Eventually the infinite set of generalized connections and tensor fields is constructed. The operators corresponding to the infinite set of generalized connections form two copies of the super-Virasoro algebra. The representation of the operators on the tensor fields is given and explicite expressions of the first few tensor fields are listed. In two appendices frequently used formulae for the manipulation of γ-matrices and for the super-Beltrami parametrization are collected.

Chapter 2

BRST Cohomology

2.1 BRST Cohomology on local total forms

The local BRST cohomology of a particular theory is defined by the BRST operator s and the space in which its cohomology is to be computed. The BRST operator is required to be nilpotent and to commute with the spacetime partial derivative ∂_m

$$s^2 = s\partial_m - \partial_m s = \partial_m \partial_n - \partial_n \partial_m = 0. \tag{2.1}$$

A basic concept used in this context is the jet bundle approach [sa89]. In the jet bundle language the fields and their partial derivatives

$$[\Phi^i] := \{\Phi^i, \partial_m \Phi^i, \partial_m \partial_n \Phi^i, \ldots\} \tag{2.2}$$

are considered as local coordinates of an infinite jet space. They are regarded to be independent apart from the obvious identification

$$\partial_{m_1} \ldots \partial_{m_i} \ldots \partial_{m_j} \ldots \partial_{m_k} \Phi^i = \partial_{m_1} \ldots \partial_{m_j} \ldots \partial_{m_i} \ldots \partial_{m_k} \Phi^i. \tag{2.3}$$

By convention the coordinates x^m and the differentials dx^m are also counted among the local jet coordinates. This turns out to be convenient since the coordinates and the diferentials are always BRST invariant

$$s\, x^m = 0, \qquad s\, dx^m = 0. \tag{2.4}$$

The derivatives ∂_m are defined as total derivative operators in the jet space

$$\partial_m = \frac{\partial}{\partial x^m} + \sum_{k \geq 0} \partial_{m m_1 \ldots \partial_{m_k}} \Phi^i \frac{\partial}{\partial \partial_{m_1} \ldots \partial_{m_k} \Phi^i}. \tag{2.5}$$

They become usual partial derivatives on the local sections of the jet bundle. Due to the invariance of the differentials, which are treated as anticommuting objects the spacetime exterior derivative

$$d = dx^m \partial_m \tag{2.6}$$

anticommutes with the BRST operator

$$sd + ds = 0, \qquad d^2 = 0, \tag{2.7}$$

which is equivalent to the second and third relation in (2.1).

2.1.1 Descent equations

A local functional is an integrated local volume form

$$W = \int \omega_D, \tag{2.8}$$

where D denotes the the spacetime dimension. In general local p-forms are

$$\omega_p = \frac{1}{p!} dx^{m_1} \dots dx^{m_p} \omega_{m_1 \dots m_p}(x, [\Phi^i]). \tag{2.9}$$

They are local in the sense that they are polynomial in the derivatives $[\partial_m \Phi^i]$ but may have an arbitrary dependence on the coordinates x^m and the undifferentiated fields Φ^i. They are not required to be globally well defined. A local functional is called BRST invariant if the BRST transformation of its integrand is d-exact in the space of local forms

$$s\,\omega_D = d\,\omega_{D-1}. \tag{2.10}$$

A local functional is called BRST-exact or trivial if $\omega_D = s\eta_D + d\eta_{D-1}$. Two solutions are called equivalent if they differ by a trivial solution. Thus the local BRST cohomology is the cohomology of s modulo d on local volume forms and is denoted by $H(s|d)$. The cohomology is well defined due to the nilpotency of s and d and the fact that they anticommute. $H(s|d)$ is represented by solutions

$$s\omega_D + d\omega_{D-1} = 0, \qquad \omega_D \neq s\eta_D + d\eta_{D-1}. \tag{2.11}$$

From this the descent equations are derived via the algebraic Poincaré lemma. It states that locally any d-closed local p-form is d-exact for $0 < p < D$ and constant for $p = 0$, while local volume forms are locally d-exact if and only if they have vanishing Euler-Lagrange derivative with respect to all the fields. Thus acting with s on (2.11) one arrives at $d(s\omega_{D-1}) = 0$. The algebraic Poincaré lemma now implies the existence of a (possibly vanishing) local $(D-2)$-form ω_{D-2} satisfying

$$s\,\omega_{D-1} + d\,\omega_{D-2} = 0. \tag{2.12}$$

Iterating the arguments one derives the so-called descent equations

$$\sigma\,\omega_p - d\,\omega_{p-1} = 0, \quad D \geq p \geq p_0; \quad s\omega_{p_0} = 0^1. \tag{2.13}$$

2.1.2 The $\tilde{s}$ cohomology

The descent equations are compactly written introducing the new nilpotent operator

$$\tilde{s} = s + d, \quad \tilde{s}^2 = 0. \tag{2.14}$$

The cohomology of $\tilde{s}$ on local total forms is defined by the condition

$$\tilde{s}\,\tilde{\omega} = 0, \quad \tilde{\omega} \neq \tilde{s}\tilde{\eta} + constant \tag{2.15}$$

where a local total form is the formal sum of local forms with different form degrees,

$$\tilde{\omega} = \sum_{p=0}^{D} \omega_p. \tag{2.16}$$

It is natural to introduce the *total degree* as the sum of the ghost number and the form degree

$$totdeg = gh + formdeg. \tag{2.17}$$

Since $\tilde{s}$ has total degree 1 it maps a local total form with total degree G into a local total form with total degree $G+1$. The BRST cohomology on local functionals with ghost number g is locally isomorphic to the $\tilde{s}$-cohomology on local total forms with total degree $G = g + D$ [br96]

$$H(s|d, \Omega^{g,D}) \cong H(\tilde{s}, \tilde{\Omega}^G), \quad \tilde{\Omega}^G = \bigoplus_{p=0}^{D} \Omega^{G-p,p}. \tag{2.18}$$

where $\Omega^{g,D}$ denotes the space of local D-forms with ghost number g.

[1] In fact for $p = 0$ the algebraic Poincaré lemma implies only $s\,\omega_{p_0} = constant$. For possible extensions see for instance [bh96].

2.2 Contracting homotopies

The computation of the $\tilde{s}$ cohomology is considerably simplyfied by switching to a new set of jet coordinates better adapted for the cohomological analysis. Following [br96] the new set of jet coordinates is denoted by $\mathcal{B} = \{\mathcal{U}^l, \mathcal{V}^l, \mathcal{W}^i\}$ and is required to have the following properties

$$\tilde{s}\mathcal{U}^l = \mathcal{V}^l, \qquad \tilde{s}\mathcal{W}^i = \mathcal{R}^i(\mathcal{W}), \tag{2.19}$$

where the $\mathcal{R}^i$ are functions of the $\mathcal{W}$'s only. The change of jet coordinates is required to be compatible with the locality requirement imposed on the cohomological problem, i.e. the $\mathcal{U}^l$'s, $\mathcal{V}^l$ and the $\mathcal{W}^i$'s have to be local total forms. In this case $\tilde{\Omega}^*$ factorizes into two independent $\tilde{s}$-invariant subspaces

$$\tilde{\Omega}^* = \tilde{\Omega}^*_{\mathcal{U},\mathcal{V}} \times \tilde{\Omega}^*_{\mathcal{W}}. \tag{2.20}$$

As a consequence the cohomology of $\tilde{s}$ also factorizes due to Künneth's formula

$$H(\tilde{s}, \tilde{\Omega}^G) = \bigoplus_{G'} H(\tilde{s}, \tilde{\Omega}^{G'}_{\mathcal{U},\mathcal{V}}) \times H(\tilde{s}, \tilde{\Omega}^{G-G'}_{\mathcal{W}}), \tag{2.21}$$

where $H(\tilde{s}, \tilde{\Omega}^{G'}_{\mathcal{U},\mathcal{V}})$ is contractible, since the $\mathcal{U}$'s and $\mathcal{V}$'s form $\tilde{s}$-doublets satisfying $\tilde{s}\mathcal{U}^i = \mathcal{V}^i$. Therefore one is left with the cohomology of $\tilde{s}$ on local total forms constructed solely of the $\mathcal{W}$'s. The $\mathcal{U}$'s and $\mathcal{V}$'s are called trivial pairs, while the $\mathcal{W}$'s are interpreted as generalized connections and tensor fields.

Chapter 3

Spinors

In this chapter some well known properties of clifford algebras and spinors [ta98], [to83], [kr96] are revisited. The conventions used in the following work are given.

3.1 Clifford algebras

Considering a metric of the form

$$\eta_{ab} = \eta^{ab} = diag(+, \ldots, +, -, \ldots, -) \tag{3.1}$$

with p positive and $q = D - p$ negative entries the generators of the Clifford algebra are objects γ^a, which satisfy the relation

$$\{\gamma^a, \gamma^b\} = \gamma^a\gamma^b + \gamma^b\gamma^a = 2\eta^{ab}\mathbf{1} \tag{3.2}$$

The products $\pm\Gamma^{a_1,\ldots,a_i}$

$$\Gamma^{a_1,\ldots,a_i} := \gamma^{a_1}\ldots\gamma^{a_i} \quad with \quad a_1 < a_2 < \ldots < a_i \tag{3.3}$$

form a finite group with 2^{D+1} elements. The Clifford algebra $\mathcal{C}_{\mathbb{K}}(p,q)$ is the algebra formed by the linear combinations of the products $\Gamma^{a_1,\ldots,a_i}$ with coefficients in some field $\mathbb{K}$. Its generators anticommute and p of them square to $\mathbf{1}$ while the remaining q square to $-\mathbf{1}$. The smallest matrices satisfying the anticommutation relation are $2^{[\frac{D}{2}]} \times 2^{[\frac{D}{2}]}$, where $[x]$ is the largest integer not larger than x. For $D = 2$ an explicit representation can be constructed using any two of the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_i^2 = \mathbf{1}, \quad \sigma_1\sigma_2\sigma_3 = i\mathbf{1} \tag{3.4}$$

to represent $\mathcal{C}(2,0)$ and imaginary multiples to obtain the other signatures. Representations of higher dimensions are constructed by tensor products of representations, which corresponds to 'matrices of matrices' [kr96], [ta98].

The antisymmetrized bilinears

$$\Sigma_{ab} := \frac{1}{4}[\gamma_a, \gamma_b] \tag{3.5}$$

provide a representation of the Lorentz algebra on the representation space of the Clifford algebra, since the commutator of Σ_{ab} with a gamma matrix γ_c gives

$$[\Sigma_{ab}, \gamma_c] = \eta_{ac}\gamma_b - \eta_{bc}\gamma_a, \tag{3.6}$$

which is of the same form as the Lorentz transformation

$$l_{ab}V_c = \eta_{bc}V_a - \eta_{ac}V_b, \qquad [l_{ab}, l_{cd}] = \eta_{bc}l_{ad} - \eta_{bd}l_{ac} - \eta_{ac}l_{bd} + \eta_{ad}l_{bc} \tag{3.7}$$

of the vector γ_c. Hence, the Σ_{ab}'s provide a representation of the Lorentz algebra

$$[\Sigma_{ab}, \Sigma_{cd}] = \eta_{bc}\Sigma_{ad} - \eta_{bd}\Sigma_{ac} - \eta_{ac}\Sigma_{bd} + \eta_{ad}\Sigma_{bc} \tag{3.8}$$

I choose as a representation of $\mathcal{C}(1,1)$ the two dimensional matrices

$$(\gamma^0)_\alpha{}^\beta = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad (\gamma^1)_\alpha{}^\beta = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \tag{3.9}$$

which satisfy the definition (3.1) with

$$\eta^{ab} = \eta_{ab} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.10}$$

The generator of Lorentz transformations then reads

$$\Sigma_{ab} = \frac{1}{4}\left(\gamma_a\gamma_b - \gamma_b\gamma_a\right) = \frac{1}{2}\varepsilon_{ab}\gamma_*, \qquad \gamma_* := \gamma^0\gamma^1 \tag{3.11}$$

where ε^{ab} denotes the totally antisymmetric tensor

$$\varepsilon_{ab} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{3.12}$$

and γ_* takes the explicit form

$$(\gamma_*)_\alpha{}^\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.13}$$

The spinor indices are often suppressed assuming the summation convention 'ten to four'. The totally antisymmetric tensor ε_{ab} satisfies the relations

$$\varepsilon_{ab} = -\varepsilon^{ab}, \qquad \varepsilon_{ab}\varepsilon^{bc} = \delta_a{}^c, \qquad \varepsilon_{ab}\varepsilon^{cd} = -\delta_a{}^c\delta_b{}^d + \delta_b{}^c\delta_a{}^d. \tag{3.14}$$

The Lorentz transformations in two dimensions explicitely read

$$l_{ab} = \varepsilon_{ab} l, \qquad\qquad l v_a = \varepsilon_a{}^b v_b, \tag{3.15}$$

$$l\psi_\alpha = \frac{1}{2}(\gamma_*)_\alpha{}^\beta \psi_\beta = \frac{1}{2}(\gamma\psi)_\alpha, \qquad l\psi^\alpha = -\tfrac{1}{2}\psi^\beta(\gamma_*)_\beta{}^\alpha = -\tfrac{1}{2}(\psi\gamma)^\alpha \tag{3.16}$$

where I used

$$l_{ab}\psi_\alpha = \frac{1}{2}\varepsilon_{ab}(\gamma_*)_\alpha{}^\beta \Delta_\beta{}^\alpha \psi_\gamma, \qquad \Delta_\alpha{}^\beta \psi_\gamma = \delta_\gamma{}^\beta \psi_\alpha, \qquad \Delta_\alpha{}^\beta \psi^\gamma = -\delta_\alpha{}^\gamma \psi^\beta. \tag{3.17}$$

In two dimensions the γ matrices satisfy the relation

$$\gamma^a\gamma^b = \eta^{ab}\mathbf{1} + \varepsilon^{ab}\gamma_* \tag{3.18}$$

and the Fierz identity

$$\delta_\alpha{}^\beta \delta_\gamma{}^\delta + (\gamma^a)_\alpha{}^\beta(\gamma_a)_\gamma{}^\delta + (\gamma_*)_\alpha{}^\beta(\gamma_*)_\gamma{}^\delta = 2\delta_\alpha{}^\delta\delta_\gamma{}^\beta. \tag{3.19}$$

Other formulas in the calculations frequently used are collected in the appendix (A.1). The totally antisymmetric tensor with spinor indices is denoted by

$$\varepsilon_{\alpha\beta} = -\varepsilon^{\alpha\beta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{3.20}$$

It has the properties

$$\varepsilon_{\alpha\beta}\varepsilon^{\beta\gamma} = \delta_\alpha{}^\gamma, \qquad \varepsilon_{\alpha\beta}\varepsilon^{\gamma\delta} = -\delta_\alpha{}^\gamma\delta_\beta{}^\delta + \delta_\beta{}^\gamma\delta_\alpha{}^\delta. \tag{3.21}$$

The charge conjugation matrix and its invers are fixed to be

$$C_{\alpha\beta} = i\varepsilon_{\alpha\beta} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad C^{\alpha\beta} = -i\varepsilon^{\alpha\beta} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \tag{3.22}$$

satisfying the relations

$$CC^\dagger = \mathbf{1}, \qquad -\gamma^{aT} = C^{-1}\gamma^a C. \tag{3.23}$$

In general spinors of $SO(p,q)$ have $2^{[\frac{D}{2}]}$ independent complex components, which may be reduced by imposing Weyl and/or Majorana conditions. General spinors without any conditions imposed are called Dirac spinors.

In even dimensions there are two inequivalent irreducible representations of the Lie algebra $so(p,q)$, which are of dimension $2^{D/2-1}$ and whose elements are called Weyl spinors. The chiral projectors $P_\pm$, which decompose the representation read in our case

$$P_\pm = \frac{1}{2}\left(\mathbf{1} \pm \gamma_*\right). \tag{3.24}$$

Therefore the first (second) component of a Dirac spinor corresponds to a right (left) chiral Weyl spinor

$$\psi_\pm = P_\pm \psi, \quad \psi_+ = \begin{pmatrix} \psi_1 \\ 0 \end{pmatrix}, \quad \psi_- = \begin{pmatrix} 0 \\ \psi_2 \end{pmatrix}. \tag{3.25}$$

Majorana spinors satisfy a certain reality condition. The Majorana condition requires that a spinor is equal to its charge conjugate

$$\psi^c = \psi. \tag{3.26}$$

Charge conjugation should change the sign of the coupling of the elektromagnetic gauge field to the fermion in the Dirac equation

$$(i\gamma^m\partial_m - e\,\gamma^m A_m - m)\psi = 0 \quad \rightarrow \quad (i\gamma^m\partial_m + e\,\gamma^m A_m - m)\psi^c = 0. \tag{3.27}$$

This equation is obtained by transposition of the Dirac equation

$$\Big((-\gamma^m)^T(i\partial_m + e\,A_m) - m\Big)\,\overline{\psi}^T = C^{-1}(i\gamma^m\partial_m + e\,\gamma^m A_m - m)C\overline{\psi}^T = 0, \qquad \psi^c := C\,\overline{\psi}^T, \tag{3.28}$$

where $\overline{\psi}$ denotes the Dirac adjoint spinor

$$\overline{\psi} = \psi^\dagger A. \tag{3.29}$$

which transforms contragradient to ψ under Lorentz transformations. In our case we have $A = \gamma^0$. The charge conjugate spinor is related to its Dirac conjugate by the charge conjugation matrix

$$\psi^c_\alpha = C_{\alpha\beta}\overline{\psi}^\beta. \tag{3.30}$$

Chapter 4

Gauge covariant algebras

Gauge theories may be characterized by the symmetry (or gauge) algebra of the infinitesimal symmetry transformations. Although the approach of exploiting the structure of gauge algebras can be extended to open and reducible symmetry algebras [br96] I restrict the investigation to irreducible and closed algebras.

In the first part of this chapter I analyse the general structure of closed irreducible gauge covariant algebras and give some definitions and conventions. In the second part of the chapter I set up the BRST formalism. The BRST transformations of tensor fields, connections and ghost fields are given.

4.1 Algebra on tensor fields

First I discuss the structure of closed irreducibel gauge algebras realized on tensor fields $\mathcal{T}^i$, which are functions of some elemtary fields φ. A precise definition of tensor fields is given later. The infinitesimal symmetry transformations are implemented by operators Δ_M, which have an algebra on the tensor fields of the form

$$[\Delta_M, \Delta_N\} = -\mathcal{F}_{MN}{}^K(\mathcal{T})\Delta_K, \tag{4.1}$$

where $[.,.\}$ denotes the graded commutator

$$[\Delta_M, \Delta_N\} = \Delta_M, \Delta_N - (-)^{MN}\Delta_N, \Delta_M. \tag{4.2}$$

The structure functions are in general field dependent. According to (4.1) they are graded antisymmetric

$$\mathcal{F}_{MN}{}^K = -(-)^{MN}\mathcal{F}_{NM}{}^K \tag{4.3}$$

depending on the grading of the corresponding operators. The grading of the structure functions is given by

$$| \mathcal{F}_{MN}{}^{K} | = | M | + | N | + | K | \qquad mod\ 2. \tag{4.4}$$

They satisfy the Bianchi identities which arise from the Jacobi identity for graded commutators

$$\sum\!\!\!\!\!\!\bigcirc_{MNP} (-)^{MP} \left(\Delta_M \mathcal{F}_{NP}{}^{K} + \mathcal{F}_{MN}{}^{R} \mathcal{F}_{RP}{}^{K} \right) = 0 \tag{4.5}$$

and guarantee the consistency of the algebra. The cyclic sum in (4.5) is defined by

$$\sum\!\!\!\!\!\!\bigcirc_{MNP} (-)^{MP} X_{MNP} = (-)^{MP} X_{MNP} + (-)^{NM} X_{NPM} + (-)^{PN} X_{PMN}. \tag{4.6}$$

The sign convention in (4.6) is consistent with the fact that the graded commutator $[A, .\}$ acts on the commutator product $[B, C\}$ as a graded derivative and therefore according to

$$[A, [B, C\}\} = [[A, B\}, C\} + (-)^{AB} [B, [A, C\}\}, \tag{4.7}$$

which is equivalent to the Jacobi identity for graded commutators. The operators Δ_M are covariant in the sense that they map tensor fields to tensor fields. They are in general not defined on all fields so that (4.1) holds. In particular they are not defined on the gauge fields $\mathcal{A}_m^N$ and the ghost fields $\hat{C}^N$ which are introduced requiring that the exterior derivative d and the BRST-operator s act on tensor fields $\mathcal{T}$ according to

$$\begin{aligned} d\mathcal{T}^i &= \mathcal{A}^M \Delta_M \mathcal{T}^i & (4.8)\\ s\mathcal{T}^i &= \hat{C}^M \Delta_M \mathcal{T}^i. & (4.9) \end{aligned}$$

Inserting for the exterior derivative $d = dx^m \partial_m$ and the connection 1-forms $\mathcal{A}^M = dx^m \mathcal{A}_m^N$ in the first equation (4.8) the partial derivative reads

$$\partial_m \mathcal{T}^i = \mathcal{A}_m^M \Delta_M \mathcal{T}^i. \tag{4.10}$$

which holds identically in the fields and their derivatives. In general this requires the existence of a locally invertible subset of $\{\mathcal{A}_m^M\}$. This defines the Δ's corresponding to the invertible subset in terms of the partial derivative and the remaining Δ's

$$\begin{aligned} \{\mathcal{A}_m^N\} = \{e_m^a, \mathcal{A}_m^\mu\}, \qquad \{\Delta_M\} &= \{\mathcal{D}_a, \Delta_\mu\} & (4.11)\\ \mathcal{D}_a &= E_a{}^m \left(\partial_m - \mathcal{A}_m^\mu \Delta_\mu\right). & (4.12) \end{aligned}$$

This can be regarded as a definition of covariant derivatives. The invertible subset of $\{\mathcal{A}_m^M\}$ is identified with the vielbein and is denoted by e_m^a and its invers by $E_a{}^m$

$$e_m^a E_a{}^n = \delta_m^n, \qquad E_a{}^m e_m^b = \delta_a{}^b. \tag{4.13}$$

The commutator of partial derivatives is computed using equation (4.10) and the fact that $\Delta_M \mathcal{T}^i$ is a tensor field

$$\begin{aligned}
[\partial_m, \partial_n]\mathcal{T}^i &= \partial_m \left(\mathcal{A}_n{}^N \Delta_N \mathcal{T}^i\right) - (m \leftrightarrow n) \\
&= (\partial_m \mathcal{A}_n{}^M)\Delta_M \mathcal{T}^i + \mathcal{A}_n{}^N \mathcal{A}_m{}^M \Delta_M \Delta_N \mathcal{T}^i - (m \leftrightarrow n) \\
&= (\partial_m \mathcal{A}_n{}^P - \partial_n \mathcal{A}_m{}^P + \mathcal{A}_m{}^M \mathcal{A}_n{}^N \mathcal{F}_{NM}{}^P)\Delta_K \mathcal{T}^i. \qquad (4.14)
\end{aligned}$$

Since the partial derivatives commute and the Δ_M are assumed to be independent we find

$$\partial_m \mathcal{A}_n{}^P - \partial_n \mathcal{A}_m{}^P + \mathcal{A}_m{}^M \mathcal{A}_n{}^N \mathcal{F}_{NM}{}^P = 0, \qquad (4.15)$$

which can be solved for the field strengths $\mathcal{F}_{ab}{}^P$ using the invers vielbein

$$\mathcal{F}_{ab}{}^P = E_a{}^n E_b{}^m (\partial_n \mathcal{A}_m{}^P - \partial_m \mathcal{A}_n{}^P - e_m{}^c \mathcal{A}_n{}^\mu \mathcal{F}_{\mu c}{}^P - \mathcal{A}_m{}^\mu e_n{}^c \mathcal{F}_{c\mu}{}^P - \mathcal{A}_m{}^\mu \mathcal{A}_n{}^\nu \mathcal{F}_{\nu\mu}{}^P). \quad (4.16)$$

As a consequence of the commutativity of the partial derivative the $\mathcal{F}_{ab}{}^P$ do not count among the elementary fields but are given by the antisymmetrized first derivatives of the gauge fields $\mathcal{A}_m{}^P$ and the remaining $\mathcal{F}_{MN}{}^P$. Equation (4.16) must not imply any differential equations for the fields but holds algebraically in the elementary fields and their derivatives.

In order to make this formalism more explicit I split the operators Δ_M into the covariant derivative $\mathcal{D}_a$, the supersymmetry transformations $\mathcal{D}_{\underline{\alpha}}$[1] and internal symmetries δ_I including Lorentz transformations l_{ab}, Yang-Mills group actions δ_i, dilatations δ_W and R transformations δ_R (acting on $\mathcal{D}_{\underline{\alpha}}$ only):

$$\{\Delta_M\} = \{\mathcal{D}_A, \delta_I\}, \quad \{\mathcal{D}_A\} = \{\mathcal{D}_a, \mathcal{D}_{\underline{\alpha}}\}, \quad \{\delta_I\} = \{l_{ab}, \delta_i, \delta_W, \delta_R\} \qquad (4.17)$$

The algebra then reads

$$[\mathcal{D}_A, \mathcal{D}_B\} = -T_{AB}{}^C \mathcal{D}_C - F_{AB}{}^I \delta_I \qquad (4.18)$$

$$[\delta_I, \mathcal{D}_A\} = -g_{IA}{}^B \mathcal{D}_B \qquad (4.19)$$

$$[\delta_I, \delta_J] = f_{IJ}{}^K. \qquad (4.20)$$

This way the structure functions achieve different interpretations. The $T_{AB}{}^C$ are the torsions, the $F_{AB}{}^I$ are the field strenghts. The field strenghts corresponding to Lorentz transformations are called curvatures. The $g_{IA}{}^B$ are the representation matrices of the

[1]The underlined index denotes Dirac spinors. In D=2 dimensions this convention is dropped.

internal symmetries and the $f_{IJ}{}^K$ denote the structure constants of the Lie algebra. The structure functions and the corresponding symbols are collected in the following table :

$\mathcal{F}_{AB}{}^C = T_{AB}{}^C$	$\mathcal{F}_{IB}{}^C = g_{IB}{}^C$	$\mathcal{F}_{IJ}{}^C = 0$
$\mathcal{F}_{AB}{}^I = F_{AB}{}^I$	$\mathcal{F}_{IB}{}^J = 0$	$\mathcal{F}_{IJ}{}^K = -f_{IJ}{}^K$

(4.21)

Inserting this in the Bianchi identities (4.5) one obtains the first Bianchi identity as the coefficient of the space-time symmetry transformations $\mathcal{D}_A$ and the second Bianchi identity as the coefficient of the internal symmertry transformations δ_I

$$BI\ 1: \qquad \sum\!\!\!\!\!\!\bigcirc_{ABC} (-)^{AC} \left(\mathcal{D}_A T_{BC}{}^D + T_{AB}{}^E T_{EC}{}^D + F_{AB}{}^I g_{IC}{}^D\right) = 0 \tag{4.22}$$

$$BI\ 2: \qquad \sum\!\!\!\!\!\!\bigcirc_{ABC} (-)^{AC} \left(\mathcal{D}_A F_{BC}{}^I + T_{AB}{}^E F_{EC}{}^I\right), \tag{4.23}$$

where the cyclic sum is given as in (4.6). The remaining identities are

$$\delta_I F_{AB}{}^J = -g_{IA}{}^E F_{EB}{}^J - (-)^{AB} g_{IB}{}^E F_{EA}{}^J + (-)^{AI+BI} F_{AB}{}^K f_{KI}{}^J \tag{4.24}$$

$$\delta_I T_{AB}{}^C = -g_{IA}{}^E T_{EB}{}^C - (-)^{AB} g_{IB}{}^E T_{EA}{}^C - (-)^{IA+IB+IE} T_{AB}{}^E g_{IE}{}^C. \tag{4.25}$$

This means that the torsion tensor $T_{AB}{}^C$ and the field strenght $F_{AB}{}^J$ transform under δ_I according to their indices. Furthermore we find that the structure constant $f_{IJ}{}^K$ is an invariant tensor

$$\delta_I f_{JK}{}^L = 0. \tag{4.26}$$

This holds since I assume that the structure constants fulfill the Jacobi identity

$$\sum\!\!\!\!\!\!\bigcirc_{IJK} f_{IJ}{}^H f_{HK}{}^L. \tag{4.27}$$

Analogous the representation matrices $g_{IA}{}^B$ are invariant tensors due to the representation property of $g_{IA}{}^B$.

4.2 BRST operator

Introducing a ghost field $\hat{C}^M$ for every infinitesimal transformation Δ_M the BRST operator s is defined to act on tensor fields according to

$$s\mathcal{T}^i = \hat{C}^M \Delta_M \mathcal{T}^i \tag{4.28}$$

as already stated in the second equation of (4.8). The grading of the ghost fields is given by

$$|C^M| = |\Delta_M| + 1 \quad mod\ 2. \tag{4.29}$$

The BRST operator is an odd graded, nilpotent operator

$$\begin{aligned} |s| &= 1 && (4.30)\\ s^2 &= 0. && (4.31) \end{aligned}$$

Furthermore I assume that s commutes with the partial derivative and that it acts trivially on the coordinates x^m and the differentials dx^m

$$\begin{aligned} [s, \partial_m] &= 0 && (4.32)\\ sx^m &= 0 && (4.33)\\ sdx^m &= 0. && (4.34) \end{aligned}$$

The nilpotency of the BRST operator fixes the BRST transformations of the ghost fields $\hat{C}^P$ uniquely to

$$s\hat{C}^P = -(-)^M \frac{1}{2} \hat{C}^M \hat{C}^N \mathcal{F}_{NM}{}^P, \tag{4.35}$$

which may be checked for any closed and irreducible gauge algebra. The first equation in (4.32) allows to compute the BRST transformations of the connections $\mathcal{A}_m^M$

$$s\mathcal{A}_m{}^P = \partial_m \hat{C}^P - \hat{C}^M \mathcal{A}_m{}^N \mathcal{F}_{NM}{}^P. \tag{4.36}$$

The consistency of these transformations with $s^2 = 0$ is easily proved. Both $s^2 C^P = 0$ and $s^2 \mathcal{A}_m{}^P = 0$ are fulfilled due to the Bianchi identities.

4.3 Generalized connections

Due to the commutativity of the BRST operator s and the partial derivative ∂_m and the fact that s acts trivially on the differentials dx^m the BRST operater anticommutes with the exterior derivative

$$\{s, d\} = 0. \tag{4.37}$$

This may be used to define the new nilpotent operator $\tilde{s}$

$$\tilde{s} := s + d. \tag{4.38}$$

Introducing the new ghost variables

$$\tilde{C}^M = \hat{C}^M + \mathcal{A}^M \tag{4.39}$$

allows us to write

$$\tilde{s}\mathcal{T}^i = \tilde{C}^m \Delta_M \mathcal{T}^i \tag{4.40}$$

for $\tilde{s}$ acting on tensor fields. The formal identity of the algebras of s and $\tilde{s}$ implies

$$\tilde{s}\tilde{C}^P = -(-)^M \frac{1}{2} \tilde{C}^M \tilde{C}^N \mathcal{F}_{NM}{}^P . \tag{4.41}$$

Splitting (4.41) into parts with ghost number 0,1 and 2 yields

$$s\hat{C}^P = -(-)^M \frac{1}{2} \hat{C}^M \hat{C}^N \mathcal{F}_{NM}{}^P \tag{4.42}$$

$$s\mathcal{A}^{\ P} + d\hat{C}^P = -\hat{C}^M \mathcal{A}_m^{\ N} \mathcal{F}_{NM}{}^P \tag{4.43}$$

$$d\mathcal{A}^{\ P} = -(-)^M \frac{1}{2} \mathcal{A}^M \mathcal{A}^N \mathcal{F}_{NM}{}^P \tag{4.44}$$

which reproduces the BRST transformations of the ghosts $\hat{C}^P$ and the connections $\mathcal{A}_m^{\ P}$. The last equation is equivalent to (4.15).

I now make contact to the introduction where I introduced a new set of jet coordinates $\mathcal{B} = \{\mathcal{U}^i, \mathcal{V}^i, \mathcal{W}^l\}$. As already mentioned the $\mathcal{W}^i$'s are interpreted as generalized connections and tensor fields, while the $\mathcal{U}$'s and $\mathcal{V}$'s are called trivial pairs. Tensor fields are identified with those $\mathcal{W}^i$'s with vanishing total degree. Therefore tensor fields have necessarily vanishing ghost number and form degree. The $\mathcal{W}$'s with nonvanishing total degree are called generalized connections. Those with total degree 1 correspond to the $\tilde{C}$'s defined above.

Chapter 5

Bianchi identities

The next step following the discussion of the structure of gauge covariant algebras is to impose constraints. Supersymmetric gauge theories necessarily have constraints imposed. This is so, since the connections introduced so far yield highly reducible theories. There are two types of constraints to be distinguished namely conventional constraints and non-conventional constraints. Conventional constraints are innocent in the sense that they can be reached by covariant redefinitions of the connections (and ghost fields). They are used to bring the gauge algebra in a standard form. Different types of theories are obtained by a particular choice of non-conventional constraints. In four dimensions there are three known supergravity theories, called old minimal, new minimal and non-minimal supergravity. Consistency of the constraints requires that the Bianchi identities are fulfilled. They usually imply additional constraints and the general parametrization of the allowed field strenghts and torsions requires the introduction of auxiliary fields.

The choice of constraints and the evaluation of their consequences is a crucial point in the construction of supersymmetric theories, since the constraints must not imply equations of motion for the fields. On the other hand underconstrained theories do not allow equations of motion for the matter fields of the type we expect, have higher spins than desired, etc.

I discuss the procedure of imposing constraints and evaluating the Bianchi identities for the case of two dimensional super-Yang-Mills theory and for pure two dimensional supergravity seperately before coupling the $U(1)$ symmetry to supergravity. The tedious consistency check of the constraints is somewhat shortened using the fact that not all of the Bianchi identities are algebraically independent [so81]. One of these 'identities for identities' arises in supergravity and is known as a theorem of N. Dragon [dr79]. A sketch of the proof of this theorem is given.

5.1 Bianchi identities in D=2 Super-Yang-Mills theory

Starting with the covariant derivative and the well known covariantization of the spinor derivative

$$\mathcal{D}_a = \partial_a + A_a \delta_g, \qquad \mathcal{D}_\alpha = D_\alpha + A_\alpha \delta_g \tag{5.1}$$

we investigate the algebra of $\mathcal{D}_a, \mathcal{D}_\alpha$ and δ_g. The covariant derivatives do not commute but give rise to field strenghts

$$[\mathcal{D}_A, \mathcal{D}_B\} = -T_{AB}{}^C \mathcal{D}_C - F_{AB}\delta_g, \tag{5.2}$$

where as in the previous chapter $[.,.\}$ denotes the graded commutator, while the commutation relations

$$[\delta_g, \mathcal{D}_a] = [\delta_g, \mathcal{D}_\alpha] = 0 \tag{5.3}$$

correspond to the fact that space time symmetries are inert to gauge transformations. The only non vanishing torsion in flat space is given by

$$T_{\alpha\beta}{}^a = -2i(\gamma^a C)_{\alpha\beta}. \tag{5.4}$$

We now study the Bianchi identities to fix the algebra. The first Bianchi identity

$$\sum\!\!\!\!\!\!\bigcirc (-)^{AC}\left(\mathcal{D}_A T_{BC}{}^D + T_{AB}{}^E T_{EC}{}^D + F_{AB}{}^I g_{IC}{}^D\right) = 0 \tag{5.5}$$

is trivial in flat space with only internal symmetries. Before we study the second Bianchi identity, we observe that we can absorb parts of the field strength $F_{\alpha\beta}$ by a redefinition of the Yang-Mills field A_a. Computing the field strength in terms of the connections we find

$$F_{\alpha\beta} = -D_\alpha A_\beta - D_\beta A_\alpha + 2i(\gamma^a C)_{\alpha\beta} A_a \tag{5.6}$$

so that

$$F_{\alpha\beta} = 2i(\gamma_* C)_{\alpha\beta}\phi \tag{5.7}$$

can be imposed as conventional constraint. Since A_a can therefore be written in terms of covariant derivatives of A_α, the A_α's are called prepotentials. We will find that no additional constraint is needed.

The Bianchi identities for the field strenghts are

$$\sum(-)^{AC}\left(\mathcal{D}_A F_{BC}{}^i + T_{AB}{}^D F_{DC}{}^i\right) = 0. \tag{5.8}$$

These are called the second BI's. Explicitly the BI's read

$$\begin{aligned}
dim[3/2]: &\qquad \sum\left(\mathcal{D}_\alpha F_{\beta\gamma} + T_{\alpha\beta}{}^c F_{c\gamma}\right) &=& \; 0 \\
dim[2]: &\qquad \mathcal{D}_a F_{\alpha\beta} + \mathcal{D}_\alpha F_{\beta a} + \mathcal{D}_\beta F_{\alpha a} + T_{\alpha\beta}{}^b F_{ba} &=& \; 0 \\
dim[5/2]: &\qquad \mathcal{D}_a F_{b\alpha} + \mathcal{D}_b F_{\alpha a} + \mathcal{D}_\alpha F_{ab} &=& \; 0 \\
dim[3]: &\qquad \sum \mathcal{D}_a F_{bc} &=& \; 0
\end{aligned} \tag{5.9}$$

To solve the BI with dimension [3/2] we decompose the field strength $F_{a\alpha}$ into a spinor λ and a spin-vector χ

$$F_{a\alpha} = (\gamma_a)_\alpha{}^\beta \lambda_\beta + \chi_{a\alpha} \tag{5.10}$$

where χ carries the spin 3/2 component and satisfies the Rarita- Schwinger condition

$$(\gamma^a)_\alpha{}^\beta \chi_{a\beta} = 0 \tag{5.11}$$

Further we use our expression (5.7) for $F_{\alpha\beta}$ to find that $F_{a\alpha}$ contains no spin 3/2 component:

$$F_{a\alpha} = (\gamma_a)_\alpha{}^\beta \lambda_\beta \tag{5.12}$$

where λ_β is given as $\mathcal{D}_\alpha\phi = (\gamma_*\lambda)_\alpha$.[1] As a next step we solve the identity with dimension [2]. First we observe by contracting with $(\gamma^a)^{\alpha\beta}$ that $\mathcal{D}_\alpha\lambda^\alpha = 0$. Therefore $\mathcal{D}_\alpha\lambda_\beta$ may be decomposed in the basis of (γ)-matrices into

$$\mathcal{D}_\alpha\lambda_\beta = (\gamma_* C)_{\alpha\beta}\psi + (\gamma_b C)_{\alpha\beta} v^b. \tag{5.13}$$

Using this decomposition we find $\psi = i/2\varepsilon^{ab}F_{ab}$ and $v_a = i\varepsilon_a{}^b\mathcal{D}_b\phi$. Putting the pieces together we finally obtain

$$\mathcal{D}_\alpha\lambda_\beta = i/2(\gamma_* C)_{\alpha\beta}\varepsilon^{ab}F_{ab} - i(\gamma_*\gamma^a C)_{\alpha\beta}\mathcal{D}_a\phi. \tag{5.14}$$

To complete our results we find from the identity with dimension [5/2]

$$\mathcal{D}_\alpha F_{ab} = -(\gamma_b \mathcal{D}_a\lambda)_\alpha + (\gamma_a\mathcal{D}_b\lambda)_\alpha. \tag{5.15}$$

The identity with dimension [3] contains no further information since it is fulfilled identically. This completes the results of the investigation of the Bianchi identities.

[1] Here we introduced the shorthand $\phi\psi := \phi^\alpha\psi_\alpha$.

5.2 Bianchi identities in D=2 supergravity

We start our investigation of the Bianchi identities with the covariant derivative in curved space-time

$$\mathcal{D}_a = E_a^m(\partial_m - \frac{1}{2}\omega_m{}^{ab}l_{ab} - \chi_m^\alpha \mathcal{D}_\alpha) \tag{5.16}$$

where $E_a{}^m$ denotes the inverse vielbein

$$E_a{}^m e_m{}^b = \delta_a{}^b, \qquad e_m{}^a E_a{}^n = \delta_m{}^n, \tag{5.17}$$

which allows us to change freely between 'world-' and 'Lorentz-indices'. Lorentz transformations contribute to the covariant derivative through the Lorentz generators l_{ab}. The 'Lorentz-' or 'spin-connection' is denoted by ω_m^{ab}. χ_m^α denotes the Rarita-Schwinger field and describes the gravitino. The covariant derivative not containing the spinor derivatives

$$\hat{\mathcal{D}}_a = E_a^m(\partial_m - \frac{1}{2}\omega_m{}^{ab}l_{ab}). \tag{5.18}$$

may be used alternatively[2] but whether we use this form of the covariant derivative or the one given in (5.16) is a question of convenience [dr87]. The covariant derivative given in (5.16) is usually called supercovariant derivative. We start the investigation of the algebra of $\mathcal{D}_a, \mathcal{D}_\alpha, l_{ab}$ by imposing a set of constraints.

5.2.1 Constraints

The usual set of standard constraints reads

$$T_{\alpha\beta}{}^a = -2i(\gamma^a C)_{\alpha\beta}, \qquad T_{ab}{}^c = T_{\alpha\beta}{}^\gamma = 0. \tag{5.19}$$

All of them are conventional constraints, since they can be achieved by redefinitions of the connections. Especially the constraint $T_{ab}{}^c = 0$ is well known from general relativity and allows us to compute the spin-connection entirely in terms of the vielbein and the Rarita-Schwinger field.

Before we are going to solve the Bianchi identities with the constraints given in (5.19) we turn to a peculiarity in supergravity. Due to a theorem stated by N. Dragon [dr79] it is sufficient in supergravity to consider only the identities which arise as the coefficient of the covariant derivative, i.e. the first Bianchi identities. The second Bianchi identities then hold on account of the first ones and on account of the Ricci identity (5.20). Furthermore the curvature is completely expressible in terms of torsions and its covariant derivatives.

[2] Later I will use this form of the covariant derivative denoted with the symbol $\nabla_m = e_m^a \hat{\mathcal{D}}_a$ to express supertransformations of certain quantities.

5.2.2 Dragons theorem

The general commutator relation or Ricci identity

$$[\mathcal{D}_A, \mathcal{D}_B] = -T_{AB}{}^C \mathcal{D}_C - F_{AB}{}^I \delta_I \tag{5.20}$$

implies the Bianchi identities via the Jacobi identity for graded commutators. The first Bianchi identity arises as the coefficient of the covariant derivative, while the second Bianchi identity arises as the coefficient of the structure group transformation. To give an idea of the proof we first introduce convenient shorthands for the identities

$$\oint\!\!\!\!\!\!\sum (-)^{AC} \left(\mathcal{D}_A T_{BC}{}^D + T_{AB}{}^E T_{EC}{}^D + R_{ABC}{}^D \right) = I^{(1)}_{ABC}{}^D \tag{5.21}$$

$$\oint\!\!\!\!\!\!\sum (-)^{AC} \left(\mathcal{D}_A R_{BCD}{}^E + T_{AB}{}^F R_{FCD}{}^E \right) = I^{(2)}_{ABCD}{}^E \tag{5.22}$$

where the field strenghts corresponding to Lorentz transformations, i.e. the curvatures, are denoted by $R_{AB}{}^{cd} g_{(ab)C}{}^D = R_{ABC}{}^D$. From the second Bianchi identity we take the part that is antisymmetric in all lower indices

$$I^{(2)}_{ABCD}{}^E - I^{(2)}_{BCDA}{}^E + I^{(2)}_{CDAB}{}^E - I^{(2)}_{DABC}{}^E = 0. \tag{5.23}$$

This equation holds on account of the first Bianchi identity and the Ricci identity (5.20) applied to the torsion

$$[\mathcal{D}_A, \mathcal{D}_B] T_{CD}{}^E = -T_{AB}{}^F \mathcal{D}_F T_{CD}{}^E + R_{ABC}{}^F T_{FD}{}^E + R_{ABD}{}^F T_{CF}{}^E - R_{ABF}{}^E T_{CD}{}^F. \tag{5.24}$$

This is a nontrivial statement although equation (5.24) is a special form of the general commutator relation (5.20) which implies all Bianchi identities. Nevertheless equation (5.24) does in general not imply the second Bianchi identity, for example not in Riemannian geometry. To proove Dragons theorem one groups together cyclic sums of curvatures in equation (5.23) and replaces them by making use of the first Bianchi identity. In this way second derivatives of the torsion appear in terms of commutators, which one replaces by using the Ricci identity (5.24). Then one ends up with the equation

$$\begin{aligned} &T_{AB}{}^F I^{(1)}_{FCD}{}^E - T_{BC}{}^F I^{(1)}_{FDA}{}^E + T_{CD}{}^F I^{(1)}_{FAB}{}^E - T_{DA}{}^F I^{(1)}_{FBC}{}^E + T_{AC}{}^F I^{(1)}_{FDB}{}^E + \\ &T_{DB}{}^F I^{(1)}_{FAC}{}^E + I^{(1)}_{ABC}{}^F T_{FD}{}^E - I^{(1)}_{BCD}{}^F T_{FA}{}^E + I^{(1)}_{CDA}{}^F T_{FB}{}^E - I^{(1)}_{DAB}{}^F T_{FC}{}^E = 0 \end{aligned} \tag{5.25}$$

which holds on account of the first Bianchi identity.

Now one has to use the restrictions on the curvature to show that equation (5.23) already implies the second Bianchi identity

$$I^{(2)}_{ABCD}{}^E = 0. \tag{5.26}$$

First we observe that the curvature is a generator-valued two form [wz77]. In N-extended supergravity the gauge group is restricted to be $SL(2,C) \times G$, where $G \subset U(N)$. Since spinor and vector components are not mixed by gauge transformations all components of the curvature $R_{ABC}{}^{D}$ where C and D belong to different representations of $SL(2,C)$ have to vanish. This restriction makes it possible to express all components of the curvature in terms of torsions without the use of torsion constraints. Furthermore it is straightforward to check that the second Bianchi identity follows from equation (5.23) since the last index pair in $I^{(2)}_{ABCD}{}^{E}$ has the properties of a generator. For example if D, E are given to be vectorial chosing A, B, C to be spinorial gives immediately

$$I^{(2)}_{\alpha\beta\gamma a}{}^{b} - I^{(2)}_{\beta\gamma a\alpha}{}^{b} + I^{(2)}_{\gamma a\alpha\beta}{}^{b} - I^{(2)}_{a\alpha\beta\gamma}{}^{b} = I^{(2)}_{\alpha\beta\gamma a}{}^{b} = 0 \tag{5.27}$$

Similar calculations for all other index combinations complete the proof. The arguments given here hold for N-extended supergravity in arbitrary dimensions.

This theorem shortens the consistency checks in supergravity and furthermore shows that constraints can be formulated as conditions for the torsions since the curvatures are completely expressible in terms of torsions.

5.2.3 Solution of the Bianchi Identities

The first BI's read

$$\sum (-)^{AC} \left(\mathcal{D}_A T_{BC}{}^{D} + T_{AB}{}^{E} T_{EC}{}^{D} + F_{AB}{}^{I} g_{IC}{}^{D} \right) = 0. \tag{5.28}$$

This equation determines the curvature completely in terms of the torsion and its covariant derivatives. In our case the generators $g_{IC}{}^{D}$ take the following simple form

$$(g_{ab})_c{}^{d} = \varepsilon_{ab}\varepsilon_c{}^{d}, \qquad (g_{ab})_\alpha{}^{\beta} = -\frac{1}{2}\varepsilon_{ab}(\gamma_*)_\alpha{}^{\beta}. \tag{5.29}$$

Writing $R_{ABC}{}^{D}$ for $R_{AB}{}^{I} g_{IC}{}^{D}$ the curvatures read

$$R_{ABc}{}^{d} = -R_{AB}\frac{1}{2}\varepsilon^{ab}\varepsilon_{ab}\varepsilon_c{}^{d} = R_{AB}\varepsilon_c{}^{d} \tag{5.30}$$

$$R_{AB\alpha}{}^{\beta} = R_{AB}\frac{1}{2}\varepsilon^{ab}\frac{1}{2}\varepsilon_{ab}(\gamma_*)_\alpha{}^{\beta} = -\frac{1}{2}R_{AB}(\gamma_*)_\alpha{}^{\beta} \tag{5.31}$$

where $R_{AB}{}^{I} = R_{AB}{}^{ab} = -R_{AB}\frac{1}{2}\varepsilon^{ab}$ was used. Then the Bianchi identities take the following form, where we already used our set of constraints given in (5.19) :

$$
\begin{array}{llrcl}
\dim[1/2]: & {}_{\alpha\beta\gamma}{}^{d} \Rightarrow & \sum T_{\alpha\beta}{}^{e}T_{e\gamma}{}^{d} & = & 0 \\
\dim[1]: & {}_{\alpha\beta\gamma}{}^{\delta} \Rightarrow & \sum T_{\alpha\beta}{}^{e}T_{e\gamma}{}^{\delta} + R_{\alpha\beta\gamma}{}^{\delta} & = & 0 \\
 & {}_{\alpha\beta c}{}^{d} \Rightarrow & R_{\alpha\beta c}{}^{d} + T_{c\alpha}{}^{\gamma}T_{\gamma\beta}{}^{d} + T_{c\beta}{}^{\gamma}T_{\gamma\alpha}{}^{d} & = & 0 \\
\dim[3/2]: & {}_{\alpha ab}{}^{c} \Rightarrow & R_{\alpha ab}{}^{c} + R_{b\alpha a}{}^{c} + T_{ab}{}^{\varepsilon}T_{\varepsilon\alpha}{}^{c} & = & 0 \\
 & {}_{\alpha a\beta}{}^{\gamma} \Rightarrow & \mathcal{D}_{\alpha}T_{a\beta}{}^{\gamma} + \mathcal{D}_{\beta}T_{a\alpha}{}^{\gamma} - R_{a\alpha\beta}{}^{\gamma} - R_{a\beta\alpha}{}^{\gamma} - T_{\beta\alpha}{}^{e}T_{ea}{}^{\gamma} & = & 0 \\
\dim[2]: & {}_{abc}{}^{d} \Rightarrow & R_{abc}{}^{d} + R_{bca}{}^{d} + R_{cab}{}^{d} & = & 0 \\
 & {}_{ab\alpha}{}^{\beta} \Rightarrow & \mathcal{D}_{a}T_{b\alpha}{}^{\beta} - \mathcal{D}_{b}T_{a\alpha}{}^{\beta} + T_{a\alpha}{}^{\gamma}T_{b\gamma}{}^{\beta} - T_{b\alpha}{}^{\gamma}T_{a\gamma}{}^{\beta} + \mathcal{D}_{\alpha}T_{ab}{}^{\beta} + R_{ab\alpha}{}^{\beta} & = & 0 \\
\dim[5/2]: & {}_{abc}{}^{\delta} \Rightarrow & \sum \mathcal{D}_{a}T_{bc}{}^{\delta} + T_{ab}{}^{\gamma}T_{\gamma c}{}^{\delta} & = & 0
\end{array}
\tag{5.32}
$$

From the BI for dim=[1/2] we learn that $T_{\alpha a}{}^{b} = 0$. In order to solve the Bianchi identities with dim=[1] we decompose $R_{\alpha\beta}$ and $T_{a\alpha}{}^{\beta}$ in our basis of γ-matrices :

$$R_{\alpha\beta} = R_a(\gamma^a C)_{\alpha\beta} + R^*(\gamma_* C)_{\alpha\beta} \tag{5.33}$$
$$T_{a\alpha}{}^{\beta} = t_a\delta_{a}{}^{\beta} + t_{a}{}^{b}(\gamma_b)_{\alpha}{}^{\beta} + t^*_a(\gamma_*)_{\alpha}{}^{\beta} \tag{5.34}$$

Here we used $R_{\alpha\beta} = R_{\beta\alpha}$ in the first equation, hence $R_{\alpha\beta}$ contains no term proportional to $\varepsilon_{\alpha\beta}$. After some algebra one ends up with the equation

$$R^* = -2it_{c}{}^{c} \tag{5.35}$$

and all other terms vanishing. Parametrizing the solution by a scalar field $R^* = iS$ we find

$$R_{\alpha\beta} = iS(\gamma_* C)_{\alpha\beta} \tag{5.36}$$
$$T_{a\alpha}{}^{\beta} = \frac{1}{4}S(\gamma_a)_{\alpha}{}^{\beta}. \tag{5.37}$$

From the equations for dim[3/2] one obtains

$$R_{a\alpha} = 2iT^{\beta}(\gamma_a C)_{\beta\alpha} \tag{5.38}$$

by inserting the constraints into the Bianchi identity with the index picture ${}_{\alpha ab}{}^{c}$. This yields

$$\varepsilon_{bc}R_{\alpha a} + \varepsilon_{ac}R_{b\alpha} = 2i\varepsilon_{ab}T^{\varepsilon}(\gamma_c C)_{\varepsilon\alpha}. \tag{5.39}$$

where we used the antisymmetry of the torsion $T_{ab}{}^{\alpha} = \varepsilon_{ab}T^{\alpha}$ and the generator valuedness of the curvature. By contraction with ε^{ab} and use of $R_{\alpha a} = -R_{a\alpha}$ we obtain the result

$$R_{a\alpha} = 2iT^{\beta}(\gamma_a C)_{\beta\alpha} \tag{5.40}$$

Inserting this equation into the Bianchi identity with the index picture ${}_{a\beta\alpha}{}^{\gamma}$ we obtain

$$T_{\alpha} = \frac{1}{4}(\gamma_*)_{\alpha}{}^{\beta}\mathcal{D}_{\beta}S \tag{5.41}$$

This is easily checked using $R_{a\alpha\beta}{}^{\gamma} = -1/2(\gamma_*)_{\beta}{}^{\gamma}R_{a\alpha}$ and the second equation in (5.36). At dimension 2 the identity with the index picture ${}_{abc}{}^{d}$ reads

$$\varepsilon_{cd}R_{ab} + \varepsilon_{ad}R_{bc} + \varepsilon_{bd}R_{ca} = 0. \tag{5.42}$$

Again using $R_{ab} = \varepsilon_{ab}R$ we find that this identity is satisfied on account of the identity in two dimensions

$$\varepsilon_{ab}\varepsilon_{cd} + \varepsilon_{bc}\varepsilon_{ad} + \varepsilon_{ca}\varepsilon_{bd} = 0. \tag{5.43}$$

Further we obtain from the bianchi identity with index picture ${}_{ab\alpha}{}^{\beta}$

$$R = \frac{1}{4}(S^2 + \mathcal{D}^2 S), \tag{5.44}$$

where $\mathcal{D}^2$ denotes $\mathcal{D}^{\alpha}\mathcal{D}_{\alpha}$.

Therefore as the result of the investigation of the Bianchi identities our non-vanishing torsions and curvatures are given as

$$\begin{aligned}
T_{\alpha\beta}{}^{a} &= 2i(\gamma^a C)_{\alpha\beta} \\
T_{a\alpha}{}^{\beta} &= \frac{1}{4}S(\gamma_a)_{\alpha}{}^{\beta} \\
R_{\alpha\beta} &= iS(\gamma_* C)_{\alpha\beta} \\
T_{ab}{}^{\alpha} &= \frac{1}{4}\varepsilon_{ab}(\gamma *)^{\alpha\beta}\mathcal{D}_{\beta}S \\
R_{a\alpha} &= -\frac{1}{2}(\gamma_a\gamma_*)_{\alpha}{}^{\beta}\mathcal{D}_{\beta}S \\
R_{ab} &= \frac{1}{4}\varepsilon_{ab}(S^2 + \mathcal{D}^2 S)
\end{aligned} \tag{5.45}$$

5.3 Supergravity with U(1) symmetry

Considering supergravity with additional U(1) internal symmetry changes our previous results obtained for pure supergravity and the Yang-Mills case seperatly in an obvious way. So the additional torsions ${T_{a\alpha}}^{\beta}$ and ${T_{ab}}^{\alpha}$ appear in the Bianchi identities for the field strengths (5.8). We find

$$\begin{aligned}
dim[3/2]: \quad & \textstyle\sum\!\!\!\!\!\bigcirc \left(\mathcal{D}_\alpha F_{\beta\gamma} + {T_{\alpha\beta}}^{c} F_{c\gamma}\right) &=& \ 0 \\
dim[2]: \quad & \mathcal{D}_a F_{\alpha\beta} + \mathcal{D}_\alpha F_{\beta a} + \mathcal{D}_\beta F_{\alpha a} + {T_{\alpha\beta}}^{b} F_{ba} + {T_{a\alpha}}^{\gamma} F_{\gamma\beta} + {T_{a\beta}}^{\gamma} F_{\gamma\alpha} &=& \ 0 \\
dim[5/2]: \quad & \mathcal{D}_a F_{b\alpha} + \mathcal{D}_b F_{\alpha a} + \mathcal{D}_\alpha F_{ab} + {T_{ab}}^{\beta} F_{\beta\alpha} + {T_{b\alpha}}^{\gamma} F_{\gamma a} - {T_{a\alpha}}^{\gamma} F_{\gamma b} &=& \ 0 \\
dim[3]: \quad & \textstyle\sum\!\!\!\!\!\bigcirc\, \mathcal{D}_a F_{bc} + {T_{ab}}^{\gamma} F_{\gamma c} &=& \ 0
\end{aligned} \tag{5.46}$$

This gives additional contributions to equations (5.14) and (5.15)

$$\mathcal{D}_\alpha \lambda_\beta \ = \ i(\gamma^a \gamma_* C)_{\alpha\beta} \mathcal{D}_a \phi + i/2(\gamma_* C)_{\alpha\beta} \varepsilon^{ab} F_{ab} + i(\gamma_* C)_{\alpha\beta} S\phi \tag{5.47}$$

$$\mathcal{D}_\alpha F_{ab} \ = \ -(\gamma_b \mathcal{D}_a \lambda)_\alpha + (\gamma_a \mathcal{D}_b \lambda)_\alpha - 2i{T_{ab}}^{\beta}(\gamma_* C)_{\beta\alpha}\phi + \frac{1}{2}\varepsilon_{ab} S(\gamma_* \lambda)_\alpha \tag{5.48}$$

The Bianchi identity with dimension [3] remains trivial. The results of the supergravity part remain unchanged, since the commutators of δ_g with $\mathcal{D}_a$ and $\mathcal{D}_\alpha$ and therefore the corresponding ${g_{IA}}^{B}$'s all vanish.

Chapter 6

Field content and BRST transformations

6.1 Field content

As already mentioned in section (4.1) the invertibility of the vielbein allows it to solve the equation

$$[\partial_m, \partial_n] = 0 = [\mathcal{A}_m{}^M \Delta_M, \mathcal{A}_n{}^N \Delta_N], \tag{6.1}$$

for the field strengths $\mathcal{F}_{ab}{}^N$. As a consequence the field strengths with lower Lorentz indices do not count among the elementary fields. From equation (6.1) we obtain

$$\partial_m \mathcal{A}_n{}^P - \partial_n \mathcal{A}_m{}^P + \mathcal{A}_m{}^M \mathcal{A}_n{}^N \mathcal{F}_{NM}{}^P = 0. \tag{6.2}$$

Using the inverse vielbein we find

$$\mathcal{F}_{ab}{}^N = E_a{}^n E_b{}^m \left(\partial_n \mathcal{A}_m{}^N - \partial_m \mathcal{A}_n{}^N - e_m{}^c \mathcal{A}_n{}^\nu \mathcal{F}_{\nu c}{}^N - \mathcal{A}_m{}^\mu e_n{}^c \mathcal{F}_{c\mu}{}^N - \mathcal{A}_m{}^\mu \mathcal{A}_n{}^\nu \mathcal{F}_{\nu\mu}{}^N\right), \tag{6.3}$$

where we again used the notation introduced in section (4.1)

$$\{\mathcal{A}_m{}^N\} = \{e_m{}^a, \mathcal{A}_m{}^\mu\}, \qquad \{\Delta_N\} = \{\mathcal{D}_m, \Delta_\mu\} \tag{6.4}$$

for our set of connections and Δ-operations. This allows us together with the constraint $T_{ab}{}^c = 0$ to compute the spin connection in terms of the vielbein and the gravitino

$$\begin{aligned} \omega_m{}^{ab} &= E^{an} E^{bk} \left(\omega_{[mn]k} - \omega_{[nk]m} + \omega_{[km]n}\right) \\ \omega_{[mn]k} &= e_{kd} \partial_{[n} e_{m]}{}^d - i \chi_n \gamma_k \chi_m. \end{aligned} \tag{6.5}$$

For the other field strenghts we find

$$\begin{aligned} T_{ab}{}^{\alpha} &= E_a{}^n E_b{}^m \left(\nabla_n \chi_m{}^{\alpha} - \nabla_m \chi_n{}^{\alpha} + (e_m^c \chi_n{}^{\beta} - \chi_m{}^{\beta} e_n{}^c) T_{c\beta}{}^{\alpha}\right) \\ &= E_a{}^n E_b{}^m \left(\nabla_n \chi_m{}^{\alpha} - \nabla_m \chi_n{}^{\alpha} + \frac{1}{4} S(\chi_n \gamma_m)^{\alpha} - \frac{1}{4} S(\chi_m \gamma_n)^{\alpha}\right) \qquad (6.6) \\ R_{ab}{}^{cd} &= E_a{}^n E_b{}^m \Big(\partial_n \omega_m{}^{cd} - \partial_m \omega_n{}^{cd} - \omega_n{}^{ce} \omega_{me}{}^d + \omega_m{}^{ce} \omega_{ne}{}^d \\ &\qquad - (\chi_m^{\alpha} e_n{}^e - e_m^e \chi_n{}^{\alpha}) R_{e\alpha}{}^{cd} - \chi_m{}^{\alpha} \chi_n{}^{\beta} R_{\beta\alpha}{}^{cd}\Big) \qquad (6.7) \\ F_{ab} &= E_a{}^n E_b{}^m \left(\partial_n A_m - \partial_m A_n - (\chi_m^{\alpha} e_n{}^c - e_m^c \chi_n{}^{\alpha}) F_{c\alpha} - \chi_m^{\beta} \chi_n{}^{\alpha} F_{\alpha\beta}\right) \\ &= E_a{}^n E_b{}^m \left(\partial_n A_m - \partial_m A_n - (\chi_m \gamma_n \lambda) + (\chi_n \gamma_m \lambda) - 2i(\chi_m \gamma_* C \chi_n)\phi\right) \quad (6.8) \end{aligned}$$

with

$$\nabla_n \chi_m{}^{\alpha} = \partial_n \chi_m{}^{\alpha} - \frac{1}{2} \omega_n{}^{ab} l_{ab} \chi_m{}^{\alpha}. \qquad (6.9)$$

Together with the results of the previous chapter equation (6.6) allows us to solve equation (5.45) for the supersymmetry transformation of the auxiliary field S. Using

$$T_{ab}{}^{\alpha} = \frac{1}{4} \varepsilon_{ab} (\gamma *)^{\alpha\beta} \mathcal{D}_{\beta} S \qquad (6.10)$$

one obtains

$$\mathcal{D}_{\alpha} S = -4(\gamma_* C)_{\alpha\beta} \varepsilon^{nm} \nabla_n \chi_m{}^{\beta} + i(\gamma^m C)_{\alpha\beta} \chi_m^{\beta} S. \qquad (6.11)$$

6.2 BRST-Transfomations

The BRST-operator s was introduced in section (4.1) requiring that s acts on tensor fields according to

$$s\mathcal{T}^i = \hat{C}^M \Delta_M \mathcal{T}^i, \qquad (6.12)$$

where $\hat{C}^M$ denotes the ghosts introduced for every infinitesimal symmetry transformation Δ_M. Further we require the nilpotency of the BRST-operator and that it commutes with the partial derivative

$$s^2 = [s, \partial_m] = 0. \qquad (6.13)$$

From $s^2 = 0$ we obtain the BRST-transformations of the ghosts and $[s, \partial_m] = 0$ gives the BRS-transformations of the connections $\mathcal{A}_m{}^M$, since the partial derivative on tensor fields reads

$$\partial_m \mathcal{T} = \mathcal{A}_m{}^M \Delta_M \mathcal{T} \qquad (6.14)$$

Thus we obtain

$$
\begin{aligned}
s\hat{C}^P &= -\frac{1}{2}(-)^N \hat{C}^N \hat{C}^M \mathcal{F}_{MN}{}^P && (6.15)\\
s\mathcal{A}_m{}^P &= \partial_m \hat{C}^P - \hat{C}^M \mathcal{A}_m{}^N \mathcal{F}_{NM}{}^P . && (6.16)
\end{aligned}
$$

A more familiar form of the transformations is obtained by a redefinition of the ghosts

$$
\hat{C}^\mu = C^\mu + C^m \mathcal{A}_m{}^\mu, \qquad C^m = \hat{C}^a E_a{}^m. \tag{6.17}
$$

The BRST-transformations then read

$$
\begin{aligned}
s\mathcal{A}_m{}^P &= C^n \partial_n \mathcal{A}_m{}^P + (\partial_m C^n)\mathcal{A}_n{}^P + \partial_m C^P - C^N \mathcal{A}_m{}^M \mathcal{F}_{MN}{}^P && (6.18)\\
sC^n &= C^m \partial_m C^n - \frac{1}{2}(-)^M C^M C^N \mathcal{F}_{NM}{}^n && (6.19)\\
sC^\mu &= C^m \partial_m C^\mu - \frac{1}{2}(-)^M C^M C^N \left(\mathcal{F}_{NM}{}^\mu - \mathcal{F}_{NM}{}^m \mathcal{A}_m{}^\mu\right) && (6.20)
\end{aligned}
$$

Since the ghost C^m corresponds to the vector field entering the Lie derivative we rename the diffeomorphism ghost and the supersymmetry ghost

$$
C^m \equiv \xi^m, \qquad C^\alpha \equiv \xi^\alpha \tag{6.21}
$$

Applying these results to the case of D=2 supergravity including a $U(1)$-transformation we find for the BRST-transformations of the connections

$$
\begin{aligned}
se_m{}^a &= \xi^n \partial_n e_m{}^a + (\partial_m \xi^n) e_n{}^a - \xi^\alpha \chi_m{}^\beta T_{\alpha\beta}{}^a + C_b{}^a e_m{}^b\\
s\chi_m{}^\alpha &= \xi^n \partial_n \chi_m{}^\alpha + (\partial_m \xi^n)\chi_n{}^\alpha + \partial_m \xi^\alpha - \xi^\beta e_m{}^a T_{a\beta}{}^\alpha\\
&\quad + \frac{1}{4}\xi^\beta \omega_m{}^{ab} \varepsilon_{ab} (\gamma_*)_\beta{}^\alpha - \frac{1}{4} C^{ab} \chi_m{}^\beta \varepsilon_{ab} (\gamma_*)_\beta{}^\alpha\\
s\omega_m{}^{ab} &= \xi^n \partial_n \omega_m{}^{ab} + (\partial_m \xi^n)\omega_n{}^{ab} + \partial_m C^{ab} - \xi^\beta \chi_m{}^\alpha R_{\alpha\beta}{}^{ab}\\
&\quad - \xi^\alpha e_m{}^c R_{c\alpha}{}^{ab} + C^{ac}\omega_{mc}{}^b - C^{bc}\omega_{mc}{}^a\\
sA_m &= \xi^n \partial_n A_m + (\partial_m \xi^n) A_n + \partial_m C\\
&\quad - \xi^\alpha \chi_m{}^\beta F_{\beta\alpha} - \xi^\alpha e_m{}^a F_{a\alpha} && (6.22)
\end{aligned}
$$

The transformations of the ghosts then read

$$
\begin{aligned}
s\xi^n &= \xi^m \partial_m \xi^n + \frac{1}{2}\xi^\alpha \xi^\beta T_{\beta\alpha}{}^n\\
s\xi^\alpha &= \xi^m \partial_m \xi^\alpha - \frac{1}{2}\xi^\gamma \xi^\beta T_{\beta\gamma}{}^m \chi_m{}^\alpha - \frac{1}{4} C^{ab} \xi^\beta \varepsilon_{ab} (\gamma_*)_\beta{}^\alpha\\
sC^{ab} &= \xi^m \partial_m C^{ab} + \frac{1}{2}\xi^\alpha \xi^\beta R_{\beta\alpha}{}^{ab} - \frac{1}{2}\xi^\alpha \xi^\beta T_{\beta\alpha}{}^m \omega_m{}^{ab}\\
sC &= \xi^m \partial_m C + \frac{1}{2}\xi^\alpha \xi^\beta F_{\beta\alpha} - \frac{1}{2}\xi^\alpha \xi^\beta T_{\beta\alpha}{}^m A_m && (6.23)
\end{aligned}
$$

The BRST transformation of the gravitational auxiliary field S is obtained from (6.12) by using (6.11)

$$sS = \xi^n \partial_n S - 4\xi^\gamma (\gamma_* C)_{\gamma\alpha} \varepsilon^{nm} \Delta_n \chi_m{}^\alpha + i\,\xi^\gamma (\gamma^m C)_{\gamma\alpha} \chi_m{}^\alpha S. \tag{6.24}$$

Eventually the BRST transformations of the gaugino λ_b, the Yang-Mills auxiliary field ϕ and the field strength F_{ab} are obtained from (6.12). The supersymmetry transformations contained in the BRST transformations are taken from the solutions of the Bianchi identities

$$\begin{aligned}
s\phi &= \xi^n \partial_n \phi + \xi^\alpha (\gamma_*)_\alpha{}^\beta \lambda_\beta \\
s\lambda_\beta &= \xi^n \partial_n \lambda_\beta + \xi^\alpha \left(\frac{i}{2} (\gamma_* C)_{\alpha\beta} \varepsilon^{ab} F_{ab} - i(\gamma_* \gamma^a C)_{\alpha\beta} \mathcal{D}_a \phi + i(\gamma_* C)_{\alpha\beta} S\phi \right) \\
&\quad + \frac{1}{4} C^{ab} \varepsilon_{ab} (\gamma_*)_\beta{}^\gamma \lambda_\gamma \\
sF_{ab} &= \xi^n \partial_n F_{ab} + \xi^\alpha \left((\gamma_a \mathcal{D}_b \lambda)_\alpha - (\gamma_b \mathcal{D}_a \lambda)_\alpha - 2iT_{ab}{}^\beta (\gamma_* C)_{\beta\alpha} \phi + \frac{1}{2} \varepsilon_{ab} S (\gamma_* \lambda)_\alpha \right) \\
&\quad - \frac{1}{2} C_a{}^e F_{eb} - \frac{1}{2} C_b{}^e F_{ae}
\end{aligned} \tag{6.25}$$

Chapter 7

Beltrami Parametrization

Beltrami differentials parametrize conformal classes of two-dimensional metrics. Therefore conformal properties of two-dimensional field theories coupled to gravity are most conveniently discussed in terms of these quantities.

In the Beltrami parametrization the BRST algebra factorizes into two independent substructures. It is the aim of this chapter to line out the crucial steps in the construction of the Beltrami differentials and the Beltrami ghost variables found by Becchi [be88]. In the supersymmetric case the Beltrami variables acquire suitable supersymmetric partners such that the factorization of the BRST algebra remains manifestly realized [dg90].

7.1 Two approaches

Beltrami coefficients may be introduced in two different though equivalent ways. In the so called Riemannian surface formalism they parametrize complex structures. In the 'metric approach' the Beltrami differentials parametrize conformal classes of metrics. First a short introduction to both approaches is given following closely the lines of [dg90], where a few remarks on light-cone coordinates are included. Then before turning to the supersymmetric generalization in the following chapter the bosonic case is investigated in some detail using the metric approach.

7.1.1 Parametrization of complex structures

In this approach one works on a Riemann surface M, i.e. a connected, topological 2-manifold which is equipped with a complex structure or equivalently with a conformal class of metrics. A complex structure is a collection of complex local coordinates. Those local coordinates have the property that two sets of such local coordinates are related by a holomorphic coordinate transformation. We consider the line element ds^2 in terms of isothermal coordinates dZ

$$\begin{aligned} dZ &= \lambda(z,\bar{z})[dz + d\bar{z}\,\mu(z,\bar{z})] && (7.1)\\ d\bar{Z} &= \bar{\lambda}(z,\bar{z})[d\bar{z} + dz\,\bar{\mu}(z,\bar{z})] && (7.2)\\ ds^2 &\propto |dZ|^2, && (7.3) \end{aligned}$$

where $(z,\bar{z})$ denotes a reference system of holomorphic coordinates. The functions $\lambda(z,\bar{z})$ and $\mu(z,\bar{z})$ are smooth complex valued. The condition $d(dZ) = 0$ implies that $\lambda(z,\bar{z})$ satisfies the differential equation

$$(\bar{\partial} - \mu\partial)\lambda = (\partial\mu)\lambda, \qquad \bar{\partial} := \partial_{\bar{z}}, \qquad \partial := \partial_z \tag{7.4}$$

and may therefore be viewed as an integrating factor for the relation $d(dZ) = 0$. The function μ is called Beltrami differential.

Considering infinitesimal changes of coordinates generated by a vector field $(\xi,\bar{\xi})$ the induced variaton of $Z(z,\bar{z})$ is described by the Lie derivative

$$\delta_\xi Z = L_\xi Z = i_\xi(dZ) = \lambda[\xi + \bar{\xi}\mu]. \tag{7.5}$$

The transformation laws for λ and μ are obtained by evaluating the variation of dZ in two different ways

$$\delta_\xi(dZ) = d(\delta_\xi Z) \tag{7.6}$$

and comparing the coefficients of dz and $d\bar{z}$. This yields [dg90]

$$\begin{aligned} \delta_\xi\mu &= (\bar{\partial} - \mu\partial + (\partial\mu))(\xi + \bar{\xi}\mu) && (7.7)\\ \delta_\xi\lambda &= \partial((\xi + \bar{\xi}\mu)\lambda). && (7.8) \end{aligned}$$

The algebra of the BRST operator is obtained by replacing the vector field $(\xi,\bar{\xi})$ by the diffeomorphism ghosts $(c,\bar{c})$ and replacing δ_ξ by s. The nilpotency of s requires

$$s^2 Z = 0 = (sC - C\partial C)\lambda \tag{7.9}$$

and thereby

$$sC = C\partial C \tag{7.10}$$

where Becchi's [be88] reparametrization of the ghost fields is used

$$C = c + \bar{c}\,\mu \tag{7.11}$$

which ensures the holomorphic factorization of the BRST variations of μ and λ.

7.1.2 The 'metric approach'

The metric approach works on a Riemannian 2-manifold (M,g). The line element can always be written as

$$ds^2 = |\rho|^2 |dz + d\bar{z}\mu|^2, \tag{7.12}$$

where the complex notation

$$dz = dx + idy, \qquad d\bar{z} = dx - idy, \tag{7.13}$$

has been introduced. The smooth and complex valued functions $\rho(z,\bar{z})$ and $\mu(z,\bar{z})$ are called the conformal factor and the Beltrami coefficient. Equation (7.12) is referred to as the Beltrami parametrization of the metric. Since the Beltrami coefficient is inert under structure group transformations consisting of Weyl and Lorentz transformations it parametrizes conformal classes of metrics. Weyl transformations of the metric are simply rescalings of the conformal factor ρ.

Considering infinitesimal reparametrizations of the manifold one determines the transformation laws of ρ and μ to be [dg90]

$$\delta_\xi \mu = (\bar{\partial} - \mu\partial + (\partial\mu))(\xi + \mu\,\bar{\xi}) \tag{7.14}$$

$$\delta_\xi \rho = (\xi\partial + \bar{\xi}\bar{\partial})\rho + \rho(\partial\xi + \mu\,\partial\bar{\xi}) \tag{7.15}$$

In this case the holomorphic factorization is realized for the Beltrami coefficient μ but not for the conformal factor ρ. Comparing these results with the parametrization of complex structures one finds combining the differential equation for λ (7.4) and the transformation law given in the second equation (7.7)

$$\delta_\xi \lambda = (\xi\partial + \bar{\xi}\bar{\partial})\lambda + \lambda(\partial\xi + \mu\,\partial\bar{\xi}), \tag{7.16}$$

which is of the same form as the transformation law for the conformal factor. Since the conformal factor ρ does not satisfy any differential equation it is not possible to achieve the holomorphic factorization.

7.1.3 Light-cone coordinates

Considering a metric with Lorentzian signature one has two linearly independent natural vector fields defined by the two independent null vectors at each point. In a corresponding coordinate system (Σ^+, Σ^-) the metric is off diagonal. The line element is therefore given as

$$ds^2 \propto d\Sigma^+ d\Sigma^-. \tag{7.17}$$

Given a reference coordinate system (σ^+, σ^-) the light-cone coordinates are given by the relation

$$d\Sigma^+ = \lambda(d\sigma^+ + h\, d\sigma^-), \tag{7.18}$$

$$d\Sigma^- = \bar{\lambda}(d\sigma^- + \bar{h}\, d\sigma^+). \tag{7.19}$$

In this case h parametrizes the contribution of $d\sigma^-$ to $d\sigma^+$ in the direction of Σ^+ and therefore has a simple geometrical interpretation. Now I compare the line elements represented in the different coordinate systems (σ^+, σ^-) and (Σ^+, Σ^-) where the metric of the reference coordinate system is denoted by

$$g_{mn} = \begin{pmatrix} g_{++} & g_{+-} \\ g_{+-} & g_{--} \end{pmatrix}. \tag{7.20}$$

This yields

$$(d\Sigma^+, d\Sigma^-) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} d\Sigma^+ \\ d\Sigma^- \end{pmatrix} \propto (d\sigma^+, d\sigma^-) \begin{pmatrix} g_{++} & g_{+-} \\ g_{+-} & g_{--} \end{pmatrix} \begin{pmatrix} d\sigma^+ \\ d\sigma^- \end{pmatrix}, \tag{7.21}$$

and thereby

$$H^T \cdot \Lambda^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Lambda \cdot H \propto \begin{pmatrix} g_{++} & g_{+-} \\ g_{+-} & g_{--} \end{pmatrix}, \tag{7.22}$$

where I introduced the notation

$$\Lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{pmatrix}, \quad H = \begin{pmatrix} 1 & h \\ \bar{h} & 1 \end{pmatrix}. \tag{7.23}$$

If one restricts the procedure to conformal classes of metrics it is possible to promote the '$\propto$' sign to a '$=$' by dividing the right hand side of (7.22) by $g = \det g_{mn}$ and suppressing the scaling factor $\lambda\bar{\lambda}$ on the left hand side. Then h and $\bar{h}$ are expressible in terms of components of the metric

$$h = \frac{g_{--}}{g_{+-} + \sqrt{g}}, \tag{7.24}$$

$$\bar{h} = \frac{g_{++}}{g_{+-} + \sqrt{g}}. \tag{7.25}$$

I consider a conformally invariant two dimensional model where the BRST operator acts on the metric corresponding to the transformation under two dimensional diffeomorphisms and local dilatations

$$sg_{mn} = \xi^l \partial_l g_{mn} + g_{ln}(\partial_m \xi^l) + g_{ml}(\partial_n \xi^l) + c g_{mn}, \tag{7.26}$$

where the indices take the values $\{+,-\}$. ξ^m denotes the diffeomorphism ghosts and c is the ghost for local dilatations. The BRST transformations of the ghosts then read

$$\begin{aligned} s\xi^n &= \xi^l \partial_l \xi^n && (7.27) \\ sc &= \xi^l \partial_l c. && (7.28) \end{aligned}$$

It is then easily verified that h as given in 7.24 transforms according to

$$sh = (\bar{\partial} - h\partial + (\partial h))C, \qquad \partial := \partial_+, \qquad \bar{\partial} := \partial_-, \tag{7.29}$$

where $C = \xi^+ + h\xi^-$ denotes a combination of the diffeomorphism ghosts. The transformation law of the new ghost variables is given by

$$sC = C\partial C. \tag{7.30}$$

Thus the BRST algebra in the new set of variables $\{h, \bar{h}, C, \bar{C}\}$ factorizes into two independent substructures. Therefore $\{h, \bar{h}\}$ may be viewed as Beltrami variables.

7.2 Bosonic theory

Before I turn to the supersymmetric case I want to introduce the main concepts used later in the discussion of the supersymmetric theory based on a discussion of the purely bosonic case.

Since local frames are a useful tool in describing gravity theories I introduce a parametrization of the vielbein rather than a parametrization of the metric. Working with the vielbein one requires Lorentz invariance in addition and thus the symmetry transformations include diffeomorphisms, local Lorentz and local Weyl transformations. The symmetries entail the corresponding ghosts, i.e. the diffeomorphism ghosts, Lorentz ghosts and Weyl ghosts.

The conformal properties of two dimensions are conveniently described in terms of coordinates (σ^+, σ^-) [gr90]

$$\sigma^{\pm} = \sigma^0 \pm \sigma^1 \tag{7.31}$$

and derivatives

$$\partial_\pm = \frac{1}{2}\left(\partial_0 \pm \partial_1\right). \tag{7.32}$$

I introduce a parametrization of the moving frame

$$e^\pm = e^0 \pm e^1 \tag{7.33}$$

in the form of

$$\begin{aligned} e^+ &= (d\sigma^+ + d\sigma^- \mu_-{}^+) e_+{}^+ && (7.34)\\ e^- &= (d\sigma^- + d\sigma^+ \mu_+{}^-) e_-{}^-. && (7.35)\end{aligned}$$

The coefficients

$$\mu_-{}^+ = \frac{e_-{}^+}{e_+{}^+}, \qquad \mu_+{}^- = \frac{e_+{}^-}{e_-{}^-}, \tag{7.36}$$

are called the Beltrami differentials. The conformal factors are $e_+{}^+$ and $e_-{}^-$. The Beltrami differentials are inert under structure group transformations. The structure group transformations are entirely carried by the conformal factors. Following section (6.2) the BRST transformation of the vielbein reads in the bosonic case

$$s e_m{}^a = \xi^n \partial_n e_m{}^a + (\partial_m \xi^n) e_n{}^a + C_b{}^a e_m{}^b + c e_m{}^a, \tag{7.37}$$

where ξ^n denotes the diffeomorphism ghosts, C^{ab} is the Lorentz ghost and c denotes the Weyl ghost respectively. The BRST transformations of the ghosts read

$$\begin{aligned} s\xi^n &= \xi^m \partial_m \xi^n \\ sC^{ab} &= \xi^m \partial_m C^{ab} \\ sc &= \xi^m \partial_m c. \end{aligned} \tag{7.38}$$

The BRST transformations of the Beltrami differentials are then easily checked to be

$$\begin{aligned} s\mu_-{}^+ &= (\partial_- - \mu_-{}^+\partial_+ + (\partial_+\mu_-{}^+))(\xi^+ + \mu_-{}^+\xi^-), \\ s\mu_+{}^- &= (\partial_+ - \mu_+{}^-\partial_- + (\partial_-\mu_+{}^-))(\xi^- + \mu_+{}^-\xi^+). \end{aligned} \tag{7.39}$$

Here again Becchi's reparametrization of the ghost fields turns up. The new ghost fields

$$\eta = (\xi^+ + \mu_-{}^+\xi^-), \qquad \bar\eta = (\xi^- + \mu_+{}^-\xi^+) \tag{7.40}$$

transform according to

$$\begin{aligned} s\eta &= \eta\partial\eta, \qquad \partial \equiv \partial_+ \\ s\bar\eta &= \bar\eta\bar\partial\bar\eta, \qquad \bar\partial \equiv \partial_-. \end{aligned} \tag{7.41}$$

The conformal factors $e_+^{\ +}$ and $e_-^{\ -}$ form trivial pairs with appropriate redefinitions of the Weyl and Lorentz ghost. Thus they drop completely out of the cohomological analysis, since they count among the $\mathcal{U}$'s and $\mathcal{V}$'s.

Introducing an additional $U(1)$ symmetry the BRST transformations of the gauge field A_m an the abelian ghost C^a read

$$\begin{aligned} sA_m &= \xi^n\partial_n A_m + (\partial_m\xi^n)A_n + \partial_m C^a \\ sC^a &= \xi^n\partial_n C^a \end{aligned} \tag{7.42}$$

Redefining the abelian ghost

$$C^a \;\to\; C^a + \xi^n A_n \tag{7.43}$$

the BRST transformations read in the new variables

$$\begin{aligned} sA &= \partial C^a - (\bar\eta - \bar\mu\,\eta)F^a \\ s\bar A &= \bar\partial C^a + (\eta - \mu\,\bar\eta)F^a \\ sC^a &= \eta\bar\eta F^a. \end{aligned} \tag{7.44}$$

where

$$F^a = \frac{1}{2}\frac{1}{1-\mu\bar\mu}\varepsilon^{nm}F^a_{nm}, \quad F^a_{nm} = \partial_n A_m - \partial_m A_n \tag{7.45}$$

denotes the $U(1)$ field strength. The abelian gauge fields form trivial pairs with derivatives of the abelian ghost, while the undifferentiated abelian ghost, the field strength and its covariant derivatives enter the cohomological analysis as generalized connections and tensor fields. For a detailed analysis of the bosonic D-string see [br97].

Chapter 8

Dirichlet-Superstrings

In this chapter I work out the supersymmetric extension to the concepts introduced in the previous chapter for the purely bosonic case. The Dirichlet-Superstring can be formulated as a $D = 2$ supergravity coupled to string coordinates with an additional $U(1)$ symmetry on the world sheet. Thus the starting point will be two dimensional $(1, 1)$ supergravity superconformally coupled to string coordinates. The model is characterized by the field content and its symmetry transformations. In this case the set of fields consists of the gravitational multiplet, i.e. the vielbein $e_m{}^a$, the gravitino or Rarita-Schwinger field $\chi_m{}^\alpha$ and the auxiliary field S to close the algebra off-shell and the matter multiplet $\phi^M = \{X^M, \psi_\alpha^M, F^M\}$. The gauge invariances of the theory are invariance under local diffeomorphisms, local Lorentz and super-Weyl invariance and local supersymmetry. As we will see, the off-shell nilpotency of the BRST-operator requires an additional bosonic symmetry transformation acting on the auxiliary field S, which we will refer to as 'auxiliary Weyl transformation', since it ensures the off-shell nilpotency of the BRST transformations in the super-Weyl ghost sector.The ghost fields associated with the gauge symmetries are denoted by ξ^m for the diffeomorphism ghosts, C^{ab} for the Lorentz ghost, C^W for the Weyl ghost, η^α for the super-Weyl ghosts and ξ^α for the supersymmetry ghosts. The ghost for the 'auxiliary Weyl transformation' is denoted by W. This fixes the field content to

$$\Psi^A = \{e_m{}^a, \chi_m{}^\alpha, S, X^M, \psi_\alpha^M, F^M, \xi^m, \xi^\alpha, C^{ab}, C^W, \eta^\alpha, W\} \tag{8.1}$$

The starting point will be the BRST transformations of D=2 supergravity without super-Weyl invariance, which we computed in section (6.2). Then I add the corresponding Weyl transformations of the fields and require the off-shell nilpotency of the BRST transformations. This will lead to the introduction of a fermionic and an additional bosonic symmetry transformation, which together with the Weyl transformation imply the super-Weyl invariance of the theory. As the next step I introduce the super-Beltrami parametrization for two dimensional super-Weyl invariant supergravity, which establishes the factorization of the BRST algebra. Then I map the cohomological problem from the space of local functionals to the space of local total forms. This will be the first step in the construction of contracting homotopies. The new local jet coordinates for the supergravity part are constructed and a list of the first few tensor fields is given. Then essentially the same procedure is carried out for the $U(1)$ sector. In the last section a compilation of the main results is given.

8.1 Super-Weyl invariance

As already computed in section (6.2) the BRST transformations of two dimensional supergravity are

$$\begin{aligned}
se_m{}^a &= \xi^n\partial_n e_m{}^a + (\partial_m\xi^n)e_n{}^a - \xi^\alpha\chi_m{}^\beta T_{\alpha\beta}{}^a + C_b{}^a e_m{}^b \\
s\chi_m{}^\alpha &= \xi^n\partial_n\chi_m{}^\alpha + (\partial_m\xi^n)\chi_n{}^\alpha + \partial_m\xi^\alpha - \xi^\beta e_m{}^a T_{a\beta}{}^\alpha \\
&\quad +\frac{1}{4}\xi^\beta\omega_m{}^{ab}\varepsilon_{ab}(\gamma_*)_\beta{}^\alpha - \frac{1}{4}C^{ab}\chi_m{}^\beta\varepsilon_{ab}(\gamma_*)_\beta{}^\alpha \\
sS &= \xi^n\partial_n S - 4\xi^\gamma(\gamma_* C)_{\gamma\alpha}\varepsilon^{nm}\nabla_n\chi_m{}^\alpha + i\,\xi^\gamma(\gamma^m C)_{\gamma\alpha}\chi_m{}^\alpha S \\
s\xi^n &= \xi^m\partial_m\xi^n + \frac{1}{2}\xi^\alpha\xi^\beta T_{\alpha\beta}{}^n \\
s\xi^\alpha &= \xi^n\partial_n\xi^\alpha - \frac{1}{2}\xi^\gamma\xi^\beta T_{\beta\gamma}{}^m\chi_m{}^\alpha - \frac{1}{4}C^{ab}\xi^\beta\varepsilon_{ab}(\gamma_*)_\beta{}^\alpha \\
sC^{ab} &= \xi^m\partial_m C^{ab} + \frac{1}{2}\xi^\alpha\xi^\beta R_{\alpha\beta}{}^{ab} - \frac{1}{2}\xi^\alpha\xi^\beta T_{\beta\alpha}{}^m\omega_m{}^{ab}.
\end{aligned} \tag{8.2}$$

The torsions and field strenghts arising in these equations are listed in equation (5.45). In order to describe matter fields I introduce the multiplet $\{X^M, \psi_\alpha^M, F^F\}$, which transforms under local supersymmetry according to

$$\begin{aligned}
\mathcal{D}_\alpha X^M &= \psi_\alpha^M \\
\mathcal{D}_\alpha\psi_\beta^M &= \mathcal{D}_\alpha\mathcal{D}_\beta X^M = \frac{1}{2}\{\mathcal{D}_\alpha,\mathcal{D}_\beta\}X^M + \frac{1}{2}[\mathcal{D}_\alpha,\mathcal{D}_\beta]X^M \\
&= i(\gamma^a C)_{\alpha\beta}\mathcal{D}_a X^M + C_{\alpha\beta}F^M \\
\mathcal{D}_\alpha F^M &= (\gamma^a)_\alpha{}^\beta\mathcal{D}_a\psi_\beta^M
\end{aligned} \tag{8.3}$$

The BRST-operator acts on these fields by

$$sX^M = \xi^m\partial_m X^M + \xi^\alpha\psi_\alpha^M$$

$$
\begin{aligned}
s\psi_\alpha^M &= \xi^m\partial_m\psi_\alpha^M - \frac{1}{2}\xi^\beta T_{\beta\alpha}{}^m(\partial_m X^M - \chi_m{}^\gamma\psi_\gamma^M) + \xi^\beta C_{\beta\alpha}F^M + \frac{1}{4}C^{ab}\varepsilon_{ab}(\gamma_*)_\alpha{}^\beta\psi_\beta^M \\
sF^M &= \xi^m\partial_m F^M + \xi^\alpha(\gamma^m)_\alpha{}^\beta\Big\{\partial_m\psi_\beta^M - \frac{1}{4}\omega_m{}^{ab}\varepsilon_{ab}(\gamma_*)_\beta{}^\gamma\psi_\gamma^M \\
&\quad + \frac{1}{2}\chi_m{}^\gamma T_{\gamma\beta}{}^n(\partial_n X^M - \chi_n{}^\delta\psi_\delta^M) - \chi_m{}^\gamma C_{\gamma\beta}F^M\Big\}
\end{aligned} \tag{8.4}
$$

Now I introduce the Weyl transformation as a new symmetry transformation. The infinitesimal local Weyl transformation of the vielbein is given by

$$\delta_W e_m{}^a = \sigma\, e_m{}^a, \tag{8.5}$$

where σ denotes the transformation parameter. The transformation of the invers vielbein is immediatly obtained from

$$\delta_W\, \delta_a{}^b = \delta_W\,(E_a{}^m e_m{}^b) = 0 \quad\Rightarrow\quad \delta_W E_a{}^m = -\sigma E_a{}^m \tag{8.6}$$

The Weyl weights of the other fields are listed in the following table

	$e_m{}^a$	$\chi_m{}^\alpha$	S	X^M	ψ_α^M	F^M
Weyl weight	1	$\frac{1}{2}$	-1	0	$-\frac{1}{2}$	-1

(8.7)

The introduction of the Weyl rescaling gives rise to additional terms in the BRST transformation of the fields according to their Weyl wheigt, with the transformation parameter σ replaced by the Weyl ghost C^W.

I then require the off-shell nilpotency of the BRST-operator. As might be already expected from counting the bosonic and fermionic degrees of freedom the off-shell nilpotency requires an additional fermionic symmetry. This manifests in the fact that the BRST-operator is not nilpotent on the gravitino. The additional transformation introduced in this way is a local fermionic transformation acting on the gravitino [wi]

$$\delta_{SW}\chi_m{}^\alpha = i\eta^\beta(\gamma_m)_\beta{}^\alpha, \tag{8.8}$$

where η is an arbitrary Majorana spinor. Together with this fermionic symmetry comes another local bosonic symmetry [fu95], which balances the bosonic and fermionic degrees of freedom and ensures the off-shell nilpotency of the BRST-operator on the auxiliary field of the gravitational multiplet. I refer to this bosonic transformation as 'auxiliary Weyl transformation'. It is associated with a shift of the auxiliary field S by an arbitrary function Λ

$$S \to S + \Lambda. \tag{8.9}$$

Altogether these transformations are called super-Weyl transformations. The complete list of BRST transformations including super-Weyl transformations reads

$$\begin{aligned}
se_m{}^a &= \xi^n\partial_n e_m{}^a + (\partial_m\xi^n)e_n{}^a - \xi^\alpha\chi_m{}^\beta T_{\alpha\beta}{}^a + C_b{}^a e_m{}^b + C^W e_m{}^a \\
s\chi_m{}^\alpha &= \xi^n\partial_n\chi_m{}^\alpha + (\partial_m\xi^n)\chi_n{}^\alpha + \partial_m\xi^\alpha - \xi^\beta e_m{}^a T_{a\beta}{}^\alpha + \frac{1}{2}C^W\chi_m{}^\alpha + i\eta^\beta(\gamma_m)_\beta{}^\alpha \\
&\quad + \frac{1}{4}\xi^\beta\omega_m{}^{ab}\varepsilon_{ab}(\gamma_*)_\beta{}^\alpha - \frac{1}{4}C^{ab}\chi_m{}^\beta\varepsilon_{ab}(\gamma_*)_\beta{}^\alpha \\
sS &= \xi^n\partial_n S - 4\xi^\gamma(\gamma_* C)_{\gamma\alpha}\varepsilon^{nm}\nabla_n\chi_m{}^\alpha + i\,\xi^\gamma(\gamma^m C)_{\gamma\alpha}\chi_m{}^\alpha S - C^W S + W \\
s\xi^n &= \xi^m\partial_m\xi^n + \frac{1}{2}\xi^\alpha\xi^\beta T_{\alpha\beta}{}^n \\
s\xi^\alpha &= \xi^n\partial_n\xi^\alpha - \frac{1}{2}\xi^\gamma\xi^\beta T_{\beta\gamma}{}^m\chi_m{}^\alpha - \frac{1}{4}C^{ab}\xi^\beta\varepsilon_{ab}(\gamma_*)_\beta{}^\alpha + \frac{1}{2}C^W\xi^\alpha \\
sC^{ab} &= \xi^m\partial_m C^{ab} + \frac{1}{2}\xi^\alpha\xi^\beta R_{\alpha\beta}{}^{ab} - \frac{1}{2}\xi^\alpha\xi^\beta T_{\beta\alpha}{}^m\omega_m{}^{ab} - 2\eta^\beta\xi^\alpha(\gamma_* C)_{\alpha\beta}\varepsilon^{ab} \\
sC^W &= \xi^n\partial_n C^W + 2\eta^\beta\xi_\beta \\
s\eta^\alpha &= \xi^n\partial_n\eta^\alpha - \frac{1}{4}C^{ab}\eta^\beta\varepsilon_{ab}(\gamma_*)_\beta{}^\alpha + i\xi^\beta(\gamma^n)_\beta{}^\alpha\left(\frac{1}{2}\partial_n C^W - \eta^\gamma\chi_{n\gamma}\right) - \frac{1}{2}C^W\eta^\alpha + \xi^\alpha W \\
sW &= \xi^n\partial_n W - 4i\xi^\beta(\gamma^m C)_{\beta\alpha}\left(\nabla_m\eta^\alpha - \frac{1}{4}\chi_m{}^\alpha W - \frac{i}{2}\chi_m{}^\gamma(\gamma^n)_\gamma{}^\alpha(\partial_n C^W)\right) \\
&\quad -4\xi^\beta\chi_m{}^\alpha(\gamma^m\gamma^n C)_{\alpha\beta}\eta^\gamma\chi_{n\gamma} - C^W W.
\end{aligned} \tag{8.10}$$

In these equations the ghosts corresponding to the additional symmetries are denoted by η^α for the fermionic symmetry and W for the 'auxiliary Weyl transformation'. Since the spin connection depends on the vielbein and the gravitino it also transforms under super-Weyl transformations

$$\begin{aligned}
\delta_W\omega_m{}^{ab} &= -\varepsilon_m{}^n(\partial_n\sigma)\varepsilon^{ab} \\
\delta\omega_m{}^{ab} &= \left(-2\eta^\alpha\chi_m^\beta(\gamma_* C)_{\beta\alpha} + 2\varepsilon_m{}^n\eta^\alpha\chi_{n\alpha}\right)\varepsilon^{ab}
\end{aligned} \tag{8.11}$$

The BRST transformation of the spin connection then reads

$$\begin{aligned}
s\omega_m{}^{ab} &= \xi^n\partial_n\omega_m{}^{ab} + (\partial_m\xi^n)\omega_n{}^{ab} + \partial_m C^{ab} - \xi^\beta\chi_m{}^\alpha R_{\alpha\beta}{}^{ab} - \xi^\alpha e_m{}^c R_{c\alpha}{}^{ab} \\
&\quad + C^{ac}\omega_{mc}{}^b - C^{bc}\omega_{mc}{}^a - \varepsilon_m{}^n(\partial_n C^W)\varepsilon^{ab} \\
&\quad + \left(-2\eta^\alpha\chi_m^\beta(\gamma_* C)_{\beta\alpha} + 2\varepsilon_m{}^n\eta^\alpha\chi_{n\alpha}\right)\varepsilon^{ab}
\end{aligned} \tag{8.12}$$

The BRST transformations of the matter multiplet including the Weyl transformations are given by

$$sX^M = \xi^m\partial_m X^M + \xi^\alpha\psi_\alpha^M$$

$$\begin{aligned}
s\psi_\alpha^M &= \xi^m\partial_m\psi_\alpha^M - \frac{1}{2}\xi^\beta T_{\beta\alpha}{}^m(\partial_m X^M - \chi_m{}^\gamma\psi_\gamma^M) + \xi^\beta C_{\beta\alpha}F^M \\
&\quad + \frac{1}{4}C^{ab}\varepsilon_{ab}(\gamma_*)_\alpha{}^\beta\psi_\beta^M - \frac{1}{2}C^W\psi_\alpha^M \\
sF^M &= \xi^m\partial_m F^M + \xi^\alpha(\gamma^m)_\alpha{}^\beta\Big\{\partial_m\psi_\beta^M - \frac{1}{4}\omega_m{}^{ab}\varepsilon_{ab}(\gamma_*)_\beta{}^\gamma\psi_\gamma^M \\
&\quad + \frac{1}{2}\chi_m{}^\gamma T_{\gamma\beta}{}^n(\partial_n X^M - \chi_n{}^\delta\psi_\delta^M) - \chi_m{}^\gamma C_{\gamma\beta}F^M\Big\} - C^W F^M \qquad (8.13)
\end{aligned}$$

8.2 Factorization of the BRST algebra

As already shown in the non-supersymmetric case the BRST algebra factorizes in the Beltrami parametrization in two independent substructures. These are described by the Beltrami differentials μ and suitable combinations of the diffeomorphism ghosts ξ^m and their conjugates.

As it is not hard to guess in the supersymmetric generalization the Beltrami differential μ acquires a fermionic partner α, the Beltramino. The super-Beltrami parametrization is therefore characterized by the doublets (μ, α) and $(\bar{\mu}, \bar{\alpha})$. As in the bosonic case the parametrization of the vielbein is given by

$$e^+ = (d\sigma^+ + d\sigma^-\mu_-{}^+)e_+{}^+ \qquad (8.14)$$

$$e^- = (d\sigma^- + d\sigma^+\mu_+{}^-)e_-{}^-. \qquad (8.15)$$

The coefficients $\mu_-{}^+$ and $\mu_+{}^-$ are the Beltrami differentials

$$\mu := \mu_-{}^+ = \frac{e_-{}^+}{e_+{}^+}, \qquad (8.16)$$

$$\bar{\mu} := \mu_+{}^- = \frac{e_+{}^-}{e_-{}^-}, \qquad (8.17)$$

where I introduced a different notation for the sake of brevity. The fermionic superpartners are suitable combinations of the gravitino fields

$$\alpha := i\sqrt{\frac{8}{e_+{}^+}}\left(\chi_-{}^2 - \mu\chi_+{}^2\right) \qquad (8.18)$$

$$\bar{\alpha} := i\sqrt{\frac{8}{e_-{}^-}}\left(\chi_+{}^1 - \bar{\mu}\chi_-{}^1\right). \qquad (8.19)$$

The numerical factor in the definition of the Beltraminos ensures that the BRST algebra takes the same form as in [dg90]. Again the super-Beltrami parametrization has the important feature that the Beltrami differential and its superpartner are inert under structure group transformations, i.e. under Lorentz and Weyl transformations

$$\begin{array}{ll} \delta_L\mu = \delta_L\bar{\mu} = 0 & \delta_W\mu = \delta_W\bar{\mu} = 0 \\ \delta_L\alpha = \delta_L\bar{\alpha} = 0 & \delta_W\alpha = \delta_W\bar{\alpha} = 0 \end{array} \tag{8.20}$$

and super-Weyl transformations

$$\begin{aligned} \delta_{SW}\alpha &= \sqrt{\frac{8}{e_+^{\;+}}}\eta^\gamma \left(e_-^{\;a}(\gamma_a)_\gamma^{\;2} - \mu e_+^{\;a}(\gamma_a)_\gamma^{\;2}\right) = 0 \\ \delta_{SW}\bar{\alpha} &= \sqrt{\frac{8}{e_-^{\;-}}}\eta^\gamma \left(e_+^{\;a}(\gamma_a)_\gamma^{\;1} - \bar{\mu} e_-^{\;a}(\gamma_a)_\gamma^{\;1}\right) = 0, \end{aligned} \tag{8.21}$$

where I used the expressions for γ_+ and γ_+ as given in appendix (B). There I collected some useful formulas on super-Beltrami variables. The new ghost variables, which replace the diffeomorphism ghosts and the supersymmetry ghosts are

$$\eta := (\xi^+ + \mu\xi^-) \tag{8.22}$$

$$\bar{\eta} := (\xi^- + \bar{\mu}\xi^+) \tag{8.23}$$

$$\varepsilon := (\hat{\xi}^2 + \xi^-\alpha), \qquad \hat{\xi}^2 := \sqrt{\frac{8}{e_+^{\;+}}}\xi^2 \tag{8.24}$$

$$\bar{\varepsilon} := (\hat{\xi}^1 + \xi^+\bar{\alpha}), \qquad \hat{\xi}^1 := \sqrt{\frac{8}{e_-^{\;-}}}\xi^1 \tag{8.25}$$

Using the new variables it is readily verified that the BRST algebra factorizes in two independent substructures, if one keeps in mind that the torsion $T_{\alpha\beta}^{\;\;\;a}$ is given by

$$T_{22}^{\;\;\;+} = 4i, \qquad T_{11}^{\;\;\;-} = 4i. \tag{8.26}$$

The BRST transformations of the Beltrami differential, the Beltramino and the new ghosts reads

$$\begin{aligned} s\mu &= \left(\bar{\partial} - \mu\partial + (\partial\mu)\right)\eta - \frac{1}{2}\alpha\varepsilon \\ s\bar{\mu} &= \left(\partial - \bar{\mu}\bar{\partial} + (\bar{\partial}\bar{\mu})\right)\bar{\eta} - \frac{1}{2}\bar{\alpha}\bar{\varepsilon} \\ s\alpha &= \left(\bar{\partial} - \mu\partial + \frac{1}{2}(\partial\mu)\right)\varepsilon + \eta\partial\alpha + \frac{1}{2}\alpha\partial\eta \\ s\bar{\alpha} &= \left(\partial - \bar{\mu}\bar{\partial} + \frac{1}{2}(\bar{\partial}\bar{\mu})\right)\bar{\varepsilon} + \bar{\eta}\bar{\partial}\bar{\alpha} + \frac{1}{2}\bar{\alpha}\bar{\partial}\bar{\eta} \\ s\eta &= \eta\partial\eta - \frac{1}{4}\varepsilon\varepsilon \end{aligned}$$

$$
\begin{aligned}
s\bar{\eta} &= \bar{\eta}\bar{\partial}\bar{\eta} - \frac{1}{4}\bar{\varepsilon}\bar{\varepsilon} \\
s\varepsilon &= \eta\partial\varepsilon - \frac{1}{2}\varepsilon\partial\eta \\
s\bar{\varepsilon} &= \bar{\eta}\bar{\partial}\bar{\varepsilon} - \frac{1}{2}\bar{\varepsilon}\bar{\partial}\bar{\eta}
\end{aligned} \tag{8.27}
$$

Expressing the BRST transformations of the diffeomorphism ghosts ξ^m and the supersymmetry ghosts ξ^α in terms of the Beltrami differential and the Beltramino one obtains

$$
s\xi^+ = (\xi^+\partial_+ + \xi^-\partial_-)\xi^+ + \frac{i}{4}\frac{1}{(1-\mu\bar{\mu})}(\hat{\xi}^2\hat{\xi}^2 + \mu\hat{\xi}^1\hat{\xi}^1), \tag{8.28}
$$

$$
s\hat{\xi}^2 = (\xi^m\partial_m)\,\hat{\xi}^2 - \frac{1}{2}\hat{\xi}^2\left(\partial\xi^+ + \mu\bar{\partial}\xi^-\right) - \frac{i}{4}\frac{1}{(1-\mu\bar{\mu})}\left(\hat{\xi}^1\hat{\xi}^1 - \bar{\mu}\hat{\xi}^2\hat{\xi}^2\right)\alpha. \tag{8.29}
$$

These are the structure relations of the field dependent Lie algebra as found in [dg90] by projection from superspace and using a Wess-Zumino supergauge.

After the factorization of the BRST algebra is established the next step in the computation of the covariant variables is to remove the trivial pairs from the chomology.

8.3 Generalized connections and tensor fields

The construction of contracting homotopies which reduce the cohomological problem considerably is a crucial step in the computation of the cohomology. The first step is to map the cohomological problem from the space of local functionals to the cohomology of $\tilde{s}$

$$
\tilde{s} = s + d \tag{8.30}
$$

in the space of local total forms. A total form is simply a sum of local forms with different form degrees and ghost numbers [br96]. The switch from s to $\tilde{s}$ is straihtforward since the s transformations turn into the $\tilde{s}$ transformations by the replacement

$$
\xi^n \to \tilde{\xi}^n = \xi^n + d\sigma^n. \tag{8.31}
$$

The switch from s to $\tilde{s}$ simplifies some arguments such as the proof that the solutions of the cohomology do not depend explicitly on the coordinates. As a consequence of the substitution rule (8.31) the new ghost variables read

$$
\begin{aligned}
\eta &\to \tilde{\eta} = \eta + d\sigma^+ + \mu d\sigma^- \\
\bar{\eta} &\to \tilde{\bar{\eta}} = \bar{\eta} + d\sigma^- + \bar{\mu} d\sigma^+ \\
\varepsilon &\to \tilde{\varepsilon} = \varepsilon + d\sigma^-\alpha \\
\bar{\varepsilon} &\to \tilde{\bar{\varepsilon}} = \bar{\varepsilon} + d\sigma^+\bar{\alpha}.
\end{aligned} \tag{8.32}
$$

The $\tilde{s}$ transformations of the fields then read

$$\begin{aligned}
\tilde{s}\mu &= \left(\bar{\partial} - \mu\partial + (\partial\mu)\right)\tilde{\eta} - \frac{1}{2}\alpha\tilde{\varepsilon} \\
\tilde{s}\alpha &= \left(\bar{\partial} - \mu\partial + \frac{1}{2}(\partial\mu)\right)\tilde{\varepsilon} + \tilde{\eta}\partial\alpha + \frac{1}{2}\alpha\partial\tilde{\eta} \\
\tilde{s}\tilde{\eta} &= \tilde{\eta}\partial\tilde{\eta} - \frac{1}{4}\tilde{\varepsilon}\tilde{\varepsilon} \\
\tilde{s}\tilde{\varepsilon} &= \tilde{\eta}\partial\tilde{\varepsilon} - \frac{1}{2}\tilde{\varepsilon}\partial\tilde{\eta}
\end{aligned} \tag{8.33}$$

with analogous transformations for $\{\bar{\mu}, \bar{\alpha}, \tilde{\bar{\eta}}, \tilde{\bar{\varepsilon}}\}$. The $\mathcal{U}^i$'s are

$$\{\mathcal{U}^i\} = \{\sigma^m, \partial^m\bar{\partial}^n\mu, \partial^m\bar{\partial}^n\alpha, \partial^m\bar{\partial}^n\bar{\mu}, \partial^m\bar{\partial}^n\bar{\alpha}, \partial^m\bar{\partial}^n e_{\pm}^{\pm} \ : m, n = 0, 1, \ldots\} \tag{8.34}$$

The corresponding $\mathcal{V}^i$'s replace one by one the $d\sigma^m$, the Lorentz and Weyl ghosts, the $\partial\tilde{\bar{\eta}}, \bar{\partial}\tilde{\eta}$ and the $\partial\tilde{\bar{\varepsilon}}, \bar{\partial}\tilde{\varepsilon}$ and all their derivatives due to

$$\begin{aligned}
\tilde{s}\sigma^m &= d\sigma^m \\
\tilde{s}\mu &= \bar{\partial}\tilde{\eta} + \ldots \\
\tilde{s}\bar{\mu} &= \partial\tilde{\bar{\eta}} + \ldots \\
\tilde{s}\alpha &= \bar{\partial}\tilde{\varepsilon} + \ldots \\
\tilde{s}\bar{\alpha} &= \partial\tilde{\bar{\varepsilon}} + \ldots \\
&\ldots
\end{aligned} \tag{8.35}$$

The infinite set of generalized connections is then given by $\{\tilde{\eta}, \tilde{\bar{\eta}}, \tilde{\varepsilon}, \tilde{\bar{\varepsilon}}\}$ and the remaining derivatives. I introduce the following notation for the generalized connections

$$\{\tilde{C}^N\} = \{\tilde{\eta}^p, \tilde{\bar{\eta}}^p, \tilde{\varepsilon}^{p+\frac{1}{2}}, \tilde{\bar{\varepsilon}}^{p+\frac{1}{2}} : p = -1, 0, 1, \ldots\}, \tag{8.36}$$

with

$$\begin{aligned}
\tilde{\eta}^p &= \frac{1}{(p+1)!}\,\partial^{p+1}\tilde{\eta} \\
\tilde{\bar{\eta}}^p &= \frac{1}{(p+1)!}\,\bar{\partial}^{p+1}\tilde{\bar{\eta}} \\
\tilde{\varepsilon}^{p+\frac{1}{2}} &= \frac{1}{2}\,\frac{1}{(p+1)!}\,\partial^{p+1}\tilde{\varepsilon} \\
\tilde{\bar{\varepsilon}}^{p+\frac{1}{2}} &= \frac{1}{2}\,\frac{1}{(p+1)!}\,\bar{\partial}^{p+1}\tilde{\bar{\eta}}.
\end{aligned} \tag{8.37}$$

The infinite set of the correponding Δ's is denoted by

$$\{\Delta_N\} = \{L_p, \bar{L}_p, G_{p+\frac{1}{2}}, \bar{G}_{p+\frac{1}{2}} : p = -1, 0, 1, \ldots\}. \tag{8.38}$$

Hence, the $\tilde{s}$ transformation on tensor fields may be expressed as

$$\tilde{s}\mathcal{T}^i = \sum_{p\geq -1}\left(\tilde{\eta}^p L_p + \tilde{\bar{\eta}}^p \bar{L}_p + \tilde{\varepsilon}^{p+\frac{1}{2}} G_{p+\frac{1}{2}} + \tilde{\bar{\varepsilon}}^{p+\frac{1}{2}} \bar{G}_{p+\frac{1}{2}}\right)\mathcal{T}^i. \tag{8.39}$$

The $\tilde{s}$ transformation of the generalized connections contains the the structure functions of the algebra of the corresponding Δ's. The $\tilde{s}$ transformations implied by the transformations (8.33) read

$$\begin{aligned} \tilde{s}\tilde{\eta}^p &= -\frac{1}{2}\tilde{\eta}^q\tilde{\eta}^r f_{rq}{}^p - \frac{1}{2}\tilde{\varepsilon}^a\tilde{\varepsilon}^b f_{ab}{}^p \\ &= -\frac{1}{2}\tilde{\eta}^q\tilde{\eta}^r (r-q)\delta^p_{r+q} - \frac{1}{2}\tilde{\varepsilon}^a\tilde{\varepsilon}^b 2\delta^p_{a+b} \end{aligned} \tag{8.40}$$

$$\begin{aligned} \tilde{s}\tilde{\varepsilon}^a &= -\frac{1}{2}\tilde{\eta}^p\tilde{\varepsilon}^c f_{cp}{}^a - \frac{1}{2}\tilde{\varepsilon}^c\tilde{\eta}^p f_{pc}{}^a \\ &= -\tilde{\varepsilon}^c\tilde{\eta}^p\left(\frac{p}{2}-c\right)\delta^a_{p+b}. \end{aligned} \tag{8.41}$$

Hence the the structure functions are all constants and the algebra of the Δ's forms two copies of the super-Virasoro algebra [ta96]

$$[L_p, L_q] = (p-q)L_{p+q}, \quad \{G_a, G_b\} = 2L_{a+b}, \quad [L_p, G_a] = \left(\frac{p}{2}-a\right)G_{p+a}, \tag{8.42}$$

with the analogous formulas for the $\bar{L}$'s and $\bar{G}$'s and furthermore

$$[L_p, \bar{L}_q] = 0, \qquad \{G_a, \bar{G}_b\} = 0, \tag{8.43}$$

$$[L_p, \bar{G}_a] = 0, \qquad [\bar{L}_p, G_a] = 0. \tag{8.44}$$

This algebra is realized on the set of tensor fields given by

$$\{\mathcal{T}^i\} = \{T^i_{m,n} : m,n = 0,1,2,\ldots\} \tag{8.45}$$

where the $T^i_{m,n}$ are

$$T^i_{m,n} = (L_{-1})^m(\bar{L}_{-1})^n\Psi^i. \tag{8.46}$$

A list of the explicit expressions for the first few tensor fields $T^i_{m,n}$ is given later. The algebraic representation of the Δ's on the tensor fields is derived by the use of the algebra and the fact that

$$L_p T^i_{0,0} = \bar{L}_p T^i_{0,0} = 0, \qquad p \geq 0 \tag{8.47}$$

$$G_a T^i_{0,0} = \bar{G}_a T^i_{0,0} = 0, \qquad a \geq \frac{1}{2} \tag{8.48}$$

This follows from the characterization of tensor fields by $\tilde{s}T^i_{0,0} = \tilde{C}^N \Delta_N T^i_{0,0}$. Further I define the matter multiplet $\Psi^i = \{X^M, \psi^M, \bar{\psi}^M, F^M\}$ according to

$$G_{-\frac{1}{2}} X^M = \psi^M, \quad \bar{G}_{-\frac{1}{2}} X^M = \bar{\psi}^M, \quad G_{-\frac{1}{2}} \bar{G}_{-\frac{1}{2}} X^M = iF^M \tag{8.49}$$

One can show by explicit calculation that the so defined matter multiplet corresponds to the one introduced in equation (8.3) by a mere redefinition with conformal factors $e_+{}^+, e_-{}^-$ in order to make them invariant under structure group transformations. In general a field Ψ^i carrying the chiral weights (r, l) is redefined by

$$\Psi^i \ \rightarrow \ (e_+{}^+)^r (e_-{}^-)^l \Psi^i. \tag{8.50}$$

According to this the matter multiplet of equation (8.3) is redefined by

$$\begin{array}{ccl} X^M & \rightarrow & X^M \\ \psi^M & \rightarrow & (e_+{}^+)^{\frac{1}{2}} \psi_2^M \\ \bar{\psi}^M & \rightarrow & (e_-{}^-)^{\frac{1}{2}} \psi_1^M \\ F^M & \rightarrow & (e_+{}^+)^{\frac{1}{2}} (e_-{}^-)^{\frac{1}{2}} F^M \end{array} \tag{8.51}$$

On these fields the algebra of the Δ's is represented by

$$L_p X^M_{m,n} = \begin{cases} (p+1)! \sum_{k=1}^{m-p} \begin{pmatrix} k+p-1 \\ p \end{pmatrix} X^M_{m-p,n} = \frac{m!}{(m-p-1)!} X^M_{m-p,n} & \text{for } p < m, \\ 0 & \text{for } p \geq m. \end{cases} \tag{8.52}$$

$$\bar{L}_q X^M_{m,n} = \begin{cases} (q+1)! \sum_{k=1}^{n-q} \begin{pmatrix} k+q-1 \\ q \end{pmatrix} X^M_{m,n-q} = \frac{n!}{(n-q-1)!} X^M_{m,n-q} & \text{for } q < n, \\ 0 & \text{for } q \geq n. \end{cases} \tag{8.53}$$

The action of G_a and $\bar{G}_a$ is given by

$$G_{p+\frac{1}{2}} X^M_{m,n} = \begin{cases} \frac{m!}{(m-p-1)!} \psi^M_{m-p-1,n} & \text{for } p < m, \\ 0 & \text{for } p \geq m. \end{cases} \tag{8.54}$$

$$\bar{G}_{q+\frac{1}{2}} X^M_{m,n} = \begin{cases} \frac{n!}{(n-q-1)!} \bar{\psi}^M_{m,n-q-1} & \text{for } q < n, \\ 0 & \text{for } q \geq n. \end{cases} \tag{8.55}$$

The action on the other fields is then easily obtained using the relations

$$[L_p, G_{-\frac{1}{2}}] = \frac{1}{2}(p+1)\, G_{p-\frac{1}{2}}, \quad \{G_{p+\frac{1}{2}}, G_{-\frac{1}{2}}\} = 2L_p \tag{8.56}$$

and the analogous formalas for $\bar{L}$ and $\bar{G}$ which are special cases of the general commutation relations (8.42). One then obtaines

$$L_p \psi^M_{m,n} = \begin{cases} \frac{m!}{(m-p)!} \left(m - p + \frac{1}{2}(p+1)\right) \psi^M_{m-p,n} & \text{for } p < m, \\ 0 & \text{for } p \geq m. \end{cases} \tag{8.57}$$

$$G_{p+\frac{1}{2}} \psi^M_{m,n} = \begin{cases} \frac{m!}{(m-p-1)!} X^M_{m-p,n} & \text{for } p < m, \\ 0 & \text{for } p \geq m. \end{cases} \tag{8.58}$$

with the obvious formulas for $\bar{L}$ and $\bar{G}$ acting on $\bar{\psi}^M_{m,n}$. Furthermore one obtaines by using the algebra

$$\bar{G}_{q+\frac{1}{2}}\lambda^M_{m,n} = \begin{cases} i\frac{n!}{(n-q-1)!}F^M_{m,n-p-1} & \text{for } q < n, \\ 0 & \text{for } q \geq n. \end{cases} \tag{8.59}$$

again with the obvious result for $G_{p+\frac{1}{2}}$ acting on $\bar{\psi}^M_{m,n}$. And finally

$$L_p F^M_{m,n} = \begin{cases} \frac{m!}{(m-p)!}\left(m-p+\frac{1}{2}(p+1)\right) F_{m-p,n} & \text{for } p < m, \\ 0 & \text{for } p \geq m. \end{cases} \tag{8.60}$$

$$G_{p+\frac{1}{2}} F^M_{m,n} = \begin{cases} \frac{m!}{(m-n-1)!}\bar{\psi}^M_{m-p,n} & \text{for } p < m, \\ 0 & \text{for } p \geq m. \end{cases} \tag{8.61}$$

This completes the representation of the super-Virasoro algebra on the matter fields $\Psi^i = \{X^M, \psi^M, \bar{\psi}^M, F^M\}$. Due to these formulas the $\tilde{s}$ transformations of the matter multiplets and moreover of the $\mathcal{W}$'s constructed out of them are local expressions, since only finitely many summands contribute to the transformations.

8.4 Covariant derivatives

As may already been observed the operators L_{-1} and $\bar{L}_{-1}$ alredy serve as covariant derivatives on the matter fields. It is easy to make contact to the results in literature by pulling out the 1-forms contained in the generalized connections. From the equations (8.32) and (8.3) one reads off

$$\begin{aligned} \tilde{\eta}^p &= \eta^p + \mathcal{A}^p, & \tilde{\bar{\eta}}^p &= \bar{\eta}^p + \bar{\mathcal{A}}^p, \\ \tilde{\varepsilon}^{p+\frac{1}{2}} &= \varepsilon^{p+\frac{1}{2}} + \mathcal{A}^{p+\frac{1}{2}}, & \tilde{\bar{\varepsilon}}^{p+\frac{1}{2}} &= \bar{\varepsilon}^{p+\frac{1}{2}} + \bar{\mathcal{A}}^{p+\frac{1}{2}}, \end{aligned} \tag{8.62}$$

and thus gets explicitely

$$\begin{aligned} \mathcal{A}^p &= \begin{pmatrix} \delta^{\ p}_{-1} \\ M^p \end{pmatrix} & \bar{\mathcal{A}}^p &= \begin{pmatrix} \bar{M}^p \\ \delta^{\ p}_{-1} \end{pmatrix}, \\ \mathcal{A}^{p+\frac{1}{2}} &= \begin{pmatrix} 0 \\ A^{p+\frac{1}{2}} \end{pmatrix} & \bar{\mathcal{A}}^{p+\frac{1}{2}} &= \begin{pmatrix} \bar{A}^{p+\frac{1}{2}} \\ 0 \end{pmatrix}, \\ M^p &= \frac{1}{(p+1)!}\partial^{p+1}\mu & \bar{M}^p &= \frac{1}{(p+1)!}\partial^{p+1}\bar{\mu}, \\ A^{p+\frac{1}{2}} &= \frac{1}{2}\frac{1}{(p+1)!}\partial^{p+1}\alpha & \bar{A}^{p+\frac{1}{2}} &= \frac{1}{2}\frac{1}{(p+1)!}\partial^{p+1}\bar{\alpha}. \end{aligned} \tag{8.63}$$

Using the representation of the partial derivative on the tensor fields $\partial_m = \mathcal{A}_m^{\ M}\Delta_M$ one obtains

$$\begin{aligned} \begin{pmatrix} \partial \\ \bar\partial \end{pmatrix} &= \begin{pmatrix} 1 & \bar\mu \\ \mu & 1 \end{pmatrix} \begin{pmatrix} L_{-1} \\ \bar L_{-1} \end{pmatrix} + \sum_{p\geq 0} \begin{pmatrix} 0 & \bar M^p \\ M^p & 0 \end{pmatrix} \begin{pmatrix} L_p \\ \bar L_p \end{pmatrix} + \\ & \sum_{p\geq -1} \begin{pmatrix} 0 & \bar A^{p+\frac12} \\ A^{p+\frac12} & 0 \end{pmatrix} \begin{pmatrix} G_{p+\frac12} \\ \bar G_{p+\frac12} \end{pmatrix}. \end{aligned} \tag{8.64}$$

Inverting the first factor on the right hand side one obtains the explicit form of the supercovariant derivative

$$\mathcal{D} \equiv L_{-1}, \qquad \bar{\mathcal{D}} \equiv \bar L_{-1}, \tag{8.65}$$

$$\begin{aligned} \mathcal{D} &= \frac{1}{1-\mu\bar\mu}\left(\partial - \bar\mu\bar\partial - \sum_{p\geq 0}(\bar M^p \bar L_p - \bar\mu M^p L_p) - \sum_{p\geq -1}(\bar A^{p+\frac12}\bar G_{p+\frac12} - \bar\mu A^{p+\frac12} G_{p+\frac12})\right) \\ \bar{\mathcal{D}} &= \frac{1}{1-\mu\bar\mu}\left(\bar\partial - \mu\partial - \sum_{p\geq 0}(M^p L_p - \mu\bar M^p \bar L_p) - \sum_{p\geq -1}(A^{p+\frac12} G_{p+\frac12} - \mu\bar A^{p+\frac12}\bar G_{p+\frac12})\right) \end{aligned} \tag{8.66}$$

The explicit expressions for the supercovariant derivatives of the matter fields are then easily obtained using the representation of the algebra on Ψ^i. The final results read

$$\begin{aligned} \mathcal{D}X^M &= \frac{1}{1-\mu\bar\mu}\left((\partial-\bar\mu\bar\partial)X^M - \frac12\bar\alpha\bar\psi^M + \frac12\bar\mu\alpha\psi^M\right) \\ \bar{\mathcal{D}}X^M &= \frac{1}{1-\mu\bar\mu}\left((\bar\partial-\mu\partial)X^M - \frac12\alpha\psi^M + \frac12\mu\bar\alpha\bar\psi^M\right) \\ \mathcal{D}\psi^M &= \frac{1}{1-\mu\bar\mu}\left(\left((\partial-\bar\mu\bar\partial + \frac12\bar\mu(\partial\mu)\right)\psi^M + \frac12\bar\mu\alpha\mathcal{D}X^M - \frac{i}{2}\bar\alpha F^M\right) \\ \bar{\mathcal{D}}\psi^M &= \frac{1}{1-\mu\bar\mu}\left(\left((\bar\partial-\mu\partial - \frac12(\partial\mu)\right)\psi^M - \frac12\alpha\mathcal{D}X^M + \frac{i}{2}\mu\bar\alpha F^M\right) \\ \mathcal{D}\bar\psi^M &= \frac{1}{1-\mu\bar\mu}\left(\left((\partial-\bar\mu\bar\partial - \frac12(\bar\partial\bar\mu)\right)\bar\psi^M - \frac12\bar\alpha\bar{\mathcal{D}}X^M + \frac{i}{2}\bar\mu\alpha F^M\right) \\ \bar{\mathcal{D}}\bar\psi^M &= \frac{1}{1-\mu\bar\mu}\left(\left((\bar\partial-\mu\partial + \frac12\mu(\bar\partial\bar\mu)\right)\bar\psi^M + \frac12\mu\bar\alpha\bar{\mathcal{D}}X^M - \frac{i}{2}\alpha F^M\right) \\ \mathcal{D}F^M &= \frac{1}{1-\mu\bar\mu}\left(\left(\partial-\bar\mu\bar\partial - \frac12(\bar\partial\bar\mu) + \frac12\bar\mu(\partial\mu)\right)F^M + \frac{i}{2}\bar\alpha\bar{\mathcal{D}}\psi^M - \frac{i}{2}\bar\mu\alpha\mathcal{D}\bar\psi^M\right) \\ \bar{\mathcal{D}}F^M &= \frac{1}{1-\mu\bar\mu}\left(\left(\bar\partial-\mu\partial - \frac12(\partial\mu) + \frac12\mu(\bar\partial\bar\mu)\right)F^M + \frac{i}{2}\alpha\mathcal{D}\bar\psi^M - \frac{i}{2}\mu\bar\alpha\bar{\mathcal{D}}\psi^M\right) \end{aligned} \tag{8.67}$$

and coincide with the results given in the literature [dg90]. According to the algebra (8.42) the operators L_p and $\bar L_p$ commute and hence the supercovariant derivatives commute

$$[\mathcal{D}, \bar{\mathcal{D}}] = 0 \tag{8.68}$$

and one obtains for $\bar{\mathcal{D}}\mathcal{D}X^M$

$$\bar{\mathcal{D}}\mathcal{D}X^M = \frac{1}{1-\mu\bar{\mu}}\left(\left(\bar{\partial} - \mu\partial - (\partial\mu)\right)\mathcal{D}X^M - \frac{1}{2}\alpha\mathcal{D}\psi^M + \mu\bar{\alpha}\mathcal{D}\bar{\psi}^M - \frac{1}{2}(\partial\alpha)\psi^M\right). \quad (8.69)$$

This completes the list of the first few tensor fields $T^i_{m,n}$.

8.5 $U(1)$ sector

Let me recall the results obtained in section (6.2) for the BRST transformations of the $U(1)$ gauge field and the abelian ghost

$$\begin{aligned} sA_m &= \xi^n\partial_n A_m + (\partial_m\xi^n)A_n + \partial_m C \\ &\quad -\xi^\alpha\chi_m{}^\beta F_{\beta\alpha} - \xi^\alpha e_m{}^a F_{a\alpha} \\ sC &= \xi^m\partial_m C + \frac{1}{2}\xi^\alpha\xi^\beta F_{\beta\alpha} - \frac{1}{2}\xi^\alpha\xi^\beta T_{\beta\alpha}{}^m A_m. \end{aligned} \quad (8.70)$$

The BRST transformation of the gauge field already suggests that the gauge field and its symmetrized derivatives form trivial pairs with derivatives of the abelian ghost. Indeed the $\mathcal{U}$'s are typically components of gauge fields and their derivatives while the $\mathcal{V}$'s contain the corresponding derivatives of the ghosts. To find the complementary $\mathcal{W}$'s the BRST transformations of the gauge field and the abelian ghost have to be supplemented with the BRST transformations of the gaugino λ_β, the auxiliary field ϕ and the field strenght F_{ab}

$$\begin{aligned} s\phi &= \xi^n\partial_n\phi + \xi^\alpha(\gamma_*)_\alpha{}^\beta\lambda_\beta \\ s\lambda_\beta &= \xi^n\partial_n\lambda_\beta + \xi^\alpha\left(\frac{i}{2}(\gamma_* C)_{\alpha\beta}\varepsilon^{ab}F_{ab} - i(\gamma_*\gamma^a C)_{\alpha\beta}\mathcal{D}_a\phi + i(\gamma_* C)_{\alpha\beta}S\phi\right) \\ &\quad +\frac{1}{4}C^{ab}\varepsilon_{ab}(\gamma_*)_\beta{}^\gamma\lambda_\gamma \\ sF_{ab} &= \xi^n\partial_n F_{ab} + \xi^\alpha\left((\gamma_a\mathcal{D}_b\lambda)_\alpha - (\gamma_b\mathcal{D}_a\lambda)_\alpha - 2iT_{ab}{}^\beta(\gamma_* C)_{\beta\alpha}\phi + \frac{1}{2}\varepsilon_{ab}S(\gamma_*\lambda)_\alpha\right) \\ &\quad -\frac{1}{2}C_a{}^e F_{eb} - \frac{1}{2}C_b{}^e F_{ae} \end{aligned} \quad (8.71)$$

As in the supergravity part I add the Weyl transformations according to the Weyl weight of the fields:

	ϕ	λ_β	F_{ab}	A_m
Weyl weight	-1	$-\frac{3}{2}$	-2	0

(8.72)

To ensure the off-shell nilpotency of the BRST operator the super-Weyl transformation of the gaugino has to be introduced. The gaugino transforms under the fermionic symmetry transformation according to

$$\delta_{SW}\lambda_\beta = 2\eta^\alpha_{SW}(\gamma_* C)_{\alpha\beta}\phi. \tag{8.73}$$

This implies that the field strenghts, which are expected to count among the $\mathcal{W}$'s,

$$\begin{aligned} F_{\alpha\beta} &= 2i(\gamma_* C)_{\alpha\beta}\phi \\ F_{a\alpha} &= (\gamma_a)_\alpha{}^\beta \lambda_\beta \\ F_{ab} &= E_a{}^n E_b{}^m \left(\partial_n A_m - \partial_m A_n - (\chi_m\gamma_n\lambda) + (\chi_n\gamma_m\lambda) - 2i(\chi_m\gamma_* C\chi_n)\phi\right) \end{aligned} \tag{8.74}$$

are not invariant under super-Weyl transformations. Thus they have to be replaced by suitable redefinitions.

As in the supergravity part the cohomological problem is mapped from the space of local functionals to the cohomology of $\tilde{s}$ in the space local total forms. To switch from s to $\tilde{s}$ I replace the ghosts according to

$$\begin{aligned} \xi^m &\rightarrow \tilde{\xi}^m = \xi^m + dx^m \\ C &\rightarrow \tilde{C} = C + \tilde{\xi}^m A_m. \end{aligned} \tag{8.75}$$

After this replacement the $\tilde{s}$ transformations of the gauge field and the abelian ghost read

$$\begin{aligned} \tilde{s}A_m &= \tilde{\xi}^n(\partial_n A_m - \partial_m A_n) + \partial_m\tilde{C} - \xi^\alpha\chi_m{}^\beta F_{\alpha\beta} - \xi^\alpha e_m{}^a F_{a\alpha} \\ \tilde{s}\tilde{C} &= -\tilde{\xi}^m\tilde{\xi}^n(\partial_n A_m - \partial_m A_n) + \frac{1}{2}\xi^\alpha\xi^\beta + \tilde{\xi}^m\xi^\alpha\chi_m{}^\beta F_{\alpha\beta} + \tilde{\xi}^m\xi^\alpha F_{m\alpha} \end{aligned} \tag{8.76}$$

Since I expect the abelian ghost $\tilde{C}$ to count among the generalized connections it should have a $\tilde{s}$ transformation involving only generalized connections and tensor fields. Indeed the $\tilde{s}$ transformation of the redefined abelian ghost $\tilde{C}$ may be rewritten in terms of the generalized connections $\{\tilde{\eta}, \tilde{\bar{\eta}}, \tilde{\varepsilon}, \tilde{\bar{\varepsilon}}\}$ introduced in the previous section and super-weyl invariant redefinitions of the field strenghts. After a somewhat tedious computation one obtains

$$\tilde{s}\tilde{C} = \tilde{\eta}\tilde{\bar{\eta}}F^a + \frac{1}{2}\tilde{\eta}\tilde{\bar{\varepsilon}}\lambda^a + \frac{1}{2}\tilde{\bar{\eta}}\tilde{\varepsilon}\bar{\lambda}^a + \frac{1}{4}\tilde{\varepsilon}\tilde{\bar{\varepsilon}}\phi^a, \tag{8.77}$$

where the super-Weyl invariant field strenghts are given by

$$\phi^a = \sqrt{\frac{e_+{}^+ e_-{}^-}{64}}\phi$$

$$
\begin{aligned}
\lambda^a &= \frac{e_+^{\ +}}{8}\sqrt{\frac{e_-^{\ -}}{8}}\left(\lambda_2+\chi_+^2\phi\right)\\
\bar\lambda^a &= \frac{e_-^{\ -}}{8}\sqrt{\frac{e_+^{\ +}}{8}}\left(\lambda_1+\chi_-^1\phi\right)\\
F^a &= \frac{1}{1-\mu\bar\mu}\left(\varepsilon^{mn}(\partial_m A_n-\partial_n A_m)+\frac{1}{2}\mu\lambda^a-\frac{1}{2}\bar\mu\bar\lambda^a-\frac{1}{4}\alpha\bar\alpha\phi^a\right).
\end{aligned} \tag{8.78}
$$

These field strenghts count among the tensor fields. Thus their $\tilde s$ transformations are of the form

$$
\tilde s\mathcal{T}^i=\sum_{p\geq -1}\left(\tilde\eta^p L_p+\tilde{\bar\eta}^p\bar L_p+\tilde\varepsilon^{p+\frac{1}{2}}G_{p+\frac{1}{2}}+\tilde{\bar\varepsilon}^{p+\frac{1}{2}}\bar G_{p+\frac{1}{2}}\right)\mathcal{T}^i. \tag{8.79}
$$

As a first step in the computation of the $\tilde s$ transformations of the field strengths one observes

$$
\begin{aligned}
\lambda^a &= G_{-\frac{1}{2}}\phi^a,\\
\bar\lambda^a &= \bar G_{-\frac{1}{2}}\phi^a\\
F^a &= \bar G_{-\frac{1}{2}}G_{-\frac{1}{2}}\phi^a
\end{aligned} \tag{8.80}
$$

The algebra of the operators $\{L_p,\bar L_q,G_{p+\frac{1}{2}},\bar G_{q+\frac{1}{2}}\}$ is represented on the tensor fields $\{\phi^a,\lambda^a,\bar\lambda^a,F^a\}$ by

$$
\begin{aligned}
L_p\phi^a_{m,n} &= \begin{cases}\frac{m!}{(m-p)!}\left(m-p+\frac{1}{2}(p+1)\right)\phi^a_{m-p,n} & \text{for } p\leq m,\\ 0 & \text{for } p>m.\end{cases}\\
\bar L_q\phi^a_{m,n} &= \begin{cases}\frac{n!}{(n-q)!}\left(n-q+\frac{1}{2}(q+1)\right)\phi^a_{m,n-q} & \text{for } q\leq n,\\ 0 & \text{for } q>n.\end{cases}\\
G_{p+\frac{1}{2}}\phi^a_{m,n} &= \begin{cases}\frac{m!}{(m-p-1)!}\lambda^a_{m-p-1,n} & \text{for } p<m,\\ 0 & \text{for } p\geq m.\end{cases}\\
\bar G_{q+\frac{1}{2}}\phi^a_{m,n} &= \begin{cases}\frac{n!}{(n-q-1)!}\bar\lambda^a_{m,n-q-1} & \text{for } q<n,\\ 0 & \text{for } q\geq n.\end{cases}\\
L_p\lambda^a_{m,n} &= \begin{cases}\frac{(m+1)!}{(m-p)!}\lambda^a_{m-p,n} & \text{for } p\leq m,\\ 0 & \text{for } p>m.\end{cases}\\
\bar L_q\lambda^a_{m,n} &= \begin{cases}\frac{n!}{(n-q)!}\left(n-q+\frac{1}{2}(q+1)\right)\lambda^a_{m,n-q} & \text{for } q\leq n,\\ 0 & \text{for } q>n.\end{cases}\\
\bar L_q\bar\lambda^a_{m,n} &= \begin{cases}\frac{(n+1)!}{(n-q)!}\bar\lambda^a_{m,n-q} & \text{for } q\leq n,\\ 0 & \text{for } q>n.\end{cases}\\
L_p\bar\lambda^a_{m,n} &= \begin{cases}\frac{m!}{(m-p)!}\left(m-p+\frac{1}{2}(p+1)\right)\bar\lambda^a_{m-p,n} & \text{for } p\leq m,\\ 0 & \text{for } p>m.\end{cases}\\
G_{p+\frac{1}{2}}\lambda^a_{m,n} &= \begin{cases}\frac{(m+1)!}{(m-p)!}\phi^a_{m-p,n} & \text{for } p\leq m,\\ 0 & \text{for } p>m.\end{cases}
\end{aligned}
$$

$$\begin{aligned}
\bar{G}_{q+\frac{1}{2}}\lambda^a_{m,n} &= \begin{cases} \frac{n!}{(n-q-1)!}F^a_{m,n-q-1} & \text{for } q<n, \\ 0 & \text{for } q\geq n. \end{cases} \\
\bar{G}_{q+\frac{1}{2}}\bar{\lambda}^a_{m,n} &= \begin{cases} \frac{(n+1)!}{(n-q)!}\phi^a_{m,n-q} & \text{for } q\leq n, \\ 0 & \text{for } q> n. \end{cases} \\
G_{p+\frac{1}{2}}\bar{\lambda}^a_{m,n} &= \begin{cases} -\frac{m!}{(m-p-1)!}F^a_{m-p-1,n} & \text{for } q<n, \\ 0 & \text{for } q\geq n. \end{cases} \\
L_pF^a_{m,n} &= \begin{cases} \frac{(m+1)!}{(m-p)!}F^a_{m-p,n} & \text{for } p\leq m, \\ 0 & \text{for } p> m. \end{cases} \\
\bar{L}_qF^a_{m,n} &= \begin{cases} \frac{(n+1)!}{(n-q)!}F^a_{m,n-q} & \text{for } q\leq n, \\ 0 & \text{for } q> n. \end{cases} \\
G_{p+\frac{1}{2}}F^a_{m,n} &= \begin{cases} -\frac{(m+1)!}{(m-p)!}\bar{\lambda}^a_{m-p,n} & \text{for } p\leq m, \\ 0 & \text{for } p> m. \end{cases} \\
\bar{G}_{q+\frac{1}{2}}F^a_{m,n} &= \begin{cases} \frac{(n+1)!}{(n-q)!}\lambda^a_{m,n-q} & \text{for } q\leq n, \\ 0 & \text{for } q> n. \end{cases}
\end{aligned} \tag{8.81}$$

Thus the $\tilde{s}$ transformations explicitely read

$$\begin{aligned}
\tilde{s}\phi^a &= \tilde{\eta}\mathcal{D}\phi^a + \tilde{\bar{\eta}}\bar{\mathcal{D}}\phi^a + \frac{1}{2}\tilde{\varepsilon}\lambda^a + \frac{1}{2}\tilde{\bar{\varepsilon}}\bar{\lambda}^a + \frac{1}{2}(\partial\tilde{\eta})\phi^a + \frac{1}{2}(\bar{\partial}\tilde{\bar{\eta}})\phi^a \\
\tilde{s}\lambda^a &= \tilde{\eta}\mathcal{D}\lambda^a + \tilde{\bar{\eta}}\bar{\mathcal{D}}\lambda^a + \frac{1}{2}\tilde{\varepsilon}\mathcal{D}\phi^a + \frac{1}{2}\tilde{\bar{\varepsilon}}F^a + (\partial\tilde{\eta})\lambda^a + \frac{1}{2}(\bar{\partial}\tilde{\bar{\eta}})\lambda^a + \frac{1}{2}(\partial\tilde{\varepsilon})\phi^a \\
\tilde{s}\bar{\lambda}^a &= \tilde{\eta}\mathcal{D}\bar{\lambda}^a + \tilde{\bar{\eta}}\bar{\mathcal{D}}\bar{\lambda}^a + \frac{1}{2}\tilde{\bar{\varepsilon}}\bar{\mathcal{D}}\phi^a - \frac{1}{2}\tilde{\varepsilon}F^a + \frac{1}{2}(\partial\tilde{\eta})\bar{\lambda}^a + (\bar{\partial}\tilde{\bar{\eta}})\bar{\lambda}^a + \frac{1}{2}(\bar{\partial}\tilde{\bar{\varepsilon}})\phi^a \\
\tilde{s}F^a &= \tilde{\eta}\mathcal{D}F^a + \tilde{\bar{\eta}}\bar{\mathcal{D}}F^a - \frac{1}{2}\tilde{\varepsilon}\mathcal{D}\bar{\lambda}^a + \frac{1}{2}\tilde{\bar{\varepsilon}}\bar{\mathcal{D}}\lambda^a + (\partial\tilde{\eta})F^a + (\bar{\partial}\tilde{\bar{\eta}})F^a - \frac{1}{2}(\partial\tilde{\varepsilon})\bar{\lambda}^a + \frac{1}{2}(\bar{\partial}\tilde{\bar{\varepsilon}})\lambda^a,
\end{aligned} \tag{8.82}$$

where I used the fact that L_{-1} and $\bar{L}_{-1}$ serve as supercovariant derivative

$$L_{-1}\equiv\mathcal{D}, \quad \bar{L}_{-1}\equiv\bar{\mathcal{D}}. \tag{8.83}$$

With these transformations one can easyly check the nilpotency of the $\tilde{s}$-operator. Especially one verifies

$$\tilde{s}^2\tilde{C}=0. \tag{8.84}$$

8.6 Results

In this section I want to gather the main results of the previous sections. The new set of local jet coordinates for the Dirichlet-Superstring is given by

$$\{\mathcal{U}^i\} = \{\sigma^m, \partial^m\bar{\partial}^n\mu, \partial^m\bar{\partial}^n\alpha, \partial^m\bar{\partial}^n\bar{\mu}, \partial^m\bar{\partial}^n\bar{\alpha}, \partial^m\bar{\partial}^n e_\pm^{\,\pm},$$

$$
\begin{aligned}
& \partial^m \bar{\partial}^n A_-, \partial^m A_+ \ : m,n = 0,1,\ldots\} \\
\{\mathcal{V}^i\} &= \{\tilde{s}\mathcal{U}^i\} \\
\{\mathcal{W}_1^i\} &= \{\tilde{C}^N\} = \{\tilde{\eta}^p, \tilde{\bar{\eta}}^p, \tilde{\varepsilon}^{p+\frac{1}{2}}, \tilde{\bar{\varepsilon}}^{p+\frac{1}{2}}, \tilde{C} : p = -1,0,1,\ldots\} \\
\{\mathcal{W}_0^i\} &= \{\mathcal{T}_{m,n}^i\} = \{\mathcal{D}^m\bar{\mathcal{D}}^n X^M, \mathcal{D}^m\bar{\mathcal{D}}^n\psi^M, \mathcal{D}^m\bar{\mathcal{D}}^n\bar{\psi}^M, \mathcal{D}^m\bar{\mathcal{D}}^n F^M, \\
& \qquad \mathcal{D}^m\bar{\mathcal{D}}^n\phi^a, \mathcal{D}^m\bar{\mathcal{D}}^n\lambda^a, \mathcal{D}^m\bar{\mathcal{D}}^n\bar{\lambda}^a, \mathcal{D}^m\bar{\mathcal{D}}^n F^a \ : m,n = 0,1,\ldots\}
\end{aligned}
\tag{8.85}
$$

where the $\mathcal{W}_1^i$ are the generalized connections and the $\mathcal{W}_0^i$ are the tensor fields according to the definition

$$
\begin{aligned}
\mathcal{T}_{m,n}^i &= L_{-1}^m \bar{L}_{-1}^n \mathcal{T}^i = \mathcal{D}^m\bar{\mathcal{D}}^n\mathcal{T}^i \\
\{\mathcal{T}^i\} &= \{X^M, \psi^M, \bar{\psi}^M, F^M, \phi^a, \lambda^a, \bar{\lambda}^a, F^a\}
\end{aligned}
\tag{8.86}
$$

In order to split the variables of the $U(1)$ sector in trivial pairs, generalized connections and tensor fields I changed from the set

$$\{C, A_m, \partial_m C, \partial_n A_m, \partial_n\partial_m C, \ldots\} \tag{8.87}$$

to

$$\{C, A_m, \partial_m C, \partial_{(n}A_{m)}, \partial_{[n}A_{m]}\partial_n\partial_m C, \ldots\}, \tag{8.88}$$

where the brackets denote symmetrization and antisymmetrization of the indices.

The $\mathcal{V}^i$'s especially replace the Lorentz and the Weyl, super-Weyl and 'auxiliary Weyl' ghost fields. Furthermore they replace certain derivatives of the diffeomorphism and supersymmetry ghosts. The $\mathcal{W}_0^i$'s can be viewed as superconformal tensor fields.

Chapter 9

Summary and Outlook

I constructed suitable local jet coordinates for the cohomological analysis of Dirichlet-superstrings. In the case of IIB D-Branes it has been recently suggested that, for manifestly $SL(2, Z)$ invariant formulation, the action should contain a world volume field for every background gauge potential [ce97]. In the case of the D-string one then should introduce two $U(1)$ gauge fields on the world-sheet. Thus I studied $D = 2$ supergravity coupled to bosonic and fermionic string coordinates including an abelian gauge transformation. The model is characterized by its field content, its gauge symmetries and a locality requirement. The class of models I considered include D-superstrings but are not restricted to them. I used a formalism suugested by F. Brandt [br96], which relates the BRST cohomology to an underlying gauge covariant algebra. The construction of an suitable set of local jet coordinates uses among others the technique of contracting homotopies. In spite of the conceptual simplicity the construction is not straightforward, since it requires the splitting of the local jet coordinates into two subsets. One subset contains trivial pairs whereas the other is required to generate an invariant subalgebra. The elements of this complementary subset are interpreted as generalized connections and tensor fields. This characterization of tensor fields and generalized connections is purely algebraic and physically maeningful, since they provide the building blocks of gauge invariant actions, Noether currents, anomalies and the equations of motion. I stess that the existence of a pair of jet coordinates stisfying the condition for trivial pairs does in general not guarantee the existence of complementary jet coordinates. The difficulty in the construction of the new set of jet coordinates $\{\mathcal{U}^i, \mathcal{V}^i, \mathcal{W}^l\}$ is not the finding of the trivial pairs $\{\mathcal{U}^i, \mathcal{V}^i\}$ but the construction of the complementary $\mathcal{W}^l$'s.

The construction of the new set of jet coordinates for the Dirichlet-superstring is based on the super-Beltrami parametrization of two dimensional supergravity. This parametrization is introduced completely in the component field formalism. The results are shown to coincide with the results obtained by projection from superspace [dg90]. The beltrami differential and the Beltramino form trivial pairs with certain derivatives of the diffeomorphism and supersymmetry ghosts. The undifferentiated diffeomorphism and supersymmetry ghosts and their remaining derivatives count among the generalized connections. The operators corresponding to this infinite set of generalized connections build two copies of the super-Virasoro algebra. Super-Weyl invariant redefinitions of the string coordinates and their covariant derivatives are interpreted as tensor fields. In the $U(1)$ sector the abelian gauge field and its symmetrized derivatives form trivial pairs with derivatives of the abelian ghosts. The undifferentiated abelian ghost counts among the generalized connections. Its $\tilde{s}$ transformation contains the super-Weyl invariant $U(1)$ field strenghts which are interpreted as tensor fields. Thus a complete set of new local jet coordinates is constucted.

Having found suitable jet coordinates one might intend to tackle the BRST cohomology itself. As already illustrated in the investigation of the BRST cohomology of the superstring [ta96] one obtains a basis for the BRST cohomology group $H(\tilde{s})$ with an infinite number of terms. This is an essential difference to the purely bosonic case. For the case of $D = 2$ gravity it has been shown [br95] that the basis of the BRST cohomology contains only a finite number of terms, which is also true for the bosonic D-string. In the supersymmetric case this is not true essentially due to the presence of the bosonic ghosts $\tilde{\varepsilon}$ and $\tilde{\bar{\varepsilon}}$. To the best of my knowlege to this day no attempt has been made to a systematic investigation of the BRST cohomology of Dirichlet-superstring.

Appendix A

Spinor space

A.1 γ-matrices in 2-D

A.1.1 Definitions

The flat metric and basic anticommutation relation of the γ-matrices are given by

$$\eta_{ab} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{A.1}$$

and

$$\{\gamma^a, \gamma^b\} = 2\eta^{ab} \tag{A.2}$$

I choose the following set of γ-matrices

$$(\gamma^0)_\alpha{}^\beta = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad (\gamma^1)_\alpha{}^\beta = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad (\gamma_*)_\alpha{}^\beta = (\gamma^0)_\alpha{}^\delta (\gamma^1)_\delta{}^\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{A.3}$$

A.1.2 Manipulation of γ-matrices

Throughout the following formulas the summation convention 'ten to four' is assumed

$$\gamma^0\gamma^1 \;=\; -\gamma^1\gamma^0 = \gamma_*, \tag{A.4}$$

$$\gamma_*^2 = \mathbf{1}, \qquad \gamma^0\gamma^0 = -\gamma^1\gamma^1 = \mathbf{1} \tag{A.5}$$

$$\gamma^0 = \gamma_0, \qquad \gamma^1 = -\gamma_1 \tag{A.6}$$

$$\gamma_a\gamma_b = \eta_{ab}\mathbf{1} + \varepsilon_{ab}\gamma_* \tag{A.7}$$

$$(\gamma^a\gamma_*) = -\varepsilon^{ab}\gamma_b \qquad (\gamma_*\gamma_a) = \varepsilon_{ab}\gamma^b \tag{A.8}$$

The following identities for γ-matrices in two dimensions are frequently used

$$\gamma^a\gamma^b\gamma_a = 0 \tag{A.9}$$

$$Tr(\gamma^a\gamma^b) = 2\eta^{ab} \tag{A.10}$$

The expressions for the torsions, field strengths and curvatures contain the contractions

$$(\gamma^0 C)_{\alpha\beta} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (\gamma^1 C)_{\alpha\beta} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (\gamma_* C)_{\alpha\beta} = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \tag{A.11}$$

The γ-matrices satisfy the Fierz identity

$$\delta_\alpha{}^\beta\delta_\gamma{}^\delta + (\gamma^a)_\alpha{}^\beta(\gamma_a)_\gamma{}^\delta + (\gamma_*)_\alpha{}^\beta(\gamma_*)_\gamma{}^\delta = 2\delta_\alpha{}^\delta\delta_\gamma{}^\beta. \tag{A.12}$$

The following two identities are equivalent formulations of the above identity

$$2(\gamma_*)_\alpha{}^\gamma(\gamma_*)_\beta{}^\delta = \delta_\alpha{}^\delta\delta_\beta{}^\gamma + (\gamma_*)_\alpha{}^\delta(\gamma_*)_\beta{}^\gamma - (\gamma_a)_\alpha{}^\delta(\gamma^a)_\beta{}^\gamma \tag{A.13}$$

$$(\gamma_a)_\alpha{}^\gamma(\gamma^a)_\alpha{}^\gamma = \delta_\alpha{}^\delta\delta_\beta{}^\gamma - (\gamma_*)_\alpha{}^\delta(\gamma_*)_\beta{}^\gamma \tag{A.14}$$

A.1.3 Light-cone coordinates

The metric in light-cone coordinates reads

$$\eta_{\pm\pm} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{A.15}$$

The γ-matrices in light-cone coordinates are

$$\begin{aligned} \gamma_+ &= \gamma_0 + \gamma_1 = \begin{pmatrix} 0 & -2i \\ 0 & 0 \end{pmatrix} \\ \gamma_- &= \gamma_0 - \gamma_1 = \begin{pmatrix} 0 & 0 \\ 2i & 0 \end{pmatrix} \end{aligned} \tag{A.16}$$

and

$$\gamma^+ = \gamma_-, \qquad \gamma^- = \gamma_+. \tag{A.17}$$

The totally antisymmetric tensor $\varepsilon_{\pm,\pm}$ reads

$$\varepsilon_{\pm\pm} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \varepsilon_\pm{}^\pm = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{A.18}$$

Appendix B

Super-Beltrami variables

In this appendix I collect the frequently used formulae of the Beltrami parametrization. The Beltrami differentials and the Beltramino are

$$\mu = \frac{e_-{}^+}{e_+{}^+}, \qquad \bar{\mu} = \frac{e_+{}^-}{e_-{}^-}$$
$$\alpha = i(\hat{\chi}_-{}^2 - \mu\,\hat{\chi}_+{}^2) \qquad \bar{\alpha} = i(\hat{\chi}_+{}^1 - \bar{\mu}\,\hat{\chi}_-{}^1) \tag{B.1}$$

with

$$\hat{\chi}_m{}^2 := \sqrt{\frac{8}{e_+{}^+}}\,\chi_m{}^2, \quad \hat{\chi}_m{}^1 := \sqrt{\frac{8}{e_-{}^-}}\,\chi_m{}^1. \tag{B.2}$$

The Beltrami ghost fields are

$$\begin{aligned} \eta &= \xi^+ + \mu\,\xi^-, \\ \bar{\eta} &= \xi^- + \bar{\mu}\,\xi^+, \\ \varepsilon &= \hat{\xi}^2 + \xi^-\alpha, \\ \bar{\varepsilon} &= \hat{\xi}^1 + \xi^+\bar{\alpha}, \end{aligned} \tag{B.3}$$

where

$$\begin{aligned} \hat{\xi}^1 &= \sqrt{\frac{8}{e_+{}^+}}\,\xi^1, \\ \hat{\xi}^2 &= \sqrt{\frac{8}{e_+{}^+}}\,\xi^2. \end{aligned} \tag{B.4}$$

I frequently used the following conversions of the diffeomorphism ghosts

$$\xi^+ = \frac{\eta - \mu\,\bar{\eta}}{1 - \mu\bar{\mu}}, \quad \xi^- = \frac{\bar{\eta} - \bar{\mu}\,\eta}{1 - \mu\bar{\mu}} \tag{B.5}$$

and of the supersymmetry ghosts

$$\begin{aligned}\hat{\xi}^1 &= \bar{\varepsilon} - \xi^+ \bar{\alpha} = \bar{\varepsilon} - \frac{\eta - \mu\,\bar{\eta}}{1 - \mu\bar{\mu}}\,\bar{\alpha} \\ \hat{\xi}^2 &= \varepsilon - \xi^- \alpha = \varepsilon - \frac{\bar{\eta} - \bar{\mu}\,\eta}{1 - \mu\bar{\mu}}\,\alpha \end{aligned} \tag{B.6}$$

and

$$\xi^+ \xi^- = \frac{\eta\bar{\eta}}{1 - \mu\bar{\mu}}. \tag{B.7}$$

Bibliography

[be58] L. Bers, *Riemann Surfaces*, Lecture notes B39 (New York University, Courant Institute of Mathematical Sciences, 1958)

[be88] C. Becchi, Nucl. Phys. **B304** (1988), 513

[bh96] F. Brandt, M. Henneaux and A. Wilch, Phys. Lett. **387B** (1996), 320; *Ward identities for rigid symmetries of higher order*, UB-ECM-PF 96/18, ULB-TH 96/17, **hep-th/9611056**

[br91] F. Brandt, *Lagrangedichten und Anomalien in vierdimensionalen supersymmetrischen Theorien*, Ph.D. thesis, unpublished, Hannover 1991

[br93] F. Brandt, *Anomaly candidates and invariants of D=4, N=1 supergravity theories*, **hep-th/9306054**

[Br93] F. Brandt, *Structure of BRS-Invariant Local Functionals*, **hep-th/9310123**

[br95] F. Brandt, W. Troost, A. van Proyen, *The BRST-antibracket cohomology of 2D gravity conformally coupled to scalar matter*, **hep-th/9509035**

[br96] F. Brandt, *Local BRST Cohomology and Covariance*, **hep-th/9604025**

[Br97] F. Brandt, *Gauge covariant algebras and local BRST cohomology*, **hep-th/9711171**

[br97] F. Brandt, *Cohomological analysis of bosonic D-strings* **hep-th/9712125**

[brs74] C. Becchi, A. Rouet, R. Stora, Phys. Lett. **52B** (1974), 344; Commun. Math. Phys. **42** (1975), 127; Ann. Phys.**98** (1976), 287

[ce97] M. Cederwall, *The manifestly $Sl(2;Z)$-covariant Superstring*, **hep-th/9709002**

[dg90] F. Delduc, F. Gieres, *Beltrami differentials, conformal models and their supersymmetric generalizations*, Class. Quant. Grav. **7**, (1990) 1907

[gd96] F. Delduc, F. Gieres, S. Gourmelen, *d=2, N=2 Superconformal Symmetries and Models*, **hep-th/9609182**

[dr87] N. Dragon, U. Ellwanger, M. Schmidt, *Supersymmetry and Supergravity*, Progress in Particle and Nuclear Physics, **18** (1987) 1-91

[dr79] N. Dragon, *Torsion and Curvature in Extended Supersymmetry*, Z. Physik G, **2**, (1979), 29-32

[fu95] T. Fujiwara, H. Igarashi, R. Kuriki, T. Tabei, *BFV-BRST Quantization of 2D Supergravity*, **hep-th/9502127**

[gr90] R. Grimm,*Left and right decomposition of two dimensional superspace geometry and its BRS structure*, Ann. Phys., **NY 200** (1990), 49

[ho79] P. S. Howe, *Super Weyl transformations in two dimensions*, J. Phys. A.: Math. Gen. 12 (1979), 393

[kr96] M. Kreuzer, *Geometrische Methoden der theoretischen Physik*, Vorlesungsskriptum

[ns71] A. Neveu, J. H. Schwarz, *Factorizable dual model of pions*, Phys. Rev. **D4** (1971) 1109

[po96] J. Polchinski, *TASI Lectures on D-branes*, **hep-th/9611050**

[ra71] P. Ramond, *Dual theory for free fermions*, Phys. Rev. **D3** (11971) 2415

[sa89] D. J. Saunders, *The Geometry of Jet Bundles*, London Mathematical Society Lecture Note Series 142 (Cambridge University Press, Cambridge, 1989)

[sc74] J. Scherk, J. H. Schwarz, *Dual models for non-hadrons*, Nucl. Phys. **B81** (1974), 118

[sc93] A. Boresch, M. Schweda, S. P. Sorella, *Vector supersymmetry of the superstring in the super-Beltrami parametrization*

[so81] M. F. Sohnius, *Identities for Bianchi identities*, in 'Superspace and Supergravity', eds. S. Hawking, M. Rocek (Cambridge University Press, Cambridge 1981), p. 469

[ta95] L. Tataru, I. V. Vancea, *BRST cohomology in Beltrami parametrization*, **hep-th/9504036** ;P. A. Blaga, L. Tataru, I. V. Vancea, *BRST cohomology for 2D gravity*, **hep-th/9504037**

[ta96] L. Tataru, I. V. Vancea, *BRST cohomology of the superstring in super-Beltrami parametrization*, **hep-th/9604088**

[ta98] Y. Tanii, *Introduction to supergravities in diverse dimensions*, **hep-th/9802138**

[to83] P. Townsend, T. Kugo, Nucl. Phys. **B221** (1983), 357

[wi] M. G. Green, J. H. Schwarz, E. Witten, *Superstring theory*, Vol. 1, Cambridge University Press (1987)

[wz77] J. Wess, B. Zumino, *Superspace Formulation of Supergravity*, Phys. Lett. **66B**, (1977), 361-364