DIPLOMARBEIT

Covariant Quantization of the Superstring

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Ao. Univ. Prof. Doz. DI Dr. techn. Maximilian Kreuzer

durch

Johanna Knapp
Haitzendorf 112
A–3485 Haitzendorf

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to my parents

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Chapter 1

Introduction

1.1 Motivation

There exist two basic formalisms in superstring theory, the RNS formalism and the Green–Schwarz formalism. In the RNS formalism space time supersymmetry is not manifest. The theory is worldsheet–supersymmetric, but target space supersymmetry only comes in after GSO projection, which eliminates the tachyon from this theory and at the same times yields a space–time supersymmetric physical spectrum.
The Green–Schwarz (GS) formalism, on the other hand, is manifestly target–space supersymmetric by construction. The physical spectrum of this theory is equivalent to the spectrum of the RNS formalism. There is, however, a long standing problem. Quantization has so far only been possible in light–cone gauge. A covariant quantization prescription is not known.

In spite of the tremendous problems with covariant quantization new interest has been laid in the Green–Schwarz formalism. The reason for that are some inconvenient features of the RNS formalism. For example, amplitudes with more than four external fermions are difficult to compute in a Lorentz covariant manner because picture–changing operators and bosonization are needed.
Furthermore, the Green–Schwarz formalism provides the natural setup to describe the superstring in supergravity backgrounds. The motivation to consider this is the AdS–CFT correspondence, which is a conjectured duality between type IIB string theory on an $AdS_5 \otimes S^5$ background and $N = 4$, $D = 4$ super Yang–Mills theory. The $AdS_5 \otimes S^5$–space is equipped with a fermionic 5–form field, the Ramond–Ramond background, which cannot be described in the RNS formalism in a straight forward way. In the GS formalism the theory can be easily coupled to Ramond–Ramond backgrounds.

In the past five years some new approaches concerning the covariant quantization of the Green–Schwarz superstring have been developed. Already in the early 80s Siegel related the GS string to a free field theory via a constraint. In 2000 Berkovits presented his pure spinor formalism, where this constraint is implemented cohomologically. Covariant quantization of this model is possible, but there is the "pure spinor constraint", which has to be satisfied by the ghost field. The effect of this constraint is that not all components of the ghost are independent, which complicates many calculations.
To avoid these difficulties Grassi, van Nieuwenhuizen and collaborators presented a new ap-

proach, where the pure spinor constraint is no longer needed. Instead, new ghosts and auxiliary fields have to be introduced. A relation of this approach to Wess–Zumino–Novikov–Witten (WZNW) models was found. This model for the superstring is the basis of this diploma thesis.

1.2 Outline of the Thesis

The paper is organized as follows:
In chapter 2 we review string models with manifest target space supersymmetry. An introduction to the GS superstring and Siegel's free field approach is given. Then we present Berkovits' pure spinor formalism and cite the most important results that can be obtained from this model.
Chapter 3, which is the central chapter of this diploma thesis, deals with the covariant quantization of the superstring without pure spinor constraints and is based on the works of van Niewenhuizen et al. In section 3.2 we give a short introduction to WZNW models and point out the WZNW properties of the superstring model. We also point out some problems of the WZNW formulation, in particular when it is applied to the type II superstring.
In section 3.3 we derive a WZNW action for the heterotic and the type II superstring using the Noether method to gauge the free field action, and we perform BRST quantization.
In section 3.4 we review the operator algebra of the WZNW model, which is a Kazama algebra, and revisit the idea to use a topological quartet to turn this algebra into a twisted $N = 2$ superconformal algebra.
Section 3.5 is dedicated to the BRST operator and the cohomology. We summarize some older approaches to the cohomology problem and then follow the ideas of van Nieuwenhuizen and collaborators to define a cohomology for the superstring as a WZNW model with two BRST operators.
In section 3.6 we present a review of the construction of a twisted $N = 4$ superconformal algebra out of the fields and currents of the model.
Section 3.7 deals with the worldsheet covariant formulation of our model, which we implement by gauging diffeomorphisms and the fermionic symmetry that is found in WZNW models.
In chapter 4 we summarize our results and discuss some open problems.
In the appendices we explain our conventions and discuss gamma matrices and spinors in ten dimensions. We give a summary of the properties the fields and ghosts in our model and the most important operator products. Furthermore we discuss the Beltrami parameterization and some useful identities which hold in two dimensions. Finally we give an extract of a Mathematica file that was written to compute the products.

Some of the results presented in this thesis were published in [1].

Chapter 2

String Models with Manifest Target Space Supersymmetry

2.1 The Green-Schwarz Superstring

In this section we construct a classical string action with manifest target space supersymmetry. We follow the arguments of [2]. We start the discussion with the massless superparticle. A simple worldline action is given by:

$$S = \frac{1}{2}\int \mathrm{d}\tau e^{-1}\dot{x}^2, \tag{2.1}$$

where e is the square root of a one–dimensional metric. This action is invariant under local reparameterizations $\tau \to f(\tau)$ and global Poincaré transformations:

$$\begin{aligned} \delta x^m &= a^m + b^m{}_n x^n \\ \delta e &= 0, \end{aligned} \tag{2.2}$$

where $b^\mu{}_\nu$ is antisymmetric.

Now we extend this action such that it is invariant under N supersymmetries. We introduce N anticommuting space–time spinor coordinates $\theta^{A\alpha}(\tau)$ with $A = 1, \ldots, N$ and $\alpha = 1 \ldots 2^{D/2}$. To construct supersymmetry transformations we introduce infinitesimal Grassmann parameters ϵ^A. The SUSY variations are then given by:

$$\begin{aligned} \delta x^m &= i\bar{\epsilon}^A \Gamma^m \theta^A \\ \delta\theta^A &= \epsilon^A \\ \delta\bar{\theta^A} &= \bar{\epsilon}^A \\ \delta e &= 0 \end{aligned} \tag{2.3}$$

It is easy to check that the following Poincaré invariant action is invariant under these transformations:

$$S = \frac{1}{2}\int \mathrm{d}\tau e^{-1}\left(\dot{x}^m - i\bar{\theta}^A\Gamma^m\dot{\theta}^A\right)^2 \tag{2.4}$$

The equations of motion are given by:

$$p^2 = 0 \qquad \dot{p}^m = 0 \qquad \Gamma^m p_m \dot{\theta} = 0, \tag{2.5}$$

where we defined:

$$p^m = \dot{x}^m - i\bar{\theta}^A\Gamma^m\dot{\theta}^A \tag{2.6}$$

The action (2.4) has an additional local fermionic symmetry. With the parameter $\kappa^{A\alpha}(\tau)$ the corresponding transformations read:

$$\begin{aligned} \delta\theta^A &= i\Gamma^m p_m \kappa^A \\ \delta x^m &= i\bar{\theta}^A\Gamma^m\delta\theta^A \\ \delta e &= 4e\dot{\bar{\theta}}^A\kappa^A \end{aligned} \tag{2.7}$$

Computing the anticommutator of two κ variations we find that the symmetry algebra only closes on–shell:

$$[\delta_1, \delta_2]\theta^A = \left(2i\Gamma_m\kappa_2^A\dot{\bar{\theta}}^B\Gamma^n p_n\Gamma^m\kappa_1^B + 4i\Gamma^m p_m\kappa_2^A\dot{\bar{\theta}}^B\kappa_1^B\right) - (1 \leftrightarrow 2) \tag{2.8}$$

The first term is proportional to the equations of motion, the second term is again a κ symmetry transformation.

There is another bosonic symmetry of the superparticle action. The transformations with scalar parameter $\lambda(\tau)$ read:

$$\begin{aligned} \delta\theta^A &= \lambda\dot{\theta}^A \\ \delta x^m &= i\bar{\theta}^A\Gamma^m\delta\theta^A \\ \delta e &= 0 \end{aligned} \tag{2.9}$$

Superstring

We generalize the results of the point particle to the superstring. The action for the bosonic string is:

$$S = -\frac{1}{8\pi}\int \mathrm{d}^2\sigma\sqrt{g}g^{\mu\nu}\partial_\mu x^m\partial_\nu x_m \tag{2.10}$$

In analogy to the superparticle we make the following guess for a superstring action which is diffeomorphism invariant and has N global supersymmetries.

$$S_1 = -\frac{1}{8\pi}\int \mathrm{d}^2\sigma\sqrt{g}g^{\mu\nu}\Pi_{\mu m}\Pi_\nu^m, \tag{2.11}$$

with

$$\Pi_\mu^m = \partial_\mu x^m - i\bar{\theta}^A\Gamma^m\partial_\mu\theta^A \tag{2.12}$$

κ–symmetry of this action is lost but it can be recovered for $N \leq 2$ by adding another term, the Wess–Zumino (WZ) term, to the action. It can be constructed as follows (We follow the line of [4]): We specialize to $D = 10$, see Appendix A.3 for gamma matrix conventions and spinor properties. First we consider the term $\partial_\mu x^m(\theta\gamma_m\partial^\mu\theta)$ in (2.11). In light–cone gauge $x^+ = x_0^+ + p^+t$ one finds that this expression contains a term $p^+\theta\gamma_+\partial_t\theta$ which is a candidate for a kinetic term for the θ^A. But for a kinetic term we would also need $p^+\theta\gamma_+\partial_\sigma\theta$. Such a term could be obtained if the action contained a term of the form $(\partial_t x^+)\theta\gamma_+\partial_\sigma\theta$, which reads $\varepsilon^{\mu\nu}\partial_\mu x^m \theta\gamma_m\partial_\nu\theta$ in covariant form, but this expression is not supersymmetric. A supersymmetric Lagrangian $\mathcal{L}_{WZ}$ that is proportional to $\varepsilon^{\mu\nu}$ can be written as a two–form which should be SUSY invariant up to a total derivative:

$$\omega_2 \;=\; \mathcal{L}_{WZ}\mathrm{d}^2x \qquad\qquad \delta\omega_2 \;=\; \mathrm{d}X \tag{2.13}$$

$\mathrm{d}^2 = 0$ implies that $\delta\mathrm{d}\omega_2 = \mathrm{d}\delta\omega_2 = 0$. Thus, we can define a three–form as $\omega_3 = \mathrm{d}\omega_2$, which has the properties $\delta\omega_3 = 0$, $\mathrm{d}\omega_3 = 0$. To construct a SUSY invariant ω_3 we have the invariant one–forms Π^m and $\mathrm{d}\theta^A$ at our disposal. The only Lorentz invariant quantity that can be written down is:

$$\omega_3 \;=\; a_{AB}\Pi^m\mathrm{d}\theta^A\gamma_m\mathrm{d}\theta^B, \tag{2.14}$$

where a_{AB} is a real symmetric $N\times N$ matrix. Diagonalizing a_{AB} by a real orthogonal transformation yields:

$$\mathrm{d}\omega_3 \;=\; -i\left(\sum_A \mathrm{d}\theta^A\gamma^m\mathrm{d}\theta^A\right)\left(\sum_B a_B\mathrm{d}\theta^B\gamma_m\mathrm{d}\theta^B\right) \tag{2.15}$$

The direct terms cancel using the Fierz identity $\gamma^m\mathrm{d}\theta^1(\mathrm{d}\theta^1\gamma_m\mathrm{d}\theta^1) = 0$ whereas the cross terms only vanish if $N = 2$ and the matrix a_{AB} has entries $(+1,-1)$. Thus, we have:

$$\omega_3 \;=\; i\Pi^m\left(\mathrm{d}\theta\gamma_m\mathrm{d}\theta - \mathrm{d}\hat{\theta}\gamma_m\mathrm{d}\hat{\theta}\right) \tag{2.16}$$

Computing the inverse of this expression yields the WZ–term up to an overall constant. Thus, we get for the complete type II Green–Schwarz action:

$$\begin{aligned}
S^{GS} &= \frac{1}{8\pi}\int \mathrm{d}^2\sigma\sqrt{-g} - \frac{1}{2}\Pi^m_\mu\Pi^\mu_m + \mathcal{L}_{WZ}\\
\mathcal{L}_{WZ} &= -i\varepsilon^{\mu\nu}\Pi^m_\mu\left((\theta\gamma_m\partial_\nu\theta) - (\hat{\theta}\gamma_m\partial_\nu\hat{\theta})\right) - \varepsilon^{\mu\nu}(\theta\gamma^m\partial_\mu\theta)(\hat{\theta}\gamma_m\partial_\nu\hat{\theta})\\
\Pi^m_\mu &= \partial_\mu x^m - i\theta\gamma^m\partial_\mu\theta - i\hat{\theta}\gamma^m\partial_\mu\hat{\theta}
\end{aligned} \tag{2.17}$$

A Green–Schwarz action that is supersymmetric can also be defined in $D = 3$ where θ is Majorana, in $D = 4$ for θ Majorana or Weyl and in $D = 6$ where θ is a Weyl spinor [2]. In a lengthy calculation it can be shown [2] that the action is invariant under the following κ symmetry:

$$\begin{aligned}
\delta\theta^\alpha &= 2i\gamma^m\Pi_{\mu m}\kappa^{\alpha\mu}\\
\delta\hat{\theta}^{\hat{\alpha}} &= 2i\gamma^m\Pi_{\mu m}\hat{\kappa}^{\hat{\alpha}\mu}\\
\delta x^m &= i\theta\gamma^m\delta\theta + i\hat{\theta}\gamma^m\delta\hat{\theta}\\
\delta\Pi^m_\mu &= 2i\partial_\mu\theta\gamma^m\delta\theta + 2i\partial_\mu\hat{\theta}\gamma^m\delta\hat{\theta}\\
\delta(\sqrt{g}g^{\mu\nu}) &= -32\sqrt{g}\left(P^{\mu\lambda}\kappa^\nu\partial_\lambda\theta + \bar{P}^{\alpha\lambda}\hat{\kappa}^\nu\partial_\lambda\hat{\theta}\right)
\end{aligned} \tag{2.18}$$

Note that the transformation parameter κ now gets an additional worldsheet index as compared to the superparticle. For the definition of the chiral projectors $P^{\mu\nu}$ and $\bar{P}^{\mu\nu}$ we refer

to Appendix C.
The supersymmetry transformations for the superstring in $D = 10$ read:

$$\begin{aligned} \delta\theta^{\alpha} &= \epsilon^{\alpha} \\ \delta\hat{\theta}^{\hat{\alpha}} &= \hat{\epsilon}^{\hat{\alpha}} \\ \delta x^m &= i\epsilon\gamma^m\theta + i\hat{\epsilon}\gamma^m\hat{\theta} \end{aligned} \tag{2.19}$$

Furthermore it can be shown that the superstring action is invariant under the following local bosonic transformations:

$$\begin{aligned} \delta\theta^{\alpha} &= \sqrt{g}P^{\mu\nu}\partial_\nu\theta^{\alpha}\lambda_\mu \\ \delta\hat{\theta}^{\hat{\alpha}} &= \sqrt{g}\bar{P}^{\mu\nu}\partial_\nu\hat{\theta}^{\hat{\alpha}}\lambda_\mu \\ \delta x^m &= i\theta\gamma^m\delta\theta + i\hat{\theta}\gamma^m\delta\hat{\theta} \end{aligned} \tag{2.20}$$

One can find this symmetry by considering the algebra of κ–transformations. Its closure requires the transformations above.
Quantization of the Green–Schwarz string has only been possible in light–cone gauge where manifest covariance is lost.
We can have the following types of superstrings:

1. Type I. For the open superstring we only have $N = 1$ supersymmetry, i.e. $\hat{\theta} = 0$. In this case suitable boundary conditions have to be satisfied.

2. Type II A/B. Each of the spinors θ and $\hat{\theta}$ can be either chiral or antichiral. Chiral spinors will be denoted with contravariant indices θ^{α} and antichiral spinors with covariant indices θ_{α}. We will introduce hatted indices for the right–moving sector in order to treat both cases at the same time:

$$\hat{\theta}^{\hat{\alpha}} = \begin{cases} \hat{\theta}_{\alpha} & \text{for type IIA} \\ \hat{\theta}^{\alpha} & \text{for type IIB} \end{cases} \tag{2.21}$$

3. Heterotic String. As in the open string case we only have $N = 1$ supersymmetry but still a left–moving and a right–moving sector and the critical dimension is 10. If we only have $\theta^{\alpha}(z)$ there is no supersymmetry for the right–moving sector. Since there are only ten x^m the Virasoro anomaly in the right moving sector is not 0. To compensate this one has to introduce 32 additional Majorana–Weyl fermions $\hat{\lambda}^A$ to cancel this anomaly. This leads to the gauge groups $SO(32)$ and $E_8 \times E_8$ depending on which boundary conditions these fields obey [2]. Whenever we refer to the heterotic string in the following we will ignore these additional fields.

2.1.1 Free Field Action and Constraints

In [5] Siegel proposed to relate the Green–Schwarz string to a free field theory via a constraint. In that paper a Poisson bracket algebra for a free theory was derived from the Green–Schwarz superstring. Our discussion follows [4]. One can rewrite the Green–Schwarz action (2.17) in terms of chiral derivatives ∂ and $\bar{\partial}$:

$$\mathcal{L} \propto -\frac{1}{2}\partial x^m\bar{\partial}x_m + i\partial x^m\theta\gamma_m\bar{\partial}\theta + i\bar{\partial}x^m\hat{\theta}\gamma_m\partial\hat{\theta}$$

$$
\begin{aligned}
&+\frac{1}{2}(\theta\gamma^m\bar{\partial}\theta)(\theta\gamma_m\partial\theta+\hat{\theta}\gamma_m\partial\hat{\theta})+\frac{1}{2}(\hat{\theta}\gamma^m\partial\hat{\theta})(\theta\gamma_m\bar{\partial}\theta+\hat{\theta}\gamma_m\bar{\partial}\hat{\theta}) \\
= &-\frac{1}{2}\partial x^m\bar{\partial}x_m-\left(i\partial x^m(\gamma_m\theta)_\alpha+\frac{1}{2}(\gamma_m\theta)_\alpha\left(\theta\gamma^m\partial\theta+\hat{\theta}\gamma^m\partial\hat{\theta}\right)\right)\bar{\partial}\theta^\alpha \\
&-\left(i\bar{\partial}x^m(\gamma_m\hat{\theta})_{\hat{\alpha}}+\frac{1}{2}(\gamma_m\hat{\theta})_{\hat{\alpha}}\left(\theta\gamma^m\bar{\partial}\theta+\hat{\theta}\gamma^m\bar{\partial}\hat{\theta}\right)\right)\partial\hat{\theta}^{\hat{\alpha}} \\
= &-\frac{1}{2}\partial x^m\bar{\partial}x_m+(p_{z\alpha})_{\mathrm{Sol}}\bar{\partial}\theta^\alpha+(\hat{p}_{\bar{z}\hat{\alpha}})_{\mathrm{Sol}}\partial\hat{\theta}^{\hat{\alpha}}
\end{aligned}
\tag{2.22}
$$

Now one can obtain a free field Lagrangian by introducing new elementary fields $p_{z\alpha}$ and $\hat{p}_{\bar{z}\hat{\alpha}}$ which become the conjugate momenta of θ^α and $\hat{\theta}^{\hat{\alpha}}$, respectively. To get back to the original Green–Schwarz action one imposes the constraints that $d_{z\alpha}=p_{z\alpha}-(p_{z\alpha})_{\mathrm{Sol}}$ and $\hat{d}_{\bar{z}\hat{\alpha}}=\hat{p}_{\bar{z}\hat{\alpha}}-(\hat{p}_{\bar{z}\hat{\alpha}})_{\mathrm{Sol}}$ vanish. The complete expressions are given by:

$$
\begin{aligned}
d_{z\alpha} &= p_{z\alpha}-(\gamma_m\theta)_\alpha\left(i\partial x^m+\frac{1}{2}\theta\gamma^m\partial\theta+\frac{1}{2}\hat{\theta}\gamma^m\partial\hat{\theta}\right) \\
\hat{d}_{\bar{z}\hat{\alpha}} &= \hat{p}_{\bar{z}\hat{\alpha}}-(\gamma_m\hat{\theta})_{\hat{\alpha}}\left(i\bar{\partial}x^m+\frac{1}{2}\theta\gamma^m\bar{\partial}\theta+\frac{1}{2}\hat{\theta}\gamma^m\bar{\partial}\hat{\theta}\right)
\end{aligned}
\tag{2.23}
$$

The free field action now reads:

$$
\begin{aligned}
S &= \frac{1}{8\pi}\int \mathrm{d}^2\sigma\sqrt{g}-\frac{1}{2}\partial x^m\bar{\partial}x_m+p_{z\alpha}\bar{\partial}\theta^\alpha+\hat{p}_{\bar{z}\hat{\alpha}}\partial\hat{\theta}^{\hat{\alpha}} \\
&= \frac{1}{8\pi}\int \mathrm{d}^2\sigma\sqrt{g}-\frac{1}{2}\Pi_z^m\Pi_{\bar{z}m}+\mathcal{L}_{WZ}+d_{z\alpha}\bar{\partial}\theta^\alpha+\hat{d}_{\bar{z}\hat{\alpha}}\partial\hat{\theta}^{\hat{\alpha}},
\end{aligned}
\tag{2.24}
$$

where $\mathcal{L}_{WZ}$ is given in (2.17). The constraint $d_{z\alpha}$ is part of the following closed algebra:

$$
\begin{aligned}
id_{z\alpha}(z)id_{z\beta}(w) &\sim -2i\frac{\gamma^m_{\alpha\beta}\Pi_{zm}(w)}{z-w} \\
id_{z\alpha}(z)\Pi_{zm}(w) &\sim -2\frac{\gamma_{m\,\alpha\beta}\partial\theta^\beta(w)}{z-w} \\
\Pi_{zm}(z)\Pi_{zm}(w) &\sim -\frac{\eta_{mn}}{(z-w)^2} \\
id_{z\alpha}(z)\partial\theta^\beta(w) &\sim -\frac{i\delta_\alpha^\beta}{(z-w)^2}
\end{aligned}
\tag{2.25}
$$

For the right–moving algebra one has to replace $z\to\bar{z}$, $\theta\to\hat{\theta}$ and $p\to\hat{p}$.
The constraints $d_{z\alpha}=0$ and $\hat{d}_{\bar{z}\hat{\alpha}}=0$ have to be implemented cohomologically but the standard BRST procedure does not work because the $d_{z\alpha}$ are mixed first and second class constraints. The first class constraints correspond to the κ symmetry. In the free theory the second class property of $d_{z\alpha}$ is reflected by the fact that the OPE of $d_{z\alpha}$ with itself does not form a closed subalgebra.

2.2 Pure Spinor Formalism

In the previous section we found that in order to relate the free field action (2.24) to the Green–Schwarz superstring one has to implement the constraints $d_{z\alpha}=0$ and $\hat{d}_{\bar{z}\alpha}=0$. In a series of papers [6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16] Berkovits set up a formalism, the pure spinor formalism for the superstring, where these constraints are implemented cohomologically. Physical states are defined as elements in the cohomology of the following BRST–like operator:

$$Q = -\oint i\lambda^\alpha d_{z\alpha}, \tag{2.26}$$

where λ^α is a commuting ghost. Since the $d_{z\alpha}$ do not have vanishing OPEs with themselves this BRST operator is not nilpotent unless one imposes the pure spinor condition for the λ^α:

$$\lambda^\alpha \gamma^m_{\alpha\beta} \lambda^\beta = 0 \tag{2.27}$$

Due to this equation only eleven of the sixteen components of this spinor are independent. To obey this condition the λ^α must be complex. The equation can be solved by decomposing λ^α with respect to a $U(5)$ subgroup of the Wick–rotated Lorentz group $SO(10)$. For a detailed description of this decomposition we refer to Appendix D in [4]. A decomposition in terms of irreducible $U(5)$ components is given by [10]:

$$\lambda^+ = e^s \qquad \lambda_{ab} = u_{ab} \qquad \lambda^a = -\frac{1}{8} e^{-s} \varepsilon^{abcde} u_{bc} u_{de}, \tag{2.28}$$

where $a = 1, \ldots, 5$ and $u_{ab} = -u_{ba}$. These expressions transform as $(1_{\frac{5}{2}}, \bar{10}_{\frac{1}{2}}, 5_{-\frac{3}{2}})$ representations of $SU(5)_{U(1)}$. For practical calculations all the expressions that involve ghosts and their conjugate momenta $\omega_{z\alpha}$ have to be decomposed via (2.28) and the conjugate momenta t and v^{ab} of s and u_{ab}. After the calculations the results can then be written again in terms of ten–dimensional covariant quantities.

In this formalism there is a subtlety concerning the Lorentz currents. In the RNS string the fermionic contribution to the Lorentz current is given by $\hat{M}^{mn} = \psi^m \psi^n$ [17]. These currents satisfy a Lorentz algebra which corresponds to the following OPE:

$$\hat{M}^{kl}(z)\hat{M}^{mn}(w) \sim \frac{\eta^{m[l}\hat{M}^{k]n}(w) - \eta^{n[l}\hat{M}^{k]m}(w)}{z-w} + \frac{\eta^{kn}\eta^{lm} - \eta^{km}\eta^{ln}}{(z-w)^2} \tag{2.29}$$

The fermionic contribution to the Lorentz current of the superstring is by naive considerations $M^{mn} = -\frac{1}{2} p\gamma^{mn}\theta$ where γ^{mn} is the antisymmetrized product of two Gamma matrices. The double pole contribution of the OPE of M^{mn} with itself yields $\frac{16}{4}(\eta^{kn}\eta^{lm} - \eta^{km}\eta^{ln})$. At the quantum level vertex operators for the superstring can only be equivalent to vertex operators of the RNS string if one defines the Lorentz current as follows:

$$M^{mn} = -\frac{1}{2} p\gamma^{mn}\theta + N^{mn}, \tag{2.30}$$

where

$$N^{kl}(z)N^{mn}(w) \sim \frac{\eta^{m[l}N^{k]n}(w) - \eta^{n[l}N^{k]m}(w)}{z-w} - 3\frac{\eta^{kn}\eta^{lm} - \eta^{km}\eta^{ln}}{(z-w)^2} \tag{2.31}$$

N^{mn} can be explicitly constructed from the pure spinor λ^α:

$$N^{mn} = -\frac{1}{2}\omega\gamma^{mn}\lambda \tag{2.32}$$

N^{mn} has a non–vanishing OPE with λ^α:

$$N^{mn}(z)\lambda^\alpha(w) \quad \sim \quad \frac{1}{2}(\gamma^{mn}\lambda)^\alpha \tag{2.33}$$

The most general massless vertex operator is:

$$U \quad = \quad \lambda^\alpha A_\alpha(x,\theta) \tag{2.34}$$

The conditions $QU = 0$ and $\delta U = Q\Omega$, where δ and Ω are the gauge variation and the gauge parameter respectively, imply:

$$\begin{aligned} \gamma^{\alpha\beta}_{[mnpqr]} D_\alpha A_\beta &= 0 \\ \delta A_\alpha &= D_\alpha \Omega \end{aligned} \tag{2.35}$$

The first equation is derived as follows: Applying Q to the vertex operator yields $\lambda^\alpha\lambda^\beta D_\alpha A_\beta = 0$. It follows from the Fierz identity that in ten dimensions every bispinor $f_{\alpha\beta}$ can be decomposed as $f_{\alpha\beta} = \gamma^m_{\alpha\beta} f_m + \gamma^{mnp}_{\alpha\beta} f_{mnp} + \gamma^{mnpqr}_{\alpha\beta} f_{mnpqr}$. For a symmetric bispinor the term with the three gamma matrices vanishes. Using the pure spinor constraint only the expression with five gamma matrices is left. One can define field strengths from A_α by:

$$\begin{aligned} A_m &= \frac{1}{8}\gamma^{\alpha\beta}_m D_\alpha A_\beta \\ A^\alpha &= \frac{1}{10}\gamma^{\alpha\beta}_m \left(D_\beta A^m - \partial^m A_\beta\right) \\ F_{mn} &= \partial_{[m}A_{n]} \;=\; \frac{1}{8}(\gamma_{mn})_\alpha{}^\beta \left(D_\beta A^\alpha\right) \end{aligned} \tag{2.36}$$

Equations (2.35) are the super–Maxwell equations and gauge invariances written in terms of a spinor superfield. $D_\alpha = \frac{\partial}{\partial\theta^\alpha} + 2\theta^\beta\gamma^m_{\alpha\beta}\frac{\partial}{\partial x^m}$ is the supersymmetric derivative. It can be shown [10] that there exists a gauge choice such that A_α can be decomposed as follows:

$$A_\alpha(x,\theta) \quad = \quad c_1(\gamma^m\theta)_\alpha a_m(x) + c_2(\theta\gamma^{mnp}\theta)(\gamma_{mnp})_{\alpha\beta}\chi^\beta(x) + c_3\partial_{[m}a_{n]}(\theta\gamma^{mnp}\theta)(\gamma_p\theta)_\alpha + \ldots, \tag{2.37}$$

where $c_1, \ldots, c_3$ are numerical constants. $a_m(x)$ can be identified with the gluon and $\chi^\beta(x)$ with the gluino. Their equations of motion describe on–shell super Yang–Mills theory.
To compute scattering amplitudes one also needs integrated vertex operators V. In the RNS formalism V can be obtained from the unintegrated vertex operator U by computing its anticommutator with the b ghost. There is no ghost of conformal weight 2 in this theory. Thus one makes a general ansatz for a massless integrated vertex operator:

$$V \quad = \quad \partial\theta^\alpha A_\alpha(x,\theta) + \Pi^m A_m(x,\theta) + d_\alpha A^\alpha(x,\theta) + \frac{1}{2}N_{mn}F^{mn} \tag{2.38}$$

Note that the Lorentz current N^{mn} appears here since it has a non–vanishing OPE with the pure spinor that appears in the BRST charge. Massive states were described in [14]. The most general vertex operator at the first mass level is given by:

$$\begin{aligned} U \quad = \quad & \partial\lambda^\alpha A_\alpha(x,\theta) + :\partial\theta^\beta\lambda^\alpha B_{\alpha\beta}(x,\theta): + :d_\beta\lambda^\alpha C^\beta{}_\alpha(x,\theta): + :\Pi^m\lambda^\alpha H_{m\alpha}(x,\theta): \\ & + :J\lambda^\alpha E_\alpha(x,\theta): + :N^{mn}\lambda^\alpha F_{\alpha mn}(x,\theta): \end{aligned} \tag{2.39}$$

$A_\alpha \dots F_{\alpha mn}$ are superfields, $J = \omega_\alpha \lambda^\alpha$ is the ghost current, the normal ordered product is defined by : $O^A \lambda^\alpha \Phi_{\alpha A}(z) := \oint \frac{\mathrm{d}w}{w-z} O^A(w) \lambda^\alpha(z) \Phi_{\alpha A}(z)$. It was shown in [14] that the physical states form a massive spin 2 multiplet containing 128 bosons and 128 fermions.

One can define N–point tree level amplitudes as the correlation function of three unintegrated vertex operators (2.34) and $N-3$ integrated vertex operators (2.38):

$$\mathcal{A} \;=\; \langle U_1(z_1) U_2(z_2) U_3(z_3) \int \mathrm{d}z_4 V_4(z_4) \dots \int \mathrm{d}z_N V_N(z_N) \rangle \tag{2.40}$$

At first one eliminates all worldsheet fields of non–zero dimension, i.e. ∂x^m, $\partial\theta^\alpha$, d_α and N^{mn}, by computing the OPEs with the other worldsheet fields. After integrating over the zero modes of x^m one gets:

$$\mathcal{A} \;=\; \int \mathrm{d}z_4 \dots \mathrm{d}z_N \langle \lambda^\alpha \lambda^\beta \lambda^\gamma f_{\alpha\beta\gamma}(z_r, k_r, \eta_r, \theta) \rangle, \tag{2.41}$$

where $\lambda^\alpha \lambda^\beta \lambda^\gamma$ comes from the three unintegrated vertex operators and $f_{\alpha\beta\gamma}$ is a function of the z_r, the momenta k_r, the polarizations η_r and the θ zero modes. It is reasonable to define a correlation function $\langle \lambda^\alpha \lambda^\beta \lambda^\gamma f_{\alpha\beta\gamma} \rangle$ such that $Y = \lambda^\alpha \lambda^\beta \lambda^\gamma f_{\alpha\beta\gamma}$ is supersymmetric and gauge invariant, which means that $QY = 0$ and $\langle Y \rangle = 0$ if $Y = Q\Omega$. The only state at zero momentum and ghost number three in the cohomology is $(\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma^p\theta)(\theta\gamma_{mnp}\theta)$. Thus, for

$$f_{\alpha\beta\gamma}(\theta) \;=\; A_{\alpha\beta\gamma} + \theta^\delta B_{\alpha\beta\gamma\delta} + \dots + (\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma^p\theta)(\theta\gamma_{mnp}\theta) F + \dots \tag{2.42}$$

one defines:

$$\langle \lambda^\alpha \lambda^\beta \lambda^\gamma f_{\alpha\beta\gamma}(z_r, k_r, \eta_r, \theta) \rangle \;=\; F(z_r, k_r, \eta_r) \tag{2.43}$$

For three–point scattering one finds, using (2.37) and (2.43), that $\langle \lambda^\alpha A^1_\alpha \lambda^\beta A^2_\beta \lambda^\gamma A^3_\gamma \rangle$ reproduces the super Yang-Mills vertex: Each of the A_α contributes one, two or three θs. If the five θs are distributed as $(1,1,3)$ one gets the three–gluon vertex $a^1_m a^2_n \partial^{[m} a^{3n]}$, whereas if they are distributed as $(2,2,1)$ one gets the gluon–gluino–gluino vertex $(\chi^1 \gamma^m \chi^2) a^3_m$. In [11] a prescription for functional integration over the zero modes for a surface of arbitrary genus was given that consistently incorporates (2.43).

To compute amplitudes on genus g surfaces one has to count the zero–modes of the fields on this surface. λ^α has eleven independent zero modes, θ^α has sixteen. N_{mn} and J have $11g$ independent zero modes on a genus g surface. For the definition of an integral measure it is useful to define a Lorentz invariant, gamma matrix–traceless tensor $\mathcal{T}_{((\alpha_1\alpha_2\alpha_3))[\delta_1\delta_2\delta_3\delta_4\delta_5]}$. This expression is unique up to rescaling and can be constructed out of $\gamma^m_{\alpha_1\delta_1} \gamma^n_{\alpha_2\delta_2} \gamma^p_{\alpha_3\delta_3} (\gamma_{mnp})_{\delta_4\delta_5}$ by symmetrizing with respect to the alpha indices, antisymmetrizing with respect to the delta indices and subtracting off the gamma matrix trace in the alpha indices. The measure factor $[\mathcal{D}\lambda]$ for the ghosts is defined as follows:

$$(\mathrm{d}^{11}\lambda)^{[\alpha_1 \dots \alpha_{11}]} \;=\; [\mathcal{D}\lambda] (\epsilon\mathcal{T})^{[\alpha_1 \dots \alpha_{11}]}_{((\beta_1\beta_2\beta_3))} \lambda^{\beta_1} \lambda^{\beta_2} \lambda^{\beta_3} \tag{2.44}$$

with $(\epsilon\mathcal{T})^{[\alpha_1 \dots \alpha_{11}]}_{((\beta_1\beta_2\beta_3))} = \epsilon^{\alpha_1 \dots \alpha_{16}} \mathcal{T}_{((\beta_1\beta_2\beta_3))[\alpha_{12} \dots \alpha_{16}]}$. Measure factors for the N^{mn} and for J can be constructed analogously [11].

To take care of the zero mode integration of θ, Berkovits defines three picture changing operators:

$$\begin{aligned} Y_C &= C_\alpha \theta^\alpha \delta(C_\beta \lambda^\beta) \\ Z_B &= \frac{1}{2} B_{mn} (\lambda \gamma^{mn} d) \delta(B^{pq} N_{pq}) \\ Z_J &= (\lambda^\alpha d_\alpha) \delta(J) \end{aligned} \quad (2.45)$$

C_α is a constant spinor, B_{mn} is a constant antisymmetric tensor. These operators are BRST invariant and their derivative is BRST exact. In addition, their supersymmetry variations are BRST trivial. The tree level amplitude is now defined as:

$$\mathcal{A} = \langle U_1(z_1) U_2(z_2) U_3(z_3) \int \mathrm{d}z_4 V_4(z_4) \ldots \int \mathrm{d}z_N V_N(z_N) Y_{C_1}(y_1) \ldots Y_{C_{11}}(y_{11}) \rangle, \quad (2.46)$$

where $Y_{C_I}(y_I) = C_{I\alpha} \theta^\alpha(y_I) \delta(C_I \lambda(y_I))$. Integration over the non–zero modes yields:

$$\mathcal{A} = \langle \lambda^\alpha \lambda^\beta \lambda^\gamma f_{\alpha\beta\gamma}(\theta) (C_1 \theta) \ldots (C_{11} \theta) \delta(C_1 \lambda) \ldots \delta(C_{11} \lambda) \rangle \quad (2.47)$$

With the measure factors the amplitude reads:

$$\begin{aligned} \mathcal{A} &= \int \mathrm{d}^{16}\theta \int [\mathcal{D}\lambda] \lambda^\alpha \lambda^\beta \lambda^\gamma f_{\alpha\beta\gamma} (C_1\theta) \ldots (C_{11}\theta) \delta(C_1\lambda) \ldots \delta(C_{11}\lambda) \\ &= \int \mathrm{d}^{16}\theta (\epsilon \mathcal{T}^{-1})^{((\alpha\beta\gamma))}_{[\rho_1 \ldots \rho_{11}]} \int \mathrm{d}\lambda^{\rho_1} \ldots \mathrm{d}\lambda^{\rho_{11}} f_{\alpha\beta\gamma} (C_1\theta) \ldots (C_{11}\theta) \delta(C_1\lambda) \ldots \delta(C_{11}\lambda) \end{aligned} \quad (2.48)$$

Lorentz invariance implies that the amplitude must be independent of the C_I. The λ integration can be carried out and the result is [11]:

$$\mathcal{A} = c (\epsilon \mathcal{T}^{-1})^{((\alpha\beta\gamma))}_{[\kappa_1 \ldots \kappa_{11}]} \int \mathrm{d}^{16}\theta \theta^{\kappa_1} \ldots \theta^{\kappa_{11}} f_{\alpha\beta\gamma}(\theta) \quad (2.49)$$

c is a normalization constant.
To compute g–loop amplitudes one needs to insert $(3g-3)$ b_{zz} ghosts of ghost number -1 that satisfy

$$[Q, b(z)] = T(z) \quad (2.50)$$

into the correlation function, where $T(z)$ is the energy momentum tensor. There is no such ghost in the theory but one can construct an expression such that:

$$\begin{aligned} [Q, \tilde{b}_B(z,w)] &= T(z) Z_B(w) \\ [Q, b_B(z)] &= T(z) Z_B(z), \end{aligned} \quad (2.51)$$

with

$$\tilde{b}_B(z,w) = b_B(z) + T(z) \int_z^w \mathrm{d}v B_{pq} \partial N^{pq}(v) \delta(BN(v)) \quad (2.52)$$

The local expression $b_B(z)$ can be constructed by an iterative procedure [11]. Now it is possible to compute N–point g–loop closed superstring amplitudes. For $g > 1$ this amplitude is defined as:

$$
\begin{aligned}
\mathcal{A} &= \int \mathrm{d}^2\tau_1 \ldots \mathrm{d}^2\tau_{3g-3} \langle | \prod_{P=1}^{3g-3} \int \mathrm{d}^2 u_P \mu_P(u_P) \tilde{b}_{B_p}(u_P, z_P) \\
&\qquad \prod_{P=3g-2}^{10g} Z_{B_P}(z_P) \prod_{R=1}^{g} Z_J(v_R) \prod_{I=1}^{11} Y_{C_I}(y_I)|^2 \prod_{T=1}^{N} \int \mathrm{d}^2 t_T V_T(t_T) \rangle
\end{aligned} \tag{2.53}
$$

$V_T(t_T)$ are dimension $(1,1)$ closed string vertex operators for the N external states (see below). $\mu_P(u_P)$ are Beltrami differentials (cf. Appendix C.1), the τ_P are the Teichmüller parameters associated to the Beltramis, $|\quad|^2$ signifies the left–right product. For $g = 1$ the amplitude is given by:

$$
\mathcal{A} = \int \mathrm{d}^2\tau \langle | \int \mathrm{d}^2 u \mu(u) \tilde{b}_{B_1}(u, z_1) \prod_{P=2}^{10} Z_{B_P}(z_P) Z_J(v) \prod_{I=1}^{11} Y_{C_I}(y_I)|^2 U_1(t_1) \prod_{T=2}^{N} \int \mathrm{d}^2 t_T V_T(t_T) \rangle \tag{2.54}
$$

For the closed superstring the unintegrated vertex operator is defined as:

$$
U = \lambda^\alpha \hat{\lambda}^{\hat{\beta}} A_{\alpha\hat{\beta}}(x, \theta, \hat{\theta}) \tag{2.55}
$$

Physical states satisfy:

$$
QU = \hat{Q}U = 0 \qquad\qquad \delta U = Q\Omega + \hat{Q}\hat{\Omega} \tag{2.56}
$$

where $\hat{Q} = -\oint i\hat{\lambda}^{\hat{\alpha}} \hat{d}_{\bar{z}\hat{\alpha}}$ and $\hat{Q}\Omega = Q\hat{\Omega} = 0$. This implies the following equations of motion and gauge transformations [10]:

$$
\begin{aligned}
D_\gamma A_{\alpha\hat{\beta}} + D_\alpha A_{\gamma\hat{\beta}} &= 0 \\
\hat{D}_{\hat{\gamma}} A_{\alpha\hat{\beta}} + \hat{D}_{\hat{\beta}} A_{\alpha\hat{\gamma}} &= 0
\end{aligned} \tag{2.57}
$$

$$
\delta A_{\alpha\hat{\beta}} = D_\alpha \hat{\Omega}_{\hat{\beta}} + \hat{D}_{\hat{\beta}} \Omega_\alpha \tag{2.58}
$$

$A_{\alpha\hat{\beta}}$ can be gauged to the following form [11]:

$$
\begin{aligned}
A_{\alpha\hat{\beta}}(x, \theta, \hat{\theta}) &= e^{ikx} \Big(h_{mn} (\gamma^m \theta)_\alpha (\gamma^n \hat{\theta})_{\hat{\beta}} + \hat{\psi}_m^{\hat{\gamma}} (\gamma^m \theta)_\alpha (\gamma^n \hat{\theta})_{\hat{\beta}} (\gamma_n \hat{\theta})_{\hat{\gamma}} \\
&\quad + \psi_n^\gamma (\gamma^m \theta)_\alpha (\gamma_m \theta)_\gamma (\gamma^n \hat{\theta})_{\hat{\beta}} + F^{\gamma\hat{\delta}} (\gamma^m \theta)_\alpha (\gamma_m \theta)_\gamma (\gamma^n \hat{\theta})_{\hat{\beta}} (\gamma_n \hat{\theta})_{\hat{\delta}} + \ldots \Big)
\end{aligned} \tag{2.59}
$$

with

$$
\begin{aligned}
k^2 \;=\; k^m h_{mn} \;=\; k^n h_{mn} \;=\; k^m \hat{\psi}_m^{\hat{\alpha}} \;=\; k^m \psi_m^\alpha &= 0 \\
k^n (\gamma_n \hat{\psi}_m)_{\hat{\alpha}} \;=\; k^n (\gamma_n \psi_m)_\alpha \;=\; k_m \gamma^m_{\alpha\gamma} F^{\gamma\hat{\delta}} \;=\; k_m \gamma^m_{\hat{\alpha}\hat{\delta}} F^{\gamma\hat{\delta}} &= 0
\end{aligned} \tag{2.60}
$$

$A_{\alpha\hat{\beta}}$ describes the on–shell type IIB supergravity multiplet where h_{mn} comprises the graviton, the antisymmetric tensor and the dilaton, and ψ_m^α and $\hat{\psi}_m^{\hat{\alpha}}$ denote gravitini and dilatini. $F^{\alpha\hat{\beta}}$ are the Ramond–Ramond field strengths.
The integrated vertex operator for the closed string is given by:

$$\begin{aligned} V &= \partial\theta^\alpha \bar{\partial}\hat{\theta}^{\hat{\beta}} A_{\alpha\hat{\beta}} + \partial\theta^\alpha \bar{\Pi}^m A_{\alpha m} + \Pi^m \bar{\partial}\hat{\theta}^{\hat{\alpha}} \hat{A}_{m\hat{\alpha}} + \Pi^m \bar{\Pi}^n A_{mn} \\ &+ d_\alpha \left(\bar{\partial}\hat{\theta}^{\hat{\beta}} E^\alpha_{\hat{\beta}} + \bar{\Pi}^m E^\alpha_m \right) + \hat{d}_{\hat{\alpha}} \left(\partial\theta^\beta \hat{E}^{\hat{\alpha}}_\beta + \Pi^m \hat{E}^{\hat{\alpha}}_m \right) \\ &+ \frac{1}{2} N_{mn} \left(\bar{\partial}\hat{\theta}^{\hat{\beta}} \hat{\Omega}^{mn}_{\hat{\beta}} + \bar{\Pi}^p \Omega^{mn}_p \right) + \frac{1}{2} \hat{N}_{mn} \left(\partial\theta^\beta \Omega^{mn}_\beta + \Pi^p \hat{\Omega}^{mn}_p \right) \\ &+ d_\alpha \hat{d}_{\hat{\beta}} P^{\alpha\hat{\beta}} + N_{mn} \hat{d}_{\hat{\alpha}} \hat{C}^{mn\hat{\alpha}} + d_\alpha \hat{N}_{mn} C^{\alpha mn} + N_{mn} \hat{N}_{pq} S^{mnpq} \end{aligned} \tag{2.61}$$

A systematic approach to compute the equations of motion and the gauge transformations as well as the component expansions was given in [18].

There are some further constructions related to the Green–Schwarz superstring and the pure spinor formalism that should be mentioned.
In [19] a superembedding formulation for the superstring is related to the GS string and the pure spinor formalism.
In [20, 21] the pure spinor constraint is implemented via a BRST double complex.

Chapter 3

Covariant Quantization of the Superstring without Pure Spinor Constraints

3.1 Introduction

This is the central chapter of this thesis. In the following we will discuss the covariant quantization of the superstring without pure spinor constraints as it was introduced by Grassi, van Nieuwenhuizen and collaborators. First we will give a short introduction to WZNW models and discuss the relation of this covariant formulation of the superstring to these models. Some difficulties concerning the generalization of the WZNW model to the type two superstring will lead to a different approach that involves the Noether method to construct an action for type II. The construction of the BRST operator and the problems concerning the definition of physical states will be investigated. It will be shown how the fields in the model can be used to construct a twisted $N = 2$ superconformal algebra and in the subsequently an $N = 4$ algebra. Finally it will be shown how to implement worldsheet diffeomorphism invariance into this model.

3.2 The Superstring as a Gauged WZNW Model

In this section we discuss the relation of the covariant quantization of the superstring to Wess-Zumino-Novikov-Witten (WZNW) models. First we will give an introduction to WZNW models. Then we will show that the heterotic superstring can be written as a WZNW model. Nilpotency of the BRST transformations will imply that we have a gauged WZNW model. Finally we discuss the problems that arise when we use the WZNW technology for the type II superstring, which will be the motivation for a different approach to the problem in section 3.3.

3.2.1 WZNW Models

WZNW models were first introduced by Witten [22] in the context of the bosonization of fermions with non–abelian symmetries in $1 + 1$ dimensions. In this paper an action was derived based on a current algebra that was obtained from the bosonization of a fermionic

Lagrangian. In this short introduction we will mostly follow the lines of [23].
The two dimensional WZNW model is a classical conformally invariant field theory whose basic fields are harmonic maps g from a Riemann surface Σ, the worldsheet, to a bosonic Lie group G. We denote the generators of the Lie algebra by T_M. The pullback of the Maurer Cartan form Θ, i.e. the left invariant Lie algebra valued one–from, to the worldsheet is given by:

$$g^*\Theta \;:=\; g^{-1}\mathrm{d}g \;=\; \mathrm{d}\sigma^\mu g^{-1}\partial_\mu g \;=\; (-)^M T_M \Theta^M \tag{3.1}$$

Analogously we can define the pullback of the right–invariant one–form Θ^R:

$$g^*\Theta^R \;:=\; \mathrm{d}g g^{-1} \;=\; (-)^M T_M \Theta^{R\,M} \tag{3.2}$$

Now we define the following action:

$$\begin{aligned} S_{\mathrm{kin}}[g] &= \frac{1}{4\lambda^2}||g^*\Theta||^2 \\ &= \frac{1}{4\lambda^2}\int_\Sigma \langle g^{-1}\mathrm{d}g, *(g^{-1}\mathrm{d}g)\rangle \\ &= -\frac{1}{4\lambda^2}\int_\Sigma \mathrm{d}^2\sigma\sqrt{h}\langle g^{-1}\partial_\mu g, g^{-1}\partial^\mu g\rangle \\ &= -\frac{1}{4\lambda^2}\int_\Sigma \mathrm{d}^2\sigma\sqrt{h}\Theta^{B\mu}\Theta^A_\mu \mathcal{H}_{AB} \end{aligned} \tag{3.3}$$

Here we called the determinant of the metric h to avoid confusion with the group element g. In the last line the scalar product $\langle,\rangle$ was computed explicitly using:

$$\langle T_M, T_N\rangle \;=\; \mathcal{H}_{MN}, \tag{3.4}$$

where $\mathcal{H}_{MN}$ is the Killing metric.
Now we add a non–local term to the action, the Wess–Zumino term. This term is defined on a three dimensional manifold B with $\partial B = \Sigma$. The map g is extended to $\tilde{g}$ with $\tilde{g} : B \to G$ such that $\tilde{g}|_\Sigma = g$. The Wess–Zumino term of the action is now defined as follows:

$$\begin{aligned} S_{WZ}[g] &= -\frac{1}{24\pi}\int_B \langle \tilde{g}^{-1}\mathrm{d}\tilde{g}, \mathrm{d}(\tilde{g}^{-1}\mathrm{d}\tilde{g})\rangle \\ &= -\frac{1}{24\pi}\int_B \mathrm{d}^3x\varepsilon^{ijk}\langle \tilde{g}^{-1}\partial_i\tilde{g}, \partial_j(\tilde{g}^{-1}\partial_k\tilde{g})\rangle \\ &= -\frac{1}{24\pi}\int_B \langle \tilde{g}^*\Theta, \mathrm{d}(\tilde{g}^*\Theta)\rangle \\ &= -\frac{1}{24\pi}\int_B \mathrm{d}^3x\varepsilon^{ijk}\mathcal{H}_{MN}\partial_j\tilde{\Theta}^N_k\tilde{\Theta}^M_i \\ &= \frac{1}{48\pi}\int_B \mathrm{d}^3x\varepsilon^{ijk}\mathcal{H}_{MR}f^R{}_{NP}\tilde{\Theta}^P_k\tilde{\Theta}^N_j\tilde{\Theta}^M_i \end{aligned} \tag{3.5}$$

The last line can be computed if one starts with $S_{WZ}[g] = -\frac{1}{24\pi}\int_B\langle\tilde{g}^{-1}\mathrm{d}\tilde{g}, \tilde{g}^{-1}\mathrm{d}\tilde{g}\tilde{g}^{-1}\mathrm{d}\tilde{g}\rangle$ and uses $[T_M, T_N] = T_R f^R{}_{MN}$.
The complete action is defined as:

$$S[g] \;=\; S_{\mathrm{kin}}[g] + n\, S_{WZ}[g], \tag{3.6}$$

where the integer n is called the level of the WZNW model. For our purposes the level will be set to -2 and the constant λ is given by $\lambda^2 = |\frac{4\pi}{n}|$.
It can be shown that the action is invariant under the action of the infinite dimensional group $G(\sigma^+) \times G(\sigma^-)$ with:

$$g(\sigma^-, \sigma^+) \mapsto \tilde{\Omega}^{-1}(\sigma^+) g(\sigma^-, \sigma^+) \Omega(\sigma^-) \tag{3.7}$$

We can split this symmetry into a chiral and an antichiral contribution:

$$\begin{aligned} \delta_c g(\sigma^-, \sigma^+) &= g(\sigma^-, \sigma^+) \omega(\sigma^-) \\ \delta_a g(\sigma^-, \sigma^+) &= -\tilde{\omega}(\sigma^+) g(\sigma^-, \sigma^+) \end{aligned} \tag{3.8}$$

In the following we will refer to these "semilocal" chiral symmetries as global symmetries. The corresponding conserved currents can be computed to be[1]:

$$\begin{aligned} J^L &= -\frac{n}{8\pi} g^{-1} \partial g = -\frac{n}{8\pi} \Theta_- \\ J^R &= -\frac{n}{8\pi} \bar{\partial} g g^{-1} = -\frac{n}{8\pi} \Theta_+^R \end{aligned} \tag{3.9}$$

The conservation laws are $\partial J^L = 0$ and $\bar{\partial} J^R = 0$. The currents satisfy the following Poisson bracket algebra:

$$\begin{aligned} \{J_A^L(\sigma^-), J_B^L(\sigma'^-)\} &= J_C^L(\sigma^-) f^C{}_{AB} \delta(\sigma^1 - \sigma'^1) + \frac{n}{4\pi} \mathcal{H}_{AB} \partial_1 \delta(\sigma^1 - \sigma'^1) \\ \{J_A^R(\sigma^-), J_B^R(\sigma'^-)\} &= J_C^R(\sigma^-) f^C{}_{AB} \delta(\sigma^1 - \sigma'^1) - \frac{n}{4\pi} \mathcal{H}_{AB} \partial_1 \delta(\sigma^1 - \sigma'^1) \end{aligned} \tag{3.10}$$

This corresponds to the following OPEs at the quantum level, expressed in terms of the Θ with $n = -2$:

$$\begin{aligned} \Theta^L_{-A}(\sigma^-) \Theta^L_{-B}(\sigma'^-) &\sim -\frac{\Theta^L_{-C}(\sigma^-) f^C{}_{AB}}{z - w} - \frac{\mathcal{H}_{AB}}{(z-w)^2} \\ \Theta^R_{-A}(\sigma^-) \Theta^R_{-B}(\sigma'^-) &\sim -\frac{\Theta^R_{-C}(\sigma^-) f^C{}_{AB}}{z - w} + \frac{\mathcal{H}_{AB}}{(z-w)^2} \end{aligned} \tag{3.11}$$

Gauged WZNW Models

Now we make the symmetry transformation (3.7) local by gauging a subgroup $H \subset G \times G$:

$$g(\sigma^-, \sigma^+) \mapsto \lambda^{-1}(\sigma^-, \sigma^+) g(\sigma^-, \sigma^+) \rho(\sigma^-, \sigma^+), \tag{3.12}$$

where $\lambda, \rho : \Sigma \to H$ are arbitrary smooth maps. We will focus on the gauging of a diagonal subgroup H. In this special case we set $\rho = \lambda$ so that the transformations read:

$$g(\sigma^-, \sigma^+) \mapsto \lambda^{-1}(\sigma^-, \sigma^+) g(\sigma^-, \sigma^+) \lambda(\sigma^-, \sigma^+) \tag{3.13}$$

We introduce gauge fields with components A and $\bar{A}$ which transform as follows under gauge transformations:

$$\begin{aligned} A &\rightarrow \lambda^{-1}(\partial + A)\lambda \\ \bar{A} &\rightarrow \lambda^{-1}(\bar{\partial} + \bar{A})\lambda \end{aligned} \tag{3.14}$$

[1]Note that we use ∂ and ∂_- synonymously. See Appendix A for our conventions.

It can be shown [23] that the following extended action is invariant under the local symmetry:

$$S_{\mathcal{H}}[g,A,\bar{A}] \;=\; S_{\mathcal{H}}[g] - \int_{\Sigma} \langle A, J^R\rangle + \langle J, \bar{A}\rangle - \frac{n}{8\pi}\langle A, g^{-1}\bar{A}g\rangle + \frac{n}{8\pi}\langle A, \bar{A}\rangle \tag{3.15}$$

Here we indicated explicitly the dependence on the Killing metric $\mathcal{H}$ of G.
Now we switch to the path integral formalism and perform the Faddeev–Popov procedure:

$$Z \;=\; \int [\mathrm{d}g][\mathrm{d}A][\mathrm{d}\bar{A}]e^{-S_{\mathcal{H}}[g,A,\bar{A}]} \tag{3.16}$$

We choose the holomorphic gauge $\bar{A}=0$. Assuming the absence of gauge anomalies the gauge fixed path integral reads:

$$Z \;=\; \int [\mathrm{d}g][\mathrm{d}A](\det\bar{\partial})e^{-S_{\mathcal{H}}[g,A,0]}, \tag{3.17}$$

where

$$\det\bar{\partial} \;=\; \int [\mathrm{d}b][\mathrm{d}c]e^{-\int_{\Sigma}\langle b,\bar{\partial}c\rangle} \tag{3.18}$$

is the Faddeev–Popov determinant and (b,c) are the Faddeev–Popov ghosts. The remaining gauge fields A can be parameterized by $A=-\partial hh^{-1}$ with a smooth function $h:\Sigma\to H$. The Polyakov–Wiegmann identity [23] states that the gauge fixed action can be written in terms of the original WZNW action:

$$S_{\mathcal{H}}[g,A,0] \;=\; S_{\mathcal{H}}[gh] - S_{\mathcal{H}}[h] \tag{3.19}$$

At the quantum level the change of variables from A to h incurs in a Jacobian factor for the functional measure of the path integral. An arbitrary infinitesimal variation of the gauge field A can be written as:

$$\delta A \;=\; -\partial(\delta hh^{-1}) - [A,\delta hh^{-1}] \;=\; -D(\delta hh^{-1}), \tag{3.20}$$

where D is the holomorphic component of the covariant derivative. From this it can be deduced that the Jacobian is given by: $[\mathrm{d}A]=(\det D)[\mathrm{d}h]$. This determinant can be expressed via an integral over fermionic fields:

$$\det D \;=\; \int [\mathrm{d}\bar{b}][\mathrm{d}\bar{c}]e^{-\int_{\Sigma}\langle\bar{b},D\bar{c}\rangle} \tag{3.21}$$

The path integral now reads:

$$Z \;=\; \int [\mathrm{d}g][\mathrm{d}h](\det D)(\det\bar{\partial})e^{-S_{\mathcal{H}}[gh]+S_{\mathcal{H}}[h]} \tag{3.22}$$

To compute the determinants we have to make a short detour. We compute a more general expression:

$$\det D\det\bar{D} \;=\; \int [\mathrm{d}b][\mathrm{d}c][\mathrm{d}\bar{b}][\mathrm{d}\bar{c}]e^{-\int_{\Sigma}\langle\bar{b},D\bar{c}\rangle-\int_{\Sigma}\langle b,\bar{D}c\rangle} \tag{3.23}$$

Now we use the following relation:

$$\det D \, \det \bar{D} \;\; = \;\; e^{-W[A,\bar{A}]} \det\partial \det\bar{\partial}, \tag{3.24}$$

where $W[A, \bar{A}]$ is the integrated anomaly. Using $A = -\partial h h^{-1}$ and $\bar{A} = -\bar{\partial}\bar{h}\bar{h}^{-1}$ it can be shown that the anomaly has the form of a WZNW action:

$$e^{-W[A,\bar{A}]} \;\; = \;\; e^{S_{\mathcal{H}'}[\bar{h}^{-1}h]}, \tag{3.25}$$

where $\mathcal{H}'$ is the Killing metric of the subgroup H. For the gauged WZNW model we have to set $\bar{A} = 0$ and $\bar{h} = 1$. Then the path integral can be written as:

$$Z \;\; = \;\; \int [\mathrm{d}g][\mathrm{d}h][\mathrm{d}b][\mathrm{d}c][\mathrm{d}\bar{b}][\mathrm{d}\bar{c}] e^{-S_{\mathcal{H}}[gh]+S_{\mathcal{H}+\mathcal{H}'}[h]} e^{-\int_\Sigma \langle b,\bar{\partial}c\rangle + \langle \bar{b},\partial\bar{c}\rangle} \tag{3.26}$$

Finally we can make a change of variables $g \to gh^{-1}$. Absence of gauge anomalies implies that the Jacobian is trivial and we finally arrive at the path integral formulation for the gauged WZNW model:

$$Z \;\; = \;\; \int [\mathrm{d}g][\mathrm{d}h][\mathrm{d}b][\mathrm{d}c][\mathrm{d}\bar{b}][\mathrm{d}\bar{c}] e^{-S_{\mathcal{H}}[g]+S_{\mathcal{H}+\mathcal{H}'}[h]} e^{-\int_\Sigma \langle b,\bar{\partial}c\rangle + \langle \bar{b},\partial\bar{c}\rangle} \tag{3.27}$$

We observe that the action of a gauged WZNW model consists of three blocks: two independent WZNW models based on G and H, respectively, and a ghost sector. The h–action has the opposite sign as compared to the g–action. This implies in particular that the currents J^h and $J^{h\,R}$ have opposite signs:

$$\begin{aligned} J^{h\,L} &= \frac{n}{8\pi} h^{-1}\partial h = \frac{n}{8\pi}\Theta^h_- \\ J^{h\,R} &= \frac{n}{8\pi}\bar{\partial} h h^{-1} = \frac{n}{8\pi}\Theta^{h\,R}_+ \end{aligned} \tag{3.28}$$

In the case we are interested in, the WZNW model $S_{\mathcal{H}'}$ is degenerate with $\mathcal{H}' = 0$. Then the OPEs for the Θ^h read:

$$\begin{aligned} \Theta^{h\,L}_{-A}(\sigma^-)\Theta^{h\,L}_{-B}(\sigma'^-) &\sim -\frac{\Theta^{h\,L}_{-C}(\sigma^-) f^C{}_{AB}}{z-w} + \frac{\mathcal{H}_{AB}}{(z-w)^2} \\ \Theta^{h\,R}_{-A}(\sigma^-)\Theta^{h\,R}_{-B}(\sigma'^-) &\sim \frac{\Theta^{h\,R}_{-C}(\sigma^-) f^C{}_{AB}}{z-w} - \frac{\mathcal{H}_{AB}}{(z-w)^2} \end{aligned} \tag{3.29}$$

For the currents J^h this means that the central terms change their signs. In the following we will be interested in gauging the full diagonal subgroup $H = G \subset G \times G$ with $h = g$.
In the following we will mostly deal with the left–moving currents and skip the superscript L.

3.2.2 The Superstring as a WZNW model

Relaxing the Pure Spinor Constraint

In their works [4][24, 25, 26, 27, 28] van Nieuwenhuizen and collaborators presented a way to covariantly quantize the superstring without the pure spinor constraint

$$\lambda\gamma^m\lambda \;\; = \;\; 0. \tag{3.30}$$

Only the chiral sector will be considered here. The starting point for this construction is Berkovits' BRST charge:

$$Q = -\int i\lambda^\alpha d_{z\alpha} \tag{3.31}$$

Now we check nilpotency on the fields using the OPEs collected in Appendix B. Acting with the BRST operator on θ^α and λ^α yields $s\theta^\alpha = i\lambda^\alpha$ and $s\lambda^\alpha = 0$. Thus, nilpotency on θ^α and λ^α can be achieved. Next we compute $sx^m = \lambda\gamma^m\theta$. Without the pure spinor constraint $s^2x^m = i\lambda\gamma^m\lambda$ does not vanish. Therefore one introduces a real anticommuting ghost ξ^m and and sets $sx^m = \lambda\gamma_m\theta\xi^m$. The transformations of the new ghost are chosen such that the BRST transformation on x^m is nilpotent, which yields $s\xi^m = -i\lambda\gamma^m\lambda$. In order to obtain these altered transformations a new $Q' = \int \Pi_{zm}\xi^m$ is added. It would have been sufficient to write ∂x_m instead of Π_{zm} but then Q' would not be supersymmetric. In a next step we construct nilpotent transformations on $d_{z\alpha}$. We have $sd_{z\alpha} = 2(\gamma^m\lambda)_\alpha\Pi_{zm}$, $s'd_{z\alpha} = 2i\xi^m(\gamma_m\partial\theta)_\alpha$ and $(s+s')^2 d_{z\alpha} = \partial(2\xi^m(\gamma_m\lambda)_\alpha)$. Thus, we introduce a new BRST transformation such that $s''d_{z\alpha} = -\partial\chi_\alpha$. Nilpotency on $d_{z\alpha}$ is achieved if we define $s\chi_\alpha = 2\xi^m(\gamma_m\lambda)_\alpha$. Using Fierz rearrangement it can be verified that $s^2\chi_\alpha = 0$. Thus we have obtained nilpotency on all the fields and ghosts. We introduce antighosts β_{zm}, $\omega_{z\alpha}$ and κ_z^α with the OPEs:

$$\begin{aligned}
\xi^m(z)\beta_m(w) &\sim -\frac{1}{z-w} \\
\lambda^\alpha(z)\omega_\alpha(w) &\sim -\frac{1}{z-w} \\
\chi_\alpha(z)\kappa^\alpha(w) &\sim -\frac{1}{z-w}
\end{aligned} \tag{3.32}$$

β_{zm} is anticommuting, whereas $\omega_{z\alpha}$ and κ_z^α are commuting antighosts.
The BRST charge is now given by:

$$Q = \oint -id_{z\alpha}\lambda^\alpha - \partial\theta^\alpha\chi_\alpha - \Pi_{zm}\xi^m + i\beta_{zm}(\lambda\gamma^m\lambda) + 2\xi^m(\kappa_z\gamma_m\lambda) \tag{3.33}$$

Unfortunately, the BRST transformations on the antighosts are not nilpotent:

$$\begin{aligned}
s\beta_{zm} &= \Pi_{zm} - 2(\kappa_z\gamma_m\lambda) \\
s\omega_{z\alpha} &= id_{z\alpha} - 2i\beta_{zm}(\gamma^m\lambda)_\alpha - 2(\gamma_m\kappa_z)_\alpha\xi^m \\
s\kappa_z^\alpha &= \partial\theta^\alpha
\end{aligned} \tag{3.34}$$

There are now two ways to proceed:

1. Continue this procedure by requiring nilpotency on the antighosts and introducing new fields. This will eventually lead to the WZNW model.

2. Terminate the procedure by hand by adding a ghost pair (b, c_z) with $c(z)b(w) = -1/(z-w)$. In the following we will refer to this as the "old approach" [4].

Before we consider the WZNW model we give a brief review of the "old approach". Computing the square of (3.33) yields:

$$[Q,Q] = \int A_z = \int \xi_m\partial\xi^m + i\lambda^\alpha\partial\chi_\alpha - i\chi_\alpha\partial\lambda^\alpha \tag{3.35}$$

This expression is BRST invariant and we have $[Q, A_z] = \partial Y$ with $Y = i\xi_m \lambda\gamma^m\lambda$. Now we define:

$$Q' = Q + \int (c_z + bB_z) \tag{3.36}$$

We find:

$$[Q', Q'] = \int (A_z - 2B_z) - b[Q, B_z] \tag{3.37}$$

Requiring Q' to be nilpotent yields:

$$[Q, B_z] = 0 \qquad B_z = \frac{1}{2}(A_z + \partial X) \qquad [Q, X] = -Y \tag{3.38}$$

with $X = -\frac{i}{2}\chi_\alpha\lambda^\alpha$. With that we obtain the BRST charge of the "old approach":

$$\begin{aligned} Q = \oint & -id_{z\alpha}\lambda^\alpha - \Pi_{zm}\xi^m - \partial\theta^\alpha\chi_\alpha + i\beta_{zm}(\lambda\gamma^m\lambda) + 2(\kappa_z\gamma_m\lambda)\xi^m \\ & + c_z + b\left(\frac{1}{2}\xi_m\partial\xi^m + \frac{i}{4}\lambda^\alpha\partial\chi_\alpha - \frac{3i}{4}\chi_\alpha\partial\lambda^\alpha\right) \end{aligned} \tag{3.39}$$

The altered transformations of the antighosts and the BRST transformations of the new ghost pair can be computed using the OPEs of Q with these fields. There are, however, some unattractive features concerning this approach.

- The introduction of the ghost pair (b, c_z) changes the central charge to $c = 20$. In order to obtain a theory with vanishing central charge another pair (ω^m, η_z^m) is introduced which cancels the central charge. The new fields are defined to be BRST inert.
- The cohomology of Q is trivial, i.e. all the currents are set to 0 cohomologically. To fix this one has to introduce a grading condition to get the correct physics.

For a review of the cohomology computations in the "old approach"we refer to section 3.5.1.

Now we come to the WZNW approach that was first presented in [28]. We add "currents" $J_M^h = -(\Pi_{zm}^h, id_{z\alpha}, \partial\theta^\alpha)$ to the BRST transformations of the antighosts:

$$\begin{aligned} s\beta_{zm} &= \Pi_{zm} - \Pi_{zm}^h - 2(\kappa_z\gamma_m\lambda) \\ s\omega_{z\alpha} &= id_{z\alpha} - id_{z\alpha}^h - 2i\beta_{zm}(\gamma^m\lambda)_\alpha - 2(\gamma_m\kappa_z)_\alpha\xi^m \\ s\kappa_z^\alpha &= \partial\theta^\alpha - \partial\theta^{\alpha h} \end{aligned} \tag{3.40}$$

Demanding nilpotency of these transformations yields the following BRST transformations for the h–currents:

$$\begin{aligned} s\Pi_z^{mh} &= \partial\xi^m + 2(\lambda\gamma^m\partial\theta^h) \\ s\theta^{\alpha h} &= -i\lambda^\alpha \\ sd_{z\alpha}^h &= \partial\chi_\alpha + 2(\gamma_m\lambda)_\alpha\Pi_z^{mh} + 2i\xi^m(\gamma_m\partial\theta^h)_\alpha \end{aligned} \tag{3.41}$$

This construction implies the following BRST operator:

$$
\begin{aligned}
Q &= \oint -(\Pi_{zm}-\Pi^h_{zm})\xi^m - (id_{z\alpha}-id^h_{z\alpha})\lambda^\alpha - (\partial\theta^\alpha - \partial\theta^{\alpha h})\chi_\alpha \\
&\quad + i\beta_{zm}(\lambda\gamma^m\lambda) + 2(\kappa_z\gamma_m\lambda)\xi^m,
\end{aligned} \tag{3.42}
$$

which is consistent if the h–currents satisfy the following OPEs:

$$
\begin{aligned}
id^h_{z\alpha}(z)id^h_{z\beta}(w) &\sim -2i\frac{\gamma^m_{\alpha\beta}\Pi^h_{zm}(w)}{z-w} & id^h_{z\alpha}(z)\Pi^h_{zm}(w) &\sim -2\frac{\gamma_{m\,\alpha\beta}\partial\theta^{\beta h}}{z-w} \\
\Pi^h_{zm}(z)\Pi^h_{zn}(w) &\sim \frac{\eta_{mn}}{(z-w)^2} & id^h_{z\alpha}(z)\partial\theta^{\beta h}(w) &\sim \frac{i\delta^\beta_\alpha}{(z-w)^2}
\end{aligned} \tag{3.43}
$$

The central terms in the OPEs change their signs as compared those of the original currents. This is exactly the behavior of the J^h_M of the gauged WZNW model.

WZNW action and Currents for the Heterotic String

Having established a first contact between the covariant formulation of the superstring and WZNW models we will now stress some more WZNW properties of this model. First we introduce the following currents with capital indices: $J_M = (\Pi_{zm}, id_{z\alpha}, \partial\theta^\alpha)$, the corresponding h–currents have already been introduced above. The Maurer–Cartan forms are $\Theta_{\mu M} = (\Pi_{\mu m}, id_{\mu\alpha}, \partial_\mu\theta^\alpha)$. Defining the structure constants ${f^m}_{\alpha\beta} = 2i\gamma^m_{\alpha\beta}$ and ${f^\alpha}_{\beta m} = 2\gamma_{m\,\alpha\beta}$ and a metric

$$
\mathcal{H}_{MN} = \begin{pmatrix} \eta_{mn} & 0 & 0 \\ 0 & 0 & i\delta^\beta_\alpha \\ 0 & -i\delta^\alpha_\beta & 0 \end{pmatrix}, \tag{3.44}
$$

we get the following OPEs:

$$
\begin{aligned}
J_M(z)J_N(w) &\sim -\frac{J_K {f^K}_{MN}}{z-w} - \frac{\mathcal{H}_{MN}}{(z-w)^2} \\
J^h_M(z)J^h_N(w) &\sim \frac{J^h_K {f^K}_{MN}}{z-w} + \frac{\mathcal{H}_{MN}}{(z-w)^2}
\end{aligned} \tag{3.45}
$$

These are consistent with (3.11) and (3.29).
Now we define super Lie algebra generators[2]:

$$
T_M = (P_m, Q_\alpha, K^\alpha) \tag{3.46}
$$

They satisfy the following (anti–)commutation relations:

$$
\begin{aligned}
&[T_M, T_N] = T_P {f^P}_{MN} \\
&[Q_\alpha, Q_\beta] = 2i\gamma^m_{\alpha\beta}P_m \qquad [Q_\alpha, P_m] = 2\gamma_{m\,\alpha\beta}K^\beta
\end{aligned} \tag{3.47}
$$

Using these generators we define a parameterization for the group manifold:

$$
g = e^{P_m x^m} e^{iQ_\alpha\theta^\alpha} e^{iK^\alpha\phi_\alpha} \tag{3.48}
$$

[2]This ansatz goes back to Siegel [29].

This is a generalization of the WZNW models presented in [23] since in this case not all the Lie algebra generators are bosonic.
With the formula

$$\mathrm{d}e^B = e^B \mathrm{d}B + \frac{1}{2}e^B[\mathrm{d}B, B] + \frac{1}{6}e^B[[\mathrm{d}B, B]B] + O([[[\mathrm{d}B, B], B], B]), \tag{3.49}$$

one can easily verify that the correct chiral currents $\partial\Pi$ and $\partial\theta$ are produced[3]:

$$\begin{aligned} g^{-1}\partial g &= T_M J^M \\ &= P_m\left(\partial x^m - i\theta\gamma^m\partial\theta\right) + Q_\alpha i\partial\theta^\alpha + K^\alpha\left(i\partial\phi_\alpha - 2i(\gamma_m\theta)_\alpha\partial x^m - \frac{2}{3}(\gamma_m\theta)_\alpha(\theta\gamma^m\partial\theta)\right) \\ &= P_m\Pi^m_z + iQ_\alpha\partial\theta^\alpha + K^\alpha d^{(\phi)}_{z\alpha} \end{aligned} \tag{3.50}$$

We could not take $p_{z\alpha}$ as elementary field but had to use ϕ_α instead. To establish the connection to $p_{z\alpha}$ we have to define:

$$p^{(\phi)}_{z\alpha} = -\partial\phi_\alpha - 2i(\gamma_m\theta)_\alpha\partial x^m - \frac{2}{3}(\gamma_m\theta)_\alpha(\theta\gamma^m\partial\theta) \tag{3.51}$$

The WZNW action without the h–currents can be written as ($n = -2$):

$$S[g] = -\frac{1}{8\pi}\int_\Sigma \mathrm{d}^2\sigma\Theta^\mu_M\Theta^M_\mu - \frac{1}{24\pi}\int_B \mathrm{d}^3x\varepsilon^{ijk}\tilde{\Theta}_{iA}f^A{}_{BC}\tilde{\Theta}^C_k\tilde{\Theta}^B_j, \tag{3.52}$$

where $\tilde{\Theta}_M$ is the extension of the Maurer–Cartan form to the three dimensional manifold B. Using the Maurer–Cartan equations

$$\begin{aligned} \mathrm{d}\Theta &= \mathrm{d}(g^{-1}\mathrm{d}g) = -g^{-1}\mathrm{d}gg^{-1}\mathrm{d}g = -\Theta\wedge\Theta \\ \mathrm{d}\Theta^C &= (-)^A\frac{1}{2}f^C{}_{BA}\Theta^A\Theta^B, \end{aligned} \tag{3.53}$$

we can rewrite the WZ term as an integral over a total derivative and we find:

$$S = \frac{1}{4\pi}\int_\Sigma \mathrm{d}^2\sigma\left(-\frac{1}{2}\Pi^{m\mu}\Pi_{m\mu} + d^{(\phi)\mu}_\alpha\partial_\mu\theta^\alpha - \frac{1}{2}\varepsilon^{\mu\nu}d^{(\phi)}_{\mu\alpha}\partial_\nu\theta^\alpha\right) \tag{3.54}$$

Partial integration and the Maurer–Cartan equations then yield the final well–known result for the action:

$$S = \frac{1}{4\pi}\int_\Sigma \mathrm{d}^2\sigma\left(-\frac{1}{2}\Pi^{m\mu}\Pi_{m\mu} + P^{\mu\nu}d^{(\phi)}_{\mu\alpha}\partial_\nu\theta^\alpha - i\varepsilon^{\mu\nu}\Pi^m_\mu(\theta\gamma_m\partial_\nu\theta)\right) \tag{3.55}$$

3.2.3 Problems using WZNW technology for the Superstring

Despite the discussion in the previous section there are some problems with the WZNW description of the superstring. In particular we focus on some complications for the generalization to the type II superstring which kept us from finding a standard WZNW formulation for this case and eventually led us to the unconventional approach that will be presented in section 3.3.

[3]Indices are lifted with the inverse of the metric, see Appendix B.

Parameterization and Doubling of the Currents

If we only consider the chiral case the WZNW model yields antiholomorphic currents that do not appear in the model for the superstring. We find:

$$\bar{\partial} g g^{-1} \quad = \quad P_m \left(\bar{\partial} x^m + i\theta\gamma^m \bar{\partial}\theta \right) + iQ_\alpha \bar{\partial}\theta^\alpha + K^\alpha \left(i\bar{\partial}\phi_\alpha - 2ix^m (\gamma_m \bar{\partial}\theta)_\alpha - \frac{2}{3}(\gamma_m\theta)_\alpha (\theta\gamma^m\bar{\partial}\theta) \right) \tag{3.56}$$

There is no interpretation for these currents for the heterotic string. It is also not possible to interpret the antichiral currents as the currents of the right–moving sector for the type II superstring.

Another problem is the fact that we have ϕ_α as an elementary field in the WZNW approach and not $p_{z\alpha}$. The action (3.55) still contains the ϕ_α–field. It can be shown that variation of the action with respect to the elementary fields x^m, θ^α, ϕ_α does not yield free field equations.

Generalization to the Type II Superstring

In order to introduce rightmovers into the WZNW model we need new Lie algebra generators with the following commutators:

$$\begin{aligned} [\hat{Q}_{\hat{\alpha}}, \hat{Q}_{\hat{\beta}}] &= -2i\gamma^m_{\alpha\beta} P_m \\ [\hat{Q}_{\hat{\alpha}}, P_m] &= -2\gamma_{m\,\hat{\alpha}\hat{\beta}} \hat{K}^{\hat{\beta}} \end{aligned} \tag{3.57}$$

We are forced by the Jacobi identity to introduce some further non–vanishing commutators:

$$\begin{aligned} [Q_\alpha, \hat{Q}_{\hat{\beta}}] &=: iR_{\alpha\hat{\beta}} = iR_{\hat{\beta}\alpha} := [\hat{Q}_{\hat{\beta}}, Q_\alpha] \\ [R_{\alpha\hat{\beta}}, \hat{Q}_{\hat{\gamma}}] &= -2\gamma^m_{\hat{\beta}\hat{\gamma}} \gamma_{m\,\alpha\delta} K^\delta \\ [R_{\alpha\hat{\beta}}, Q_\gamma] &= -2\gamma^m_{\alpha\gamma} \gamma_{m\,\hat{\beta}\hat{\delta}} \hat{K}^{\hat{\delta}} \end{aligned} \tag{3.58}$$

These commutators imply that left– and right–moving sectors mix. All the other commutators between generators of the left– and right–moving sector can be chosen to be 0.
We introduce a new field $y^{\alpha\hat{\alpha}}$ for the generator $R_{\alpha\hat{\alpha}}$ and choose the following parameterization for the group element:

$$g \quad = \quad e^{P_m x^m + iQ_\alpha\theta^\alpha + iK^\alpha\phi_\alpha + i\hat{Q}_{\hat{\alpha}}\hat{\theta}^{\hat{\alpha}} + i\hat{K}^{\hat{\alpha}}\hat{\phi}_{\hat{\alpha}} + R_{\alpha\hat{\alpha}} y^{\alpha\hat{\alpha}}} \tag{3.59}$$

The problem would simplify if it were possible to express $y^{\alpha\hat{\alpha}}$ in terms of the fields x^m, θ^α, $\hat{\theta}^\alpha$, ϕ_α and $\hat{\phi}_{\hat{\alpha}}$. For this purpose we compute the composition law of two group elements g_1 and g_2 using the Baker–Campbell–Hausdorff formula [30]:

$$e^A e^B \quad = \quad e^{A+B+\frac{1}{2}[A,B]+\frac{1}{12}[A,[A,B]]-\frac{1}{12}[B,[B,A]]+\ldots} \tag{3.60}$$

After a lengthy calculation we get the following contributions for the respective generators:

1. Terms proportional to P_m:

$$\begin{aligned} g_1 g_2 \Big|_{P_m} &= \exp\left[P_m(x_1^m + x_2^m) + \frac{1}{2} P_m \left(2i\gamma^m_{\alpha\beta}\theta_1^\alpha\theta_2^\beta - 2i\gamma^m_{\hat{\alpha}\hat{\beta}}\hat{\theta}_1^{\hat{\alpha}}\hat{\theta}_2^{\hat{\beta}} \right) \right] \\ &= \exp\left[P_m(x_1^m + x_2^m) + iP_m \left((\theta_1\gamma^m\theta_2) - (\hat{\theta}_1\gamma^m\hat{\theta}_2) \right) \right] \end{aligned} \tag{3.61}$$

2. Terms proportional to Q_α:

$$g_1 g_2 \Big|_{Q_\alpha} = \exp\Big[iQ_\alpha(\theta_1^\alpha + \theta_2^\alpha)\Big] \tag{3.62}$$

3. Terms proportional to $\hat{Q}_{\hat{\alpha}}$:

$$g_1 g_2 \Big|_{\hat{Q}_{\hat{\alpha}}} = \exp\Big[i\hat{Q}_{\hat{\alpha}}(\hat{\theta}_1^{\hat{\alpha}} + \hat{\theta}_2^{\hat{\alpha}})\Big] \tag{3.63}$$

4. Terms proportional to $R_{\alpha\hat{\beta}}$:

$$g_1 g_2 \Big|_{R_{\alpha\hat{\beta}}} = \exp\Big[R_{\alpha\hat{\beta}}(y_1^{\alpha\hat{\beta}} + y_2^{\alpha\hat{\beta}}) + \frac{i}{2}R_{\alpha\hat{\beta}}(\theta_1^\alpha\hat{\theta}_2^{\hat{\beta}} + \hat{\theta}_1^{\hat{\beta}}\theta_2^\alpha)\Big] \tag{3.64}$$

5. Terms proportional to K^δ:

$$\begin{aligned} g_1 g_2 \Big|_{K^\delta} = \exp\Big[& iK^\delta(\phi_{1\delta} + \phi_{2\delta}) + iK^\delta\gamma_{m\,\delta\alpha}\Big(- x_1^m\theta_2^\alpha + x_2^m\theta_1^\alpha + \gamma^m_{\hat{\alpha}\hat{\gamma}}\hat{\theta}_1^{\hat{\gamma}}y_2^{\alpha\hat{\alpha}} - \gamma^m_{\hat{\gamma}\hat{\alpha}}\hat{\theta}_2^{\hat{\gamma}}y_1^{\alpha\hat{\alpha}}\Big) \\ & + \frac{1}{3}K^\delta\Big(- \gamma_{m\,\delta\alpha}\gamma^m_{\beta\gamma}\theta_1^\alpha\theta_1^\beta\theta_2^\gamma + \gamma_{m\,\alpha\delta}\gamma^m_{\hat{\beta}\hat{\gamma}}\theta_1^\alpha\hat{\theta}_1^{\hat{\beta}}\hat{\theta}_2^{\hat{\gamma}} - \frac{1}{2}\gamma_{m\,\delta\beta}\gamma^m_{\hat{\alpha}\hat{\beta}}\hat{\theta}_1^{\hat{\alpha}}(\theta_1^\beta\hat{\theta}_2^{\hat{\beta}} + \hat{\theta}_1^{\hat{\beta}}\theta_2^\beta)\Big) \\ & - \frac{1}{3}K^\delta\Big(\gamma_{m\,\delta\alpha}\gamma^m_{\beta\gamma}\theta_2^\alpha\theta_1^\beta\theta_2^\gamma - \gamma_{m\,\alpha\delta}\gamma^m_{\hat{\beta}\hat{\gamma}}\theta_2^\alpha\hat{\theta}_1^{\hat{\beta}}\hat{\theta}_2^{\hat{\gamma}} + \frac{1}{2}\gamma_{m\,\delta\beta}\gamma^m_{\hat{\alpha}\hat{\beta}}\hat{\theta}_2^{\hat{\alpha}}(\theta_1^\beta\hat{\theta}_2^{\hat{\beta}} + \hat{\theta}_1^{\hat{\beta}}\theta_2^\beta)\Big)\Big] \end{aligned} \tag{3.65}$$

6. Terms proportional to $\hat{K}^{\hat{\delta}}$:

$$\begin{aligned} g_1 g_2 \Big|_{\hat{K}^{\hat{\delta}}} = \exp\Big[& i\hat{K}^{\hat{\delta}}(\hat{\phi}_{1\hat{\delta}} + \hat{\phi}_{2\hat{\delta}}) + i\hat{K}^{\hat{\delta}}\gamma_{m\,\hat{\delta}\hat{\alpha}}\Big(x_1^m\hat{\theta}_2^{\hat{\alpha}} - x_2^m\hat{\theta}_1^{\hat{\alpha}} + \gamma^m_{\alpha\gamma}\theta_1^\gamma y_2^{\alpha\hat{\alpha}} - \gamma^m_{\alpha\gamma}\theta_2^\gamma y_1^{\alpha\hat{\alpha}}\Big) \\ & + \frac{1}{3}\hat{K}^{\hat{\delta}}\Big(\gamma_{m\,\hat{\delta}\hat{\alpha}}\gamma^m_{\beta\gamma}\hat{\theta}_1^{\hat{\alpha}}\theta_1^\beta\theta_2^\gamma - \gamma_{m\,\hat{\delta}\hat{\alpha}}\gamma^m_{\hat{\beta}\hat{\gamma}}\hat{\theta}_1^{\hat{\alpha}}\hat{\theta}_1^{\hat{\beta}}\hat{\theta}_2^{\hat{\gamma}} - \frac{1}{2}\gamma_{m\,\hat{\delta}\hat{\beta}}\gamma^m_{\alpha\beta}\theta_1^\alpha(\theta_1^\beta\hat{\theta}_2^{\hat{\beta}} + \hat{\theta}_1^{\hat{\beta}}\theta_2^\beta)\Big) \\ & - \frac{1}{3}\hat{K}^{\hat{\delta}}\Big(- \gamma_{m\,\hat{\delta}\hat{\alpha}}\gamma^m_{\beta\gamma}\hat{\theta}_2^{\hat{\alpha}}\theta_1^\beta\theta_2^\gamma + \gamma_{m\,\hat{\delta}\hat{\alpha}}\gamma^m_{\hat{\beta}\hat{\gamma}}\hat{\theta}_2^{\hat{\alpha}}\hat{\theta}_1^{\hat{\beta}}\hat{\theta}_2^{\hat{\gamma}} + \frac{1}{2}\gamma_{m\,\hat{\delta}\hat{\beta}}\gamma^m_{\alpha\beta}\theta_2^\alpha(\theta_1^\beta\hat{\theta}_2^{\hat{\beta}} + \hat{\theta}_1^{\hat{\beta}}\theta_2^\beta)\Big)\Big] \end{aligned} \tag{3.66}$$

In order to get a proper composition law we must have:

$$e^{R_{\alpha\hat{\beta}}y_1'^{\alpha\hat{\beta}}}e^{R_{\alpha\hat{\beta}}y_2'^{\alpha\hat{\beta}}} \stackrel{!}{=} e^{R_{\alpha\hat{\beta}}y_3'^{\alpha\hat{\beta}}} \tag{3.67}$$

Considering equation (3.64) the only reasonable choice seems to be $y^{\alpha\hat{\beta}} = \frac{i}{2}\theta^\alpha\hat{\theta}^{\hat{\beta}}$ but unfortunately we get a wrong sign. What we want to get is:

$$\begin{aligned} e^{R_{\alpha\hat{\beta}}\theta_1^\alpha\hat{\theta}_1^{\hat{\beta}}}e^{R_{\alpha\hat{\beta}}\theta_1^\alpha\hat{\theta}_1^{\hat{\beta}}} & \stackrel{!}{=} \exp\Big[(\theta_1^\alpha + \theta_2^\alpha)(\hat{\theta}_1^{\hat{\beta}} + \hat{\theta}_2^{\hat{\beta}})\Big] \\ & = \exp\Big[\frac{i}{2}\Big(\theta_1^\alpha\hat{\theta}_1^{\hat{\beta}} + \theta_2^\alpha\hat{\theta}_2^{\hat{\beta}} + \theta_1^\alpha\hat{\theta}_2^{\hat{\beta}} + \theta_2^\alpha\hat{\theta}_1^{\hat{\beta}}\Big)\Big] \end{aligned} \tag{3.68}$$

Using (3.64) to compute the product we get a negative sign in the last term as compared to the equation above. Thus it is not possible to express $y^{\alpha\hat{\alpha}}$ in terms of the other fields and

therefore it has to be made dynamical.
Now we compute the left– and right–moving currents from the parameterization (3.59). The chiral currents from $g^{-1}\partial g$ are:

$$\begin{aligned}
J_z^m &= \partial x^m - i(\theta\gamma^m\partial\theta) + i(\hat\theta\gamma^m\partial\hat\theta)\\
J_z^\alpha &= i\partial\theta^\alpha\\
d_{z\alpha}^{(\phi)} = J_{z\alpha} &= i\partial\phi_\alpha + ix^m(\gamma_m\partial\theta)_\alpha - i\partial x^m(\gamma_m\theta)_\alpha - \frac{2}{3}(\gamma_m\theta)_\alpha(\theta\gamma^m\partial\theta)\\
&\quad +(\gamma_m\theta)_\alpha(\hat\theta\gamma^m\partial\hat\theta) - i\gamma_{m\,\alpha\beta}\partial y^{\beta\hat\beta}\gamma^m_{\hat\beta\hat\alpha}\hat\theta^{\hat\alpha} + i\gamma_{m\,\alpha\beta}y^{\beta\hat\beta}\gamma^m_{\hat\beta\hat\alpha}\partial\hat\theta^{\hat\alpha}\\
J_z^{\hat\alpha} &= i\partial\hat\theta^{\hat\alpha}\\
J_{z\hat\alpha} &= i\partial\hat\phi_{\hat\alpha} - ix^m(\gamma_m\partial\hat\theta)_{\hat\alpha} + i\partial x^m(\gamma_m\hat\theta)_{\hat\alpha} - \frac{2}{3}(\gamma_m\hat\theta)_{\hat\alpha}(\hat\theta\gamma^m\partial\hat\theta)\\
&\quad +(\gamma_m\hat\theta)_{\hat\alpha}(\theta\gamma^m\partial\theta) - i\gamma_{m\,\hat\alpha\hat\beta}\partial y^{\hat\beta\beta}\gamma^m_{\beta\alpha}\theta^\alpha + i\gamma_{m\,\hat\alpha\hat\beta}y^{\hat\beta\beta}\gamma^m_{\beta\alpha}\partial\theta^\alpha\\
J_z^{\alpha\hat\alpha} &= \partial y^{\alpha\hat\alpha} - \frac{i}{2}\left(\hat\theta^{\hat\alpha}\partial\theta^\alpha + \theta^\alpha\partial\hat\theta^{\hat\alpha}\right)
\end{aligned} \tag{3.69}$$

We choose a metric where the mixing of right– and left–moving sector is minimal to pull Lie algebra indices:

$$H_{MN} = \begin{pmatrix}
\eta_{mn} & 0 & 0 & 0 & 0 & 0\\
0 & 0 & i\delta_\alpha^\beta & 0 & 0 & 0\\
0 & -i\delta_\beta^\alpha & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & i\delta_{\hat\alpha}^{\hat\beta} & 0\\
0 & 0 & 0 & -i\delta_{\hat\beta}^{\hat\alpha} & 0 & 0\\
0 & 0 & 0 & 0 & 0 & h\gamma^m_{\alpha\beta}\gamma_{m\,\hat\alpha\hat\beta}
\end{pmatrix} \tag{3.70}$$

The right–moving currents $\bar\partial g g^{-1}$ can be computed out of the left–moving ones via:

$$\bar\partial g g^{-1} = -g\bar\partial g^{-1} = -\left(g(-\phi^M)\right)^{-1}\bar\partial\left(g(\phi^M)\right), \tag{3.71}$$

with $\phi^M = (x^m, i\theta^\alpha, i\phi_\alpha, i\hat\theta^{\hat\alpha}, i\hat\phi_{\hat\alpha}, y^{\alpha\hat\alpha})$. For the calculation this means that we take the chiral currents and exchange ∂ by $\bar\partial$ and change the sign in every term that is quadratic in ϕ^M. Thus, we get:

$$\begin{aligned}
J_{\bar z}^{m\,R} &= \bar\partial x^m + i(\theta\gamma_m\bar\partial\theta) - i(\hat\theta\gamma_m\bar\partial\hat\theta)\\
J_{\bar z}^{\alpha\,R} &= i\bar\partial\theta^\alpha\\
J_{\bar z\alpha}^R &= i\bar\partial\phi_\alpha - ix^m(\gamma_m\bar\partial\theta)_\alpha + i\bar\partial x^m(\gamma_m\theta)_\alpha - \frac{2}{3}(\gamma_m\theta)_\alpha(\theta\gamma^m\bar\partial\theta)\\
&\quad +(\gamma_m\theta)_\alpha(\hat\theta\gamma^m\bar\partial\hat\theta) + i\gamma_{m\,\alpha\beta}\bar\partial y^{\beta\hat\beta}\gamma^m_{\hat\beta\hat\alpha}\hat\theta^{\hat\alpha} - i\gamma_{m\,\alpha\beta}y^{\beta\hat\beta}\gamma^m_{\hat\beta\hat\alpha}\bar\partial\hat\theta^{\hat\alpha}\\
J_{\bar z}^{\hat\alpha\,R} &= i\bar\partial\hat\theta^{\hat\alpha}\\
\hat d_{\bar z\hat\alpha}^{(\hat\phi)} = J_{\bar z\hat\alpha}^R &= i\bar\partial\hat\phi_{\hat\alpha} + ix^m(\gamma_m\bar\partial\hat\theta)_{\hat\alpha} - i\bar\partial x^m(\gamma_m\hat\theta)_{\hat\alpha} - \frac{2}{3}(\gamma_m\hat\theta)_{\hat\alpha}(\hat\theta\gamma^m\bar\partial\hat\theta)\\
&\quad +(\gamma_m\hat\theta)_{\hat\alpha}(\theta\gamma^m\bar\partial\theta) + i\gamma_{m\,\hat\alpha\hat\beta}\bar\partial y^{\hat\beta\beta}\gamma^m_{\beta\alpha}\theta^\alpha - i\gamma_{m\,\hat\alpha\hat\beta}y^{\hat\beta\beta}\gamma^m_{\beta\alpha}\bar\partial\theta^\alpha\\
J_{\bar z}^{\alpha\hat\alpha\,R} &= \bar\partial y^{\alpha\hat\alpha} + \frac{i}{2}\left(\hat\theta^{\hat\alpha}\bar\partial\theta^\alpha + \theta^\alpha\bar\partial\hat\theta^{\hat\alpha}\right)
\end{aligned} \tag{3.72}$$

We make the following observations concerning these results:

- We have a doubling of the currents. For the chiral sector we only need J_z^m, J_z^α and $J_{z\alpha}$, for the antichiral sector $J_{\bar{z}}^{m\,R}$, $J_{\bar{z}}^{\hat{\alpha}\,R}$ and $J_{\bar{z}\hat{\alpha}}^R$.

- For the currents $J_\mu^{m\,(R)}$ we would have expected $\Pi_\mu^m = \partial_\mu x^m - i(\theta\gamma^m\partial_\mu\theta) - i(\hat{\theta}\gamma^m\partial_\mu\hat{\theta})$. Using the parameterization (3.59) we can achieve on–shell, i.e. using the equations of motion $\bar{\partial}\theta = 0$ and $\partial\hat{\theta} = 0$, that the left– and right–moving sectors decouple: $J_z^m \rightarrow \partial x^m - i(\theta\gamma^m\partial\theta)$ and $J_{\bar{z}}^{m\,R} \rightarrow \bar{\partial}x^m - i(\hat{\theta}\gamma^m\bar{\partial}\hat{\theta})$.

- The d– and $\hat{d}$–currents contain terms that are not supersymmetric. Choosing a different parameterization like $e^{P_m x^m} e^{iQ_\alpha\theta^\alpha} e^{iK^\alpha\phi_\alpha} e^{i\hat{Q}_{\hat{\alpha}}\hat{\theta}^{\hat{\alpha}}} e^{i\hat{K}^{\hat{\alpha}}\hat{\phi}_{\hat{\alpha}}} e^{R_{\alpha\hat{\alpha}}y^{\alpha\hat{\alpha}}}$ does not solve this problem because it removes the non–supersymmetric terms only in one sector as it can be seen already for the heterotic case (3.56).

Considering these problems we quit the attempt to construct a WZNW action for the type II superstring via a parameterization of the group element and successfully chose a different approach that will be presented in the following section. An approach to the construction of a WZNW action based on supergroups can be found in [31].

3.3 Gauging the Action via the Noether Procedure

As we have seen in the previous section the standard WZNW approach to the superstring causes some problems:

- For the heterotic string one starts with a chiral algebra and the WZNW model produces a chiral as well as an antichiral algebra. It is not possible to interpret the antichiral algebra as the right moving sector of the type II superstring. Consequently one has to start with a chiral and an antichiral algebra and both of them double.

- For the type II superstring the central extension of the supersymmetry algebra forces one, via the Jacobi identity, to introduce an additional generator with two spinor indices. As we have seen in section 3.2.3 the corresponding field cannot be expressed in terms of the fields $x^m, \theta^\alpha, \phi_\alpha$ and thus has to be made dynamical. It has not been possible to produce a WZNW action for the type II superstring.

- In [28] the currents J_M^h of the gauged WZNW were introduced by hand in order to render the BRST transformations of the antighosts nilpotent. In this section we will present a more fundamental way to introduce these h–currents into the action.

- Some aspects concerning the relation of the WZNW field ϕ_α and the momentum $p_{z\alpha}$ of the free field action are unclear.

To circumvent these difficulties we will use the Noether method to perform the WZNW action. We start with the free field action and gauge the symmetry algebra induced by the OPEs of the currents J_M. We will mostly follow the line of [1] but we will treat the heterotic and the type II case separately with the intention to focus on the details of Noether procedure for the heterotic case. This will enable us to discuss the complications that arise for the type II string without having to distract ourselves with the technicalities that come from gauging the symmetries.

The Noether method is an elegant way to construct out of an action that has a global symmetry a new action that is invariant under the local version of this symmetry. The iterative construction works as follows:

1. Take an action S_0 that is invariant under a global symmetry such that $\delta_0 S_0 = 0$. The subscripts denote a grading that keeps track of the order of the iteration. Compute the variation of the action S_0 under the **local** variation δ_1. According to Noether's theorem this yields the currents:

$$\delta_1 S_0 \;=\; \int \frac{\delta S_0}{\delta \phi^{\mathcal{A}}} \delta_1 \phi^{\mathcal{A}} \;\equiv\; \int J_{\mu M} \partial^\mu \omega^M, \tag{3.73}$$

   where $\phi^{\mathcal{A}}$ stands for all the fields in the action and ω^M is the transformation parameter.

2. Introduce a "gauge connection" A_μ^M that transforms as $\delta_0 A_\mu^M = \partial_\mu \omega^M$. Add a new term,

$$S_1 = -\int J^\mu_M A^M_\mu, \tag{3.74}$$

   to the action. With that one achieves $\delta_1 S_0 + \delta_0 S_1 = 0$.

3. Compute $\delta_1 S_1$ and obtain $\delta_0 S_2 + \delta_1 S_1 + \delta_2 S_0 = 0$ by adding a suitable S_2 to the action and/or by altering the transformations of the fields $\phi^{\mathcal{A}}$. These computations may carry some ambiguities.

4. Continue until the procedure terminates or until sufficient steps are made to guess the form of the final action and transformations.

For the superstring we would get for example: $\delta_1 S_0 = \int J_{zM} \bar\partial \omega^M + \hat{J}_{\bar z \hat M} \partial \hat\omega^{\hat M}$ where the hatted quantities stand for the right movers.
Note that it is also possible to use the Noether theorem "backwards" by considering the fact that via partial integration we have:

$$\delta_1 S_0 \;=\; \int \frac{\delta S_0}{\delta \phi^{\mathcal{A}}} \delta_1 \phi^{\mathcal{A}} \;\equiv\; \int J_{\mu M} \partial^\mu \omega^M \;=\; -\int \partial^\mu J_{\mu M} \omega^M \tag{3.75}$$

Thus, it is possible to use the Noether theorem to compute the transformations if one knows the conserved currents.
The standard reference for the Noether method is [32] but it was already applied in earlier works, for example by Deser [33].

3.3.1 The Heterotic String

We start with the free field action for the heterotic string[4]:

$$S_0 \;=\; \int -\frac{1}{2} \partial x^m \bar\partial x_m + p_{z\alpha} \bar\partial \theta^\alpha \;=\; \int -\frac{1}{2} \Pi^m_z \Pi_{\bar z m} - i \Pi^m_z \left(\theta \gamma_m \partial \theta\right) + i \Pi^m_{\bar z} \left(\theta \gamma_m \bar\partial \theta\right) + d_{z\alpha} \bar\partial \theta^\alpha \tag{3.76}$$

The conserved currents are:

$$J_{zM} \;=\; \left(\Pi_{zm}, i d_{z\alpha}, \partial \theta^\alpha\right) \tag{3.77}$$

[4] From now on we absorb the prefactors in the action into the $\int$–sign, see also Appendix A.

The elementary fields are $\phi^{\mathcal{A}} = (x^m, \theta^\alpha, p_{z\alpha})$. In order to avoid confusing prefactors and signs in our condensed notation we use the following transformation parameters:

$$\omega^M \;=\; (\omega^m, \omega^\alpha, -i\omega_\alpha) \tag{3.78}$$

We compute the transformations of the elementary fields by computing their OPEs with the currents:

$$\delta J_z^N(w) \;=\; \operatorname*{Res}_{z\to w} J_{zM}(z)\omega^M(z)\phi^N(w) \tag{3.79}$$

Using this method of calculation we may miss off–shell contributions to the transformations. In this simple case, however, the OPEs yield the full transformations:

$$\delta J_M \;=\; -\partial\omega_M + J_P f^P{}_{MN}\omega^N \tag{3.80}$$

$$\begin{aligned}
\delta\Pi_{zm} &= -\partial\omega_m + 2\left(\omega\gamma_m\partial\theta\right)\\
\delta\partial\theta^\alpha &= i\partial\omega^\alpha\\
\delta id_{z\alpha} &= -\partial\omega_\alpha + 2i\left(\gamma_m\omega\right)_\alpha \Pi_z^m + 2\omega^m\left(\gamma_m\partial\theta\right)_\alpha
\end{aligned} \tag{3.81}$$

Since it is easier to gauge the action in terms of the elementary fields, we express the currents in terms of the elementary fields to obtain the following transformations:

$$\begin{aligned}
\delta x^m &= -\omega^m + \left(\omega\gamma^m\theta\right)\\
\delta\theta^\alpha &= i\omega^\alpha\\
\delta p_{z\alpha} &= i\partial\omega_\alpha - 2i\omega^m\left(\gamma_m\partial\theta\right)_\alpha - i\partial\omega^m\left(\gamma_m\theta\right)_\alpha + \left(\gamma_m\omega\right)_\alpha\partial x^m\\
&\quad +\frac{3i}{2}\left(\omega\gamma_m\theta\right)\left(\gamma^m\partial\theta\right)_\alpha + \frac{1}{2}\left(\partial\omega\gamma_m\theta\right)\left(\gamma^m\theta\right)_\alpha
\end{aligned} \tag{3.82}$$

The first step in the Noether procedure is to compute $\delta_1 S_0$. Since we got the transformations out of the OPEs of the currents this is merely a check if we computed them correctly. Using partial integration and Fierz rearrangement we find that this is indeed the case:

$$\delta_1 S_0 \;=\; \int \left(\partial x^m - i\theta\gamma^m\partial\theta\right)\bar\partial\omega_m + \partial\theta^\alpha\bar\partial\left(-i\omega_\alpha\right) + \left(p_\alpha - \left(\gamma_m\theta\right)_\alpha\left(i\partial x^m + \frac{1}{2}\theta\gamma^m\partial\theta\right)\right)\bar\partial\omega^\alpha \tag{3.83}$$

Introducing

$$S_1 = -\int J_{zM}A_{\bar z}^M \;=\; -\int \Pi_{zm}A_{\bar z}^m + id_{z\alpha}A_{\bar z}^\alpha + \partial\theta^\alpha A_{\bar z\alpha} \tag{3.84}$$

we have constructed $\delta_1 S_0 + \delta_0 S_1 = 0$.

In the next step we compute $\delta_1 S_1$:

$$\begin{aligned}
\delta_1 S_1 &= -\int \Pi_{zm}\left(\delta_1 A_{\bar z}^m + 2i\left(\omega\gamma^m A_{\bar z}\right)\right) + id_{z\alpha}\delta_1 A_{\bar z}^\alpha + \partial\theta^\alpha\left(\delta_1 A_{\bar z\alpha} + 2\omega^m\left(\gamma_m A_{\bar z}\right)_\alpha\right)\\
&\qquad -\partial\omega_m A_{\bar z}^m - \partial\omega^\alpha A_{\bar z\alpha} + i\partial\omega_\alpha A_{\bar z}^\alpha\\
&= -\int J_{zP}\left(f^P{}_{MN}\omega^N A_{\bar z}^M + \delta_1 A_{\bar z}^P\right) - \partial\omega_M A_{\bar z}^M
\end{aligned} \tag{3.85}$$

We can read off the local transformations of the gauge connections:

$$\delta_1 A_{\bar z}^P \;=\; f^P{}_{MN}\omega^N A_{\bar z}^M \tag{3.86}$$

There are still the $\partial\omega_M$–terms left to be absorbed. These terms come from the central extension of the current algebra. Theoretically we have two possibilities to get rid of them but unfortunately none of them work here:

- Adding a term S_2 that is quadratic in the gauge connections does not work because we would need an expression of conformal weight $(1,1)$ and there is neither a field A_{zM} in our model nor are there connections with upper indices. Even if we introduced a field A_z^M and added a term $S_2 = -\int A_{zM}A_{\bar{z}}^M$ to the action the δ_0–variation would yield $A_{zM}\bar{\partial}\omega^M$ which we do not want.

- We cannot alter the transformations of the elementary fields because the action S_0 is quadratic in the elementary fields and therefore the term we want to cancel would have to be linear in the fields which it is not.

Knowing that the procedure must terminate after the first step we use a trick which leads us to the gauged WZNW model. We double the fields and subtract from the original Lagrangian the same Lagrangian in terms of these new auxiliary fields. We demand that the double poles in the current algebra corresponding to these fields have the opposite sign as compared to those of the currents J_M. Since this is exactly the behavior of the h–currents in a gauged WZNW model we will denote the new fields with a superscript h. The new action S_0^h is separately invariant under the same chiral transformations (3.82) with the coordinates replaced by their h–analogues and with a gauge parameter ω_M^h. For our purpose it is necessary to set $\omega_M^h = \omega_M$. In the WZNW language this means that we gauge the complete diagonal subgroup $H = G\times G$ [23]. For the action we write now $S_0 \rightarrow S_0 - S_0^h \equiv S_0$. The gauge transformations of the h–currents read:

$$\delta_0 J_M^h = \partial\omega_M + J_P^h f^P{}_{MN}\omega^N \tag{3.87}$$

$$\begin{aligned}
\delta_0\Pi_{zm}^h &= -\partial\omega_m + 2\left(\omega\gamma_m\partial\theta^h\right) \\
\delta_0\partial\theta^{h\alpha} &= i\partial\omega^\alpha \\
\delta_0 i d_{z\alpha}^h &= -\partial\omega_\alpha + 2i\left(\gamma_m\omega\right)_\alpha\Pi_z^{mh} + 2\omega^m\left(\gamma_m\partial\theta^h\right)_\alpha,
\end{aligned} \tag{3.88}$$

with

$$J_M^h = \left(-\Pi_{zm}^h, -id_{z\alpha}^h, -\partial\theta^{h\alpha}\right) \tag{3.89}$$

Variation of the new action under local transformations now yields:

$$\delta_1 S_0 = \int\left(J_M + J_M^h\right)\bar{\partial}\omega^M \tag{3.90}$$

Now we add to the action a term:

$$S_1 = -\int\left(J_M + J_M^h\right)A_{\bar{z}}^M \tag{3.91}$$

Computation of $\delta_1 S_1$ now yields:

$$\delta_1 S_1 = -\int J_{zP}\left(f^P{}_{MN}\omega^N A_{\bar{z}}^M + \delta_1 A_{\bar{z}}^P\right), \tag{3.92}$$

and the procedure terminates after the first step if we define:

$$\delta_1 A^P_{\bar{z}} \quad = \quad -f^P{}_{MN}\omega^N A^M_{\bar{z}} \tag{3.93}$$

With that we have achieved $\delta S = 0$ with $\delta = \delta_0 + \delta_1$ and $S = S_0 + S_1$.

Gauge Fixing and BRST Transformation

It is possible to use the gauge freedom to put all the gauge connections to 0 again. This is done by the standard procedure of gauge fixing as it can be found for example in [34].
We introduce ghosts by writing the transformation parameters as:

$$\omega^M \quad = \quad \Lambda c^M, \tag{3.94}$$

with a global anticommuting parameter Λ. The BRST variation s on the elementary fields is then defined as:

$$\delta\phi^{\mathcal{A}} \quad = \quad \Lambda\, s\phi^{\mathcal{A}} \tag{3.95}$$

We add the usual gauge fixing term to the Lagrangian:

$$\mathcal{L}_{gf} \quad = \quad \mathcal{L} + s\left(b_M A^M_{\bar{z}}\right) \tag{3.96}$$

with

$$sb_M \; = \; \Omega_M \qquad\qquad s\Omega_M \; = \; 0, \tag{3.97}$$

where Ω_M is a Lagrange multiplier field, the Nakanishi–Lautrup field. The BRST transformations of the ghosts are defined such that s becomes nilpotent. This is achieved by demanding nilpotency of the BRST transformations on the fields (or currents) and using the Jacobi identity.
We get the BRST transformation of the gauge connection from its gauge transformation by pulling the parameter Λ out in front:

$$\begin{aligned} \delta A^P_{\bar{z}} \quad &= \quad \bar{\partial}\omega^P - f^P{}_{MN}\omega^N A^M_{\bar{z}} \\ sA^P_{\bar{z}} \quad &= \quad \bar{\partial}c^P - (-)^{N+M+P} f^P{}_{MN}c^N A^M_{\bar{z}} \end{aligned} \tag{3.98}$$

For the action we get, explicitly computing $s(b_M A^M_{\bar{z}})$:

$$S_{gf} \quad = \quad S + \int \Lambda_P A^P_{\bar{z}} - (-)^P b_P \bar{\partial} c^P + (-)^{N+M} b_P f^P{}_{MN} c^N A^M_{\bar{z}} \tag{3.99}$$

Ω_M and $A^M_{\bar{z}}$ can be integrated out because the variation with respect to those fields yields algebraic equations of motion:

$$\begin{aligned} A^M_{\bar{z}} \quad &= \quad 0 \\ \Omega_M \quad &= \quad \left(J_M + J^h_M\right) - (-)^{N+M} b_P f^P{}_{MN} c^N \end{aligned} \tag{3.100}$$

Now we define:

$$c^M \;=\; (-\xi^m, \lambda^\alpha, \chi_\alpha) \qquad\qquad c_M \;=\; (-\xi_m, i\chi_\alpha, -i\lambda^\alpha) \tag{3.101}$$

$$b_M \;=\; (\beta_{zm}, \omega_{z\alpha}, \kappa^\alpha_z) \qquad\qquad b^M \;=\; (\beta^m_z, i\kappa^\alpha_z, -i\omega_{z\alpha}) \tag{3.102}$$

With these definitions we finally arrive at the gauged and gauge fixed action of the heterotic string, which coincides with the result of [28]:

$$
\begin{aligned}
S &= \int -\frac{1}{2}\partial x^m \bar{\partial} x_m + \frac{1}{2}\partial x^{mh}\bar{\partial} x^h_m + p_{z\alpha}\bar{\partial}\theta^\alpha - p^h_{z\alpha}\bar{\partial}\theta^{\alpha h} \\
&\quad +\beta_{zm}\bar{\partial}\xi^m + \omega_{z\alpha}\bar{\partial}\lambda^\alpha + \kappa^\alpha_z\bar{\partial}\chi_\alpha \\
&= \int -\frac{1}{2}\Pi^m_z\Pi_{\bar{z}m} - i\Pi^m_z\left(\theta\gamma_m\bar{\partial}\theta\right) + i\Pi^m_{\bar{z}}\left(\theta\gamma_m\partial\theta\right) + d_{z\alpha}\bar{\partial}\theta^\alpha \\
&\quad +\frac{1}{2}\Pi^{mh}_z\Pi^h_{\bar{z}m} + i\Pi^{mh}_z\left(\theta^h\gamma_m\bar{\partial}\theta^h\right) - i\Pi^{mh}_{\bar{z}}\left(\theta^h\gamma_m\partial\theta^h\right) - d^h_{z\alpha}\bar{\partial}\theta^{\alpha h} \\
&\quad +\beta_{zm}\bar{\partial}\xi^m + \omega_{z\alpha}\bar{\partial}\lambda^\alpha + \kappa^\alpha_z\bar{\partial}\chi_\alpha
\end{aligned} \tag{3.103}
$$

To conclude this subsection we give a complete list of the BRST transformations of the fields, currents, ghosts and antighosts:

$$
\begin{aligned}
sx^m &= \xi^m + \left(\lambda\gamma^m\theta\right) \\
s\theta^\alpha &= i\lambda^\alpha \\
sp_{z\alpha} &= i\partial\chi_\alpha + 2i\xi^m\left(\gamma_m\partial\theta\right)_\alpha + i\partial\xi^m\left(\gamma_m\theta\right)_\alpha + \left(\gamma_m\lambda\right)_\alpha\partial x^m \\
&\quad +\frac{3i}{2}\left(\lambda\gamma_m\theta\right)\left(\gamma^m\partial\theta\right)_\alpha + \frac{i}{2}\left(\partial\lambda\gamma_m\theta\right)\left(\gamma^m\theta\right)_\alpha
\end{aligned} \tag{3.104}
$$

$$
sJ_M = -\partial c_M + (-)^{N+M} J_P f^P{}_{MN} c^N
$$

$$
\begin{aligned}
s\Pi_{zm} &= \partial\xi_m + 2\left(\lambda\gamma_m\partial\theta\right) \\
s\partial\theta^\alpha &= i\partial\lambda^\alpha \\
s\, id_{z\alpha} &= -i\partial\chi_\alpha + 2i\left(\gamma_m\lambda\right)_\alpha\Pi^m_z - 2\xi^m\left(\gamma_m\partial\theta\right)_\alpha
\end{aligned} \tag{3.105}
$$

One obtains the transformations of the h–fields by replacing the fields by their h–counterparts.

$$
sc^M = -(-)^K\frac{1}{2}f^M{}_{LK}c^K c^L
$$

$$
\begin{aligned}
s\xi^m &= -i\left(\lambda\gamma^m\lambda\right) \\
s\lambda^\alpha &= 0 \\
s\chi_\alpha &= 2\left(\gamma_m\lambda\right)_\alpha\xi^m
\end{aligned} \tag{3.106}
$$

$$
sb_M = \left(J_M + J^h_M\right) - (-)^{N+M} b_P f^P{}_{MN} c^N = \Omega_M
$$

$$
\begin{aligned}
s\beta_{zm} &= \left(\Pi_{zm} - \Pi^h_{zm}\right) - 2\left(\kappa_z\gamma_m\lambda\right) \\
s\omega_{z\alpha} &= \left(id_{z\alpha} - id^h_{z\alpha}\right) - 2i\beta_{zm}\left(\gamma^m\lambda\right)_\alpha - 2\left(\gamma_m\kappa_z\right)_\alpha\xi^m \\
s\kappa^\alpha_z &= \left(\partial\theta^\alpha - \partial\theta^{\alpha h}\right)
\end{aligned} \tag{3.107}
$$

BRST Current, Composite B–field and Energy–Momentum Tensor

We can read off the BRST current if we transform the action with local parameter Λ: $\delta_\Lambda(\ldots) = \Lambda\, s(\ldots)$

$$\delta_\Lambda S \;=\; \int \left(J_M + J^h_M\right) \bar\partial \left(\Lambda c^M\right) - \delta_\Lambda \left((-)^M b_M \bar\partial c^M\right) \;=\; \int \bar\partial \Lambda \left((-)^M \left(J_M + J^h_M\right) c^M + b_M\, s c^M\right) \tag{3.108}$$

We read off the following BRST current:

$$\begin{aligned} j^B_z &= (-)^M \left(J_M + J^h_M\right) c^M - (-)^K \frac{1}{2} b_M f^M{}_{LK} c^K c^L \\ &= -\left(\Pi_{zm} - \Pi^h_{zm}\right) \xi^m - \left(i d_{z\alpha} - i d^h_{z\alpha}\right) \lambda^\alpha - \left(\partial\theta^\alpha - \partial\theta^{\alpha h}\right) \chi_\alpha + i\beta_{zm} \left(\lambda\gamma^m\lambda\right) + 2\left(\kappa_z \gamma_m \lambda\right) \xi^m \end{aligned} \tag{3.109}$$

For gauged WZNW models with $H = G \times G$ there exists an operator B_{zz} which makes the energy–momentum tensor BRST exact [23]:

$$T_{zz} \;=\; [Q, B_{zz}] \qquad\qquad Q \;=\; \oint j^B_z \tag{3.110}$$

In our approach to the WZNW model it is now easy to find the symmetry corresponding to B_{zz}. Looking at the ghost action

$$S_{gh} \;=\; \int -(-)^M b_M \bar\partial c^M \;=\; \int -(-)^M c_M \bar\partial b^M \tag{3.111}$$

one sees that b_M and c^M can exchange their role as long as the conformal weight of b_M is of no importance. Thus, one can construct a new symmetry by taking $c^M \leftrightarrow b^M$, which reads in components:

$$-\xi^m \;\leftrightarrow\; \beta^m_z \qquad \lambda^\alpha \;\leftrightarrow\; i\kappa^\alpha_z \qquad \chi_\alpha \;\leftrightarrow\; -i\omega_{z\alpha} \tag{3.112}$$

Performing this exchange in all BRST transformations and in the complete BRST current would yield another nilpotent fermionic symmetry. Our aim is, however, to make the energy–momentum tensor BRST exact with respect to the generator B_{zz}. T_{zz} is basically the square of the original currents minus the square of the h–currents (Sugawara construction). The BRST current contains $(J_M + J^h_M)$–terms, thus we need $(J_M - J^h_M)$–terms in B_{zz}. Changing the relative sign of the transformation parameter for the elementary fields and for the h–fields does not affect the invariance of the matter action ("matter action" as opposed to "ghost action"), as the h–part and the original part are invariant independently. The resulting contribution to the current coming from the matter part of the action is then the difference between J_M and J^h_M.
We call Λ_B the transformation parameter corresponding to the new symmetry and t the fermionic transformation: $\delta_{\Lambda_B}(\ldots) \;=\; \Lambda_B\, t(\ldots)$. The variation of the complete action with respect to this transformation then yields:

$$\begin{aligned} \delta_{\Lambda_B} S \;&=\; \int \bar\partial \Lambda_B \left((-)^M \left(J_M - J^h_M\right) b^M + c_M\, t b^M\right) + (-)^M \Lambda_B \left(J_M - J^h_M\right) \bar\partial b^M \\ &\qquad -(-)^M \Lambda_B\, t c_M \bar\partial b^M + \Lambda_B c_M \bar\partial\, t b^M \end{aligned} \tag{3.113}$$

One can easily promote this transformation to a global symmetry of the whole action by defining:

$$tc_M = \left(J_M - J_M^h\right) \qquad tb^M = 0 \tag{3.114}$$

Note that B_{zz} is no longer nilpotent. Out of the variation of the action we can read off that the current of the new symmetry is $(-)^M \left(J_M - J_M^h\right) b^M$. In order to obtain the correct energy–momentum tensor in an OPE with the BRST current, this expression has to be multiplied with a factor $-\frac{1}{2}$. Thus, we get for B_{zz}:

$$B_{zz} = -\frac{1}{2}\left(\Pi_{zm} + \Pi_{zm}^h\right)\beta_z^m + \frac{i}{2}\left(id_{z\alpha} + id_{z\alpha}^h\right)\kappa_z^\alpha - \frac{i}{2}\left(\partial\theta^\alpha + \partial\theta^{\alpha h}\right)\omega_{z\alpha} \tag{3.115}$$

For completeness we write down the transformations of the fields and ghosts under the fermionic symmetry:

$$\begin{aligned} tx^m &= -\beta_z^m + i\left(\kappa_z\gamma^m\theta\right) \\ t\theta^\alpha &= -\kappa_z^\alpha \\ tp_{z\alpha} &= \partial\omega_{z\alpha} - 2i\beta_z^m\left(\gamma_m\partial\theta\right)_\alpha - i\partial\beta_z^m\left(\gamma_m\theta\right)_\alpha + i\left(\gamma_m\kappa_z\right)_\alpha\partial x^m \\ &\quad -\frac{3}{2}\left(\kappa_z\gamma_m\theta\right)\left(\gamma^m\partial\theta\right)_\alpha - \frac{1}{2}\left(\partial\kappa_z\gamma_m\theta\right)\left(\gamma^m\theta\right)_\alpha \end{aligned} \tag{3.116}$$

$$tJ_M = -\partial b_M + (-)^{N+M} J_P f^P{}_{MN} b^N$$

$$\begin{aligned} t\Pi_{zm} &= -\partial\beta_{zm} + 2i\left(\kappa_z\gamma_m\partial\theta\right) \\ t\partial\theta^\alpha &= -\partial\kappa_z^\alpha \\ t\, id_{z\alpha} &= \partial\omega_{z\alpha} - 2\left(\gamma_m\kappa_z\right)_\alpha\Pi_z^m + 2\beta_z^m\left(\gamma_m\partial\theta\right)_\alpha \end{aligned} \tag{3.117}$$

The corresponding h–fields and currents transform accordingly.

$$\begin{aligned} tc_M &= -\frac{1}{2}\left(J_M - J_M^h\right) \\ t\xi^m &= \frac{1}{2}\left(\Pi_z^m + \Pi_z^{mh}\right) \\ t\lambda^\alpha &= -\frac{i}{2}\left(\partial\theta^\alpha + \partial\theta^{\alpha h}\right) \\ t\chi_\alpha &= \frac{i}{2}\left(id_{z\alpha} + id_{z\alpha}^h\right) \end{aligned} \tag{3.118}$$

$$sb_M = 0 \tag{3.119}$$

To conclude this section, we verify that B_{zz} is a homotopy for the energy–momentum tensor T_{zz}:

$$\begin{aligned} sB_{zz} &= -(-)^M\frac{1}{2}\left(\left(J_P - J_P^h\right) f^P{}_{NM} c^M - 2\partial c_N\right) b^N \\ &\quad -(-)^M\frac{1}{2}\left(J^M - J^{Mh}\right)\left(\left(J_M + J_M^h\right) - (-)^{N+M} b_P f^P{}_{MN} c^N\right) \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}J_M J^M + \frac{1}{2}J^h_M J^{Mh} - (-)^M b_M \partial c^M \\
&= -\frac{1}{2}\Pi_{zm}\Pi^m_z + \frac{1}{2}\Pi^h_{zm}\Pi^{mh}_z + d_{z\alpha}\partial\theta^\alpha - d^h_{z\alpha}\partial\theta^{\alpha h} + \beta_{zm}\partial\xi^m + \omega_{z\alpha}\partial\lambda^\alpha + \kappa^\alpha_z\partial\chi_\alpha \\
&= T_{zz} \qquad (3.120)
\end{aligned}
$$

3.3.2 Type II String

Our starting point is the free field action for the type II superstring:

$$
\begin{aligned}
S &= \int -\frac{1}{2}\partial x^m \bar{\partial} x_m + p_{z\alpha}\bar{\partial}\theta^\alpha + \hat{p}_{\bar{z}\hat{\alpha}}\partial\hat{\theta}^{\hat{\alpha}} \\
&= \int -\frac{1}{2}\Pi^m_z \Pi_{\bar{z}m} + \mathcal{L}_{WZ} + d_{z\alpha}\bar{\partial}\theta^\alpha + \hat{d}_{\bar{z}\hat{\alpha}}\partial\hat{\theta}^{\hat{\alpha}}, \qquad (3.121)
\end{aligned}
$$

where

$$
\begin{aligned}
\Pi^m_\mu &= \partial_\mu x^m - i\theta\gamma^m\partial_\mu\theta - i\hat{\theta}\gamma^m\partial_\mu\hat{\theta} \qquad (3.122) \\
d_{\mu\alpha} &= p_{\mu\alpha} - (\gamma_m\theta)_\alpha \left(i\partial_\mu x^m + \frac{1}{2}\theta\gamma^m\partial_\mu\theta + \frac{1}{2}\hat{\theta}\gamma^m\partial_\mu\hat{\theta} \right) \qquad (3.123) \\
\hat{d}_{\mu\hat{\alpha}} &= \hat{p}_{\mu\hat{\alpha}} - (\gamma_m\hat{\theta})_{\hat{\alpha}} \left(i\partial_\mu x^m + \frac{1}{2}\theta\gamma^m\partial_\mu\theta + \frac{1}{2}\hat{\theta}\gamma^m\partial_\mu\hat{\theta} \right) \qquad (3.124) \\
\mathcal{L}_{WZ} &= -i\varepsilon^{\mu\nu}\Pi^m_\mu \left((\theta\gamma_m\partial_\nu\theta) - (\hat{\theta}\gamma_m\partial_\nu\hat{\theta}) \right) - \varepsilon^{\mu\nu}(\theta\gamma_m\partial_\mu\theta)(\hat{\theta}\gamma^m\partial_\nu\hat{\theta}). \qquad (3.125)
\end{aligned}
$$

Our aim is to gauge this action in a procedure similar to the heterotic case. We will, however, encounter some complications which will eventually lead us to the introduction of new auxiliary fields into the action.

Manifest Supersymmetry of the Conserved Currents

In a first try we naively extend the gauge transformations to the closed string case:

$$
\begin{aligned}
\delta x^m &= -\omega^m + (\omega\gamma^m\theta) - \hat{\omega}^m + (\hat{\omega}\gamma^m\hat{\theta}) \\
\delta\theta^\alpha &= i\omega^\alpha \\
\delta p_{z\alpha} &= i\partial\omega_\alpha - 2i\omega^m(\gamma_m\partial\theta)_\alpha - i\partial\omega^m(\gamma_m\theta)_\alpha + (\gamma_m\omega)_\alpha\partial x^m + \frac{3i}{2}(\omega\gamma_m\theta)(\gamma^m\partial\theta)_\alpha + \frac{i}{2}(\partial\omega\gamma_m\theta)(\gamma^m\theta)_\alpha \\
\delta\hat{\theta}^{\hat{\alpha}} &= i\hat{\omega}^{\hat{\alpha}} \\
\delta\hat{p}_{\bar{z}\hat{\alpha}} &= i\bar{\partial}\hat{\omega}_{\hat{\alpha}} - 2i\hat{\omega}_m(\gamma^m\bar{\partial}\hat{\theta})_{\hat{\alpha}} - i\bar{\partial}\hat{\omega}^m(\gamma_m\hat{\theta})_{\hat{\alpha}} + (\gamma_m\hat{\omega})_{\hat{\alpha}}\bar{\partial}x^m + \frac{3i}{2}(\hat{\omega}\gamma_m\hat{\theta})(\gamma^m\bar{\partial}\hat{\theta})_{\hat{\alpha}} + \frac{i}{2}(\bar{\partial}\hat{\omega}\gamma_m\hat{\theta})(\gamma^m\hat{\theta})_{\hat{\alpha}}
\end{aligned}
$$
(3.126)

If we compute the gauge variation of $d_{z\alpha}$ using these transformations we find the following result:

$$
\begin{aligned}
\delta d_{z\alpha} &= i\partial\omega_\alpha + 2i(\gamma_m\omega)_\alpha \left(\partial x^m - i\theta\gamma^m\partial\theta - \frac{i}{4}\hat{\theta}\gamma^m\partial\hat{\theta} \right) - 2i\omega^m(\gamma_m\theta)_\alpha \\
&\quad + \frac{3}{2}(\gamma_m\theta)_\alpha(\hat{\omega}\gamma^m\partial\hat{\theta}) + \frac{1}{2}(\gamma_m\theta)_\alpha(\partial\hat{\omega}\gamma^m\hat{\theta}) + i\partial\hat{\omega}^m(\gamma_m\theta)_\alpha \qquad (3.127)
\end{aligned}
$$

We get $\delta\hat{d}_{\bar{z}\hat{\alpha}}$ by replacing $\partial \leftrightarrow \bar{\partial}$ and $\omega^M \leftrightarrow \hat{\omega}^M$. Now we make two observations:

- The second summand in (3.127) is clearly not invariant under supersymmetry transformations $\delta_\varepsilon\theta^\alpha = \varepsilon^\alpha$, $\delta_\varepsilon\hat{\theta}^{\hat{\alpha}} = \hat{\varepsilon}^{\hat{\alpha}}$ and $\delta_\varepsilon x^m = i(\varepsilon\gamma^m\theta) + i(\hat{\varepsilon}\gamma^m\hat{\theta})$. In analogy to the heterotic case we would rather want this expression to be replaced by the supersymmetric $2i(\gamma_m\omega)_\alpha\Pi_z^m$.
- The last two terms in (3.127) also violate supersymmetry, but in this case this does not hurt because for the global symmetry these terms vanish since $\partial\hat{\omega}^M = 0$

We want the transformations to commute with supersymmetry because through gauging and gauge fixing they will become the BRST transformations. If we want an off–shell formalism that is manifestly supersymmetric, BRST symmetry and supersymmetry should commute.
To overcome these problems we alter the transformations of $p_{z\alpha}$ and $\hat{p}_{\bar{z}\hat{\alpha}}$ in order to obtain manifestly supersymmetric transformations for $d_{z\alpha}$ and $\hat{d}_{\bar{z}\hat{\alpha}}$. For that purpose we get rid of the transformations that vanish if we only consider the chiral sector by simply subtracting them in the p–variation. To fix the other problem we use the following trick: Remember that there exist trivial gauge transformations [34]:

$$\delta\phi^{\mathcal{A}} \;=\; (-)^{\mathcal{B}} A^{\mathcal{A}\mathcal{B}} \frac{\delta S}{\delta\phi^{\mathcal{B}}}, \tag{3.128}$$

where $A^{\mathcal{A}\mathcal{B}}$ is graded antisymmetric. This is a local symmetry of the Lagrangian that is present in any theory with more than one field. It does not imply a gauge freedom. Invariance of the action is easily checked:

$$\delta S \;=\; \int \frac{\delta S}{\delta\phi^{\mathcal{A}}}\delta\phi^{\mathcal{A}} \;=\; \int (-)^{\mathcal{B}} \frac{\delta S}{\delta\phi^{\mathcal{A}}} A^{\mathcal{A}\mathcal{B}} \frac{\delta S}{\delta\phi^{\mathcal{B}}} \;=\; 0 \tag{3.129}$$

Adding such trivial gauge transformations to $p_{z\alpha}$ and $\hat{p}_{\bar{z}\hat{\alpha}}$ does not change the form of the currents but surprisingly we can find a transformation such that the gauge variation of $d_{z\alpha}$ and $\hat{d}_{\bar{z}\hat{\alpha}}$ becomes supersymmetric. If we set $\phi^{\mathcal{A}} = (x^m, \theta^\alpha, p_{z\alpha}, \hat{\theta}^\alpha, \hat{p}_{\bar{z}\hat{\alpha}})$ we get the desired result if the matrix $A^{\mathcal{A}\mathcal{B}}$ has the following elements[5]:

$$\begin{aligned} A^{\underline{\alpha}\underline{\hat{\beta}}} &= (\gamma_m\hat{\mu})_{\hat{\beta}}(\gamma^m\theta)_\alpha + (\gamma_m\mu)_\alpha(\gamma^m\hat{\theta})_{\hat{\beta}} \\ A^{\underline{\hat{\alpha}}\underline{\beta}} &= (\gamma_m\hat{\mu})_{\hat{\alpha}}(\gamma^m\theta)_\beta + (\gamma_m\mu)_\beta(\gamma^m\hat{\theta})_{\hat{\alpha}} \end{aligned} \tag{3.130}$$

We fix the parameters μ and $\bar{\mu}$ by demanding that the $d_{z\alpha}$ and $\hat{d}_{\bar{z}\hat{\alpha}}$ transform supersymmetrically. This yields the altered transformations for $p_{z\alpha}$ and $\hat{p}_{\bar{z}\hat{\alpha}}$:

$$\begin{aligned} \delta p_{z\alpha} &= i\partial\omega_\alpha - 2i\omega_m(\gamma^m\partial\theta)_\alpha - i\partial\omega^m(\gamma_m\theta)_\alpha + (\gamma_m\omega)_\alpha\partial x^m + \frac{3i}{2}(\omega\gamma_m\theta)(\gamma^m\partial\theta)_\alpha + \frac{i}{2}(\partial\omega\gamma_m\theta)(\gamma^m\theta)_\alpha \\ &\quad -\frac{3i}{2}(\gamma_m\omega)_\alpha(\hat{\theta}\gamma^m\partial\hat{\theta}) + \frac{3i}{2}(\hat{\omega}\gamma_m\partial\hat{\theta})(\gamma^m\theta)_\alpha - i\partial\hat{\omega}_m(\gamma^m\theta)_\alpha + \frac{i}{2}(\partial\hat{\omega}\gamma^m\hat{\theta})(\gamma_m\theta)_\alpha \\ \delta\hat{p}_{\bar{z}\hat{\alpha}} &= i\bar{\partial}\hat{\omega}_{\hat{\alpha}} - 2i\hat{\omega}_m(\gamma^m\bar{\partial}\hat{\theta})_{\hat{\alpha}} - i\bar{\partial}\hat{\omega}^m(\gamma_m\hat{\theta})_{\hat{\alpha}} + (\gamma_m\hat{\omega})_{\hat{\alpha}}\bar{\partial}x^m + \frac{3i}{2}(\hat{\omega}\gamma_m\hat{\theta})(\gamma^m\bar{\partial}\hat{\theta})_{\hat{\alpha}} + \frac{i}{2}(\bar{\partial}\hat{\omega}\gamma_m\hat{\theta})(\gamma^m\hat{\theta})_{\hat{\alpha}} \\ &\quad -\frac{3i}{2}(\gamma_m\hat{\omega})_{\hat{\alpha}}(\theta\gamma^m\bar{\partial}\theta) + \frac{3i}{2}(\omega\gamma_m\bar{\partial}\theta)(\gamma^m\hat{\theta})_{\hat{\alpha}} - i\bar{\partial}\omega^m(\gamma_m\hat{\theta})_{\hat{\alpha}} + \frac{i}{2}(\bar{\partial}\omega\gamma_m\theta)(\gamma^m\hat{\theta})_{\hat{\alpha}} \end{aligned} \tag{3.131}$$

The transformations of x^m, θ^α and $\hat{\theta}^{\hat{\alpha}}$ remain unchanged. With that we obtain supersymmetric transformations of the conserved currents:

$$\delta J_M \;=\; -\partial\omega_M + J_P f^P{}_{MN}\omega^N + (\hat{\omega} - \text{terms})$$

[5]See Appendix B for an explanation of the underlined indices.

$$
\begin{aligned}
\delta\Pi_{zm} &= -\partial\omega_m + 2(\omega\gamma_m\partial\theta) - \partial\hat{\omega}_m + 2(\hat{\omega}\gamma_m\partial\hat{\theta}) \\
\delta\partial\theta^\alpha &= i\partial\omega^\alpha \\
\delta i d_{z\alpha} &= -\partial\omega_\alpha + 2i(\gamma_m\omega)_\alpha\Pi_z^m + 2\omega^m(\gamma_m\partial\theta)_\alpha
\end{aligned} \tag{3.132}
$$

$$
\delta\hat{J}_{\hat{M}} = -\bar{\partial}\hat{\omega}_{\hat{M}} + \hat{J}^{\hat{P}}{}_{\hat{M}\hat{N}}\hat{\omega}^{\hat{N}} + (\omega - \text{terms})
$$

$$
\begin{aligned}
\delta\Pi_{\bar{z}m} &= -\bar{\partial}\omega_m + 2(\omega\gamma_m\bar{\partial}\theta) - \bar{\partial}\hat{\omega}_m + 2(\hat{\omega}\gamma_m\bar{\partial}\hat{\theta}) \\
\delta\bar{\partial}\hat{\theta}^{\hat{\alpha}} &= i\bar{\partial}\hat{\omega}^{\hat{\alpha}} \\
\delta i\hat{d}_{\bar{z}\hat{\alpha}} &= -\bar{\partial}\hat{\omega}_{\hat{\alpha}} + 2i(\gamma_m\hat{\omega})_{\hat{\alpha}}\Pi_{\bar{z}}^m + 2\hat{\omega}^m(\gamma_m\bar{\partial}\hat{\theta})_{\hat{\alpha}}
\end{aligned} \tag{3.133}
$$

In the condensed notation we have $\hat{J}_{\hat{M}} = (\Pi_{\bar{z}m}, i\hat{d}_{\bar{z}\hat{\alpha}}, \bar{\partial}\hat{\theta}^{\hat{\alpha}})$ and $\hat{\omega}^{\hat{M}} = (\hat{\omega}^m, \hat{\omega}^{\hat{\alpha}}, -i\hat{\omega}_{\hat{\alpha}})$ for the right moving sector.

Non–Closure of the off–shell Algebra and P_{zm}, $P_{\bar{z}m}$

Having found supersymmetric gauge transformations unfortunately has not solved all of our problems. Computing the commutator of the gauge transformations on J_M an $\hat{J}_{\hat{M}}$ we find that the gauge algebra does not close on all the currents:

$$
\begin{aligned}
[\delta_1, \delta_2] i d_{z\alpha} &= 2i\partial\left((\gamma_m\omega_1)_\alpha\omega_2^m - \omega_1^m(\gamma_m\omega_2)_\alpha\right) + 4i(\gamma_m\partial\theta)_\alpha(\omega_1\gamma^m\omega_2) \\
&\quad + 2i(\gamma_m\omega_2)_\alpha\left(-\partial\hat{\omega}_1^m + 2(\hat{\omega}_1\gamma^m\partial\hat{\theta})\right) - 2i(\gamma_m\omega_1)_\alpha\left(-\partial\hat{\omega}_2^m + 2(\hat{\omega}_2\gamma^m\partial\hat{\theta})\right)
\end{aligned} \tag{3.134}
$$

Here the Fierz identity was used once. We find an analogous expression for the commutator acting on $i\hat{d}_{\bar{z}\hat{\alpha}}$. The first two terms correspond to transformations with the parameters ω_α and ω^m, respectively. The last two terms show the non–closure of the algebra since off–shell hatted quantities appear in the transformation of $id_{z\alpha}$. The reason for these terms to show up is the Π_{zm} in the transformation of $id_{z\alpha}$. Its gauge variation contains, in contrast to the heterotic case, also hatted variables.
Non–closure of the gauge algebra implies that at the BRST–level the transformations are not nilpotent on $id_{z\alpha}$. Lack of BRST nilpotency on this current was already encountered in [25] and it was solved by introducing an auxiliary variable P_0^m for Π_0^m. The authors wrote all BRST–transformations with only ∂_1–derivatives using the free field equations to eliminate the ∂_0–contributions. Nilpotency on all the fields was reinstalled but the price one has to pay was that the transformation rules for the heterotic string were altered.
We will now present a similar ansatz to close the algebra by manifestly separating the transformations of the chiral and the antichiral sector off-shell. This will be achieved by introducing two auxiliary fields P_z^m and $P_{\bar{z}}^m$ into the transformation of $id_{z\alpha}$ and $i\hat{d}_{\bar{z}\hat{\alpha}}$ which transform as Π_z^m and $\Pi_{\bar{z}}^m$ in the chiral and antichiral case, respectively. We will call the corresponding gauge transformation $\tilde{\delta}$:

$$
\begin{aligned}
\tilde{\delta}P_z^m &\overset{!}{=} -\partial\omega^m + 2\left(\omega\gamma^m\partial\theta\right) \\
\tilde{\delta} i d_{z\alpha} &\overset{!}{=} -\partial\omega_\alpha + 2i\left(\gamma_m\omega\right)_\alpha P_z^m + 2\omega^m\left(\gamma_m\partial\theta\right)_\alpha \\
&= \delta i d_{z\alpha} - 2i\left(\gamma_m\omega\right)_\alpha\left(\Pi_z^m - P_z^m\right),
\end{aligned} \tag{3.135}
$$

and analogously for the right movers. The transformations are supersymmetric if we define the new fields to be SUSY–inert. At the same time we have to guarantee that the equations of motion remain unchanged and that on–shell P coincides with Π. It seems obvious that this can be achieved by introducing the P into the action as a Legendre transform of the Π. This would mean that we replace $-\frac{1}{2}\Pi_z^m\Pi_{\bar{z}m}$ by $\frac{1}{2}P_z^m P_{\bar{z}m} - \frac{1}{2}P_z^m\Pi_{\bar{z}m} - \frac{1}{2}\Pi_z^m P_{\bar{z}m}$. This is a first order action and variation with respect to P_z^m and $P_{\bar{z}m}$ yields algebraic equations of motion $P_z^m = \Pi_z^m$ and $P_{\bar{z}m} = \Pi_{\bar{z}m}$ which can be reinserted to reproduce the original action. However it turns out that an action of this form is not invariant under the simple variation we postulated for $id_{z\alpha}$ in (3.135)[6].
A more general possibility to introduce the P into the action without changing the equations of motion is to add a term proportional to $(P-\Pi)^2$ to the Lagrangian (3.121). We make the following ansatz:

$$\tilde{\mathcal{L}} \;=\; \mathcal{L} + \frac{c}{2}\left(P_z^m - \Pi_z^m\right)\left(P_{\bar{z}m} - \Pi_{\bar{z}m}\right) \tag{3.136}$$

Now we fix the parameter c by the invariance condition:

$$\begin{aligned}
\tilde{\delta}\tilde{S} \;&=\; \int \tilde{\delta}\mathcal{L} + \frac{c}{2}\tilde{\delta}\left\{\left(P_z^m - \Pi_z^m\right)\left(P_{\bar{z}m} - \Pi_{\bar{z}m}\right)\right\} \\
&=\; \int \delta\mathcal{L} + \left(\tilde{\delta} - \delta\right) d_{z\alpha}\bar{\partial}\theta^\alpha + \left(\tilde{\delta} - \delta\right) \hat{d}_{\bar{z}\hat{\alpha}}\partial\hat{\theta}^{\hat{\alpha}} \\
&\qquad + c\left\{\left((\hat{\omega}\gamma^m\partial\hat{\theta}) - \frac{1}{2}\partial\hat{\omega}^m\right)\left(\Pi_{\bar{z}m} - P_{\bar{z}m}\right) + \left((\omega\gamma^m\bar{\partial}\theta) - \frac{1}{2}\bar{\partial}\omega^m\right)\left(\Pi_{zm} - P_{zm}\right)\right\} \\
&=\; \int \delta\mathcal{L} + \frac{c}{2}\bar{\partial}\omega^m\left(P_{zm} - \Pi_{zm}\right) + \frac{c}{2}\partial\hat{\omega}^m\left(P_{\bar{z}m} - \Pi_{\bar{z}m}\right) \\
&\qquad + (-2+c)(\omega\gamma^m\bar{\partial}\theta)\left(\Pi_{zm} - P_{zm}\right) + (-2+c)(\hat{\omega}\gamma^m\partial\hat{\theta})\left(\Pi_{\bar{z}m} - P_{\bar{z}m}\right)
\end{aligned} \tag{3.137}$$

For the global variation to vanish we have to choose $c = 2$. From this calculation one can see that the conserved currents Π_{zm} and $\Pi_{\bar{z}m}$ are replaced by P_{zm} and $P_{\bar{z}m}$, respectively.
The new action has now the following form (we drop the $\tilde{}$ again.):

$$S \;=\; \int P_z^m P_{\bar{z}m} - P_z^m\Pi_{\bar{z}m} - \Pi_z^m P_{\bar{z}m} + \frac{1}{2}\Pi_z^m\Pi_{\bar{z}m} + \mathcal{L}_{WZ} + d_{z\alpha}\bar{\partial}\theta^\alpha + \hat{d}_{\bar{z}\hat{\alpha}}\partial\hat{\theta}^{\hat{\alpha}} \tag{3.138}$$

Introducing $J_M = (P_{zm}, id_{z\alpha}, \partial\theta^\alpha)$ and $\hat{J}_M = (P_{\bar{z}m}, i\hat{d}_{\bar{z}\hat{\alpha}}, \bar{\partial}\hat{\theta}^{\hat{\alpha}})$ we find the following gauge transformations of the currents:

$$\begin{aligned}
\delta J_M \;&=\; -\partial\omega_M + J_P f^P{}_{MN}\omega^N \\
\\
\delta P_z^m \;&=\; -\partial\omega^m + 2i(\omega\gamma^m\partial\theta) \\
\delta\partial\theta^\alpha \;&=\; i\partial\omega^\alpha \\
\delta id_{z\alpha} \;&=\; -\partial\omega_\alpha + 2i(\gamma_m\omega)_\alpha P_z^m + 2\omega^m(\gamma_m\partial\theta)_\alpha \\
\\
\delta\hat{J}_{\hat{M}} \;&=\; -\bar{\partial}\hat{\omega}_{\hat{M}} + \hat{J}_{\hat{P}}\hat{f}^{\hat{P}}{}_{\hat{M}\hat{N}}\hat{\omega}^{\hat{N}}
\end{aligned} \tag{3.139}$$

[6]In fact, it can be checked that the first order form of the action is invariant if we choose $\tilde{\delta}id_{z\alpha} = -\partial\omega_\alpha + 2i(\gamma_m\omega)_\alpha\left(\left(1 - \frac{1}{\sqrt{2}}\right)\Pi_z^m + \frac{1}{\sqrt{2}}P_z^m\right) + 2\omega^m(\gamma_m\partial\theta)_\alpha$.

$$\begin{aligned}\delta P_{\bar z}^m &= -\bar\partial\hat\omega^m + 2i(\hat\omega\gamma^m\bar\partial\hat\theta)\\ \delta\bar\partial\hat\theta^{\hat\alpha} &= i\bar\partial\hat\omega^{\hat\alpha}\\ \delta i\hat d_{\bar z\hat\alpha} &= -\bar\partial\hat\omega_{\hat\alpha} + 2i(\gamma_m\hat\omega)_{\hat\alpha}P_{\bar z}^m + 2\hat\omega^m(\gamma_m\bar\partial\hat\theta)_{\hat\alpha}\end{aligned}\tag{3.140}$$

Noether Procedure

In principle, the procedure to implement the local gauge symmetry in order to obtain the action of a gauged WZNW model for the type II string is completely analogous to the heterotic case. The only difference is that we now have the P–fields instead of the Π. We get an additional gauge connection $\hat A_z^{\hat M}$ for the right moving sector.The h–fields are introduced in a completely analogous way. Gauge fixing and the introduction of ghosts and BRST symmetry are as in the heterotic case, apart from the fact that we need a second Nakanishi–Lautrup field $\hat\Lambda_M$ for gauge fixing. $b_M \equiv b_{zM}$ are fields of conformal weight 1 which will become the antighosts. Now we give some of the crucial results:

$$\delta_1 S_0 = \int \left(J_M + J_M^h\right)\bar\partial\omega^M + \left(\hat J_{\bar z\hat M} + \hat J_{\bar z\hat M}^h\right)\partial\hat\omega^{\hat M},\tag{3.141}$$

with $J_M^h = -(P_{zm}^h, id_{z\alpha}^h, \partial\theta^{\alpha h})$ and $\hat J_{\hat M}^h = -(P_{\bar z m}^h, i\hat d_{\bar z\hat\alpha}, \bar\partial\hat\theta^{\hat\alpha h})$.

$$S_1 = -\int \left(J_{zM} + J_{zM}^h\right)A_{\bar z}^M + \left(\hat J_{\bar z\hat M} + \hat J_{\bar z\hat M}^h\right)\hat A_z^{\hat M}\tag{3.142}$$

$$\delta_0 A_{\bar z}^M = \bar\partial\omega^M \qquad \delta_0\hat A_z^{\hat M} = \partial\hat\omega^{\hat M}\tag{3.143}$$

We get $\delta_1 S_1 = 0$ if we define:

$$\delta_1 A_{\bar z}^P = -f^P{}_{MN}\omega^N A_{\bar z}^M \qquad \delta_1\hat A_z^{\hat P} = -\hat f^{\hat P}{}_{\hat M\hat N}\hat\omega^{\hat N}\hat A_z^{\hat M}\tag{3.144}$$

We have the following new or altered BRST transformations of the fields and currents:

$$\begin{aligned}s\Pi_\mu^m &= \partial_\mu\xi^m + 2(\lambda\gamma^m\partial_\mu\theta) + \partial_\mu\hat\xi^m + 2(\hat\lambda\gamma^m\partial_\mu\hat\theta)\\[1em] sP_z^m &= \partial\xi^m + 2(\lambda\gamma^m\partial\theta)\\ s\theta^\alpha &= i\lambda^\alpha\\ sid_{z\alpha} &= -i\partial\chi_\alpha + 2i(\gamma_m\lambda)_\alpha P_z^m - 2\xi^m(\gamma_m\partial\theta)_\alpha\\[1em] sP_{\bar z}^m &= \bar\partial\hat\xi^m + 2(\hat\lambda\gamma^m\bar\partial\hat\theta)\\ s\hat\theta^{\hat\alpha} &= i\hat\lambda^{\hat\alpha}\\ si\hat d_{\bar z\hat\alpha} &= -i\bar\partial\hat\chi_{\hat\alpha} + 2i(\gamma_m\hat\lambda)_{\hat\alpha}P_{\bar z}^m - 2\hat\xi^m(\gamma_m\bar\partial\hat\theta)_{\hat\alpha}\end{aligned}\tag{3.145}$$

The corresponding h–currents transform accordingly.
The BRST transformations for the ghosts remain unchanged. For the new hatted ghosts we have $s\hat c^{\hat M} = -(-)^{\hat K}\frac{1}{2}\hat f^{\hat M}{}_{\hat L\hat K}\hat c^{\hat K}\hat c^{\hat L}$ where $\hat c^{\hat M} = (-\hat\xi^m, \hat\lambda^{\hat\alpha}, \hat\chi_{\hat\alpha})$. For the right moving antighosts the BRST transformations are $s\hat b_{\hat M} = (\hat J_{\hat M} + \hat J_{\hat M}^h) - (-)^{\hat N+\hat M}\hat b_{\hat P}\hat f^{\hat P}{}_{\hat M\hat N}\hat c^{\hat N}$ with

$\hat{b}_{\hat{M}} = (\hat{\beta}_{\bar{z}m}, \hat{\omega}_{\bar{z}\hat{\alpha}}, \hat{\kappa}_{\bar{z}}^{\hat{\alpha}})$. Note that the J contain the P-fields. In particular we have $s\beta_{zm} = (P_{zm} - P^h_{zm}) - 2(\kappa_z\gamma_m\lambda)$.
The left– and right–moving BRST currents are given by:

$$\begin{aligned} j^B_z &= -(P_{zm} - P^h_{zm})\xi^m - (id_{z\alpha} - id^h_{z\alpha})\lambda^\alpha - (\partial\theta^\alpha - \partial\theta^{\alpha h})\chi_\alpha + i\beta_{zm}(\lambda\gamma^m\lambda) + 2(\kappa_z\gamma_m\lambda)\xi^m \\ \hat{j}^B_{\bar{z}} &= -(P_{\bar{z}m} - P^h_{\bar{z}m})\hat{\xi}^m - (i\hat{d}_{\bar{z}\hat{\alpha}} - i\hat{d}^h_{\bar{z}\hat{\alpha}})\hat{\lambda}^{\hat{\alpha}} - (\bar{\partial}\hat{\theta}^{\hat{\alpha}} - \bar{\partial}\hat{\theta}^{\hat{\alpha}h})\hat{\chi}_{\hat{\alpha}} + i\hat{\beta}_{\bar{z}m}(\hat{\lambda}\gamma^m\hat{\lambda}) + 2(\hat{\kappa}_{\bar{z}}\gamma_m\hat{\lambda})\hat{\xi}^m \end{aligned} \tag{3.146}$$

For the left– and right–moving B–currents we find:

$$\begin{aligned} B_{zz} &= -\frac{1}{2}\left(P_{zm} + P^h_{zm}\right)\beta^m_z + \frac{i}{2}\left(id_{z\alpha} + id^h_{z\alpha}\right)\kappa^\alpha_z - \frac{i}{2}\left(\partial\theta^\alpha + \partial\theta^{\alpha h}\right)\omega_{z\alpha} \\ \hat{B}_{\bar{z}\bar{z}} &= -\frac{1}{2}\left(P_{\bar{z}m} + P^h_{\bar{z}m}\right)\hat{\beta}^m_{\bar{z}} + \frac{i}{2}\left(i\hat{d}_{\bar{z}\hat{\alpha}} + i\hat{d}^h_{\bar{z}\hat{\alpha}}\right)\hat{\kappa}^{\hat{\alpha}}_{\bar{z}} - \frac{i}{2}\left(\bar{\partial}\hat{\theta}^{\hat{\alpha}} + \bar{\partial}\hat{\theta}^{\hat{\alpha}h}\right)\hat{\omega}_{\bar{z}\hat{\alpha}} \end{aligned} \tag{3.147}$$

For the type II string B_{zz} is a homotopy for the energy momentum tensor T_{zz} only on the operator level and not as an off–shell current:

$$\begin{aligned} sB_{zz} &= -\frac{1}{2}P_{zm}P^m_z + \frac{1}{2}P^h_{zm}P^{mh}_z + d_{z\alpha}\partial\theta^\alpha - d^h_{z\alpha}\partial\theta^{\alpha h} + \beta_{zm}\partial\xi^m + \omega_{z\alpha}\partial\lambda^\alpha + \kappa^\alpha_z\partial\chi_\alpha \\ &\overset{\text{on shell}}{=} T_{zz}, \end{aligned} \tag{3.148}$$

where "on–shell" means $P \to \Pi$ and $\partial\hat{\theta} = \bar{\partial}\theta = 0$. In contrast to that, the off–shell holomorphic component of the energy momentum tensor reads:

$$\begin{aligned} T_{zz} &= (P_z - \Pi_z)^2 - (P^h_z - \Pi^h_z)^2 - \frac{1}{2}\Pi_{zm}\Pi^m_z + \frac{1}{2}\Pi^h_{zm}\Pi^{mh}_z + d_{z\alpha}\partial\theta^\alpha - d^h_{z\alpha}\partial\theta^{\alpha h} + \hat{d}_{z\hat{\alpha}}\partial\hat{\theta}^{\hat{\alpha}} - \hat{d}^h_{z\hat{\alpha}}\partial\hat{\theta}^{\hat{\alpha}h} \\ &\quad + \beta_{zm}\partial\xi^m + \omega_{z\alpha}\partial\lambda^\alpha + \kappa^\alpha_z\partial\chi_\alpha + \hat{\beta}_{zm}\partial\hat{\xi}^m + \hat{\omega}_{z\hat{\alpha}}\partial\hat{\lambda}^{\hat{\alpha}} + \hat{\kappa}^{\hat{\alpha}}_z\partial\hat{\chi}_{\hat{\alpha}} \end{aligned} \tag{3.149}$$

Similarly we have on–shell $sB_{\bar{z}\bar{z}} = T_{\bar{z}\bar{z}}$. The complete gauged and gauge fixed action is now given by:

$$\begin{aligned} S = \int & P^m_z P_{\bar{z}m} - P^m_z\Pi_{\bar{z}m} - \Pi^m_z P_{\bar{z}m} + \frac{1}{2}\Pi^m_z\Pi_{\bar{z}m} + \mathcal{L}_{WZ} + d_{z\alpha}\bar{\partial}\theta^\alpha + \hat{d}_{\bar{z}\hat{\alpha}}\partial\hat{\theta}^{\hat{\alpha}} \\ & - P^{mh}_z P^h_{\bar{z}m} + P^{mh}_z\Pi^h_{\bar{z}m} + \Pi^{mh}_z P^h_{\bar{z}m} - \frac{1}{2}\Pi^{mh}_z\Pi^h_{\bar{z}m} - \mathcal{L}^h_{WZ} - d^h_{z\alpha}\bar{\partial}\theta^{\alpha h} - \hat{d}^h_{\bar{z}\hat{\alpha}}\partial\hat{\theta}^{\hat{\alpha}h} \\ & + \beta_{zm}\bar{\partial}\xi^m + \omega_{z\alpha}\bar{\partial}\lambda^\alpha + \kappa^\alpha_z\bar{\partial}\chi_\alpha + \hat{\beta}_{\bar{z}m}\partial\hat{\xi}^m + \hat{\omega}_{\bar{z}\hat{\alpha}}\partial\hat{\lambda}^{\hat{\alpha}} + \hat{\kappa}^{\hat{\alpha}}_{\bar{z}}\partial\hat{\chi}_{\hat{\alpha}} \end{aligned} \tag{3.150}$$

3.4 $N = 2$ Algebra

3.4.1 Kazama Algebra

In this section we focus on the operator algebra satisfied by the energy–momentum tensor $T(z)$, the BRST current $j^B(z)$, the ghost current $j^{\text{gh}}(z)$ and the composite B–field. By the introduction of the fields P_{zm} and $P_{\bar{z}m}$ we achieved in section 3.3.2 that the left– and right–moving sector for the type II superstring decouple on–shell. Thus, we will concentrate on the holomorphic sector only and the results presented here are valid for both the heterotic and the closed string case. It turns out that the currents satisfy a Kazama algebra [35]. Using the

Mathematica–package OPE–defs.m by Kris Thielemans [36] it is straight forward to check the following relations:

$$\begin{aligned}
T(z)T(w) &\sim \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} \\
T(z)j^B(w) &\sim \frac{j^B(w)}{(z-w)^2} + \frac{\partial j^B(w)}{z-w} \\
T(z)B(w) &\sim \frac{2B(w)}{(z-w)^2} + \frac{\partial B(w)}{z-w} \\
T(z)j^{\rm gh}(w) &\sim \frac{22}{(z-w)^3} + \frac{j^{\rm gh}(w)}{(z-w)^2} + \frac{\partial j^{\rm gh}(w)}{z-w} \\
j^B(z)B(w) &\sim \frac{-22}{(z-w)^3} + \frac{j^{\rm gh}(w)}{(z-w)^2} + \frac{T(w)}{z-w} \\
j^{\rm gh}(z)j^B(w) &\sim \frac{j^B(w)}{z-w} \\
j^{\rm gh}B(w) &\sim \frac{-B(w)}{z-w} \\
j^{\rm gh}(z)j^{\rm gh}(w) &\sim \frac{-22}{(z-w)^2} \\
B(z)B(w) &\sim \frac{F(w)}{z-w} \\
j^B(z)\Phi(w) &\sim \frac{F(w)}{z-w}
\end{aligned} \tag{3.151}$$

The algebra closes on two new composite fields given by:

$$F_{zzz} = -i\beta_{zm}\left(\kappa_z\gamma^m\left(\partial\theta - \partial\theta^h\right)\right) + \frac{i}{2}\left(\kappa_z\gamma^m\kappa_z\right)\left(\Pi_{zm} - \Pi^h_{zm}\right) \tag{3.152}$$

$$\Phi_{zzz} = \frac{i}{2}\beta_{zm}\left(\kappa_z\gamma^m\kappa_z\right) \tag{3.153}$$

3.4.2 Topological Quartet and $N = 2$–Algebra

The only term in the algebra that prevents it from coinciding with an $N = 2$ twisted[7] superconformal algebra is the BRST–exact operator F_{zzz}. We know from [23] that the currents of WZNW models satisfy a Kazama algebra. It was shown in [28] how one can turn this Kazama algebra into an $N = 2$ superconformal algebra.
For this purpose we introduce a topological Koszul quartet consisting of a pair of anticommuting ghosts (b'_{zz}, c'^z) and a pair of commuting ones, (β'_{zz}, γ'^z). We assign to the anticommuting ghosts the ghost numbers -1 and 1, respectively. The commuting ghosts, however get ghost numbers -2 and 2 in order for B_{zz} to have a well–defined ghost number, as will become clear soon. Introduction of this quartet into our theory causes the following changes:

$$\mathcal{L} \rightarrow \mathcal{L} + b'_{zz}\bar\partial c'^z + \beta'_{zz}\bar\partial\gamma'^z \tag{3.154}$$

[7] The algebra is twisted because the OPE of the energy momentum tensor with itself has no central term.

$$\begin{aligned}
T_{zz} &\to T_{zz} + T^{\mathrm{top}\prime}_{zz} = T_{zz} + 2b'_{zz}\partial c'^z + \partial b'_{zz} c'^z + 2\beta'_{zz}\partial\gamma'^z + \partial\beta'_{zz}\gamma'^z && (3.155)\\
j^B_z &\to j^B_z + j^{B\,\mathrm{top}\prime}_z = j^B_z + b'_{zz}\gamma'^z && (3.156)\\
j^{\mathrm{gh}}_z &\to j^{\mathrm{gh}}_z + j^{gh\,\mathrm{top}\prime}_z = j^{\mathrm{gh}}_z + b'_{zz}c'^z + 2\beta'_{zz}\gamma'^z && (3.157)
\end{aligned}$$

The form of the BRST operator for the quartet induces:

$$sc'^z = -\gamma'^z \qquad s\beta'_{zz} = -b'_{zz} \tag{3.158}$$

Thus, the ghosts BRST–transform into one another, i.e. they form a quartet, and do not contribute to the cohomology.
There is also a B_{zz} field for the Koszul quartet:

$$B^{\mathrm{top}\prime}_{zz} = -2\beta'_{zz}\partial c'^z - c'^z\partial\beta'_{zz} - \mu' b'_{zz} \tag{3.159}$$

It can be shown that $T^{\mathrm{top}\prime}_{zz}$, $j^{B\,\mathrm{top}\prime}_z$, $j^{gh\,\mathrm{top}\prime}_z$ and $B^{\mathrm{top}\prime}_{zz}$ satisfy a twisted $N=2$ superconformal algebra with ghost number anomaly -3 for arbitrary parameter μ'. In order to get an $N=2$–algebra for our theory we have to set $\mu'=1$ and alter B_{zz} as follows:

$$B_{zz} \to B_{zz} - 2\beta'_{zz}\partial c'^z - c'^z\partial\beta'_{zz} - b'_{zz} - \frac{1}{2}c'^z F_{zzz} - \frac{1}{2}\gamma'^z\Phi_{zzz} \tag{3.160}$$

Now we see that assigning γ'^z ghost number 2 gives B_{zz} ghost number 1. With that we find that B_{zz} is now nilpotent and the twisted superconformal algebra is:

$$\begin{aligned}
T(z)T(w) &\sim \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}\\
T(z)j^B(w) &\sim \frac{j^B(w)}{(z-w)^2} + \frac{\partial j^B(w)}{z-w}\\
T(z)B(w) &\sim \frac{2B(w)}{(z-w)^2} + \frac{\partial B(w)}{z-w}\\
T(z)j^{\mathrm{gh}}(w) &\sim \frac{25}{(z-w)^3} + \frac{j^{\mathrm{gh}}(w)}{(z-w)^2} + \frac{\partial j^{\mathrm{gh}}(w)}{z-w}\\
j^B(z)B(w) &\sim \frac{-25}{(z-w)^3} + \frac{j^{\mathrm{gh}}(w)}{(z-w)^2} + \frac{T(w)}{z-w}\\
j^{\mathrm{gh}}(z)j^B(w) &\sim \frac{j^B(w)}{z-w}\\
j^{\mathrm{gh}}B(w) &\sim \frac{-B(w)}{z-w}\\
j^{\mathrm{gh}}(z)j^{\mathrm{gh}}(w) &\sim \frac{-25}{(z-w)^2}\\
B(z)B(w) &\sim 0 && (3.161)
\end{aligned}$$

We get the structure of an untwisted $N=2$–algebra by defining: $\hat{T} = T - \frac{1}{2}\partial g^{\mathrm{gh}}$, $J = j^{\mathrm{gh}}$, $G^+ = j^B$ and $G^- = 2B$ where $G^\pm$ are the supersymmetry generators and J is the $U(1)$–current. Then we find:

$$\hat{T}(z)\hat{T}(w) \sim \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}$$

$$
\begin{aligned}
\hat{T}(z)G^{\pm}(w) &\sim \frac{3/2\, G^{\pm}(w)}{(z-w)^2} + \frac{\partial G^{\pm}(w)}{z-w} \\
\hat{T}(z)J(w) &\sim \frac{J(w)}{(z-w)^2} + \frac{\partial J(w)}{z-w} \\
G^{+}(z)G^{-}(w) &\sim \frac{2c/3}{(z-w)^3} + \frac{2J(w)}{(z-w)^2} + \frac{2\hat{T}(w) + \partial J(w)}{z-w} \\
J(z)G^{\pm}(w) &\sim \frac{\pm J(w)}{z-w} \\
J(z)J(w) &\sim \frac{c/3}{(z-w)^2}
\end{aligned}
\tag{3.162}
$$

In our casae we have $c = 75$.

3.5 BRST Operator and Cohomology

In contrast to Berkovits' pure spinor approach the definition of physical states in our model presents a great difficulty and unfortunately no satisfying solution has been found to the present day. Relaxing the pure spinor constraint alters the BRST operator Q. In addition to the term $\int -id^{\alpha}_{z\alpha\lambda}$ which implements the constraint $d_{z\alpha} = 0$ we get terms proportional to all the currents Π_{zm} and $\partial\theta^{\alpha}$, see for example equation (3.109). This implies that all the currents of our WZNW model are set to 0 which renders the cohomology trivial. In the WZNW approach we have the additional complication that we have to get rid of the auxiliary h–currents. In this section we present a review of the ideas that have been suggested to fix this problem.

3.5.1 Cohomology in the Old Approach and the Grading Condition

In the "old approach" (cf. section 3.2.2) the BRST–operator for the heterotic string reads [4]:

$$
\begin{aligned}
Q &= \oint -id_{z\alpha}\lambda^{\alpha} - \Pi_{zm}\xi^{m} - \partial\theta^{\alpha}\chi_{\alpha} + i\beta_{zm}\left(\lambda\gamma^{m}\lambda\right) + 2\left(\kappa_{z}\gamma_{m}\lambda\right)\xi^{m} \\
&\qquad + c_{z} + b\left(\frac{1}{2}\xi_{m}\partial\xi^{m} + \frac{i}{4}\lambda^{\alpha}\partial\chi_{\alpha} - \frac{3i}{4}\chi_{\alpha}\partial\lambda^{\alpha}\right),
\end{aligned}
\tag{3.163}
$$

where the ghost number $(-1,1)$–pair (b, c_z) with $b(z)c_z(w) \sim -1/(z-w)$ was introduced by hand to make the BRST charge nilpotent. Now one can make an ansatz for a generic unintegrated massless vertex operator of ghost number 1 [26]:

$$
\begin{aligned}
\mathcal{U}^{(1)} &= \lambda^{\alpha}A_{\alpha} + \xi^{m}A_{m} + \chi_{\alpha}A^{\alpha} \\
&\quad + b\left(\lambda^{\alpha}\lambda^{\beta}F_{\alpha\beta} + \lambda^{\alpha}\xi^{m}F_{\alpha m} + \xi^{m}\xi^{n}F_{mn} + \lambda^{\alpha}\chi_{\beta}F_{\alpha}{}^{\beta} + \chi_{\alpha}\xi^{m}F^{\alpha}{}_{m} + \chi_{\alpha}\chi_{\beta}F^{\alpha\beta}\right)
\end{aligned}
\tag{3.164}
$$

The fields $A_{\alpha}, \ldots, F^{\alpha\beta}$ are arbitrary superfields. Note that they can only depend on x_m and θ^{α} and not on ∂x^m, $\partial\theta^{\alpha}$, $d_{z\alpha}$ or $p_{z\alpha}$ because $\mathcal{U}^{(1)}$ has to be a worldsheet scalar.

Now we demand $Q\mathcal{U}^{(1)} \stackrel{!}{=} 0$ which yields the following equations from the terms quadratic in the ghosts:

$$\begin{aligned}
\lambda\lambda: &\qquad D_{(\alpha}A_{\beta)} + \gamma^m_{\alpha\beta}A_m + iF_{\alpha\beta} \;=\; 0 \\
\lambda\xi: &\qquad \partial_m A_\alpha - iD_\alpha A_m + 2\gamma_{m\,\alpha\beta}A^\beta + F_{\alpha m} \;=\; 0 \\
\xi\xi: &\qquad \partial_{[m}A_{n]} + F_{mn} \;=\; 0 \\
\lambda\chi: &\qquad D_\beta A^\alpha + iF_\beta{}^\alpha \;=\; 0 \\
\xi\chi: &\qquad \partial_m A^\alpha + F^\alpha{}_m \;=\; 0 \\
\chi\chi: &\qquad F^{\alpha\beta} \;=\; 0,
\end{aligned} \tag{3.165}$$

where $D_\alpha = \frac{\partial}{\partial\theta^\alpha} + 2\theta^\beta\gamma^m_{\alpha\beta}\frac{\partial}{\partial x_m}$ with the normalization $D_\alpha D_\beta + D_\beta D_\alpha = 4\gamma^m_{\alpha\beta}\partial_m$. We get another set of equations cubic in the ghosts which correspond to the Bianchi identities for the curvatures $F_{\alpha\beta}$, $F_{\alpha m}$, F_{mn}, $F^\alpha{}_m$, $F_\alpha{}^\beta$ and $F^{\alpha\beta}$. Note that due to our conventions we get some different factors and signs as compared to [26], which can be absorbed into the definitions of the fields $A_\alpha \ldots F^{\alpha\beta}$.

From the first two equations one can derive the equations for $N=1$, $D=(9,1)$ linearized Yang–Mills theory,

$$\gamma^{\alpha\beta}_{[mnpqr]}D_\alpha A_\beta \;=\; 0, \tag{3.166}$$

and the definition of the vector potential A_m and the spinoral field strength A^α in terms of A_α:

$$\begin{aligned}
A_m &= \frac{1}{8}\gamma^{\alpha\beta}_m D_\alpha A_\beta \\
A^\alpha &= \frac{1}{10}\gamma^{m\,\alpha\beta}\left(D_\beta A_m - \partial_m A_\beta\right)
\end{aligned} \tag{3.167}$$

Unfortunately, *this only works if we impose* $F_{\alpha\beta} = F_{m\alpha} = 0$. Otherwise we get the wrong equations of motion. This is a symptom of the fact that our BRST operator sets more than the constraint $d_{z\alpha}$ to 0. Thus, we need to look for a motivation to set the unwanted field strengths to 0.

In [24] it was argued that $F_{\alpha\beta} = F_{m\alpha} = 0$ could be obtained in the following way: The BRST charge is deformed to $Q_{\mathcal{U}} = Q + \mathcal{U}^{(1)}$. This shifts the currents Π_{zm}, $\partial\theta^\alpha$ and $d_{z\alpha}$ by the gauge potentials A_m, A^α and A_α. We get the equations (3.165) from requiring the nilpotency of $Q_{\mathcal{U}}$ (up to terms quadratic in $\mathcal{U}$ (!)) which implies $[Q, \mathcal{U}^{(1)}] = 0$. Now the same shift is made for the energy momentum tensor:

$$\begin{aligned}
T^A_{zz} \;=\; & -\frac{1}{2}(\Pi^m_z - A_m)(\Pi_{zm} - A_m) - (-d_{z\alpha} + A_\alpha)(\partial\theta^\alpha - A^\alpha) \\
& + \beta_{zm}\partial\xi^m + \omega_{z\alpha}\partial\lambda^\alpha + \kappa^\alpha_z\partial\chi_\alpha - \partial b c_z - \partial\eta^m\omega_{zm}
\end{aligned} \tag{3.168}$$

One demands that T^A_{zz} satisfies the standard OPE:

$$T^A(z)T^A(w) \;\sim\; \frac{2T^A(w)}{(z-w)^2} + \frac{\partial T^A(w)}{z-w} \tag{3.169}$$

This yields constraints on the gauge potentials and the field strengths including the linearized super Yang–Mills equations of motion and gauge fixing conditions. It is then claimed that these equations together with the Bianchi identities imply $F_{\alpha\beta} = F_{m\alpha} = 0$.

In the following papers [26] and [25] this ansatz was abandoned and replaced by the notion of the grading. Physical vertex operators are defined as operators with non–negative grading. One starts by assigning grading -1 to $d_{z\alpha}$ and the opposite grading to the corresponding ghost λ^α. This assures that the term $-id_{z\alpha}\lambda^\alpha$ has grading 0 and therefore remains in the BRST operator. Then one demands that the grading is preserved in the operator product expansion. Thus $dd \sim \Pi$ implies that Π_{zm} has grading -2 and ξ^m has grading 2. $d\Pi \sim \partial\theta$ assigns grading -3 to $\partial\theta^\alpha$ and grading 3 to χ_α. The ghost c_z is defined to have grading 4. The antighosts are given the opposite grading of the corresponding ghosts.
One finds that with this definition the BRST operator (3.163) has non–negative grading. In the vertex operator $\mathcal{U}^{(1)}$ (3.164), however, we find two terms with negative grading: The terms $b\lambda^\alpha\lambda^\beta F_{\alpha\beta}$ and $b\lambda^\alpha\xi^m F_{\alpha m}$ have grading -2 and -1, respectively, and are, according to the new definition of physical vertex operators, not allowed. This efficiently removes the unwanted terms from (3.165) but still the grading has some unattractive features:

- The grading is introduced by hand and does not seem to come out of the theory in some fundamental way. In [37] the grading was related to general properties of Lie algebras but no explicit calculations for the superstring were given.

- From the OPEs of ghosts and antighosts it follows that the unit operator has grading 0. In contrast to that the OPEs $d\partial\theta \sim (z-w)^{-2}$ and $\Pi\Pi \sim (z-w)^{-2}$ imply that the grading of the unit operator is -4. To avoid this inconsistency it was suggested in [4] to introduce a central charge operator I for the double poles that has grading -4.

The arguments presented in this section can be generalized to the type II string which was shown in [26].

3.5.2 Coset Gauging and Second BRST Operator

In [37] it was shown for simple Lie algebras how to gauge constraints related to coset generators and not to those of a subgroup. A simple Lie algebra decomposed into the Cartan–Weyl basis reads:

$$
\begin{aligned}
[E_\alpha, E_{-\alpha}] &= \alpha^i H_i & [H_i, E_{\pm\alpha}] &= \pm\alpha_i E_{\pm\alpha} \\
[E_\alpha, E_\beta] &= N_{\alpha\beta} E_{\alpha+\beta} \quad \text{if} \quad \alpha+\beta \neq 0 & [H_i, H_j] &= 0
\end{aligned}
\tag{3.170}
$$

One can gauge the E_α alone (and not together with the H_i) by first gauging both E_α and H_i, i.e. setting them to 0 cohomologically by a BRST operator Q, and then undo the gauging of the H_i by introducing a second BRST operator Q_c. For the superstring, E_α corresponds to $d_{z\alpha}$ and the H_i correspond to Π_{zm} and $\partial\theta_\alpha$. The case of the superstring is not contained in the discussion of [37] since the string is based on a non–semisimple Lie algebra. In [1] these concepts were extended to an arbitrary set of constraints that generate a first class system and then specialized to the case of the superstring.
We consider a gauge algebra with generators $G_M = \oint J_M$ that satisfies:

$$
[G_M, G_N] = G_K f^K{}_{MN} \tag{3.171}
$$

Note that there is no central extension. Thus, the following construction only works for the superstring as a gauged WZNW model where the above algebra is satisfied for $J_M \equiv J_M + J^h_M$.

It cannot be applied to the “old approach”.
Now we assume that we only want to gauge symmetries which correspond to some subset of generators G_α that do not generate a subalgebra. We call $G_M \equiv (G_\mathfrak{a}, G_\alpha)$ where $G_\mathfrak{a}$ are the remaining generators. In order to gauge the generators G_α one first has to gauge the complete algebra and one gets the usual BRST operator of the form:

$$Q = \int (-)^M J_M c^M - (-)^K \frac{1}{2} b_M f^M{}_{LK} c^K c^L \tag{3.172}$$

Now we undo the gauging of $G_\mathfrak{a}$ by setting the corresponding ghosts $c^\mathfrak{a}$ to 0 in the cohomology. This is achieved by making them BRST exact, but the $c^\mathfrak{a}$ cannot be made exact with respect to Q because their BRST transformation is already fixed to something non–zero. Therefore a new BRST operator $Q_c = \oint j_c$ has to be introduced together with some new fields. The ghosts $c^\mathfrak{a}$ and the antighosts $b_\mathfrak{a}$ as well as the new fields will be removed from the cohomology via the following diagram:

$$\begin{array}{ccccccc} b_\mathfrak{a} & & & & & & c^\mathfrak{a} \\ & \overset{Q_c}{\searrow} & & & & \overset{Q_c}{\nearrow} & \\ \downarrow K & & \pi_\mathfrak{a} & & \varphi^\mathfrak{a} & & \uparrow -K \\ & \overset{Q}{\nearrow} & & & & \overset{Q}{\searrow} & \\ b'_\mathfrak{a} & & & & & & c'^\mathfrak{a} \end{array} \tag{3.173}$$

Here $c'^\mathfrak{a}$ and $b'_\mathfrak{a}$ are new ghosts and antighosts with grading $|\mathfrak{a}|+1$, whereas $\pi_\mathfrak{a}$ and $\varphi^\mathfrak{a}$ are fields with ghost number 0 and grading $|\mathfrak{a}|$. K is a homotopy operator.
The contribution of the new fields to the Lagrangian is:

$$\mathcal{L}' = -(-)^\mathfrak{a} b'_\mathfrak{a} \bar{\partial} c'^\mathfrak{a} + \pi_\mathfrak{a} \bar{\partial} \varphi^\mathfrak{a} \tag{3.174}$$

The on–shell energy–momentum tensor is modified to:

$$T_{zz} \rightarrow T_{zz} - (-)^\mathfrak{a} b'_\mathfrak{a} \partial c'^\mathfrak{a} + \pi_\mathfrak{a} \partial \varphi^\mathfrak{a} \tag{3.175}$$

As indicated in the diagram, Q has to be extended by a term $(-)^\mathfrak{a} \pi_\mathfrak{a} c'^\mathfrak{a}$:

$$Q = \oint (-)^m J_M c^M - (-)^K \frac{1}{2} b_M f^M{}_{LK} c^K c^L + (-)^\mathfrak{a} \pi_\mathfrak{a} c'^\mathfrak{a} \tag{3.176}$$

This yields the desired BRST transformations $s b'_\mathfrak{a} = \pi_\mathfrak{a}$ and $s\varphi^\mathfrak{a} = c'^\mathfrak{a}$.
One can construct a suitable Q_c, that anticommutes with Q, as the commutator of Q with a homotopy operator K:

$$K = \oint k = \oint (-)^\mathfrak{a} b'_\mathfrak{a} c^\mathfrak{a}, \tag{3.177}$$

with $\delta_K c'^\mathfrak{a} = -c^\mathfrak{a}$ and $\delta_K b_\mathfrak{a} = b'_\mathfrak{a}$. Now we can compute Q_c:

$$Q_c = \oint j_c = [Q, K] = \oint (-)^\mathfrak{a} \pi_\mathfrak{a} c^\mathfrak{a} + (-)^N \frac{1}{2} b'_\mathfrak{a} f^\mathfrak{a}{}_{MN} c^N c^M \tag{3.178}$$

From this we get the following transformations:

$$s_c b_M = \pi_\mathfrak{a} \delta^\mathfrak{a}_M + (-)^{N+M} b'_\mathfrak{b} f^\mathfrak{b}{}_{MN} c^N$$

$$
\begin{aligned}
s_c\varphi^{\mathfrak{a}} &= c^{\mathfrak{a}} \\
sc'^{\mathfrak{a}} &= (-)^K \frac{1}{2} f^{\mathfrak{a}}{}_{LK} c^K c^L,
\end{aligned} \tag{3.179}
$$

where $\delta^{\mathfrak{a}}_M$ means that there is only a contribution if M belongs to the subset $\mathfrak{a}$.
The Jacobi identity $\sum [Q, [Q, K]] = 0$ implies, together with the nilpotency of Q, that Q and Q_c anticommute:

$$
[Q, Q_c] = 0 \tag{3.180}
$$

Furthermore $\sum [Q_c, [Q, K]] = 0$ implies:

$$
[Q_c, Q_c] = [Q, [K, Q_c]] = 0 \tag{3.181}
$$

In the last step we used the fact that $\delta_K Q_c = 0$ trivially since it does not contain $c'^{\mathfrak{a}}$ and $b_{\mathfrak{a}}$ by construction.
Physical variables are defined to lie in the relative cohomology of Q with respect to Q_c.
Part of Q turns out to be exact with respect to Q_c. We define:

$$
\Xi = (-)^{\mathfrak{a}} b_{\mathfrak{a}} c'^{\mathfrak{a}} + \left(J_{\mathfrak{a}} - (-)^{\gamma+\mathfrak{a}} b_\alpha f^\alpha{}_{\mathfrak{a}\gamma} c^\gamma\right) \varphi^{\mathfrak{a}} \tag{3.182}
$$

Then we can write Q as follows:

$$
\begin{aligned}
Q = \oint & (-)^\alpha J_\alpha c^\alpha - (-)^\beta \frac{1}{2} b_\gamma f^\gamma{}_{\alpha\beta} c^\beta c^\alpha + s_c \Xi \\
& -(-)^{\mathfrak{b}} \frac{1}{2} b_\gamma f^\gamma{}_{\mathfrak{a}\mathfrak{b}} c^{\mathfrak{b}} c^{\mathfrak{a}} - (-)^N b'_{\mathfrak{b}} f^{\mathfrak{b}}{}_{\mathfrak{a}N} c^N c'^{\mathfrak{a}} - (-)^{\gamma+\mathfrak{a}+N+\alpha} b'_{\mathfrak{b}} f^{\mathfrak{b}}{}_{\alpha N} c^N f^\alpha{}_{\mathfrak{a}\gamma} c^\gamma \varphi^{\mathfrak{a}}
\end{aligned} \tag{3.183}
$$

For gauging the roots of a simple Lie algebra all structure constants in the second line vanish [37] and the result is a BRST charge which consists of the constraint we want to implement plus something which is Q_c–exact.

Coset Gauging for the Superstring

Now we specialize the above considerations for the type II and the heterotic superstring. We will only consider the chiral sector here. The generalization to the closed string case is straight forward. As usual one simply has to replace Π_{zm} by P_{zm} and add the corresponding hatted quantities to the expressions given here. For the superstring (3.183) reduces to:

$$
Q = \oint (-)^\alpha J_\alpha c^\alpha + s_c \Xi - (-)^\alpha b'_{\mathfrak{b}} f^{\mathfrak{b}}{}_{\mathfrak{a}\alpha} c^\alpha c'^{\mathfrak{a}} \tag{3.184}
$$

We make the following identifications for currents and ghosts:

$$
J_M = (\Pi_{zm} - \Pi^h_{zm}, id_{z\alpha} - id^h_{z\alpha}, \partial\theta^\alpha - \partial\theta^{\alpha h}) \qquad J_{\mathfrak{a}} = (\Pi_{zm} - \Pi^h_{zm}, \partial\theta^\alpha - \partial\theta^{\alpha h}) \tag{3.185}
$$

$$
c^{\mathfrak{a}} = (-\xi^m, \chi_\alpha) \qquad b_{\mathfrak{a}} = (\beta_{zm}, \kappa^\alpha_z) \tag{3.186}
$$

$$
c'^{\mathfrak{a}} = (-\xi'^m, \chi'_\alpha) \qquad b'_{\mathfrak{a}} = (\beta'_{zm}, \kappa'^\alpha_z) \tag{3.187}
$$

$$
\varphi^{\mathfrak{a}} = (\varphi^m, -i\varphi_\alpha) \qquad \pi_{z\mathfrak{a}} = (\pi_{zm}, -i\pi^\alpha_z) \tag{3.188}
$$

The additional terms in the Lagrangian are given by:

$$\mathcal{L}' = \beta'_{zm}\bar{\partial}\xi'^m + \kappa'^{\alpha}_z\bar{\partial}\chi'_\alpha + \pi_{zm}\bar{\partial}\varphi^m - \pi^\alpha_z\bar{\partial}\varphi_\alpha \tag{3.189}$$

For the type II superstring we also get a contribution from the right–movers:

$$\hat{\mathcal{L}}' = \hat{\beta}'_{\bar{z}m}\partial\hat{\xi}'^m + \hat{\kappa}^{\hat{\alpha}\prime}_{\bar{z}}\partial\hat{\chi}'_{\hat{\alpha}} + \hat{\pi}_{\bar{z}m}\partial\hat{\varphi}^m - \hat{\pi}^{\hat{\alpha}}_{\bar{z}}\partial\hat{\varphi}_{\hat{\alpha}} \tag{3.190}$$

The BRST current and the corresponding transformations for the new fields read:

$$\begin{aligned} j^B_z &= -(\Pi_{zm} - \Pi^h_{zm})\xi^m - (id_{z\alpha} - id^h_{z\alpha})\lambda^\alpha - (\partial\theta^\alpha - \partial\theta^{\alpha h})\chi_\alpha \\ &\quad + i\beta_{zm}(\lambda\gamma^m\lambda) + 2(\kappa_z\gamma_m\lambda)\xi^m - \pi_{zm}\xi'^m + i\pi^\alpha_z\chi'_\alpha \end{aligned} \tag{3.191}$$

$$\begin{aligned} s\beta'_{zm} &= \pi_{zm} & s\varphi^m &= -\xi'^m \\ s\kappa'^\alpha_z &= -i\pi^\alpha_z & s\varphi_\alpha &= i\chi'_\alpha \end{aligned} \tag{3.192}$$

The homotopy current and the δ_K–transformations are given by:

$$k_z = -\beta'_{zm}\xi^m - \kappa'^\alpha_z\chi_\alpha \tag{3.193}$$

$$\begin{aligned} \delta_K\xi'^m &= -\xi^m & \delta_K\beta_{zm} &= \beta'_{zm} \\ \delta_K\chi'_\alpha &= -\chi_\alpha & \delta_K\kappa^\alpha_z &= \kappa'^\alpha_z \end{aligned} \tag{3.194}$$

The second BRST current is computed via $j_{zc} = sk_z$:

$$j_{zc} = -\pi_{zm}\xi^m + i\pi^\alpha_z\chi_\alpha - i\beta'_{zm}(\lambda\gamma^m\lambda) - 2(\kappa'_z\gamma_m\lambda)\xi^m \tag{3.195}$$

The corresponding BRST charge is:

$$Q_c = \oint -\pi_{zm}\xi^m + i\pi^\alpha_z\chi_\alpha - i\beta'_{zm}(\lambda\gamma^m\lambda) - 2(\kappa'_z\gamma_m\lambda)\xi^m \tag{3.196}$$

$$\begin{aligned} s_c\beta_{zm} &= \pi_{zm} + 2(\kappa'_z\gamma_m\lambda) \\ s_c\kappa^\alpha_z &= -i\pi^\alpha_z \\ s_c\omega_{z\alpha} &= 2i\beta'_{zm}(\gamma^m\lambda)_\alpha + 2(\gamma_m\kappa'_z)_\alpha\xi^m \end{aligned} \tag{3.197}$$

$$\begin{aligned} s_c\varphi^m &= -\xi^m & s_c\xi'^m &= i(\lambda\gamma^m\lambda) \\ s_c\varphi_\alpha &= i\chi_\alpha & s_c\chi'_\alpha &= 2(\gamma_m\lambda)_\alpha\xi^m \end{aligned} \tag{3.198}$$

The new fields contribute to the on–shell energy–momentum tensor in the following way:

$$T_{zz} \rightarrow T_{zz} + \beta'_{zm}\partial\xi'^m + \kappa'^\alpha_z\partial\chi'_\alpha + \pi_{zm}\varphi^m - \pi^\alpha_z\varphi_\alpha \tag{3.199}$$

In order to maintain the on–shell relation $T_{zz} = sB_{zz}$ the composite B–field has to be modified to:

$$\begin{aligned} B_{zz} &\rightarrow B_{zz} + b'_{\mathfrak{a}}\partial\varphi^{\mathfrak{a}} \\ &\rightarrow B_{zz} + \beta'_{zm}\partial\varphi^m - i\kappa'^\alpha_z\partial\varphi_\alpha \end{aligned} \tag{3.200}$$

The ghost current changes as follows:

$$j_z^{\text{gh}} \quad \to \quad j_z^{gh} + \beta'_{zm}\xi'^m + \kappa_z'^\alpha \chi'_\alpha \tag{3.201}$$

The new expressions for T_{zz} j_z^B, j_z^{gh} and B_{zz} still satisfy an $N = 2$ twisted superconformal algebra (3.161) where the ghost number anomaly is changed due to $c'^{\mathfrak{a}}$ and $b'_{\mathfrak{a}}$ by an amount of -6 from -25 to -31.

3.5.3 Deforming the BRST charge Q

Some interesting results found shortly before concluding this paper show that the deformation of the BRST charge may not be necessary at all. Thus, this section is no longer essential for this thesis and may be skipped. The new results will be presented in the following section.
The BRST charge (3.184) contains the constraint–term $-id_{z\alpha}\lambda^\alpha$ plus Q_c–exact terms and an additional expression. From the construction of the relative cohomology it is expected to recover Berkovits' cohomology up to Q_c–exact terms. In order to get rid of the unwanted term we try to deform Q. Unfortunately it will turn out that all reasonable ansätze for the deformed Q make the situation worse or do not change anything.
Since all the computations in this section will be performed in condensed notation we neglect the different gradings of the fields collected by the indices M and $\mathfrak{a}$. This has no influence on our results and the signs in our calculations become more transparent. In these simplified conventions all the currents J_M are bosonic, all the ghosts are fermionic, the primed ghosts are fermionic, the other new fields $(\pi_{z\mathfrak{a}}, \varphi^{\mathfrak{a}})$ are bosonic. Q, Q_c, K and Ξ are then:

$$Q = \oint J_\alpha c^\alpha + s_c\Xi - b'_{\mathfrak{b}} f^{\mathfrak{b}}{}_{\mathfrak{a}M} c^M c'^{\mathfrak{a}} \tag{3.202}$$

$$Q_c = [Q, K] = \oint \pi_{\mathfrak{a}} c^{\mathfrak{a}} + \frac{1}{2} b'_{\mathfrak{a}} f^{\mathfrak{a}}{}_{MN} c^N c^M \tag{3.203}$$

$$K = \oint b'_{\mathfrak{a}} c^{\mathfrak{a}} \tag{3.204}$$

$$\Xi = b_{\mathfrak{a}} c'^{\mathfrak{a}} + J_{\mathfrak{a}} \varphi^{\mathfrak{a}} \tag{3.205}$$

We apply the following procedure to deform Q:

- Add some terms to Q that produce $b'_{\mathfrak{b}} f^{\mathfrak{b}}{}_{\mathfrak{a}M} c^M c'^{\mathfrak{a}}$ via $s_c\Xi = [[Q, K], \Xi]$.
- Add more terms such that the new BRST charge is nilpotent on all the fields.
- Compute $j_c = -\delta_K j$.
- Compute the Q_c–exact terms $-\delta_\Xi j_c$ and hope that the unwanted term in (3.202) is included in the result.

The K– and Ξ– variations act as follows:

$$-\delta_K : \quad \begin{array}{ccc} c'^{\mathfrak{a}} & \to & c^{\mathfrak{a}} \\ b_{\mathfrak{a}} & \to & -b'_{\mathfrak{a}} \end{array} \qquad\qquad -\delta_\Xi : \quad \begin{array}{ccc} c^{\mathfrak{a}} & \to & c'^{\mathfrak{a}} \\ b'_{\mathfrak{a}} & \to & -b_{\mathfrak{a}} \\ \pi_{\mathfrak{a}} & \to & J_{\mathfrak{a}} \end{array} \tag{3.206}$$

Now we look for terms that produce the expression $b'_{\mathfrak{b}} f^{b}{}_{\mathfrak{a}N} c^N c'^{\mathfrak{a}}$ when we apply first δ_K and then δ_Ξ. The following combinations of primed and unprimed ghosts and antighosts exist:

	K		Ξ	
$b'fc'c'$	$\rightarrow$	$b'fcc'$	$\rightarrow$	$b'fc'c'$
			$\rightarrow$	$bfcc'$
$bfc'c'$	$\rightarrow$	$b'fcc'$	$\rightarrow$	$bfcc'$
			$\rightarrow$	$b'fc'c'$
	$\rightarrow$	$b'fc'c'$	$\rightarrow$	$bfc'c'$
$b'fcc'$	$\rightarrow$	$b'fcc$	$\rightarrow$	$b'fcc'$
			$\rightarrow$	$bfcc$
$b'fcc$	$\rightarrow$	0		
$bfcc'$	$\rightarrow$	$b'fcc'$	$\rightarrow$	$bfcc'$
			$\rightarrow$	$b'fcc$
	$\rightarrow$	$bfcc$	$\rightarrow$	$bfcc'$
$bfcc$	$\rightarrow$	$b'fcc$	$\rightarrow$	$b'fcc'$
			$\rightarrow$	$bfcc$

Note that the only terms that can produce a term of the form $b'fcc'$ are the term itself and the term $bfcc$ that is the standard cubic ghost term in the BRST operator.
We make an ansatz by adding a linear combination of the deformation candidates to Q:

$$\begin{aligned} Q_{\text{def}} &= \oint J_M c^M - \frac{1}{2} b_M f^M{}_{LK} c^K c^L + \pi_{\mathfrak{a}} c'^{\mathfrak{a}} + \rho_1 b'_{\mathfrak{a}} f^{\mathfrak{a}}{}_{\mathfrak{b}N} c^N c'^{\mathfrak{b}} + \sigma_1 b_M f^M{}_{LK} c^K c^L \\ &\quad + \rho_2 \pi_{\mathfrak{c}} f^{\mathfrak{c}}{}_{\mathfrak{a}N} c^N \varphi^{\mathfrak{a}} + \sigma_2 J_M c^M \end{aligned} \tag{3.207}$$

Some comments are in order:

- ρ_1, σ_1, ρ_2 and σ_2 are arbitrary constants that will be fixed by the requirement that Q is nilpotent.
- For the σ_1–term we actually should have written $\sigma_1 b_{\mathfrak{a}} f^{\mathfrak{a}}{}_{LK} c^K c^L$ (and consequently $\sigma_2 J_{\mathfrak{a}} c^{\mathfrak{a}}$) but for the string the expressions are equivalent since there is no structure constant with an upper index α.
- The ρ_2–term and the σ_2–term are added in order to achieve nilpotency of Q.
- It does not seem useful to add any of the other terms to Q since they either reproduce themselves or one of the others.

The altered BRST transformations are given by:

$$\begin{aligned} s\varphi^{\mathfrak{a}} &= c'^{\mathfrak{a}} + \rho_2 f^{\mathfrak{a}}{}_{\mathfrak{b}N} c^N \varphi^{\mathfrak{b}} \\ sb'_{\mathfrak{a}} &= \pi_a + \rho_1 b'_{\mathfrak{c}} f^{\mathfrak{c}}{}_{\mathfrak{a}N} c^N \\ s\pi_{\mathfrak{a}} &= -\rho_2 \pi_{\mathfrak{c}} f^{\mathfrak{c}}{}_{\mathfrak{a}N} c^N \end{aligned}$$

$$
\begin{aligned}
sb_K &= (1+\sigma_2)J_K - (1+2\sigma_1)b_L f^L{}_{KN}c^N - \rho_1 b'_{\mathfrak{a}} f^{\mathfrak{a}}{}_{\mathfrak{b}K}c'^{\mathfrak{b}} + \rho_2 \pi_{\mathfrak{c}} f^{\mathfrak{c}}{}_{\mathfrak{b}K}\varphi^{\mathfrak{b}} \\
sc^M &= \left(-\frac{1}{2}+\sigma_1\right) f^M{}_{LK}c^K c^L \\
sJ_K &= (1+\sigma_2)J_M f^M{}_{KN}c^N \\
sc'^{\mathfrak{a}} &= \rho_1 f^{\mathfrak{a}}{}_{\mathfrak{b}N}c^N c'^{\mathfrak{b}}
\end{aligned} \tag{3.208}
$$

Now we check the nilpotency on all fields. For the superstring most terms that are quadratic in the structure constants vanish. The only non-zero structure constants are $f^m{}_{\alpha\beta} = 2i\gamma^m{}_{\alpha\beta}$ and $f^{\beta}_{\alpha m} = 2\gamma_{m\alpha\beta}$. The vanishing f^2 terms will not be given explicitly in the following expressions and will be denoted by '...' In particular we make use of $f^M{}_{\mathfrak{b}N}f^N{}_{LK} = 0$.

$$
s^2\varphi^{\mathfrak{a}} = \rho_1 f^{\mathfrak{a}}{}_{\mathfrak{b}N}c^N c'^{\mathfrak{b}} - \rho_2 f^{\mathfrak{a}}{}_{\mathfrak{b}N}c^N c'^{\mathfrak{b}} + \ldots \tag{3.209}
$$

In order to make this expression vanish we have to set $\rho_1 = \rho_2$:

$$
s^2 b'_{\mathfrak{a}} = -\rho_2 \pi_{\mathfrak{c}} f^{\mathfrak{c}}{}_{\mathfrak{a}N}c^N + \rho_1 \pi_{\mathfrak{c}} f^{\mathfrak{c}}{}_{\mathfrak{a}N}c^N + \ldots = 0 \tag{3.210}
$$

$$
s^2\pi_a = \ldots = 0 \tag{3.211}
$$

$$
\begin{aligned}
s^2 b_K &= (1+\sigma_2)^2 J_M f^M{}_{KN}c^N - (1+2\sigma_1)(1+\sigma_2)J_L f^L{}_{KN}c^N + (1+2\sigma_1)^2 b_R f^R{}_{LT}c^T f^L{}_{KN}c^N \\
&\quad +(1+2\sigma_1)\left(-\frac{1}{2}+\sigma_1\right) b_L f^L{}_{KN}f^N{}_{RT}c^T c^R - \rho_1 \pi_{\mathfrak{a}} f^{\mathfrak{a}}{}_{\mathfrak{b}K}c'^{\mathfrak{b}} + \rho_2 \pi_{\mathfrak{a}} f^{\mathfrak{a}}{}_{\mathfrak{b}K}c'^{\mathfrak{b}} + \ldots
\end{aligned} \tag{3.212}
$$

This expression cannot vanish unless $\sigma_1 = \sigma_2 = 0$. It follows from the Jacobi identity that $\frac{1}{2}f^L{}_{KN}f^N{}_{RT}b_L c^T c^R = b_L f^L{}_{NT}c^T f^N{}_{KR}c^R$. In order to use this relation in the expression above we would have to demand $(1+2\sigma_1)(1-2\sigma_2) \stackrel{!}{=} (1+2\sigma_1)^2$ which is of course only fulfilled when $\sigma_1 = 0$ which immediately implies that $\sigma_2 = 0$. Adding further terms to Q to make this expression nilpotent would amount to multiplying Q with a constant factor which, of course, would not change anything.
With the Jacobi identity we get:

$$
s^2 J_K = J_R f^R{}_{MT}c^T f^M{}_{KN}c^N - \frac{1}{2}J_M f^M{}_{KN}f^N{}_{RT}c^T c^R = 0, \tag{3.213}
$$

and

$$
s^2 c^M = \frac{1}{2}f^M{}_{LK}f^K{}_{RT}c^T c^R c^L + \ldots = 0. \tag{3.214}
$$

Finally we have:

$$
s^2 c'^{\mathfrak{a}} = \ldots = 0 \tag{3.215}
$$

Since we were forced to set $\sigma_1 = \sigma_2 = 0$ the deformed BRST charge now reads:

$$
Q_{\text{def}} = \oint J_M c^M - \frac{1}{2}b_M f^M{}_{LK}c^K c^L + \pi_{\mathfrak{a}} c'^{\mathfrak{a}} + \rho b'_{\mathfrak{a}} f^{\mathfrak{a}}{}_{\mathfrak{b}N}c^N c'^{\mathfrak{b}} + \rho \pi_{\mathfrak{c}} f^{\mathfrak{c}}{}_{\mathfrak{a}N}c^N \varphi^{\mathfrak{a}} \tag{3.216}
$$

Now we compute $j_c = -\delta_K j_{\text{def}}$:

$$
j_c = \frac{1}{2}b'_{\mathfrak{a}} f^{\mathfrak{a}}{}_{LK}c^K c^L + \pi_{\mathfrak{a}} c^{\mathfrak{a}} + \rho b'_{\mathfrak{a}} f^{\mathfrak{a}}{}_{\mathfrak{b}N}c^N c^{\mathfrak{b}} \tag{3.217}
$$

Next we evaluate $-\delta_\Xi j_c$:

$$
\begin{aligned}
-\delta_\Xi j_c &= J_\mathfrak{a} c^\mathfrak{a} - \frac{1}{2} b_\mathfrak{a} f^\mathfrak{a}{}_{LK} c^K c^L + \pi_\mathfrak{a} c'^\mathfrak{a} + b'_\mathfrak{a} f^\mathfrak{a}{}_{\mathfrak{b}L} c^L c'^\mathfrak{b} \\
&\quad -\rho b_\mathfrak{a} f^\mathfrak{a}{}_{\mathfrak{b}L} c^L c^\mathfrak{b} + \rho b'_\mathfrak{a} f^\mathfrak{a}{}_{\mathfrak{b}L} c^L c'^\mathfrak{b} + \underbrace{\rho b'_\mathfrak{a} f^\mathfrak{a}{}_{\mathfrak{b}\mathfrak{c}} c^\mathfrak{c} c'^\mathfrak{b}}_{0}
\end{aligned}
\tag{3.218}
$$

We draw the following conclusions from this calculation:

- The ρ–term reproduces itself but also yields further terms we did not have before.
- The term $b'_\mathfrak{a} f^\mathfrak{a}{}_{\mathfrak{b}L} c^L c'^\mathfrak{b}$ appears anyway since it comes from the term $\frac{1}{2} b_M f^M{}_{LK} c^K c^L$. Since we are not allowed to add another term of this form to Q (the σ–terms) it does not seem possible to deform Q without making matters worse.

Some attempts were made to leave Q fixed and alter the homotopy K. We found that we have to choose K such that it only affects one field in each term of Q in order to maintain nilpotency of Q_c. The following K–deformations turned out to be bad:

1. $K = \Xi = c'^\mathfrak{a} b_\mathfrak{a} + c^\mathfrak{a} b'_\mathfrak{a}$ did not work because $bfcc \to b'fc'c - b'fcc - bfcc$ under K and Ξ.
2. $K = c^\mathfrak{a} b_\mathfrak{a} + c'^\mathfrak{a} b'_\mathfrak{a}$. K acting on $-\frac{1}{2} bfcc$ yields $-\frac{1}{2} bfcc + 2 b_\mathfrak{a} f^\mathfrak{a}{}_{K\mathfrak{b}} c^\mathfrak{b} c^K$.
3. K such that $\pi_\mathfrak{a} \to \pi_\mathfrak{a}$ and $c'^\mathfrak{a} \to c^\mathfrak{a}$ yields a $J_\mathfrak{a} c'^\mathfrak{a}$ that would not be taken care of properly, i.e. yielding a $J_\mathfrak{a} c^\mathfrak{a}$, by the corresponding Ξ.
4. K such that $\pi_\mathfrak{a} \to b_\mathfrak{a}$ and $c'^\mathfrak{a} \to c^\mathfrak{a}$ would imply that $c^\mathfrak{a} \to \varphi^\mathfrak{a}$ which causes significant trouble.
5. K such that $b_\mathfrak{a} \to b'_\mathfrak{a}$ and $c'^\mathfrak{a} \to \varphi^\mathfrak{a}$. The latter implies: $\pi_\mathfrak{a} \to b'_\mathfrak{a}$ which produces a term $b'_\mathfrak{a} c^\mathfrak{a}$ which we do not want.
6. $K = c'b' + b'c$. Then $\pi_\mathfrak{a} c'^\mathfrak{a}$ transforms into itself but then we cannot find a Ξ that also produces the $J_\mathfrak{a} c^\mathfrak{a}$–term out of this expression.

The last attempt to deform Q was to give up the homotopy K, that guarantees that the two BRST charges anticommute, and to make the most general ansatz for j and j_c, which contains all the previous guesses. For ghost number 1 and conformal weight 1 we get:

$$
\begin{aligned}
\tilde{j} &= \mu_0 J_\alpha c^\alpha + \mu_1 J_\mathfrak{a} c^\mathfrak{a} + \mu_2 J_\mathfrak{a} c'^\mathfrak{a} + \mu_3 \pi_\mathfrak{a} c^\mathfrak{a} + \mu_4 \pi_\mathfrak{a} c'^\mathfrak{a} - \mu_5 b_\mathfrak{a} f^\mathfrak{a}{}_{\mathfrak{b}\alpha} c^\alpha c^\mathfrak{b} \\
&\quad -\frac{\mu_6}{2} b_\mathfrak{a} f^\mathfrak{a}{}_{\alpha\beta} c^\beta c^\alpha - \mu_7 b_\mathfrak{a} f^\mathfrak{a}{}_{\mathfrak{b}\alpha} c^\alpha c'^\mathfrak{b} - \mu_8 b'_\mathfrak{a} f^\mathfrak{a}{}_{\mathfrak{b}\alpha} c^\alpha c^\mathfrak{b} - \frac{\mu_9}{2} b'_\mathfrak{a} f^\mathfrak{a}{}_{\alpha\beta} c^\beta c^\alpha \\
&\quad -\mu_{10} b'_\mathfrak{a} f^\mathfrak{a}{}_{\mathfrak{b}\alpha} c^\alpha c'^\mathfrak{b} + \mu_{11} \pi_\mathfrak{a} f^\mathfrak{a}{}_{\mathfrak{b}\alpha} c^\alpha \varphi^\mathfrak{b} + \mu_{12} j_\mathfrak{a} f^\mathfrak{a}{}_{\mathfrak{b}\alpha} c^\alpha \varphi^\mathfrak{b}
\end{aligned}
\tag{3.219}
$$

$\tilde{j}_c$ must be of the same form but may have different numerical constants:

$$
\begin{aligned}
\tilde{Q}_c &= \lambda_0 J_\alpha c^\alpha + \lambda_1 J_\mathfrak{a} c^\mathfrak{a} + \lambda_2 J_\mathfrak{a} c'^\mathfrak{a} + \lambda_3 \pi_\mathfrak{a} c^\mathfrak{a} + \lambda_4 \pi_\mathfrak{a} c'^\mathfrak{a} - \lambda_5 b_\mathfrak{a} f^\mathfrak{a}{}_{\mathfrak{b}\alpha} c^\alpha c^\mathfrak{b} \\
&\quad -\frac{\lambda_6}{2} b_\mathfrak{a} f^\mathfrak{a}{}_{\alpha\beta} c^\beta c^\alpha - \lambda_7 b_\mathfrak{a} f^\mathfrak{a}{}_{\mathfrak{b}\alpha} c^\alpha c'^\mathfrak{b} - \lambda_8 b'_\mathfrak{a} f^\mathfrak{a}{}_{\mathfrak{b}\alpha} c^\alpha c^\mathfrak{b} - \frac{\lambda_9}{2} b'_\mathfrak{a} f^\mathfrak{a}{}_{\alpha\beta} c^\beta c^\alpha \\
&\quad -\lambda_{10} b'_\mathfrak{a} f^\mathfrak{a}{}_{\mathfrak{b}\alpha} c^\alpha c'^\mathfrak{b} + \lambda_{11} \pi_\mathfrak{a} f^\mathfrak{a}{}_{\mathfrak{b}\alpha} c^\alpha \varphi^\mathfrak{b} + \lambda_{12} j_\mathfrak{a} f^\mathfrak{a}{}_{\mathfrak{b}\alpha} c^\alpha \varphi^\mathfrak{b}
\end{aligned}
\tag{3.220}
$$

Demanding nilpotency of $\tilde{Q}$ and $\tilde{Q}_c$ yields the following equations for the μ and λ:

$$
\begin{aligned}
\mu_0^2 - \mu_1\mu_6 - \mu_2\mu_9 &= 0 \\
\mu_3\mu_5 + \mu_4\mu_8 + \mu_{11}\mu_3 &= 0 \\
\mu_3\mu_6 + \mu_4\mu_9 &= 0 \\
\mu_3\mu_7 + \mu_4\mu_{10} + \mu_{11}\mu_4 &= 0 \\
\mu_0\mu_1 - \mu_1\mu_5 - \mu_8\mu_2 - \mu_{12}\mu_3 &= 0 \\
\mu_0\mu_2 - \mu_7\mu_1 - \mu_{10}\mu_2 - \mu_{12}\mu_4 &= 0
\end{aligned}
\tag{3.221}
$$

$$
\begin{aligned}
\lambda_0^2 - \lambda_1\lambda_6 - \lambda_2\lambda_9 &= 0 \\
\lambda_3\lambda_5 + \lambda_4\lambda_8 + \lambda_{11}\lambda_3 &= 0 \\
\lambda_3\lambda_6 + \lambda_4\lambda_9 &= 0 \\
\lambda_3\lambda_7 + \lambda_4\lambda_{10} + \lambda_{11}\lambda_4 &= 0 \\
\lambda_0\lambda_1 - \lambda_1\lambda_5 - \lambda_8\lambda_2 - \lambda_{12}\lambda_3 &= 0 \\
\lambda_0\lambda_2 - \lambda_7\lambda_1 - \lambda_{10}\lambda_2 - \lambda_{12}\lambda_4 &= 0
\end{aligned}
\tag{3.222}
$$

For the most general expression for $\tilde{\tilde{\Xi}}$ with ghost number 0 and conformal weight 1 we find:

$$
\begin{aligned}
\tilde{\tilde{\Xi}} = {} & \rho_0 b_{\mathfrak{a}} c'^{\mathfrak{a}} + \rho_1 b'_{\mathfrak{a}} c^{\mathfrak{a}} + \rho_2 b_\alpha c^\alpha + \rho_3 b_{\mathfrak{a}} c^{\mathfrak{a}} + \rho_4 b'_{\mathfrak{a}} c'^{\mathfrak{a}} \\
& + \rho_5 \pi_{\mathfrak{a}} \varphi^{\mathfrak{a}} + \rho_6 J_{\mathfrak{a}} \varphi^{\mathfrak{a}} + \rho_7 b_{\mathfrak{a}} f^{\mathfrak{a}}{}_{\mathfrak{b}\alpha} c^\alpha \varphi^{\mathfrak{b}} + \rho_8 b'_{\mathfrak{a}} f^{\mathfrak{a}}{}_{\mathfrak{b}\alpha} c^\alpha \varphi^{\mathfrak{b}}
\end{aligned}
\tag{3.223}
$$

If we demand that $\tilde{s}_c\tilde{\tilde{\Xi}}$ yields the full $\tilde{Q}$ apart from the constraint term given by $\lambda_0 J_\alpha c^\alpha$ we get the following set of equations:

$$
\begin{aligned}
-\mu_1 + \rho_1\lambda_2 + \rho_6\lambda_3 + \rho_3\lambda_1 &= 0 \\
-\mu_2 + \rho_0\lambda_1 + \rho_4\lambda_2 + \rho_6\lambda_4 &= 0 \\
-\mu_3 + \rho_1\lambda_4 + \rho_3\lambda_3 + \rho_5\lambda_3 &= 0 \\
-\mu_4 + \rho_0\lambda_3 + \rho_4\lambda_4 + \rho_5\lambda_4 &= 0 \\
\mu_5 - \rho_1\lambda_7 - \rho_2\lambda_5 + \rho_7\lambda_3 + \rho_0\lambda_8 &= 0 \\
\frac{1}{2}\mu_6 + \frac{1}{2}\rho_0\lambda_9 - \rho_2\lambda_6 + \frac{1}{2}\rho_3\lambda_6 &= 0 \\
\mu_7 - \rho_0\lambda_5 + \rho_0\lambda_{10} - \rho_2\lambda_7 - \rho_3\lambda_7 + \rho_7\lambda_4 - \rho_4\lambda_7 &= 0 \\
\mu_8 - \rho_1\lambda_{10} + \rho_1\lambda_5 - \rho_2\lambda_8 - \rho_3\lambda_8 + \rho_4\lambda_8 + \rho_8\lambda_3 &= 0 \\
\frac{1}{2}\mu_9 + \frac{1}{2}\rho_1\lambda_6 - \rho_2\lambda_9 + \frac{1}{2}\rho_4\lambda_9 &= 0 \\
\mu_{10} - \rho_0\lambda_8 + \rho_1\lambda_7 - \rho_2\lambda_{10} + \rho_8\lambda_4 &= 0 \\
-\mu_{11} + \rho_2\lambda_{11} + \rho_7\lambda_3 + \rho_8\lambda_4 &= 0 \\
-\mu_{12} - \rho_2\lambda_{12} - \rho_5\lambda_{12} + \rho_6\lambda_0 + \rho_6\lambda_{11} + \rho_7\lambda_1 + \rho_8\lambda_2 &= 0 \\
\rho_2\lambda_0 &= 0
\end{aligned}
\tag{3.224}
$$

Demanding finally that Q and Q_c anticommute yields:

$$\begin{aligned}
\lambda_0\mu_0 - \frac{1}{2}\lambda_1\mu_6 - \frac{1}{2}\lambda_2\mu_9 - \frac{1}{2}\lambda_9\mu_2 - \frac{1}{2}\lambda_6\mu_1 &= 0\\
\lambda_0\mu_1 + \lambda_1\mu_0 - \lambda_1\mu_5 - \lambda_2\mu_8 - \lambda_5\mu_1 - \lambda_8\mu_2 - \lambda_3\mu_{12} - \lambda_{12}\mu_3 &= 0\\
\lambda_0\mu_2 - \lambda_1\mu_7 + \lambda_2\mu_0 - \lambda_2\mu_{10} - \lambda_4\mu_{12} - \lambda_7\mu_1 - \lambda_{10}\mu_2 - \lambda_{12}\mu_4 &= 0\\
-\lambda_3\mu_{11} + \lambda_3\mu_5 - \lambda_4\mu_8 - \lambda_5\mu_3 - \lambda_8\mu_4 - \lambda_{11}\mu_3 &= 0\\
-\frac{1}{2}\lambda_3\mu_6 - \frac{1}{2}\lambda_4\mu_9 - \frac{1}{2}\lambda_9\mu_4 - \frac{1}{2}\lambda_6\mu_3 &= 0\\
-\lambda_3\mu_7 - \lambda_4\mu_{11} - \lambda_4\mu_{10} - \lambda_7\mu_3 - \lambda_{10}\mu_4 - \lambda_{11}\mu_4 &= 0 \qquad (3.225)
\end{aligned}$$

We have 31 equations and 35 unknowns. It takes too long for Mathematica to find a solution for these equations straight away. Making some reasonable assumptions for some of the constants (preservation of the undeformed Q, unaltered transformations of the J_M, etc.) either brought us to the deformations discussed in the previous section or led to other sets of equations which also have no solution.
One could also try to deform Q such that it squares to 0 up to Q_c–exact terms.

BRST Cohomology and Vertex Operator

After many futile attempts to remove the non–Q_c–exact term from our Q we recently found out that it may not be necessary to deform the BRST charge Q (3.184) at all because the problematic term $b'_{\mathfrak{b}} f^{\mathfrak{b}}{}_{\mathfrak{a}M} c^M c'^{\mathfrak{a}}$ seems to have no influence on the physical spectrum as we will show now.
The massless vertex operator of ghost number one is:

$$\mathcal{U}^{(1)} = (-)^{\alpha} c^{\alpha} A_{\alpha}(x^m,\theta^{\alpha},\varphi^{\mathfrak{a}}) + (-)^{\mathfrak{a}} c^{\mathfrak{a}} W_{\mathfrak{a}}(x^m,\theta^{\alpha},\varphi^{\mathfrak{a}}) + (-)^{\mathfrak{a}} B_{\mathfrak{a}}(x^m,\theta^{\alpha},\varphi^{\mathfrak{a}}) c'^{\mathfrak{a}} \qquad (3.226)$$

According to [37] physical states are defined to lie in the relative cohomology $H(Q|H(Q_c))$ of Q with respect to Q_c, i.e. they are Q_c–closed and Q–closed modulo Q_c–exact terms. Acting with Q_c on the vertex operator yields:

$$\begin{aligned}
s_c\mathcal{U}^{(1)} &= (-)^{\alpha} s_c\varphi^{\mathfrak{b}} \frac{\partial}{\partial\varphi^{\mathfrak{b}}} A_{\alpha}(x^m,\theta^{\alpha},\varphi^{\mathfrak{a}}) c^{\alpha} + (-)^{\mathfrak{a}} s_c\varphi^{\mathfrak{b}} \frac{\partial}{\partial\varphi^{\mathfrak{b}}} A_{\mathfrak{a}}(x^m,\theta^{\alpha},\varphi^{\mathfrak{a}}) c^{\mathfrak{a}}\\
&\quad + (-)^{\mathfrak{a}} s_c\varphi^{\mathfrak{b}} \frac{\partial}{\partial\varphi^{\mathfrak{b}}} B_{\mathfrak{a}}(x^m,\theta^{\alpha},\varphi^{\mathfrak{a}}) c'^{\mathfrak{a}} + B_{\mathfrak{a}} s_c c'^{\mathfrak{a}}\\
&= (-)^{\alpha} c^{\mathfrak{b}} \frac{\partial}{\partial\varphi^{\mathfrak{b}}} A_{\alpha}(x^m,\theta^{\alpha},\varphi^{\mathfrak{a}}) c^{\alpha} + (-)^{\mathfrak{a}} c^{\mathfrak{b}} \frac{\partial}{\partial\varphi^{\mathfrak{b}}} A_{\mathfrak{a}}(x^m,\theta^{\alpha},\varphi^{\mathfrak{a}}) c^{\mathfrak{a}}\\
&\quad + (-)^{\mathfrak{a}} c^{\mathfrak{b}} \frac{\partial}{\partial\varphi^{\mathfrak{b}}} B_{\mathfrak{a}}(x^m,\theta^{\alpha},\varphi^{\mathfrak{a}}) c'^{\mathfrak{a}} + (-)^K \frac{1}{2} B_{\mathfrak{a}}(x^m,\theta^{\alpha},\varphi^{\mathfrak{a}}) f^{\mathfrak{a}}{}_{LK} c^K c^L \qquad (3.227)
\end{aligned}$$

From this we conclude that $B_{\mathfrak{a}} = 0$ and $\frac{\partial}{\partial\varphi^{\mathfrak{b}}} A_{\alpha} = 0$ and $\frac{\partial}{\partial\varphi^{\mathfrak{b}}} A_{\mathfrak{a}} = 0$. Using $s_c\varphi^{\mathfrak{a}} = c^{\mathfrak{a}}$ we can rewrite the vertex operator:

$$\mathcal{U}^{(1)} = (-)^{\alpha} c^{\alpha} A_{\alpha}(x^m,\theta^{\alpha}) + (-)^{\mathfrak{a}} [Q_c, \varphi^{\mathfrak{a}} W_{\mathfrak{a}}(x^m,\theta^{\alpha})] \qquad (3.228)$$

This is Berkovits' vertex operator plus a Q_c–exact term. Now we compute $[Q,\mathcal{U}^{(1)}]$:

$$\begin{aligned}
[Q,\mathcal{U}^{(1)}] &= [J_{\alpha} c^{\alpha}, c^{\beta} A_{\beta}] + (-)^{\alpha+\mathfrak{a}} [J_{\alpha} c^{\alpha}, [Q_c, \varphi^{\mathfrak{a}} W_{\mathfrak{a}}]] + (-)^{\alpha} [[Q_c, \Xi], c^{\alpha} A_{\alpha}]\\
&\quad + (-)^{\mathfrak{a}} [[Q_c, \Xi], [Q_c, \varphi^{\mathfrak{a}} W_{\mathfrak{a}}]] - [b'_{\mathfrak{b}} f^{\mathfrak{b}}{}_{\mathfrak{a}\alpha} c^{\alpha} c'^{\mathfrak{a}}, c^{\beta} A_{\beta}] - (-)^{\alpha+\mathfrak{a}} [b'_{\mathfrak{a}} f^{\mathfrak{b}}{}_{\mathfrak{a}\alpha} c^{\alpha} c'^{\mathfrak{a}}, [Q_c, \varphi^{\mathfrak{c}} W_{\mathfrak{c}}]]
\end{aligned} \qquad (3.229)$$

For the second and the third term we use the Jacobi identity to get Q_c–exact expressions:

$$\begin{aligned}
(-)^{\alpha+\mathfrak{a}}[J_\alpha c^\alpha,[Q_c,\varphi^{\mathfrak{a}}W_{\mathfrak{a}}]]+(-)^{\alpha+\mathfrak{a}+1}[Q_c,[\varphi^{\mathfrak{a}}W_{\mathfrak{a}},J_\alpha c^\alpha]]+(-)^{\alpha+\mathfrak{a}}[\varphi^{\mathfrak{a}}W_{\mathfrak{a}},[J_\alpha c^\alpha,Q_c]] &= 0\\
\rightarrow \qquad [J_\alpha c^\alpha,[Q_c,\varphi^{\mathfrak{a}}W_{\mathfrak{a}}]]-[Q_c,[\varphi^{\mathfrak{a}}W_{\mathfrak{a}},J_\alpha c^\alpha]] &= 0
\end{aligned} \tag{3.230}$$

$$\begin{aligned}
(-)^{\alpha}[[Q_c,\Xi],c^\alpha A_\alpha]+(-)^{\alpha+1}[[\Xi,c^\alpha A_\alpha],Q_c]+(-)^{\alpha+1}[[c^\alpha A_\alpha,Q_c],\Xi] &= 0\\
\rightarrow \qquad [[Q_c,\Xi],c^\alpha A_\alpha]+[Q_c,[\Xi,c^\alpha A_\alpha]] &= 0,
\end{aligned} \tag{3.231}$$

where $[Q_c,J_\alpha c^\alpha]=[Q_c,c^\alpha A_\alpha]=0$. Using $[Q_c,Q_c]=0$ the fourth term in (3.230) can be rewritten as $(-)^{\mathfrak{a}}[Q_c,[\Xi,[Q_c,\varphi^{\mathfrak{a}}W_{\mathfrak{a}}]]]$. The last two terms in (3.230) are zero since both $c^\alpha A_\alpha$ and $c^{\mathfrak{a}}W_{\mathfrak{a}}$ have vanishing commutators with $b'_{\mathfrak{b}}f^{\mathfrak{b}}{}_{\mathfrak{a}\alpha}c^\alpha c'^{\mathfrak{a}}$. Thus, we get:

$$\begin{aligned}
[Q,\mathcal{U}^{(1)}] &= [J_\alpha c^\alpha,c^\beta A_\beta]\\
&\quad +[Q_c,(-)^{\alpha+\mathfrak{a}}[\varphi^{\mathfrak{a}}W_{\mathfrak{a}},J_\alpha c^\alpha]+(-)^{\alpha+\mathfrak{a}+1}[\Xi,c^\alpha A_\alpha]+(-)^{\mathfrak{a}}[\Xi,[Q_c,\varphi^{\mathfrak{a}}W_{\mathfrak{a}}]]]
\end{aligned} \tag{3.232}$$

The first term corresponds to Berkovits' cohomology, all the other terms are Q_c exact, as it is needed to get a relative cohomology which yields the physical spectrum that comes out of the pure–spinor formalism.

3.5.4 Worldsheet Diffeomorphism Invariance

In order to implement worldsheet diffeomorphism invariance one has to gauge the symmetry corresponding to T_{zz}. To get a complete ghost system with vanishing central charge one also has to gauge the fermionic symmetry corresponding to B_{zz}. These calculations were performed explicitly and will be presented in section 3.7. We will use the results derived there since they are known in the literature [38] and because we need them to conclude this section on cohomology.
Having added a quartet $(b'_{zz},c'^z,\beta'_{zz},\gamma'^z)$ to obtain an $N=2$ algebra we add another topological quartet $(b_{zz},c^z,\beta_{zz},\gamma^z)$ to our model. This quartet has the same structure as the one used to construct the $N=2$ superconformal algebra. We have:

$$\begin{aligned}
T^{\mathrm{top}}_{zz} &= 2b_{zz}\partial c^z+\partial b_{zz}c^z+2\beta_{zz}\partial\gamma^z+\partial\beta_{zz}\gamma^z\\
j^{B\mathrm{top}}_z &= b_{zz}\gamma^z\\
B^{\mathrm{top}}_{zz} &= -2\partial c^z\beta_{zz}-c^z\partial\beta_{zz}-\mu b_{zz}\\
j^{\mathrm{gh,top}}_z &= b_{zz}c^z+2\beta_{zz}\gamma^z
\end{aligned} \tag{3.233}$$

The Lagrangian is modified to:

$$\mathcal{L} \;\rightarrow\; b_{zz}\bar\partial c^z+\beta_{zz}\bar\partial\gamma^z, \tag{3.234}$$

and $b_{\bar z\bar z}\partial c^{\bar z}+\beta_{\bar z\bar z}\partial\gamma^{\bar z}$ if one considers the type II superstring. One can add T^{top}_{zz}, $j^{B\mathrm{top}}_c$, B^{top}_{zz} and $j^{\mathrm{gh,top}}_z$ to T_{zz}, j^B_z, B_{zz} and j^{gh}_z to obtain an $N=2$ superconformal algebra with ghost number anomaly -34.
From the gauging and gauge fixing procedure we get the following BRST operator:

$$Q_V = \oint c^z\left(T_{zz}+\frac{1}{2}T^{\mathrm{top}}_{zz}\right)+\gamma^z\left(B_{zz}+\frac{1}{2}B^{\mathrm{top}}_{zz}\right) \tag{3.235}$$

This operator anticommutes with $Q + Q^{\rm top}$.
We can maintain the BRST–exactness of the new energy momentum tensor $T_{zz} + T^{\rm top}_{zz}$:

$$\left[Q + Q^{\rm top} + \oint c^c \left(T_{zz} + \frac{1}{2}T^{\rm top}_{zz}\right) + \gamma^z \left(B_{zz} + \frac{1}{2}B^{\rm top}_{zz}\right), B_{zz} + B^{\rm top}_{zz}\right] = T_{zz} + T^{\rm top}_{zz} \quad (3.236)$$

Now we finally have introduced all the fields we need in our model. Thus, we give the complete expressions for the Lagrangian, the on–shell energy momentum tensor, the composite B–field, the BRST current and the ghost current:

$$\begin{aligned}
\mathcal{L} \;=\;& P^m_z P_{\bar{z}m} - P^m_z \Pi_{\bar{z}m} - \Pi^m_z P_{\bar{z}m} + \frac{1}{2}\Pi^m_z \Pi_{\bar{z}m} + \mathcal{L}_{WZ} + d_{z\alpha}\bar{\partial}\theta^\alpha + \hat{d}_{\bar{z}\hat{\alpha}}\partial\hat{\theta}^{\hat{\alpha}} \\
& - P^{mh}_z P^h_{\bar{z}m} + P^{mh}_z \Pi^h_{\bar{z}m} + \Pi^{mh}_z P^h_{\bar{z}m} - \frac{1}{2}\Pi^{mh}_z \Pi^h_{\bar{z}m} - \mathcal{L}^h_{WZ} - d^h_{z\alpha}\bar{\partial}\theta^{\alpha h} - \hat{d}^h_{\bar{z}\hat{\alpha}}\partial\hat{\theta}^{\hat{\alpha}h} \\
& + \beta_{zm}\bar{\partial}\xi^m + \omega_{z\alpha}\bar{\partial}\lambda^\alpha + \kappa^\alpha_z\bar{\partial}\chi_\alpha + \hat{\beta}_{\bar{z}m}\partial\hat{\xi}^m + \hat{\omega}_{\bar{z}\hat{\alpha}}\partial\hat{\lambda}^{\hat{\alpha}} + \hat{\kappa}^{\hat{\alpha}}_{\bar{z}}\partial\hat{\chi}_{\hat{\alpha}} \\
& + \beta'_{zm}\bar{\partial}\xi'^m + \kappa'^\alpha_z\bar{\partial}\chi'_\alpha + \pi_{zm}\bar{\partial}\varphi^m - \pi^\alpha_z\bar{\partial}\varphi_\alpha \\
& + \hat{\beta}'_{\bar{z}m}\partial\hat{\xi}'^m + \hat{\kappa}^{\hat{\alpha}\prime}_{\bar{z}}\partial\hat{\chi}'_{\hat{\alpha}} + \hat{\pi}_{\bar{z}m}\partial\hat{\varphi}^m - \hat{\pi}^{\hat{\alpha}}_{\bar{z}}\partial\hat{\varphi}_{\hat{\alpha}} \\
& + b'_{zz}\bar{\partial}c'^z + \beta'_{zz}\bar{\partial}\gamma'^z + \hat{b}'_{\bar{z}\bar{z}}\partial\hat{c}'^{\bar{z}} + \hat{\beta}'_{\bar{z}\bar{z}}\partial\hat{\gamma}'^{\bar{z}} \\
& + b_{zz}\bar{\partial}c^z + \beta_{zz}\bar{\partial}\gamma^z + b_{\bar{z}\bar{z}}\partial c^{\bar{z}} + \beta_{\bar{z}\bar{z}}\partial\gamma^{\bar{z}}
\end{aligned} \quad (3.237)$$

$$\begin{aligned}
\mathbf{T}_{zz} \;=\;& -\frac{1}{2}\Pi_{zm}\Pi^m_z + \frac{1}{2}\Pi^h_{zm}\Pi^{mh}_z + d_{z\alpha}\partial\theta^\alpha - d^h_{z\alpha}\partial\theta^{\alpha h} + \beta_{zm}\partial\xi^m + \omega_{z\alpha}\partial\lambda^\alpha + \kappa^\alpha_z\partial\chi_\alpha \\
& + \beta'_{zm}\partial\xi'^m + \kappa'^\alpha_z\partial\chi'_\alpha + \pi_{zm}\varphi^m - \pi^\alpha_z\varphi_\alpha \\
& + 2b'_{zz}\partial c'^z + \partial b'_{zz}c'^z + 2\beta'_{zz}\partial\gamma'^z + \partial\beta'_{zz}\gamma'^z \\
& + 2b_{zz}\partial c^z + \partial b_{zz}c^z + 2\beta_{zz}\partial\gamma^z + \partial\beta_{zz}\gamma^z
\end{aligned} \quad (3.238)$$

$$\begin{aligned}
\mathbf{B}_{zz} \;=\;& -\frac{1}{2}\left(\Pi_{zm} + \Pi^h_{zm}\right)\beta^m_z + \frac{i}{2}\left(id_{z\alpha} + id^h_{z\alpha}\right)\kappa^\alpha_z - \frac{i}{2}\left(\partial\theta^\alpha + \partial\theta^{\alpha h}\right)\omega_{z\alpha} \\
& + \beta'_{zm}\partial\varphi^m - i\kappa'^\alpha_z\partial\varphi_\alpha - 2\beta'_{zz}\partial c'^z - c'^z\partial\beta'_{zz} - b'_{zz} - \frac{1}{2}c'^z F_{zzz} - \frac{1}{2}\gamma'^z\Phi_{zzz} \\
& - 2\beta_{zz}\partial c^z - c^z\partial\beta_{zz}
\end{aligned} \quad (3.239)$$

Note that we set $\mu = 0$ in order to obtain nilpotency.

$$\begin{aligned}
\mathbf{j}^B_z \;=\;& -\left(\Pi_{zm} - \Pi^h_{zm}\right)\xi^m - \left(id_{z\alpha} - id^h_{z\alpha}\right)\lambda^\alpha - \left(\partial\theta^\alpha - \partial\theta^{\alpha h}\right)\chi_\alpha + i\beta_{zm}(\lambda\gamma^m\lambda) + 2(\kappa_z\gamma_m\lambda)\xi^m \\
& - \pi_{zm}\xi'^m + i\pi^\alpha_z\chi'_\alpha + b'_{zz}\gamma'^z + b_{zz}\gamma^z \\
& + c^z\left(T_{zz} + \frac{1}{2}T^{\rm top}_{zz}\right) + \gamma^z\left(B_{zz} + \frac{1}{2}B^{\rm top}_{zz}\right)
\end{aligned} \quad (3.240)$$

$$\mathbf{j}^{\rm gh}_z = \beta_{zm}\xi^m + \omega_{z\alpha}\lambda^\alpha + \kappa^\alpha_z\chi_\alpha + \beta'_{zm}\xi'^m + \kappa'^\alpha_z\chi'_\alpha + b'_{zz}c'^z + 2\beta'_{zz}c^z + b_{zz}c^z + 2\beta_{zz}c^z \quad (3.241)$$

We have two problems to define the cohomology for this model:

1. We have to get rid of the auxiliary h–currents. In gauged WZNW models, as presented for example in [23], these currents remain in the cohomology. In [28] it was proposed to impose the condition $B_0|\mathrm{phys}\rangle = 0$ as an analogue to the Siegel gauge $b_0|\mathrm{phys}\rangle = 0$ of the RNS–string (see for example [39]). It is argued that this condition removes the dependence of the combinations $J_M - J^h_M$ which is not fixed by the BRST charge Q (which fixes $J_M + J^h_M$). Thus, for physical states J_M and J^h_M decouple.

2. We have to construct a $\mathbf{Q}_c$. Using the homotopy (3.177) to define $\mathbf{Q}_c = [\mathbf{Q}, K]$ does not yield a nilpotent $\mathbf{Q}_c$ because of the quadratic and cubic antighost terms in F_{zzz} and Φ_{zzz} in (3.239) whose contribution to $\mathbf{Q}$ does not vanish in $\mathbf{Q}_c^2 = -\frac{1}{2}[\mathbf{Q}, [K, [K, \mathbf{Q}]]]$.
In [41] it was proposed to construct a nilpotent $\mathbf{Q}_c$ by using the bosonization of one of the two quartets. We fermionize the bosonic ghosts as follows [17]:

$$\begin{aligned} \gamma'^z &= \eta'_z e^{-\phi'} & \gamma^z &= \eta_z e^{-\phi} \\ \beta'_{zz} &= \partial\xi' e^{\phi'} & \beta_{zz} &= \partial\xi e^{\phi}, \end{aligned} \qquad (3.242)$$

where the new fields satisfy the OPEs $\xi(z)\eta(w) \sim \frac{1}{z-w}$ and $\phi(z)\phi(w) \sim -\ln(z-w)$, analogously for the primed fields.
Now one can define $\mathbf{Q}_c$ via a similarity transformation:

$$\mathbf{Q}_c = e^{-R} \oint \eta'_z e^R \qquad R = [\mathbf{Q}, \oint \xi' K_z] \qquad [\mathbf{Q}, \oint K_z] = Q_c \qquad (3.243)$$

where Q_c is given by (3.196). $\mathbf{Q}$ remains unchanged under the similarity transformation and $\mathbf{Q}_c$ is of the form $\oint \eta'_z + Q_c + \ldots$. The extra terms $\ldots$ are needed in order that $\mathbf{Q}$ and $\mathbf{Q}_c$ anticommute. This construction is not unique. A different choice using both quartets is:

$$\bar{\mathbf{Q}}_c = e^{-R} \oint (r\eta_z + s\eta'_z) e^R \qquad R = [\mathbf{Q}, \oint \xi' K_z + \xi K^{\mathrm{top}}_z] \qquad K^{\mathrm{top}}_z = c'^z b_{zz} \qquad (3.244)$$

with arbitrary r, s.
The problem with this construction, apart from the fact that it is not unique, is that $\mathbf{Q}_c$ comes out of a similarity transformation of η. Similarity transformations do not change physics. Thus, it is expected that one needs a filtration similar to the grading to define the physical spectrum. Since the second BRST operator was constructed to replace the grading this result is not very satisfying.

3.6 N=4 Algebra

In [41] it was shown that one can use the two topological quartets and the fields from the WZNW model to construct a twisted $N = 4$ superconformal algebra. For this purpose we split the complete expressions for the energy momentum tensor, the composite B–field, the BRST current and the ghost current into the following components:

$$T^W_{zz} = -\frac{1}{2}\Pi_{zm}\Pi^m_z + \frac{1}{2}\Pi^h_{zm}\Pi^{mh}_z + d_{z\alpha}\partial\theta^\alpha - d^h_{z\alpha}\partial\theta^{\alpha h} + \beta_{zm}\partial\xi^m + \omega_{z\alpha}\partial\lambda^\alpha + \kappa^\alpha_z\partial\chi_\alpha$$

$$
\begin{aligned}
T^{\rm co}_{zz} &= \beta'_{zm}\partial\xi'^m + \kappa'^\alpha_z\partial\chi'_\alpha + \pi_{zm}\varphi^m - \pi^\alpha_z\varphi_\alpha \\
T^{K'}_{zz} &= 2b'_{zz}\partial c'^z + \partial b'_{zz}c'^z + 2\beta'_{zz}\partial\gamma'^z + \partial\beta'_{zz}\gamma'^z \\
T^{K}_{zz} &= 2b_{zz}\partial c^z + \partial b_{zz}c^z + 2\beta_{zz}\partial\gamma^z + \partial\beta_{zz}\gamma^z
\end{aligned} \tag{3.245}
$$

$$
\begin{aligned}
\tilde{B}^W_{zz} &= -\frac{1}{2}\left(\Pi_{zm}+\Pi^h_{zm}\right)\beta^m_z + \frac{i}{2}\left(id_{z\alpha}+id^h_{z\alpha}\right)\kappa^\alpha_z - \frac{i}{2}\left(\partial\theta^\alpha+\partial\theta^{\alpha h}\right)\omega_{z\alpha} - \frac{1}{2}c'^z F_{zzz} - \frac{1}{2}\gamma'^z\Phi_{zzz} \\
B^{\rm co}_{zz} &= \beta'_{zm}\partial\varphi^m - i\kappa'^\alpha_z\partial\varphi_\alpha \\
B^{K'}_{zz} &= -2\beta'_{zz}\partial c'^z - c'^z\partial\beta'_{zz} - b'_{zz} \\
B^{K}_{zz} &= -2\beta_{zz}\partial c^z - c^z\partial\beta_{zz} - \mu b_{zz}
\end{aligned} \tag{3.246}
$$

$$
\begin{aligned}
j^{B\,W}_z &= -\left(\Pi_{zm}-\Pi^h_{zm}\right)\xi^m - \left(id_{z\alpha}-id^h_{z\alpha}\right)\lambda^\alpha - \left(\partial\theta^\alpha-\partial\theta^{\alpha h}\right)\chi_\alpha \\
&\quad + i\beta_{zm}(\lambda\gamma^m\lambda) + 2(\kappa_z\gamma_m\lambda)\xi^m \\
j^{B\,\rm co}_z &= -\pi_{zm}\xi'^m + i\pi^\alpha_z\chi'_\alpha \\
j^{B\,K'}_z &= b'_{zz}\gamma'^z \\
j^{B\,K}_z &= b_{zz}\gamma^z
\end{aligned} \tag{3.247}
$$

$$
\begin{aligned}
j^{\rm gh\,W}_z &= \beta_{zm}\xi^m + \omega_{z\alpha}\lambda^\alpha + \kappa^\alpha_z\chi_\alpha \\
j^{\rm gh\,co}_z &= \beta'_{zm}\xi'^m + \kappa'^\alpha_z\chi'_\alpha \\
j^{\rm gh\,K'}_z &= b'_{zz}c'^z + 2\beta'_{zz}c'^z \\
j^{\rm gh\,K}_z &= b_{zz}c^z + 2\beta_{zz}c^z
\end{aligned} \tag{3.248}
$$

Out of these expressions we construct the following $N=4$ superconformal currents:

$$
\begin{aligned}
T_{zz} &= T^W_{zz} + T^{\rm co}_{zz} + T^{K'} + T^K \\
J_{z3} &= j^{\rm gh\,W}_z + j^{\rm gh\,co}_z + j^{\rm gh\,K'}_z + j^{\rm gh\,K}_z \\
G^+_z &= j^{B\,W}_z + j^{B\,\rm co}_z + j^{B\,K'}_z + j^{B\,K}_z \\
G^-_{zz} &= \tilde{B}^W + B^{\rm co} + B^{K'} + B^K \\
J^{++} &= \gamma^z\left(j^{\rm gh\,W}_z + j^{\rm gh\,co}_z + j^{\rm gh\,K'}_z + \frac{1}{2}j^{\rm gh\,K}_z\right) - c^z\left(j^{B\,W}_z + j^{B\,\rm co}_z + j^{B\,K'}_z + \frac{1}{2}j^{B\,K}_z\right) + 17\partial\gamma^z \\
J^{--}_{zz} &= -\beta_{zz} \\
\tilde{G}^+_z &= c^z\left(T^W_{zz} + T^{\rm co}_{zz} + T^{K'} + \frac{1}{2}T^K\right) + \gamma^z\left(\tilde{B}^W + B^{\rm co} + B^{K'} + \frac{1}{2}B^K\right) \\
&\quad -\mu\left(j^{B\,W}_z + j^{B\,\rm co}_z + j^{B\,K'}_z + \frac{1}{2}j^{B\,K}_z\right) - \partial\left(c^z\left(j^{\rm gh\,W}_z + j^{\rm gh\,co}_z + j^{\rm gh\,K'}_z + \frac{1}{2}j^{\rm gh\,K}_z\right)\right) - 17\partial^2 c^z \\
\tilde{G}^-_{zz} &= b_{zz}
\end{aligned} \tag{3.249}
$$

The total derivative terms in J^{++} and $\tilde{G}^+$ are improvement terms which ensure that the correct algebra relations are satisfied. Note that for $\mu = 0$ $\oint\tilde{G}^+$ is the BRST charge that comes in after gauging worldsheet diffeomorphisms and the fermionic symmetry.
These currents satisfy the following algebra[8]:

$$
J_3 J^{\pm\pm}(w) \sim \pm 2\frac{J^{\pm\pm}(w)}{z-w}
$$

[8]The relations were checked using the OPE package [36]. See also Appendix D.

$$
\begin{aligned}
J^{++}(z)J^{--}(w) &\sim \frac{-17}{(z-w)^2} + \frac{J_3(w)}{z-w} \\
J_3(z)J_3(w) &\sim \frac{-34}{(z-w)^2} \\
T(z)J_3(w) &\sim \frac{34}{(z-w)^3} + \frac{J_3(w)}{(z-w)^2} + \frac{\partial J_3(w)}{z-w} \\
J^{++}(z)G^{-}(w) &\sim \frac{-\tilde{G}^{+}(w)}{z-w} \\
J^{++}(z)\tilde{G}^{-}(w) &\sim \frac{-G^{+}(w)}{z-w} \\
J^{--}(z)\tilde{G}^{+}(w) &\sim \frac{-G^{-}(w)}{z-w} \\
J^{--}(z)G^{+}(w) &\sim \frac{-\tilde{G}^{-}(w)}{z-w} \\
J_3(z)G^{\pm}(w) &\sim \pm\frac{G^{\pm}(w)}{z-w} \\
J_3(z)\tilde{G}^{\pm}(w) &\sim \pm\frac{\tilde{G}^{\pm}(w)}{z-w} \\
T(z)\tilde{G}^{+}(w) &\sim \frac{\tilde{G}^{+}(w)}{(z-w)^2} + \frac{\partial\tilde{G}^{+}(w)}{z-w} \\
T(z)\tilde{G}^{-}(w) &\sim \frac{2\tilde{G}^{-}(w)}{(z-w)^2} + \frac{\partial\tilde{G}^{-}(w)}{z-w} \\
G^{\pm}(z)\tilde{G}^{\pm}(w) &\sim \frac{2J^{\pm\pm}(w)}{(z-w)^2} + \frac{\partial J^{\pm\pm}(w)}{z-w} \\
G^{+}(z)G^{-}(w) &\sim \frac{-34}{(z-w)^3} + \frac{J_3(w)}{(z-w)^2} + \frac{T(w)}{z-w} \\
\tilde{G}^{+}(z)\tilde{G}^{-}(w) &\sim \frac{34}{(z-w)^3} + \frac{-J_3(w)}{(z-w)^2} + \frac{-T(w)}{z-w}
\end{aligned} \tag{3.250}
$$

All the other OPEs are regular. Now we make some comments:

- The currents $J^{\pm\pm}$ and J_3 form an $SU(2)$ subalgebra. Thus, we have an $N = 4$ superconformal algebra of the "small" type [42][9].
- The algebra (3.250) is a twisted $N = 4$ superconformal algebra because the $SU(2)$ triplet has spin $(0, 1, 2)$ instead of spin 1.
- (T, J_3, G^+, G^-) and $(T, J_3, \tilde{G}^+, \tilde{G}^-)$ are $N = 2$ multiplets. Both are topological multiplets since the anomalies in TJ_3 and J_3J_3 have opposite signs.
- G^+ and $\tilde{G}^+$, which correspond to the BRST currents in our model, are nilpotent and anticommute.

[9] Apart from the small type there is a large superconformal algebra based on $su(2) \oplus su(2) \oplus u(1)$, a middle type based on $su(2) \oplus u(1) \oplus u(1) \oplus u(1) \oplus u(1)$, an asymmetric one based on $su(2) \oplus u(1) \oplus u(1)$ and a non–reductive type [43].

3.7 Worldsheet Covariant Formulation of the Superstring

In this section we will implement worldsheet diffeomorphism invariance into our action by gauging the symmetry corresponding to the energy momentum tensor via the Noether procedure. Gauge fixing then yields the ghost pair (b_{zz}, c^z). Since we want to maintain a model with vanishing central charge we also need the second pair (β_{zz}, γ^z). This is the motivation to gauge the fermionic symmetry that corresponds to B_{zz}.

The calculations turn out to be very complicated, so our starting point will be a toy model with the following action:

$$S_0^{\text{toy}} = \int -\frac{1}{2}\partial x^m \bar{\partial} x_m + \frac{1}{2}\partial x^{mh} \bar{\partial} x_m^h + \beta_{zm}\bar{\partial}\xi^m \tag{3.251}$$

This is the simplest form of an action that is invariant under global diffeomorphisms and the fermionic symmetry. The diffeomorphism transformations are given by:

$$\begin{aligned}
\delta_T x^m &= c^z \partial x^m + c^{\bar{z}} \bar{\partial} x^m \\
\delta_T x^{hm} &= c^z \partial x^{mh} + c^{\bar{z}} \bar{\partial} x^{mh} \\
\delta_T \xi^m &= c^z \partial \xi^m + c^{\bar{z}} \bar{\partial} \xi^m \\
\delta_T \beta_{zm} &= c^z \partial \beta_{zm} + c^{\bar{z}} \bar{\partial} \beta_{zm} + \partial c^z \beta_{zm} + \partial c^{\bar{z}} \beta_{\bar{z}m}.
\end{aligned} \tag{3.252}$$

Here c is still a commuting parameter.

The transformations of the elementary fields under the fermionic symmetry read:

$$\begin{aligned}
\delta_B x^m &= \frac{1}{2}\gamma^z \beta_z^m \\
\delta_B x^{mh} &= -\frac{1}{2}\gamma^z \beta_z^m \\
\delta_B \xi^m &= \frac{1}{2}\gamma^z \left(\partial x^m + \partial x^{mh}\right),
\end{aligned} \tag{3.253}$$

with anticommuting parameter γ.

The first observation we make is that we cannot gauge both symmetries in one step. The reason for that is the non–closure of the algebra. We compute the commutator of δ_K and δ_B on all the fields:

$$\begin{aligned}
[\delta_B, \delta_T] x^m &= \delta_B \left(c^z \partial x^m + c^{\bar{z}} \bar{\partial} x^m\right) - \delta_T \left(\frac{1}{2}\gamma^z \beta_z^m\right) \\
&= \frac{1}{2} c^z \partial \gamma^z \beta_z^m + \frac{1}{2} c^{\bar{z}} \bar{\partial} \gamma^z \beta_z^m - \frac{1}{2}\gamma^z \partial c^z \beta_z^m - \frac{1}{2}\gamma^z \partial c^{\bar{z}} \beta_{\bar{z}}^m
\end{aligned} \tag{3.254}$$

For $[\delta_B, \delta_T] x^{mh}$ we get the same expression with an overall minus sign. Thus, the combination $[\delta_B, \delta_T](x^m + x^{mh})$ is 0. The first three terms look like a B–transformation of x, the last one, however, is not a B–transformation since the fermionic symmetry transformations do not contain a $\beta_{\bar{z}m}$.

$$\begin{aligned}
[\delta_B, \delta_T] \xi^m &= \delta_B \left(c^z \partial \xi^m + c^{\bar{z}} \bar{\partial} \xi^m\right) - \delta_T \left(\frac{1}{2}\gamma^z (\partial x^m + \partial x^{mh})\right) \\
&= (c^z \partial \gamma^z - \gamma^z \partial c + c^{\bar{z}} \bar{\partial} \gamma^z)(\partial x^m + \partial x^{mh}) - \gamma^z \partial c^{\bar{z}} (\bar{\partial} x^m + \bar{\partial} x^{mh})
\end{aligned} \tag{3.255}$$

Finally, $[\delta_B, \delta_T]\beta_{zm} = 0$. Thus, to gauge both symmetries one has to proceed as follows:

- Gauge diffeomorphisms via the Noether procedure in order to obtain a worldsheet covariant action.
- Find a covariant form of the fermionic symmetry transformations under which the new action is invariant and gauge the covariantized fermionic symmetry.

3.7.1 Gauging Diffeomorphisms

We expect the complete covariant action of the heterotic string to be of the following form[10]:

$$\begin{aligned} S &= \int \sqrt{-g} \ -\frac{1}{2}g^{\mu\nu}\Pi^m_\mu \Pi_{\nu m} - i\varepsilon^{\mu\nu}\Pi_{\mu m}(\theta\gamma^m\partial_\nu\theta) + P^{\mu\nu}d_{\mu\alpha}\nabla_\nu\theta^\alpha \\ &\quad +\frac{1}{2}g^{\mu\nu}\Pi^{mh}_\mu \Pi^h_{\nu m} + i\varepsilon^{\mu\nu}\Pi^h_{\mu m}(\theta^h\gamma^m\partial_\nu\theta^h) + P^{\mu\nu}d^h_{\mu\alpha}\nabla_\nu\theta^{\alpha h} \\ &\quad +P^{\mu\nu}\beta_{\mu m}\nabla_\nu\xi^m + P^{\mu\nu}\omega_{\mu\alpha}\nabla_\nu\lambda^\alpha + P^{\mu\nu}\kappa^\alpha_\mu\nabla_\nu\chi_\alpha \\ &\quad +\frac{1}{2}P^{\mu\nu}P^\lambda{}_\rho\beta'_{\mu\lambda}\nabla_\nu\gamma'^\rho + \frac{1}{2}P^{\mu\nu}P^\lambda{}_\rho b'_{\mu\lambda}\nabla_\nu c'^\rho \end{aligned} \tag{3.256}$$

The $\Pi\Pi$– term has the same structure as the $\partial x\bar{\partial}x$–term of the toy model action. All the WZNW ghost terms transform like $\beta\bar{\partial}\xi$ of the toy model under diffeomorphisms. Since $\sqrt{-g}\varepsilon^{\mu\nu} \propto \epsilon^{\mu\nu}$ the $\varepsilon^{\mu\nu}$–terms in (3.256) directly translate into the flat case and do now have to be covariantized using the Noether procedure. Thus, it is sufficient to gauge the toy model action using the Noether procedure and check in addition that $b'_{zz}\bar{\partial}c'^z$ translates into $\frac{1}{2}P^{\mu\nu}P^\lambda{}_\rho b'_{\mu\lambda}\nabla_\nu c'^\rho$.
From now on we use the convention that Greek letters from the beginning of the alphabet are target space spinor indices and Greek letters from the middle of the alphabet are covariant worldsheet indices.

Worldsheet Diffeomorphism Invariance of the Toy Model

We start with the action (3.251) and compute its variation under local diffeomorphisms (3.252):

$$\delta_1 S_0^{\text{toy}} = \int \bar{\partial}c^z T_{zz} + \partial c^{\bar{z}} T_{\bar{z}\bar{z}} + \bar{\partial}\gamma^z B_{zz}, \tag{3.257}$$

where

$$\begin{aligned} T_{zz} &= -\frac{1}{2}\partial x^m\partial x_m + \frac{1}{2}\partial x^{mh}\partial x^h_m + \beta_{zm}\partial\xi^m \\ T_{\bar{z}\bar{z}} &= -\frac{1}{2}\bar{\partial} x^m\bar{\partial} x_m + \frac{1}{2}\bar{\partial} x^{mh}\bar{\partial} x^h_m + \beta_{\bar{z}m}\bar{\partial}\xi^m \end{aligned} \tag{3.258}$$

Now we introduce gauge connections which we call $\mu_{\bar{z}}{}^z \equiv \mu$ and $\bar{\mu}_z{}^{\bar{z}} \equiv \bar{\mu}$ because we will show that they can be identified with the Beltrami differentials (see Appendix C.1). We add the following term to the action:

$$S_1^{\text{toy}} = -\int \mu_{\bar{z}}{}^z T_{zz} + \bar{\mu}_z{}^{\bar{z}} T_{\bar{z}\bar{z}} \tag{3.259}$$

[10]Throughout this section we will neglect, for simplicity, fields and ghosts that come in through coset gauging. A generalization to the case where these fields are included is straight forward.

We get $\delta_1 S_0^{\text{toy}} + \delta_0 S_1^{\text{toy}} = 0$ if we define:

$$\delta_0 \mu_{\bar{z}}{}^{z} \;=\; \bar{\partial} c^{z} \qquad\qquad \delta_z \bar{\mu}_{z}{}^{\bar{z}} \;=\; \partial c^{\bar{z}} \tag{3.260}$$

Next we compute $\delta_1 S_1^{\text{toy}}$. A lengthy but simple calculation yields:

$$\begin{aligned}
\delta_1 S_1^{\text{toy}} \;=\; & \int \left(-\delta_1 \mu - 2\mu\partial c + \partial(\mu c) + \bar{\partial}(\mu\bar{c})\right)\left(-\frac{1}{2}\partial x^m \partial x_m + \frac{1}{2}\partial x^{mh}\partial x^h_m + \beta_{zm}\partial\xi^m\right)\\
& + \left(-\delta_1\bar{\mu} - 2\bar{\mu}\bar{\partial}\bar{c} + \bar{\partial}(\bar{\mu}\bar{c}) + \partial(\bar{\mu}c)\right)\left(-\frac{1}{2}\bar{\partial} x^m \bar{\partial} x_m + \frac{1}{2}\bar{\partial} x^{mh}\bar{\partial} x^h_m + \beta_{\bar{z}m}\bar{\partial}\xi^m\right)\\
& -2\left(\partial\bar{c}\mu + \bar{\partial}c\bar{\mu}\right)\left(-\frac{1}{2}\partial x^m \bar{\partial} x_m + \frac{1}{2}\partial x^{mh}\bar{\partial} x^h_m\right)\\
& -\mu\partial\bar{c}\left(\beta_{\bar{z}m}\partial\xi^m + \beta_{zm}\bar{\partial}\xi^m\right) - \bar{\mu}\bar{\partial}c\left(\beta_{\bar{z}m}\partial\xi^m + \beta_{zm}\bar{\partial}\xi^m\right)\\
=\; & -\int \left(\delta_1\mu - \partial(\mu c) - \bar{\partial}(\mu\bar{c}) + 2\mu\partial c\right) T_{zz}\\
& + \left(\delta_1\bar{\mu} - \partial(\bar{\mu}c) - \bar{\partial}(\bar{\mu}\bar{c}) + 2\bar{\mu}\bar{\partial}\bar{c}\right) T_{\bar{z}\bar{z}}\\
& + \mu\partial\bar{c}(T_{z\bar{z}} + T_{\bar{z}z}) + \bar{\mu}\bar{\partial}c(T_{z\bar{z}} + T_{\bar{z}z}),
\end{aligned} \tag{3.261}$$

where we used the following abbreviations[11]:

$$\begin{aligned}
T_{z\bar{z}} &= -\frac{1}{2}\partial x^m \bar{\partial} x_m + \frac{1}{2}\partial x^{mh}\bar{\partial} x^h_m + \beta_{zm}\bar{\partial}\xi^m && (3.262)\\
T_{\bar{z}z} &= -\frac{1}{2}\bar{\partial} x^m \partial x_m + \frac{1}{2}\bar{\partial} x^{mh}\partial x^h_m + \beta_{\bar{z}m}\partial\xi^m, && (3.263)
\end{aligned}$$

and $c^z \equiv c$, $c^{\bar{z}} \equiv \bar{c}$. From this we obtain the variations for μ and $\bar{\mu}$:

$$\begin{aligned}
\delta_1\mu &= \partial(\mu c) + \bar{\partial}(\mu\bar{c}) - 2\mu\partial c\\
\delta_1\bar{\mu} &= \partial(\bar{\mu}c) + \bar{\partial}(\bar{\mu}\bar{c}) - 2\bar{\mu}\bar{\partial}\bar{c}
\end{aligned} \tag{3.264}$$

The remaining terms can be compensated by adding the following term to the action:

$$S_2^{\text{toy}} \;=\; \int \mu\bar{\mu}\left(T_{z\bar{z}} + T_{\bar{z}z}\right), \tag{3.265}$$

With that we have achieved $\delta_1 S_1^{\text{toy}} + \delta_0 S_2^{\text{toy}} = 0$. The next order is $\delta_3 S_0^{\text{toy}} + \delta_2 S_1^{\text{toy}} + \delta_1 S_2^{\text{toy}} + \delta_0 S_3^{\text{toy}} = 0$. We find:

$$\delta_1 S_2^{\text{toy}} \;=\; \int 2\bar{\partial}c\mu\bar{\mu}T_{zz} + 2\partial\bar{c}\mu\bar{\mu}T_{\bar{z}\bar{z}} \tag{3.266}$$

Now there are two ways to proceed:

1. We cancel $\delta_1 S_2^{\text{toy}}$ with $\delta_2 S_1^{\text{toy}}$ by defining:

$$\delta_2\mu \;=\; 2\bar{\partial}c\mu\bar{\mu} \qquad\qquad \delta_2\bar{\mu} \;=\; 2\partial\bar{c}\mu\bar{\mu} \tag{3.267}$$

[11] Although it is quite intuitive and practical to use this notation it may be a little misleading since for the flat case $T_{z\bar{z}} = T_{\bar{z}z} = 0$.

With that we get $\delta_2 S_1^{\text{toy}} + \delta_1 S_2^{\text{toy}} = 0$. We get an additional term in the action at order 4 again:

$$\delta_2 S_2^{\text{toy}} = \int 2\bar{\partial}c\mu\bar{\mu}^2(T_{z\bar{z}} + T_{\bar{z}z}) + 2\partial\bar{c}\mu^2\bar{\mu}(T_{z\bar{z}} + T_{\bar{z}z}), \tag{3.268}$$

which can be compensated by $\delta_0 S_4^{\text{toy}}$ where:

$$S_4^{\text{toy}} = -\int \mu^2\bar{\mu}^2(T_{z\bar{z}} + T_{\bar{z}z}) \tag{3.269}$$

2. Introduce an S_3^{toy},

$$S_3^{\text{toy}} = -\int \mu^2\bar{\mu}T_{zz} + \bar{\mu}^2\mu T_{\bar{z}\bar{z}}, \tag{3.270}$$

such that

$$\delta_1 S_2^{\text{toy}} + \delta_0 S_3^{\text{toy}} = -\int \partial\bar{c}\mu T_{zz} + \bar{\partial}c\bar{\mu}T_{\bar{z}\bar{z}}. \tag{3.271}$$

These terms can be compensated by a $\delta_2 S_1$ if we define:

$$\delta_2\mu = -\partial\bar{c}\mu^2 \tag{3.272}$$
$$\delta_2\bar{\mu} = -\bar{\partial}c\bar{\mu}^2 \tag{3.273}$$

The second choice leads directly to the Beltrami parameterization of the covariant action (3.256), as we will show now. The matter part of the toy model action after three steps of the Noether procedure reads:

$$S_{\text{gauged},x}^{\text{toy}} = -\frac{1}{2}\int \partial x^m\bar{\partial}x_m - \mu\partial x^m\partial x_m - \bar{\mu}\bar{\partial}x^m\bar{\partial}x_m + 2\mu\bar{\mu}\partial x^m\bar{\partial}x_m$$
$$-\mu^2\bar{\mu}\partial x^m\partial x_m - \mu\bar{\mu}^2\bar{\partial}x^m\bar{\partial}x_m + \ldots \tag{3.274}$$

For the Beltrami parameterization of the covariant action for the free boson we get, using (C.17) and (C.18):

$$\begin{aligned} S^x &= -\frac{1}{2}\int \sqrt{-g}g^{\mu\nu}\partial_\mu x^m\partial_\nu x_m \\ &= -\int \frac{1}{(1-\mu\bar{\mu})}\left((1+\mu\bar{\mu})\partial x^m\bar{\partial}x_m - \mu\partial x^m\partial x_m - \bar{\mu}\bar{\partial}x^m\bar{\partial}x_m\right) \\ &= -\int \partial x^m\bar{\partial}x_m - \mu\partial x^m\partial x_m - \bar{\mu}\bar{\partial}x^m\bar{\partial}x_m + \mu\bar{\mu}\partial x^m\bar{\partial}x_m \\ &\quad -\mu^2\bar{\mu}\partial x^m\bar{\partial}x_m - \mu\bar{\mu}^2\bar{\partial}x^m\bar{\partial}x_m + O\left((\mu\bar{\mu})^2\right), \end{aligned} \tag{3.275}$$

where we used $\frac{1}{1-x} = 1 + x + x^2 + \ldots$. These results agree up to a factor 2. This factor can be explained as follows: For gauging via the Noether procedure we started with an action in light cone coordinates. The covariant action uses the coordinates (σ^0, σ^1). In our conventions

these coordinates are related by a factor 2.
For the ghost action we get from the Noether procedure:

$$
\begin{aligned}
S^{\text{toy gh}}_{\text{gauged}} &= \int \beta_{zm}\bar{\partial}\xi^m - \mu\beta_{zm}\partial\xi^m - \bar{\mu}\beta_{\bar{z}m}\bar{\partial}\xi^m + \mu\bar{\mu}\beta_{zm}\bar{\partial}\xi^m + \mu\bar{\mu}\beta_{\bar{z}m}\partial\xi^m \\
&\quad -\mu^2\bar{\mu}\beta_{zm}\partial\xi^m - \mu\bar{\mu}^2\beta_{\bar{z}m}\bar{\partial}\xi^m + \ldots \qquad (3.276)
\end{aligned}
$$

Comparing this with the covariant form using (C.19),

$$
\begin{aligned}
S^{\text{gh}} &= \int \sqrt{-g}P^{\mu\nu}\beta_\mu\partial_\nu\xi^m \\
&= \int \frac{2}{1-\mu\bar{\mu}}\left(\beta_{zm}\bar{\partial}\xi^m - \mu\beta_{zm}\partial\xi^m - \bar{\mu}\beta_{\bar{z}m}\bar{\partial}\xi^m + \mu\bar{\mu}\beta_{\bar{z}m}\partial\xi^m\right) \\
&= 2\int \beta_{zm}\bar{\partial}\xi^m - \mu\beta_{zm}\partial\xi^m - \bar{\mu}\beta_{\bar{z}m}\bar{\partial}\xi^m + \mu\bar{\mu}\beta_{zm}\bar{\partial}\xi^m \\
&\quad +\mu^2\bar{\mu}\beta_{zm}\bar{\partial}\xi^m + \mu\bar{\mu}^2\beta_{\bar{z}m}\bar{\partial}\xi^m + O\left((\mu\bar{\mu})^2\right), \qquad (3.277)
\end{aligned}
$$

we find that these results agree.

Worldsheet Diffeomorphism Invariance of the Quartet Terms

We examine the term

$$
S^{\text{quart}} = \int \frac{1}{2}\sqrt{-g}P^{\mu\nu}P^{\lambda}{}_{\rho}b'_{\mu\lambda}\nabla_\nu c'^\rho \qquad (3.278)
$$

of the covariant action. Decomposition in terms of the Beltramis yields:

$$
\begin{aligned}
\frac{1}{2}\sqrt{-g}P^{\mu\nu}P^{\lambda}{}_{\rho}b'_{\mu\lambda}\nabla_\nu c^\rho &= \frac{2}{(1-\mu\bar{\mu})^2}\Big(b'_{zz}\bar{\partial}c'^z - \mu b'_{zz}\partial c'^z + \mu b'_{zz}\bar{\partial}c'^{\bar{z}} - 2\bar{\mu}b'_{z\bar{z}}\bar{\partial}c'^z \\
&\quad -2\mu\bar{\mu}b'_{z\bar{z}}\bar{\partial}c'^{\bar{z}} + \bar{\mu}^2 b'_{\bar{z}\bar{z}}\bar{\partial}c'^z + \mu^2 b'_{zz}\partial c'^{\bar{z}} - 2\mu\bar{\mu}b'_{z\bar{z}}\partial c'^z \\
&\quad -\mu\bar{\mu}^2 b'_{\bar{z}\bar{z}}\bar{\partial}c'^{\bar{z}} + \mu\bar{\mu}^2 b'_{\bar{z}\bar{z}}\partial c'^z + 2\mu^2\bar{\mu}b'_{z\bar{z}}\partial c'^{\bar{z}} \\
&\quad +\mu^2\bar{\mu}^2 b'_{\bar{z}\bar{z}}\partial c'^{\bar{z}}\Big) + \ldots, \qquad (3.279)
\end{aligned}
$$

where ... stands for connection terms coming from the the covariant derivative which is, in contrast to the previous cases, not equal to the ordinary partial derivative.
Now we go to the other end of our problem and gauge diffeomorphisms of the action

$$
S^{\text{quart}}_0 = \int b'_{zz}\bar{\partial}c'^z \qquad (3.280)
$$

Local diffeomorphism variations of the fields are given by (we called the transformation parameters $\varepsilon^z \equiv \varepsilon$ and $\varepsilon^{\bar{z}} \equiv \bar{\varepsilon}$ to avoid confusion with the ghosts from the topological quartets.):

$$
\begin{aligned}
\delta_1 b'_{\mu\nu} &= \varepsilon^\lambda\partial_\lambda b'_{\mu\nu} + \partial_\mu\varepsilon^\lambda b'_{\lambda\nu} + \partial_\nu\varepsilon^\lambda b'_{\mu\lambda} \\
\delta_1 c'^\mu &= \varepsilon^\mu\partial_\nu c'^\mu - \partial_\nu\varepsilon^\mu c'^\nu \qquad (3.281)
\end{aligned}
$$

In particular we get:

$$
\begin{aligned}
\delta_1 b'_{zz} &= \varepsilon\partial b'_{zz} + \bar{\varepsilon}\bar{\partial}b'_{zz} + 2\partial\varepsilon b'_{zz} + 2\partial\bar{\varepsilon}b'_{z\bar{z}} \\
\delta_1 c'^z &= \varepsilon\partial c'^{\bar{z}} + \bar{\varepsilon}\bar{\partial}c'^{\bar{z}} - \partial\bar{\varepsilon}c'^z - \bar{\partial}\bar{\varepsilon}c'^{\bar{z}} \qquad (3.282)
\end{aligned}
$$

Using this we get for the variation of the action:

$$\begin{aligned}
\delta_1 S_0^{\text{quart}} &= \int \left(\varepsilon\partial b'_{zz}c'^z + \bar\varepsilon\bar\partial b'_{zz} + 2\partial\varepsilon b'_{zz} + 2\partial\bar\varepsilon b'_{z\bar z}\right)\bar\partial c'^z + b'_{zz}\bar\partial\left(\varepsilon\partial c'^z + \bar\varepsilon\bar\partial c'^z - \partial\varepsilon c'^z - \partial\varepsilon c'^{\bar z}\right) \\
&= \int \bar\partial\epsilon\left(\partial b'_{zz} + 2b'_{zz}\partial c'^z + \bar\partial b'_{zz}c'^{\bar z}\right) + 2\partial\bar\varepsilon b'_{z\bar z}\bar\partial c'^z \qquad (3.283)
\end{aligned}$$

There are some unexpected issues about this result.

- The term after $\bar\partial\varepsilon$ should yield the corresponding component of the energy-momentum tensor T_{zz}. There is, however, an additional on–shell contribution $\bar\partial b'_{zz}c'^z$.
- The $2b'_{zz}\partial c'^z$ term of T_{zz} does not appear in (3.279), instead there is $-\mu b'_{zz}\partial c'^z + \mu b'_{zz}\bar\partial c'^{\bar z}$. These terms are equivalent by partial integration up to terms with derivatives on the Beltramis. These expressions correspond to contributions from the connection terms coming from the covariant derivatives.

The first step in the Noether procedure can be performed in the standard way. We introduce Beltramis μ and $\bar\mu$ with $\delta_0\mu = \bar\partial\varepsilon$ and $\delta_0\bar\mu = \partial\bar\varepsilon$. Adding

$$S_1 = -\int \mu\left(\partial b'_{zz}c'^z + 2b'_{zz}\partial c'^z + \bar\partial b'_{zz}c'^{\bar z}\right) + 2\bar\mu\, b'_{z\bar z}\bar\partial c'^z \qquad (3.284)$$

to S_0 we obtain $\delta_1 S_0 + \delta_0 S_1 = 0$.
We will skip the second step of the Noether procedure since it turns out to be very tedious. What one can see quickly is that the term $2\bar\mu\partial\bar\varepsilon b'_{\bar z\bar z}\bar\partial c'^z$ coming from $\delta_1(2b'_{z\bar z}\bar\partial c'^z)$ can be compensated by adding a term $\bar\mu^2 b'_{\bar z\bar z}\bar\partial c'^z$ to the action. This is consistent with the $\bar\mu^2$–term in (3.279).

3.7.2 Gauging the Fermionic Symmetry

The fermionic symmetry transformations under which the complete action (3.237) (without $(b_{zz}, c^z, \beta_{zz}, \gamma^z)$ which we want to get out of this calculation) is invariant can be computed by commuting the OPEs of all fields with the complete B–current (3.239):

$$\begin{aligned}
\delta x^m &= \frac{1}{2}\gamma^z\beta^m_z - \frac{i}{2}\gamma^z(\kappa_z\gamma^m\theta) + \frac{i}{4}\gamma^z c'^z(\kappa_z\gamma^m\kappa_z) \\
\delta\Pi^m_z &= \frac{1}{2}\partial(\gamma^z\beta^m_z) - i\gamma^z(\kappa_z\gamma^m\partial\theta) + \frac{i}{4}\partial\left(\gamma^z c'^z(\kappa_z\gamma^m\kappa_z)\right) \\
\delta\partial\theta^\alpha &= \frac{1}{2}\partial(\gamma^z\kappa^\alpha_z) \\
\delta d_{z\alpha} &= -\frac{i}{2}\partial(\gamma^z\omega_{z\alpha}) - i\gamma^z(\gamma_m\kappa_z)_\alpha\Pi^m_z + i\gamma^z\beta^m(\gamma_m\partial\theta)_\alpha \\
&\quad + \frac{i}{2}\partial\left(\gamma^z c'^z(\gamma^m\kappa_z)_\alpha\beta_{zm}\right) - \frac{1}{2}\gamma^z c'^z(\gamma^m\partial\theta)_\alpha(\kappa_z\gamma_m\kappa_z) \qquad (3.285)
\end{aligned}$$

$$\begin{aligned}
\delta x^{mh} &= -\frac{1}{2}\gamma^z\beta^m_z + \frac{i}{2}\gamma^z(\kappa_z\gamma^m\theta^h) + \frac{i}{4}\gamma^z c'^z(\kappa_z\gamma^m\kappa_z) \\
\delta\Pi^{mh}_z &= -\frac{1}{2}\partial(\gamma^z\beta^m_z) + i\gamma^z(\kappa_z\gamma^m\partial\theta^h) + \frac{i}{4}\partial\left(\gamma^z c'^z(\kappa_z\gamma^m\kappa_z)\right)
\end{aligned}$$

$$
\begin{aligned}
\delta\partial\theta^{\alpha h} &= -\frac{1}{2}\partial(\gamma^z\kappa_z^\alpha) \\
\delta d_{z\alpha}^h &= \frac{i}{2}\partial(\gamma^z\omega_{z\alpha}) + i\gamma^z(\gamma_m\kappa_z)_\alpha\Pi_z^{mh} - i\gamma^z\beta^m(\gamma_m\partial\theta^h)_\alpha \\
&\quad + \frac{i}{2}\partial\left(\gamma^z c'^z(\gamma^m\kappa_z)_\alpha\beta_{zm}\right) - \frac{1}{2}\gamma^z c'^z(\gamma^m\partial\theta^h)_\alpha(\kappa_z\gamma_m\kappa_z)
\end{aligned}
\tag{3.286}
$$

For the transformation of $\Pi_{\bar{z}}^m$ one has to replace $\partial \to \bar{\partial}$ in $\delta\Pi_z^m$.

$$
\begin{aligned}
\delta\xi^m &= \frac{1}{2}\gamma^z\left(\Pi_z^m + \Pi_z^{mh}\right) + \frac{i}{2}\gamma^z c'^z\left(\kappa_z\gamma^m\left(\partial\theta - \partial\theta^h\right)\right) + \frac{i}{4}\gamma^z\gamma'^z(\kappa_z\gamma^m\kappa_z) \\
\delta\lambda^\alpha &= -\frac{i}{2}\gamma^z\left(\partial\theta^\alpha + \partial\theta^{\alpha h}\right) \\
\delta\chi_\alpha &= -\frac{1}{2}\gamma^z\left(d_{z\alpha} + d_{z\alpha}^h\right) + i\gamma^z c'^z(\gamma^m\kappa_z)_\alpha\left(\Pi_{zm} - \Pi_{zm}^h\right) - \frac{1}{2}\gamma^z c'^z\beta_{zm}\left(\gamma^m(\partial\theta - \partial\theta^h)\right)_\alpha \\
&\quad - \frac{i}{2}\gamma^z\gamma'^a\beta_{zm}(\gamma^m\kappa_z)_\alpha
\end{aligned}
\tag{3.287}
$$

$$
\delta\beta_{zm} = \delta\kappa_z^\alpha = \delta\omega_{z\alpha} = 0 \tag{3.288}
$$

$$
\begin{aligned}
\delta c'^z &= \gamma^z \\
\delta b'_{zz} &= -2\partial\gamma^z\beta'_{zz} - \gamma^z\partial\beta'_{zz} + \frac{1}{2}\gamma^z F_{zzz} \\
&= -2\partial\gamma^z\beta'_{zz} - \gamma^z\partial\beta'_{zz} - \frac{i}{2}\beta_{zm}\left(\kappa_z\gamma^m(\partial\theta - \partial\theta^h)\right) + \frac{i}{4}\gamma^z(\kappa_z\gamma^m\kappa_z)\left(\Pi_{zm} - \Pi_{zm}^h\right) \\
\delta\gamma'^z &= -\gamma^z\partial c'^z + \partial\gamma^z c'^z \\
\delta\beta'_{zz} &= -\frac{1}{2}\gamma^z\Phi_{zzz} = \frac{i}{4}\gamma^z\beta_{zm}(\kappa_z\gamma^m\kappa_z)
\end{aligned}
\tag{3.289}
$$

Now we have to find the covariant form of these transformations. We could use our intuition and guess the covariant form of the transformations above as we did for the action, but in this case there is more than one way to write them in a covariant form. The best way to get the transformations is to use the Noether theorem backwards. This method to obtain symmetry transformation from a given conserved current was briefly mentioned in section 3.3:

$$
\delta_1 S_0 = \int \frac{\delta S}{\delta\phi^{\mathcal{A}}}\delta\phi^{\mathcal{A}} \stackrel{!}{=} -\int 2\gamma^\nu\nabla^\mu B_{\mu\nu}, \tag{3.290}
$$

where S_0 is given by (3.256).

For this purpose we compute the equations of motion for all the fields. Then we take the right hand side of the above equation and manipulate it until we get an expression proportional to the equations of motion. Comparing the coefficients then yields the correct transformations. In order to perform this inverse Noether method we need the covariant form of the current B_{zz}. Fortunately it turns out that there is a unique way to write down this current in its covariant form:

$$
\begin{aligned}
B_{\mu\nu} &= -\frac{1}{8}P^\lambda{}_\mu P^\rho{}_\nu(\Pi_{\lambda m} + \Pi_{\lambda m}^h)\beta_\rho^m + \frac{i}{8}P^\lambda{}_\mu P^\rho{}_\nu(id_{\lambda\alpha} + id_{\lambda\alpha}^h)\kappa_\rho^\alpha - \frac{i}{8}P^\lambda{}_\mu P^\rho{}_\nu(\nabla_\lambda\theta^\alpha + \nabla_\lambda\theta^{\alpha h})\omega_{\rho\alpha} \\
&\quad - \frac{1}{4}P^\lambda{}_\mu P^\rho{}_\nu P^{\sigma\tau}\beta'_{\lambda\rho}\nabla_\sigma c'_\tau - \frac{1}{8}P^\lambda{}_\mu P^\rho{}_\nu P^{\sigma\tau}c'_\tau\nabla_\sigma\beta'_{\lambda\rho} - \frac{1}{4}P^\lambda{}_\mu P^\rho{}_\nu b'_{\lambda\rho} \\
&\quad - \frac{1}{16}P^\lambda{}_\mu P^\rho{}_\nu P^{\sigma\tau}c'_\tau F_{\sigma\lambda\rho} - \frac{1}{16}P^\lambda{}_\mu P^\rho{}_\nu P^{\sigma\tau}\gamma'_\tau\Phi_{\sigma\lambda\rho}
\end{aligned}
$$

$$
\begin{aligned}
F_{\sigma\lambda\rho} &= -i\beta_{\sigma m}\big(\kappa_\lambda\gamma^m(\nabla_\rho\theta-\nabla_\rho\theta^h)\big)+\frac{i}{2}(\kappa_\sigma\gamma^m\kappa_\lambda)(\Pi_{\rho m}-\Pi^h_{\rho m})\\
\Phi_{\sigma\lambda\tau} &= \frac{i}{2}\beta_{\sigma m}(\kappa_\lambda\gamma^m\kappa_\rho)
\end{aligned}
\qquad (3.291)
$$

Now we compute the equations of motion. We have the following field content:

$$
\phi^{\mathcal{A}} = \Big(x^m,\theta^\alpha,d_{\mu\alpha},x^{mh},\theta^{\alpha h},d^h_{\mu\alpha},\xi^m,\beta_{\mu m},\lambda^\alpha,\omega_{\mu\alpha},\chi_\alpha,\kappa^\alpha_\mu,c'^\rho,b'_{\mu\lambda},\gamma'^\rho,\beta'_{\mu\lambda}\Big) \qquad (3.292)
$$

Note that we used the field $d_{\mu\alpha}$ instead of the elementary field $p_{\mu\alpha}$. This can be done since these fields yield equivalent equations of motion. The advantage is that the d–transformations have a simpler form. The variation of the action with respect to those fields reads:

$$
\frac{\delta S_0}{\delta x^m} = P^{\mu\nu}\nabla_\mu\Pi_{\nu m} \qquad \frac{\delta S_0}{\delta x^{mh}} = -P^{\mu\nu}\nabla_\mu\Pi^h_{\nu m}
$$

$$
\begin{aligned}
\frac{\delta S_0}{\delta\theta^\alpha} &= -2iP^{\mu\nu}\Pi^m_\mu(\gamma_m\nabla_\nu\theta)_\alpha-iP^{\mu\nu}\nabla_\mu\Pi^m_\nu(\gamma_m\theta)_\alpha-P^{\mu\nu}\nabla_\nu d_{\mu\alpha}\\
\frac{\delta S_0}{\delta\theta^{\alpha h}} &= 2iP^{\mu\nu}\Pi^{mh}_\mu(\gamma_m\nabla_\nu\theta^h)_\alpha+iP^{\mu\nu}\nabla_\mu\Pi^{mh}_\nu(\gamma_m\theta^h)_\alpha+P^{\mu\nu}\nabla_\nu d^h_{\mu\alpha}
\end{aligned}
$$

$$
\frac{\delta S_0}{\delta d_{\mu\alpha}} = -P^{\mu\nu}\nabla_\nu\theta^\alpha \qquad \frac{\delta S_0}{\delta d^h_{\mu\alpha}} = P^{\mu\nu}\nabla_\nu\theta^{\alpha h} \qquad (3.293)
$$

$$
\begin{aligned}
\frac{\delta S_0}{\delta\xi^m} &= -P^{\mu\nu}\nabla_\nu\beta_{\mu m} &\quad \frac{\delta S_0}{\delta\lambda^\alpha} &= -P^{\mu\nu}\nabla_\nu\omega_{\mu\alpha} &\quad \frac{\delta S}{\delta\chi_\alpha} &= -P^{\mu\nu}\nabla_\nu\kappa_{\mu\alpha}\\
\frac{\delta S_0}{\delta\beta_{\mu m}} &= -P^{\mu\nu}\nabla_\nu\xi^m &\quad \frac{\delta S_0}{\delta\omega_{\mu\alpha}} &= P^{\mu\nu}\nabla_\nu\lambda^\alpha &\quad \frac{\delta S_0}{\delta\kappa_{\mu\alpha}} &= P^{\mu\nu}\nabla_\nu\chi_\alpha
\end{aligned}
\qquad (3.294)
$$

$$
\begin{aligned}
\frac{\delta S_0}{\delta c'^\rho} &= -\frac{1}{2}P^{\mu\nu}P^\lambda{}_\rho\nabla_\nu b'_{\mu\lambda} &\quad \frac{\delta S_0}{\delta\gamma'^\rho} &= -\frac{1}{2}P^{\mu\nu}P^\lambda{}_\rho\beta'_{\mu\lambda}\\
\frac{\delta S_0}{\delta b'_{\mu\lambda}} &= -\frac{1}{2}P^{\mu\nu}P^\lambda{}_\rho\nabla_\nu c'^\rho &\quad \frac{\delta S_0}{\delta\beta'_{\mu\lambda}} &= -\frac{1}{2}P^{\mu\nu}P^\lambda{}_\rho\nabla_\nu\gamma'^\rho
\end{aligned}
\qquad (3.295)
$$

At first we compute the transformations of the WZNW ghosts. For this purpose we have to collect all the terms in $B_{\mu\nu}$ that contain antighosts. The identity (C.9) was used in the following calculations:

- $\delta\xi^m$: $-\int 2\gamma^\nu\nabla^\mu B_{\mu\nu}\big|_\beta =$

$$
\begin{aligned}
&= -\int\gamma^\nu\Big[-\frac{1}{4}P^\rho{}_\mu P^\lambda{}_\nu(\Pi_{\lambda m}+\Pi^h_{\lambda m})\nabla^\mu\beta^m_\rho+\frac{i}{8}P^\lambda{}_\mu P^\rho{}_\mu P^{\sigma\tau}c'_\tau\nabla^\mu\beta_{\sigma m}\big(\kappa_\lambda\gamma^m(\partial_\rho\theta-\partial_\rho\theta^h)\big)\\
&\qquad -\frac{i}{16}P^\lambda{}_\mu P^\rho{}_\nu P^{\sigma\tau}\gamma'_\tau\nabla^\mu\beta_{\sigma m}(\kappa_\lambda\gamma^m\kappa_\rho)\Big]\\
&= -\int P^\rho{}_\mu\nabla^\mu\beta_{\rho m}\gamma^\nu\Big[\frac{1}{4}P^\lambda{}_\nu(\Pi^m_\lambda+\Pi^{mh}_\lambda)+\frac{i}{8}P^\sigma{}_\nu P^{\lambda\tau}c'_\tau\big(\kappa_\lambda\gamma^m(\partial_\sigma\theta-\partial_\sigma\theta^h)\big)\\
&\qquad +\frac{i}{16}P^\sigma{}_\nu P^{\lambda\tau}\gamma'_\tau(\kappa_\lambda\gamma^m\kappa_\sigma)\Big]
\end{aligned}
\qquad (3.296)
$$

- $\delta\lambda^\alpha$: $-\int 2\gamma^\nu\nabla^\mu B_{\mu\nu}\big|_\omega =$

$$= -\int \gamma^\nu\Big[-\frac{i}{4}P^\lambda{}_\mu P^\rho{}_\nu(\partial_\lambda\theta^\alpha+\partial_\lambda\theta^{\alpha h})\Big]$$
$$= -\int P^{\rho\mu}\nabla_\mu\omega_{\rho\alpha}\Big[-\frac{i}{4}P^\lambda{}_\nu\gamma^\nu(\nabla_\lambda\theta^\alpha+\nabla_\lambda\theta^{\alpha h})\Big] \quad (3.297)$$

- $\delta\chi_\alpha$: $-\int 2\gamma^\nu\nabla^\mu B_{\mu\nu}\big|_\kappa =$

$$= -\int P^\lambda{}_\mu\nabla^\mu\kappa^\alpha_\lambda\,\gamma^\nu\Big[-\frac{1}{4}P^\rho{}_\nu(d_{\rho\alpha}+d^h_{\rho\alpha})+\frac{i}{8}P^\rho{}_\nu P^{\sigma\tau}c'_\tau\beta_{\sigma m}(\gamma^m(\partial_\rho\theta-\partial_\rho\theta^h)_\alpha)$$
$$-\frac{i}{8}P^\rho{}_\nu P^{\sigma\tau}c'_\tau(\gamma^m\kappa_\sigma)_\alpha(\Pi_{\rho m}-\Pi^h_{\rho m})-\frac{i}{8}P^\rho{}_\nu P^{\sigma\tau}\gamma'_\tau\beta_{\sigma m}(\gamma^m\kappa_\rho)_\alpha\Big] \quad (3.298)$$

Next we evaluate the transformations for the quartet ghosts.

- $\delta c'^\nu$: $-\int 2\gamma^\nu\nabla^\mu B_{\mu\nu}\big|_{b'} =$

$$= -\int \gamma^\nu\Big[-\frac{1}{2}P^\lambda{}_\mu P^\rho{}_\nu\nabla^\mu b'_{\lambda\rho}\Big] \quad (3.299)$$

- $\delta\gamma'^\nu$: $-\int 2\gamma^\nu\nabla^\mu B_{\mu\nu}\big|_{\beta'} =$

$$= -\frac{1}{2}\int\gamma^\nu\Big[-P^\lambda{}_\mu P^\rho{}_\nu P^{\sigma\tau}\nabla^\mu\beta'_{\lambda\rho}\nabla_\sigma c'^\tau-P^\lambda{}_\mu P^\rho{}_\nu P^{\sigma\tau}\beta'_{\lambda\rho}\nabla^\mu\nabla_\sigma c'_\tau$$
$$-\frac{1}{2}P^\lambda{}_\mu P^\rho{}_\nu P^{\sigma\tau}\nabla^\mu c'_\tau\nabla_\sigma\beta'_{\lambda\rho}-\frac{1}{2}P^\lambda{}_\mu P^\rho{}_\nu P^{\sigma\tau}c'_\tau\nabla^\mu\nabla_\sigma\beta'_{\lambda\rho}\Big]$$
$$= -\frac{1}{2}\int P^\lambda{}_\mu P^\rho{}_\nu\nabla^\mu\beta'_{\lambda\rho}\Big[-P^{\sigma\tau}\gamma^\nu\nabla_\sigma c'_\tau+\frac{1}{2}P^{\sigma\tau}\nabla_\sigma(\gamma^\nu c'_\tau)\Big]$$
$$+P^\lambda{}_\nu P^{\sigma\tau}\nabla^\mu c'_\tau\Big[-P^\rho{}_\nu\nabla_\sigma(\gamma^\nu\beta'_{\lambda\rho})+\frac{1}{2}P^\rho{}_\nu\gamma^\nu\nabla_\sigma\beta'_{\lambda\rho}\Big] \quad (3.300)$$

The expression in the last line belongs to $\delta b'_{\sigma\lambda}$. It enters this calculation because we used partial integration to get the second equality. There is another difficulty in this calculation. Curvature terms arise since $\nabla^\mu\nabla_\sigma c'_\tau=\nabla_\sigma\nabla^\mu c'_\tau+[\nabla^\mu,\nabla_\sigma]c'_\tau=\nabla_\sigma\nabla^\mu c'_\tau+R_{\tau\eta}{}^\mu{}_\sigma c'^\eta$ and $\nabla^\mu\nabla_\sigma\beta'_{\lambda\rho}=\nabla_\sigma\nabla^\mu\beta'_{\lambda\rho}+R_\lambda{}^{\eta\mu}{}_\sigma\beta'_{\eta\rho}+R_\rho{}^{\eta\mu}{}_\sigma\beta'_{\lambda\eta}$ but these terms cancel. One can check this using $R^\mu{}_{\nu\lambda\rho}=-\frac{R}{2}\varepsilon^\mu{}_\nu\varepsilon_{\lambda\rho}$, which holds in two dimensions, and (C.9).

- $\delta b'_{\sigma\lambda}$: $-\int 2\gamma^\nu\nabla^\mu B_{\mu\nu}\big|_{c'} =$

$$= -\frac{1}{2}\int P^\lambda{}_\mu P^{\sigma\tau}\nabla^\mu c'_\tau\Big[-P^\rho{}_\nu\nabla_\sigma(\gamma^\nu\beta'_{\lambda\rho})+\frac{1}{2}P^\rho{}_\nu\gamma^\nu\nabla_\sigma\beta'_{\lambda\rho}+\frac{1}{4}\gamma^\nu P^\rho{}_\nu F_{\sigma\lambda\rho}\Big]$$
$$= -\frac{1}{2}\int P^\lambda{}_\mu P^{\sigma\tau}\nabla^\mu c'_\tau\Big[-P^\rho{}_\nu\nabla_\sigma(\gamma^\nu\beta'_{\lambda\rho})+\frac{1}{2}P^\rho{}_\nu\gamma^\nu\nabla_\sigma\beta'_{\lambda\rho}$$
$$-\frac{i}{4}P^\rho{}_\nu\gamma^\nu\beta_{\sigma m}\big(\kappa_\lambda\gamma^m(\partial_\rho\theta-\partial_\rho\theta^h)\big)+\frac{i}{8}P^\rho{}_\nu\gamma^\nu(\kappa_\sigma\gamma^m\kappa_\lambda)(\Pi_{\rho m}-\Pi^h_{\rho m})\Big] \quad (3.301)$$

- $\delta\beta'_{\sigma\lambda}$: $-\int 2\gamma^\nu\nabla^\mu B_{\mu\nu}\big|_{\gamma'} =$

$$
\begin{aligned}
&= \int \gamma^\nu\left[\frac{1}{8}P^\lambda{}_\mu P^\rho{}_\nu P^{\sigma\tau}\nabla^\mu\gamma'_\tau\Phi_{\sigma\lambda\rho}\right] \\
&= \frac{1}{2}\int P^\lambda{}_\mu P^{\sigma\tau}\nabla^\mu\gamma'_\tau\left[\frac{i}{8}P^\rho{}_\nu\gamma^\nu(\kappa_\lambda\gamma^m\kappa_\rho)\right] \qquad (3.302)
\end{aligned}
$$

Finally we come to the elementary fields. We start with $\delta\theta^{\alpha(h)}$ and collect all the terms that are proportional to $d^{(h)}_{\mu\alpha}$:

$$
\begin{aligned}
&-\int \gamma^\nu\frac{i}{4}P^\lambda{}_\mu P^\rho{}_\nu\nabla^\mu(id_{\lambda\alpha}+id^h_{\lambda\alpha})\kappa^\alpha_\rho \\
&= \int -P^\lambda_\mu\nabla^\mu d_{\lambda\alpha}\left(\frac{1}{4}P^\rho{}_\nu\gamma^\nu\kappa^\alpha_\rho\right)+P^\lambda_\mu\nabla^\mu d^h_{\lambda\alpha}\left(-\frac{1}{4}P^\rho{}_\nu\gamma^\nu\kappa^\alpha_\rho\right) \qquad (3.303)
\end{aligned}
$$

From this we get the correct transformations for $\theta^{\alpha(h)}$ but the prefactor that should be proportional to the equations of motion is incomplete. The missing terms are given by:

$$
\begin{aligned}
\left(\frac{\delta S_0}{\delta\theta^\alpha}+P^{\mu\nu}\nabla_\nu d_{\mu\alpha}\right)\delta\theta^\alpha &= \left(-2iP^{\mu\nu}\Pi^m_\mu(\gamma_m\partial_\nu\theta)_\alpha - iP^{\mu\nu}\nabla_\mu\Pi^m_\nu(\gamma_m\theta)_\alpha\right)\left(\frac{1}{4}P^\rho{}_\lambda\gamma^\lambda\kappa^\alpha_\rho\right) \\
&= \frac{i}{2}P^{\mu\nu}P^\rho{}_\lambda\gamma^\lambda\Pi^m_\mu(\kappa_\rho\gamma_m\nabla_\nu\theta)+\frac{i}{4}P^{\mu\nu}P^\rho{}_\lambda\gamma^\lambda\nabla_\mu\Pi^m_\nu(\kappa_\rho\gamma_m\theta)
\end{aligned}
$$
(3.304)

We add and subtract those terms. This repairs the equation for $\delta\theta^{\alpha(h)}$ but the new terms will contribute to $\delta x^{m(h)}$ and $\delta d^{(h)}_{\mu\alpha}$.
Next we consider $\delta x^{m(h)}$ where we collect the terms proportional to $P^{\mu\nu}\nabla_\mu\Pi^{m(h)}_\nu$. One of the additional terms is included in the first line of the following equation:

$$
\begin{aligned}
&\int P^{\mu\nu}\nabla_\mu\Pi^m_\nu\left(-\frac{i}{4}P^\rho{}_\lambda\gamma^\lambda(\kappa_\rho\gamma_m\theta)\right)-P^{\mu\nu}\nabla_\mu\Pi^{mh}_\nu\left(\frac{i}{4}P^\rho{}_\lambda\gamma^\lambda(\kappa_\rho\gamma_m\theta^h)\right) \\
&\quad+\frac{1}{4}P^\lambda{}_\mu P^\rho{}_\lambda\gamma^\nu\nabla^\mu(\Pi_{\lambda m}+\Pi^h_{\lambda m})\beta^m_\rho+\frac{i}{16}P^\lambda{}_\mu P^\rho{}_\lambda\gamma^\nu P^{\sigma\tau}\gamma^\nu c'_\tau(\kappa_\sigma\gamma^m\kappa_\lambda)\nabla^\mu(\Pi_{\rho m}-\Pi^h_{\rho m}) \\
=&\int P^{\mu\nu}\nabla_\mu\Pi^m_\nu\left[-\frac{i}{4}P^\rho{}_\nu\gamma^\nu(\kappa_\rho\gamma_m\theta)+\frac{1}{4}P^\rho{}_\nu\gamma^\nu\beta_{\rho m}+\frac{i}{16}P^\lambda{}_\nu P^{\sigma\tau}\gamma^\nu c'_\tau(\kappa_\sigma\gamma^m\kappa_\lambda)\right] \\
&\quad-P^{\mu\nu}\nabla_\mu\Pi^{mh}_\nu\left[\frac{i}{4}P^\rho{}_\nu\gamma^\nu(\kappa_\rho\gamma_m\theta^h)-\frac{1}{4}P^\rho{}_\nu\gamma^\nu\beta_{\rho m}+\frac{i}{16}P^\lambda{}_\nu P^{\sigma\tau}\gamma^\nu c'_\tau(\kappa_\sigma\gamma^m\kappa_\lambda)\right] \\
&\quad+\frac{i}{2}\varepsilon^{\mu\nu}P^\rho{}_\nu\gamma^\nu\left(\nabla_\mu\theta\gamma_m\nabla_\lambda\theta+\nabla_\mu\theta^h\gamma_m\nabla_\lambda\theta^h\right)\beta^m_\rho \\
&\quad-\frac{1}{8}\varepsilon^{\mu\rho}P^\lambda{}_\nu P^{\sigma\tau}\gamma^\nu c'_\tau(\kappa_\sigma\gamma^m\kappa_\lambda)\left(\nabla_\mu\theta\gamma_m\nabla_\rho\theta-\nabla_\mu\theta^h\gamma_m\nabla_\rho\theta^h\right) \qquad (3.305)
\end{aligned}
$$

The last terms had to be added in order to make the transformations complete. Finally we get for $\delta^{(h)}_{\mu\alpha}$, collecting the terms proportional to $-P^{\mu\nu}\nabla_\nu\theta^\alpha$ and subtracting the terms we added to complete the other transformations:

$$
\int -\frac{i}{2}P^{\mu\nu}P^\rho{}_\lambda\gamma^\lambda\Pi^m_\mu(\kappa_\rho\gamma_m\nabla_\nu\theta)-\frac{i}{2}P^{\mu\nu}P^\rho{}_\lambda\gamma^\lambda\Pi^{mh}_\mu(\kappa_\rho\gamma_m\nabla_\nu\theta^h)
$$

$$
\begin{aligned}
&-P^{\mu\lambda}\nabla_\lambda\theta^\alpha\left(\frac{i}{2}P^\rho{}_\nu\gamma^\nu(\gamma_m\nabla_\mu\theta)_\alpha\beta^m_\rho\right)-P^{\mu\rho}\nabla_\rho\theta^\alpha\left(\frac{1}{8}P^\lambda{}_\nu P^{\sigma\tau}\gamma^\nu c'_\tau(\kappa_\sigma\gamma^m\kappa_\lambda)(\gamma_m\nabla_\mu\theta)_\alpha\right)\\
&+P^{\mu\lambda}\nabla_\lambda\theta^{\alpha h}\left(-\frac{i}{2}P^\rho{}_\nu\gamma^\nu(\gamma_m\nabla_\mu\theta^h)_\alpha\beta^m_\rho\right)+P^{\mu\rho}\nabla_\rho\theta^{\alpha h}\left(\frac{1}{8}P^\lambda{}_\nu P^{\sigma\tau}\gamma^\nu c'_\tau(\kappa_\sigma\gamma^m\kappa_\lambda)(\gamma_m\nabla_\mu\theta^h)_\alpha\right)\\
&+\frac{i}{4}P^\lambda{}_\mu P^\rho{}_\nu\gamma^\nu\big(\nabla^\mu\nabla_\lambda\theta^\alpha+\nabla^\mu\nabla_\lambda\theta^{\alpha h}\big)\omega_{\rho\alpha}-\frac{i}{8}P^\lambda{}_\mu P^\rho{}_\nu\gamma^\nu P^{\sigma\tau}\gamma^\nu c'_\tau\beta_{\sigma m}\big(\kappa_\lambda\gamma^m(\nabla^\mu\nabla_\rho\theta-\nabla^\mu\nabla_\rho\theta^h)\big)\\
=\ &\int -P^{\mu\nu}\nabla_\nu\theta^\alpha\Big[-\frac{i}{2}P^\rho{}_\lambda\gamma^\lambda\Pi^m_\mu(\gamma_m\kappa_\rho)_\alpha-\frac{i}{2}P^\rho{}_\lambda\gamma^\lambda(\gamma^m\nabla_\mu\theta)_\alpha\beta^m_\rho-\frac{1}{8}P^\lambda{}_\rho P^{\sigma\tau}\gamma^\rho c'_\tau(\kappa_\sigma\gamma^m\kappa_\lambda)(\gamma_m\nabla_\mu\theta)_\alpha\\
&-\frac{i}{4}P^\rho{}_\nu\nabla_\mu(\gamma^\nu\omega_{\rho\alpha})+\frac{i}{8}P^\lambda{}_\nu P^{\sigma\tau}\nabla_\mu\big(\gamma^\nu c'_\tau\beta_{\sigma m}(\gamma^m\kappa_\lambda)_\alpha\big)\Big]\\
&+P^{\mu\nu}\nabla_\nu\theta^{\alpha h}\Big[\frac{i}{2}P^\rho{}_\lambda\gamma^\lambda\Pi^{mh}_\mu(\gamma_m\kappa_\rho)_\alpha+\frac{i}{2}P^\rho{}_\lambda\gamma^\lambda(\gamma^m\nabla_\mu\theta^h)_\alpha\beta^m_\rho-\frac{1}{8}P^\lambda{}_\rho P^{\sigma\tau}\gamma^\rho c'_\tau(\kappa_\sigma\gamma^m\kappa_\lambda)(\gamma_m\nabla_\mu\theta^h)_\alpha\\
&+\frac{i}{4}P^\rho{}_\nu\nabla_\mu(\gamma^\nu\omega_{\rho\alpha})+\frac{i}{8}P^\lambda{}_\nu P^{\sigma\tau}\nabla_\mu\big(\gamma^\nu c'_\tau\beta_{\sigma m}(\gamma^m\kappa_\lambda)_\alpha\big)\Big]
\end{aligned}
\tag{3.306}
$$

Now we summarize our results for the covariantized fermionic symmetry transformations:

$$
\begin{aligned}
\delta x^m &= -\frac{i}{4}P^\rho{}_\nu\gamma^\nu(\kappa_\rho\gamma^m\theta)+\frac{1}{4}P^\rho{}_\nu\gamma^\nu\beta^m_\rho+\frac{i}{16}P^\rho{}_\nu P^{\sigma\tau}\gamma^\nu c'_\tau(\kappa_\sigma\gamma^m\kappa_\rho)\\
\delta\Pi^m_\mu &= -\frac{i}{2}P^\rho{}_\nu\gamma^\nu(\kappa_\rho\gamma^m\nabla_\mu\theta)+\frac{1}{4}P^\rho{}_\nu\nabla_\mu(\gamma^\nu\beta^m_\rho)+\frac{i}{16}P^\rho{}_\nu P^{\sigma\tau}\nabla_\mu\big(\gamma^\nu c'_\tau(\kappa_\sigma\gamma^m\kappa_\rho)\big)\\
\delta\theta^\alpha &= \frac{1}{4}P^\rho{}_\nu\gamma^\nu\kappa^\alpha_\rho\\
\delta d_{\mu\alpha} &= -\frac{i}{2}P^\rho{}_\nu\gamma^\nu\Pi^m_\mu(\gamma_m\kappa_\rho)_\alpha-\frac{i}{2}P^\rho{}_\nu\gamma^\nu(\gamma_m\nabla_\mu\theta)_\alpha\beta^m_\rho-\frac{1}{8}P^\rho{}_\nu P^{\sigma\tau}\gamma^\nu c'_\tau(\kappa_\sigma\gamma^m\kappa_\rho)(\gamma_m\nabla_\mu\theta)_\alpha\\
&\quad-\frac{i}{4}P^\rho{}_\nu\nabla_\mu(\gamma^\nu\omega_{\rho\alpha})+\frac{i}{8}P^\rho{}_\nu P^{\sigma\tau}\nabla_\mu\big(\gamma^\nu c'_\tau\beta_{\sigma m}(\gamma^m\kappa_\rho)_\alpha\big)
\end{aligned}
\tag{3.307}
$$

$$
\begin{aligned}
\delta x^{mh} &= \frac{i}{4}P^\rho{}_\nu\gamma^\nu(\kappa_\rho\gamma^m\theta^h)-\frac{1}{4}P^\rho{}_\nu\gamma^\nu\beta^m_\rho+\frac{i}{16}P^\rho{}_\nu P^{\sigma\tau}\gamma^\nu c'_\tau(\kappa_\sigma\gamma^m\kappa_\rho)\\
\delta\Pi^{mh}_\mu &= \frac{i}{2}P^\rho{}_\nu\gamma^\nu(\kappa_\rho\gamma^m\nabla_\mu\theta^h)-\frac{1}{4}P^\rho{}_\nu\nabla_\mu(\gamma^\nu\beta^m_\rho)+\frac{i}{16}P^\rho{}_\nu P^{\sigma\tau}\nabla_\mu\big(\gamma^\nu c'_\tau(\kappa_\sigma\gamma^m\kappa_\rho)\big)\\
\delta\theta^{\alpha h} &= -\frac{1}{4}P^\rho{}_\nu\gamma^\nu\kappa^\alpha_\rho\\
\delta d^h_{\mu\alpha} &= \frac{i}{2}P^\rho{}_\nu\gamma^\nu\Pi^{mh}_\mu(\gamma_m\kappa_\rho)_\alpha+\frac{i}{2}P^\rho{}_\nu\gamma^\nu(\gamma_m\nabla_\mu\theta^h)_\alpha\beta^m_\rho-\frac{1}{8}P^\rho{}_\nu P^{\sigma\tau}\gamma^\nu c'_\tau(\kappa_\sigma\gamma^m\kappa_\rho(\gamma_m\nabla_\mu\theta^h)_\alpha\\
&\quad+\frac{i}{4}P^\rho{}_\nu\nabla_\mu(\gamma^\nu\omega_{\rho\alpha})+\frac{i}{8}P^\rho{}_\nu P^{\sigma\tau}\nabla_\mu\big(\gamma^\nu c'_\tau\beta_{\sigma m}(\gamma^m\kappa_\rho)_\alpha\big)
\end{aligned}
\tag{3.308}
$$

$$
\begin{aligned}
\delta\xi^m &= \frac{1}{4}P^\lambda{}_\nu\gamma^\nu(\Pi^m_\lambda+\Pi^{hm}_\lambda)+\frac{i}{8}P^\rho{}_\nu P^{\lambda\tau}\gamma^\nu c'_\tau\big(\kappa_\lambda\gamma^m(\nabla_\rho\theta-\nabla_\rho\theta^h)\big)+\frac{i}{16}P^\rho{}_\nu P^{\lambda\tau}\gamma^\nu\gamma'_\tau(\kappa_\lambda\gamma^m\kappa_\rho)\\
\delta\lambda^\alpha &= -\frac{i}{4}P^\rho{}_\nu\gamma^\nu(\nabla_\rho\theta^\alpha-\nabla_\rho\theta^{\alpha h})\\
\delta\chi_\alpha &= -\frac{1}{4}P^\rho{}_\nu\gamma^\nu(d_{\rho\alpha}+d^h_{\rho\alpha})+\frac{i}{8}P^\rho{}_\nu P^{\sigma\tau}\gamma^\nu c'_\tau\beta_{\sigma m}\big(\gamma^m(\nabla_\rho\theta-\nabla_\rho\theta^h)\big)_\alpha\\
&\quad-\frac{i}{8}P^\rho{}_\nu P^{\sigma\tau}\gamma^\nu c'_\tau(\gamma^m\kappa_\sigma)_\alpha(\Pi_{\rho m}-\Pi^h_{\rho m})-\frac{i}{8}P^\rho{}_\nu P^{\sigma\tau}\gamma^\nu\gamma'_\tau\beta_{\sigma m}(\gamma^m\kappa_\rho)_\alpha
\end{aligned}
\tag{3.309}
$$

$$
\begin{aligned}
\delta c'^{\nu} &= \gamma^{\nu} \\
\delta b'_{\sigma\lambda} &= -P^{\rho}{}_{\nu}\nabla_{\sigma}(\gamma^{\nu}\beta'_{\lambda\rho}) + \frac{1}{2}P^{\rho}{}_{\nu}\gamma^{\nu}\nabla_{\sigma}\beta'_{\lambda\rho} - \frac{i}{4}P^{\rho}{}_{\nu}\gamma^{\nu}\beta_{\sigma m}\big(\kappa_{\lambda}\gamma^{m}(\nabla_{\rho}\theta - \nabla_{\rho}\theta^{h})\big) \\
&\quad + \frac{i}{8}P^{\rho}{}_{\nu}\gamma^{\nu}(\kappa_{\sigma}\gamma^{m}\kappa_{\lambda})(\Pi_{\rho m} - \Pi^{h}_{\rho m}) \\
\delta\gamma'^{\nu} &= -P^{\sigma\tau}\gamma^{\nu}\nabla_{\sigma}c'_{\tau} + \frac{1}{2}P^{\sigma\tau}\nabla_{\sigma}(\gamma^{\nu}c'_{\tau}) \\
\delta\beta'_{\sigma\lambda} &= \frac{i}{8}P^{\rho}{}_{\nu}\gamma^{\nu}\beta_{\sigma m}(\kappa_{\lambda}\gamma^{m}\kappa_{\rho})
\end{aligned}
\tag{3.310}
$$

In a lengthy calculation one can show that $B_{\mu\nu}$ is still nilpotent, i.e. that the fermionic variation of the B–current is 0. This makes the Noether procedure very simple since it terminates after the first step. So far, the variation of the action under the fermionic transformation is:

$$
\delta S_0 = \int 2\nabla_{\mu}\gamma^{\nu}B^{\mu}{}_{\nu} \tag{3.311}
$$

We add the following term to the action:

$$
S_1 = \int -2A_{\mu}{}^{\nu}B^{\mu}{}_{\nu} \tag{3.312}
$$

The new action is invariant if we define:

$$
\delta_0 A_{\mu}{}^{\nu} = \nabla_{\mu}\gamma^{\nu} \tag{3.313}
$$

No further steps are needed. Thus, the complete gauged action is given by:

$$
\begin{aligned}
S = \int \sqrt{-g}\ & -\frac{1}{2}g^{\mu\nu}\Pi^{m}_{\mu}\Pi_{\nu m} - i\varepsilon^{\mu\nu}\Pi_{\mu m}(\theta\gamma^{m}\nabla_{\nu}\theta) + P^{\mu\nu}d_{\mu\alpha}\nabla_{\nu}\theta_{\alpha} \\
& + \frac{1}{2}g^{\mu\nu}\Pi^{mh}_{\mu}\Pi^{h}_{\nu m} + i\varepsilon^{\mu\nu}\Pi^{h}_{\mu m}(\theta^{h}\gamma^{m}\nabla_{\nu}\theta^{h}) - P^{\mu\nu}d^{h}_{\mu\alpha}\nabla_{\nu}\theta^{h}_{\alpha} \\
& + P^{\mu\nu}\beta_{\mu m}\nabla_{\nu}\xi^{m} + P^{\mu\nu}\omega_{\mu\alpha}\nabla_{\nu}\lambda^{\alpha} + P^{\mu\nu}\kappa^{\alpha}_{\mu}\nabla_{\nu}\chi_{\alpha} \\
& + \frac{1}{2}P^{\mu\nu}P^{\lambda}{}_{\rho}\beta'_{\mu\lambda}\nabla_{\nu}\gamma'^{\rho} + \frac{1}{2}P^{\mu\nu}P^{\lambda}{}_{\rho}b'_{\mu\lambda}\nabla_{\nu}c'^{\rho} - 2A_{\mu}{}^{\nu}B^{\mu}{}_{\nu}
\end{aligned}
\tag{3.314}
$$

3.7.3 WZNW BRST Symmetry

Having gauged diffeomorphisms and the fermionic symmetry the new gauged action is no longer invariant under BRST transformations. As we will demonstrate now, BRST invariance can be reestablished by introducing a BRST transformation of the metric. Due to the complexity of the calculations we only consider the toy model which we introduced above. The gauged action is given by:

$$
S^{\text{toy}}_{\text{gauged}} = \int \sqrt{-g}\left[-\frac{1}{2}g^{\mu\nu}\nabla_{\mu}x^{m}\nabla_{\nu}x_{m} + \frac{1}{2}g^{\mu\nu}\nabla_{\mu}x^{mh}\nabla_{\nu}x^{h}_{m} + P^{\mu\nu}\beta_{\mu m}\nabla_{\nu}\xi^{m} - 2A_{\nu}{}^{\rho}B_{\rho}{}^{\nu}\right] \tag{3.315}
$$

with

$$
B_{\rho\nu} = -\frac{1}{8}P^{\mu}{}_{\nu}P^{\lambda}{}_{\rho}\beta_{\mu m}\left(\nabla_{\lambda}x^{m} + \nabla_{\lambda}x^{mh}\right). \tag{3.316}
$$

Note that we do not need a topological quartet for this model since it already has an $N = 2$ superconformal algebra. The covariant form of the BRST transformations looks as follows:

$$\begin{aligned} sx^m &= \xi^m \\ sx^{mh} &= \xi^m \\ s\xi^m &= 0 \\ s\beta_{\mu m} &= \left(\nabla_\mu x_m - \nabla_\mu x^h_m\right) \end{aligned} \tag{3.317}$$

The BRST variation of $B_{\rho\nu}$ then reads:

$$sB_{\rho\nu} = \frac{1}{4}P^{(\mu}{}_\nu P^{\lambda)}{}_\rho \left(-\frac{1}{2}\nabla_{(\mu}x_m\nabla_{\lambda)}x^m + \frac{1}{2}\nabla_{(\mu}x^h_m\nabla_{\lambda)}x^{mh} + \beta_{(\mu m}\nabla_{\lambda)}\xi^m\right) \tag{3.318}$$

This expression looks similar to the energy–momentum tensor which reads:

$$\begin{aligned} T_{\sigma\tau} &= \frac{1}{\sqrt{-g}}\frac{\delta S}{\delta g^{\sigma\tau}} \\ &= \frac{1}{2}\left(2\delta^\mu_\sigma\delta^\nu_\tau - g_{\sigma\tau}g^{\mu\nu}\right)\Big(-\frac{1}{2}\nabla_\mu x^m\nabla_\nu x_m + \frac{1}{2}\nabla_\mu x^{mh}\nabla_\nu x^h_m + \beta_{\mu m}\nabla_\nu\xi^m \\ &\quad + \frac{1}{4}P^\lambda{}_\rho A_\nu{}^\rho\beta_{\mu m}(\nabla_\lambda x^m + \nabla_\lambda x^{mh})\Big) + \frac{1}{8}\left(2g_{\sigma\rho}\varepsilon^\lambda{}_\tau - g_{\sigma\tau}\varepsilon^\lambda{}_\rho\right)P^{\mu\nu}A_\nu{}^\rho\beta_{\mu m}(\nabla_\lambda x^m + \nabla_\lambda x^{mh}) \end{aligned} \tag{3.319}$$

For this calculation we made use of the following relations:

$$\begin{aligned} \delta\sqrt{-g} &= \frac{1}{2}\sqrt{-g}g^{\mu\nu}\delta g_{\mu\nu} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu} \\ \delta\varepsilon^{\mu\nu} &= -\frac{\sqrt{-g}}{(-g)}\epsilon^{\mu\nu} = -\frac{1}{2}g_{\sigma\tau}\delta g^{\sigma\tau}\varepsilon^{\mu\nu} \\ \delta P^{\mu\nu} &= \frac{1}{2}\delta g^{\sigma\tau}\left(2\delta^\mu_\sigma\delta^\nu_\tau - g_{\sigma\tau}\varepsilon^{\mu\nu}\right) \\ \delta\left(\sqrt{-g}P^{\mu\nu}\right) &= \delta\left(\sqrt{-g}g^{\mu\nu}\right) = \frac{1}{2}\sqrt{-g}\delta g^{\sigma\tau}\left(2\delta^\mu_\sigma\delta^\nu_\tau - g_{\sigma\tau}g^{\mu\nu}\right) \end{aligned} \tag{3.320}$$

To cancel the term that comes from the BRST variation of $A_\nu{}^\rho B_\rho{}^\nu$ we thus define a BRST variation of the metric. The BRST variation of the action is then given by:

$$sS^{\text{toy}}_{\text{gauged}} = \underbrace{sS}_{0} + \int\sqrt{-g}\left(\frac{1}{2}T_{\nu\rho}\,sg^{\nu\rho} - 2\,sA_\nu{}^\rho B_\rho{}^\nu + 2A_\nu{}^\rho\,sB_\rho{}^\nu\right) \tag{3.321}$$

In contrast to $T_{\mu\nu}$, which has non–vanishing components T_{zz} and $T_{\bar{z}\bar{z}}$ in conformally flat coordinates, $sB_{\nu\rho}$ only has a non–vanishing zz–component. Hence, we expect part of the BRST variation of $B_\nu{}^\rho$ to coincide with $\frac{\Lambda}{8}\bar{P}^{\nu\sigma}P_\rho{}^\tau$. With $P^{\sigma\nu}g_{\sigma\tau}P^\tau{}_\rho = 0$ the following relations hold:

$$\begin{aligned} P^{\sigma\nu}P^\tau{}_\rho\left(2\delta^\mu_\sigma\delta^\tau_\rho - g_{\sigma\tau}g^{\mu\lambda}\right) &= 2P^{(\mu\nu}P^{\lambda)}{}_\rho \\ \frac{1}{4}P^{\sigma\nu}P^\tau{}_\rho\left(2g_{\sigma\kappa}\varepsilon^\lambda{}_\tau - g_{\sigma\tau}\varepsilon^\lambda{}_\kappa\right) &= -\frac{1}{2}P_\kappa{}^\nu P^\lambda{}_\rho \end{aligned} \tag{3.322}$$

If we define

$$sg^{\sigma\tau} = -\frac{1}{2}P^{\sigma\nu}P^{\tau}{}_{\rho}A_{\nu}{}^{\rho} \tag{3.323}$$

we find after a lengthy calculation:

$$sS^{\text{toy}}_{\text{gauged}} = \int\sqrt{-g}\left(\frac{1}{4}P^{\mu\nu}P^{\lambda}{}_{\rho}\Big(sA_{\nu}{}^{\rho} - \underbrace{\delta^{\sigma}_{\tau}A_{\sigma}{}^{\rho}A_{\nu}{}^{\tau}}_{0}\Big)\beta_{\mu m}\left(\nabla_{\lambda}x^{m} + \nabla_{\lambda}x^{mh}\right)\right) \tag{3.324}$$

From this we can read off:

$$sA_{\nu}{}^{\rho} = 0 \tag{3.325}$$

Thus, we have reimplemented BRST symmetry in the gauged action.

3.7.4 Gauge Fixing

Now we perform the gauge fixing of diffeomorphism invariance and the fermionic symmetry. After integrating out all Lagrange multiplier fields we expect a second BRST operator of the following form:

$$Q_V = \oint c^{z}\left(T_{zz} + \frac{1}{2}T^{\text{top}}_{zz}\right) + \gamma^{z}\left(B_{zz} + \frac{1}{2}B^{\text{top}}_{zz}\right) \tag{3.326}$$

In order to make the new BRST operator anticommute with the WZNW BRST operator we expect:

$$Q \rightarrow Q + \oint b_{zz}\gamma^{z} \tag{3.327}$$

We add a gauge fixing term to the gauged action, fixing the metric to the flat light cone metric and all other gauge fields to zero.

$$S^{\text{toy}}_{\text{gf}} = S^{\text{toy}}_{\text{gauged}} + s_V\int\sqrt{-g}\left(b_{\mu\nu}(g^{\mu\nu} - \hat{g}^{\mu\nu}) + 2g^{\nu\lambda}\beta_{\mu\nu}A_{\lambda}{}^{\mu}\right) \tag{3.328}$$

We define the new BRST variations as follows:

$$\begin{aligned}
& s_V\, g_{\mu\nu} = \nabla_{\mu}c_{\nu} + \nabla_{\nu}c_{\mu} - 2\lambda g_{\mu\nu} \\
\leftrightarrow\ & s_V\, g^{\mu\nu} = -\nabla^{\mu}c^{\nu} - \nabla^{\nu}c^{\mu} + 2\lambda g^{\mu\nu} \\
\leftrightarrow\ & s_V\, \sqrt{-g} = \frac{1}{2}\sqrt{-g}g^{\mu\nu}s_V g_{\mu\nu} = \sqrt{-g}(\nabla_{\mu}c^{\mu} - 2\lambda) \\
\\
& s_V\, (\sqrt{-g}b_{\mu\nu}) = \sqrt{-g}\Lambda_{\mu\nu} \\
\leftrightarrow\ & s_V\, b_{\mu\nu} = \Lambda_{\mu\nu} - (\nabla_{\lambda}c^{\lambda} - 2\lambda)b_{\mu\nu} \\
\\
& s_V\, (\sqrt{-g}g^{\nu\lambda}\beta_{\mu\nu}) = \sqrt{-g}g^{\nu\lambda}\Omega_{\mu\nu} \\
\leftrightarrow\ & s_V\, \beta_{\mu\nu} = \Omega_{\mu\nu} + (2g_{\nu(\sigma}\delta^{\lambda}_{\tau)} - g_{\sigma\tau}\delta^{\lambda}_{\nu})\left(\nabla^{(\sigma}c^{\tau)} - \lambda g^{\sigma\tau}\right)\beta_{\mu\lambda}
\end{aligned}$$

$$s_V\, A_\lambda{}^\mu \;=\; \tilde{\lambda}_\lambda{}^\mu + \frac{1}{4}P^\mu{}_\sigma P_\lambda{}^\tau \left(\nabla_\tau\gamma^\sigma - \tilde{\lambda}_\lambda{}^\mu\right) + c^\kappa\nabla_\kappa A_\lambda{}^\mu + \nabla_\lambda c^\kappa A_\kappa{}^\mu - \nabla_\kappa c^\mu A_\lambda{}^\kappa \tag{3.329}$$

$(\gamma^\mu, \tilde{\lambda}_\mu{}^\nu)$ and (c^μ, λ) are now commuting and anticommuting ghosts, respectively. With that we get for the gauge fixed action:

$$\begin{aligned} S^{\text{toy}}_{\text{gf}} \;&=\; S^{\text{toy}}_{\text{gauged}} + \int\sqrt{-g}\Big[\Lambda_{\mu\nu}(g^{\mu\nu}-\hat{g}^{\mu\nu}) + 2\Omega_{\mu\nu}g^{\nu\lambda}A_\lambda{}^\mu + 2b_{\mu\nu}\left(\nabla^{(\mu}c^{\nu)} - \lambda g^{\mu\nu}\right) \\ &\quad + 2g^{\nu\lambda}\beta_{\mu\nu}\left(\tilde{\lambda}_\lambda{}^\mu + \frac{1}{4}P^\mu{}_\sigma P_\lambda{}^\tau\left(\nabla_\tau\gamma^\sigma - \tilde{\lambda}_\lambda{}^\mu\right) + c^\kappa\nabla_\kappa A_\lambda{}^\mu + \nabla_\lambda c^\kappa A_\kappa{}^\mu - \nabla_\kappa c^\mu A_\lambda{}^\kappa\right)\Big] \end{aligned} \tag{3.330}$$

Next we integrate out the Lagrange multipliers, using the following variations:

$$\begin{aligned} \frac{\delta}{\delta\tilde{\lambda}^{\tau\sigma}}: &\quad 0 &=&\; \beta_{\sigma\tau} - \frac{1}{4}P^\mu{}_\sigma P^\nu{}_\tau\beta_{\mu\nu} \\ \frac{\delta}{\delta(\beta-\frac{1}{4}PP\beta)^\lambda{}_\mu}: &\quad \tilde{\lambda}_\lambda{}^\mu &=&\; \frac{1}{4}P_\lambda{}^\tau P^\mu{}_\sigma\tilde{\lambda}_\tau{}^\sigma \\ \frac{\delta}{\delta\Omega_\mu{}^\lambda}: &\quad A_\lambda{}^\mu &=&\; 0 \\ \frac{\delta}{\delta A^{\nu\mu}}: &\quad \Omega_{\mu\nu} &=&\; -B_{\mu\nu} + \nabla_\kappa(\beta_{\mu\nu}c^\kappa) - \beta_{\mu\lambda}\nabla^\lambda c_\nu + \beta_{\kappa\nu}\nabla_\mu c^\kappa \\ \frac{\delta}{\delta\lambda}: &\quad b_{\mu\nu}g^{\mu\nu} &=&\; 0 \\ \frac{\delta}{\delta(b_{\mu\nu}g^{\mu\nu})}: &\quad \lambda &=&\; \frac{1}{2}\nabla_\mu c^\mu \\ \frac{\delta}{\delta\Lambda_{\mu\nu}}: &\quad g^{\mu\nu} &=&\; \hat{g}^{\mu\nu} = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ \frac{\delta}{\delta g^{\mu\nu}}: &\quad \Lambda_{\mu\nu} &=&\; -T_{\mu\nu} - T^{\text{top}}_{\mu\nu} \end{aligned} \tag{3.331}$$

Inserting these results back into (3.330) we end up with the following expression:

$$\begin{aligned} S^{\text{toy}} \;&=\; \int\sqrt{-g}\Big[-\frac{1}{2}g^{\mu\nu}\nabla_\mu x^m\nabla_\nu x_m + \frac{1}{2}g^{\mu\nu}\nabla_\mu x^{mh}\nabla_\nu x^h_m + P^{\mu\nu}\beta_{\mu m}\partial_\nu\xi^m \\ &\quad + b_{zz}\bar{\partial}c^z + b_{\bar{z}\bar{z}}\partial c^{\bar{z}} + \beta_{zz}\bar{\partial}\gamma^z\Big] \end{aligned} \tag{3.332}$$

Finally we compute the additional term in the WZNW BRST operator Q by demanding that the new BRST variation and the WZNW variation anticommute. Using

$$\begin{aligned} sA_\mu{}^\nu &= 0 \\ sg^{\mu\nu} &= -\frac{1}{2}P^{\mu\sigma}P^\nu{}_\tau A_\sigma{}^\tau \end{aligned} \tag{3.333}$$

we compute the WZNW BRST variation of the homotopy (with respect to s_V) of the gauge fixing term in (3.330). ($S^{\text{toy}}_{\text{gauged}}$ is already s–invariant.)

$$s\left(\sqrt{-g}b_{\mu\nu}(g^{\mu\nu}-\hat{g}^{\mu\nu}) + 2\sqrt{-g}g^{\nu\lambda}\beta_{\mu\nu}A_\lambda{}^\mu\right)$$

$$
\begin{aligned}
&= \sqrt{-g}\, sb_{\mu\nu}(g^{\mu\nu} - \hat{g}^{\mu\nu}) - \sqrt{-g}b_{\mu\nu}\, sg^{\sigma\tau}\left(\delta^{\mu}_{(\sigma}\delta^{\nu}_{\tau)} - \frac{1}{2}g_{\sigma\tau}(g^{\mu\nu} - \hat{g}^{\mu\nu})\right) \\
&\quad + \sqrt{-g}\, sg^{\sigma\tau}\left(2\delta^{\nu}_{\sigma)}\delta^{\lambda}_{\tau)} - g_{\sigma\tau}g^{\nu\lambda}\right)\beta_{\mu\nu}A_{\lambda}{}^{\mu} + 2\sqrt{-g}g^{\nu\lambda}\, s\beta_{\mu\nu}A_{\lambda}{}^{\mu} \\
&= \sqrt{-g}\, sb_{\mu\nu}(g^{\mu\nu} - \hat{g}^{\mu\nu}) + \frac{1}{2}\sqrt{-g}b_{\sigma\tau}P^{\sigma\lambda}P^{\tau}{}_{\mu}A_{\lambda}{}^{\mu} \\
&\quad - \sqrt{-g}P^{\sigma\lambda}P^{\tau}{}_{\mu}A_{\lambda}{}^{\mu}A_{\tau}{}^{\lambda}\beta_{\lambda\sigma} + 2\sqrt{-g}g^{\nu\lambda}\, s\beta_{\mu\nu}A_{\lambda}{}^{\mu} \qquad (3.334)
\end{aligned}
$$

In the last step we made use of $P^{\mu\lambda}P^{\nu}{}_{\lambda} = 0$. For the two BRST transformations to anticommute this expression has to vanish. This can be achieved by defining the following WZNW BRST variations of the antighosts:

$$
\begin{aligned}
sb_{\mu\nu} &= 0 \\
s\beta_{\mu\nu} &= -\frac{1}{4}b_{\sigma\tau}P^{\sigma}{}_{\nu}P^{\tau}{}_{\mu} - \frac{1}{2}P^{\sigma}{}_{\nu}P^{\tau}{}_{\mu}A_{\tau}{}^{\lambda}\beta_{\lambda\sigma} \qquad (3.335)
\end{aligned}
$$

After integrating out the Lagrange multipliers ($A_{\tau}{}^{\lambda} = 0$) we get the transformation

$$
s\beta_{zz} = -b_{zz} \qquad (3.336)
$$

which is implemented by adding a term $b_{zz}\gamma^{z}$ to the WZNW BRST operator:

$$
Q \rightarrow Q + \oint b_{zz}\gamma^{z} \qquad (3.337)
$$

The new BRST transformations are now given by:

$$
\begin{aligned}
s_V\, b_{zz} &= -T_{zz} - T^{\text{top}}_{zz} \\
s_V\, b_{\bar{z}\bar{z}} &= -T_{\bar{z}\bar{z}} - T^{\text{top}}_{\bar{z}\bar{z}} \\
s_V\, \beta_{zz} &= -B_{zz} - B^{\text{top}}_{zz} \qquad (3.338)
\end{aligned}
$$

with

$$
\begin{aligned}
T^{\text{top}}_{zz} &= 2b_{zz}\partial c^{z} + \partial b_{zz}c^{z} + \bar{\partial}b_{zz}c^{\bar{z}} + 2\beta_{zz}\partial\gamma^{z} + \partial\beta_{zz}\gamma^{z} + \bar{\partial}\beta_{zz}\gamma^{\bar{z}} \\
T^{\text{top}}_{\bar{z}\bar{z}} &= 2b_{\bar{z}\bar{z}}\bar{\partial}c^{\bar{z}} + \bar{\partial}b_{\bar{z}\bar{z}}c^{\bar{z}} + \partial b_{\bar{z}\bar{z}}c^{z} \\
B^{\text{top}}_{zz} &= -2\beta_{zz}\partial c^{z} - \partial\beta_{zz}c^{z} - \bar{\partial}\beta_{zz}c^{\bar{z}} \qquad (3.339)
\end{aligned}
$$

T_{zz}, $T_{\bar{z}\bar{z}}$ and B_{zz} are the currents coming from the WZNW model. These transformations correspond to the following BRST charge:

$$
Q_V = \oint c^{z}\left(T_{zz} + \frac{1}{2}T^{\text{top}}_{zz}\right) + \gamma^{z}\left(B_{zz} + \frac{1}{2}B^{\text{top}}_{zz}\right) \qquad (3.340)
$$

From our results for the toy model we conclude that the gauge fixing procedure works analogously for the heterotic superstring and yields the results we already anticipated in section 3.5.4. Note that this computation of T^{top}_{zz}, $T^{\text{top}}_{\bar{z}\bar{z}}$ and B^{top}_{zz} yields also terms that vanish on–shell which are not present in the expressions given in 3.5.4.

3.7.5 Worldsheet Covariant Formulation of the Type II Superstring

For the type II superstring the covariant form of the action reads:

$$\begin{aligned}
S \;=\; \int \sqrt{-g}\;\; & P^{\mu\nu}P_{\mu m}P^m_\nu - P^{\mu\nu}P_{\mu m}\Pi^m_\nu - P^{\mu\nu}\Pi_{\mu m}P^m_\nu + \frac{1}{2}P^{\mu\nu}\Pi_{\mu m}\Pi^m_\nu + \mathcal{L}_{WZ} \\
& -P^{\mu\nu}P^h_{\mu m}P^{mh}_\nu + P^{\mu\nu}P^h_{\mu m}\Pi^{mh}_\nu + P^{\mu\nu}\Pi^h_{\mu m}P^{mh}_\nu - \frac{1}{2}P^{\mu\nu}\Pi^h_{\mu m}\Pi^{mh}_\nu - \mathcal{L}^h_{WZ} \\
& +P^{\mu\nu}d_{\mu\alpha}\nabla_\nu\theta^\alpha + \bar{P}^{\nu\nu}\hat{d}_{\mu\hat{\alpha}}\nabla_\nu\hat{\theta}^{\hat{\alpha}} - P^{\mu\nu}d^h_{\mu\alpha}\nabla_\nu\theta^{\alpha h} - \bar{P}^{\nu\nu}\hat{d}^h_{\mu\hat{\alpha}}\nabla_\nu\hat{\theta}^{\hat{\alpha}h} \\
& +P^{\mu\nu}\beta_{\mu m}\nabla_\nu\xi^m + P^{\mu\nu}\omega_{\mu\alpha}\nabla_\nu\lambda^\alpha + P^{\mu\nu}\kappa^\alpha_\mu\nabla_\nu\chi_\alpha \\
& +\bar{P}^{\mu\nu}\hat{\beta}_{\mu m}\nabla_\nu\hat{\xi}^m + \bar{P}^{\mu\nu}\hat{\omega}_{\mu\hat{\alpha}}\nabla_\nu\hat{\lambda}^{\hat{\alpha}} + \bar{P}^{\mu\nu}\hat{\kappa}^{\hat{\alpha}}_\mu\nabla_\nu\hat{\chi}_{\hat{\alpha}} \\
& +\frac{1}{2}P^{\mu\nu}P^\lambda{}_\rho\beta'_{\mu\lambda}\nabla_\nu\gamma'^\rho + \frac{1}{2}P^{\mu\nu}P^\lambda{}_\rho b'_{\mu\lambda}\nabla_\nu c'^\rho \\
& +\frac{1}{2}\bar{P}^{\mu\nu}\bar{P}^\lambda{}_\rho\hat{\beta}'_{\mu\lambda}\nabla_\nu\hat{\gamma}'^\rho + \frac{1}{2}\bar{P}^{\mu\nu}\bar{P}^\lambda{}_\rho\hat{b}'_{\mu\lambda}\nabla_\nu\hat{c}'^\rho
\end{aligned} \tag{3.341}$$

where

$$\mathcal{L}_{WZ} \;=\; -i\varepsilon^{\mu\nu}\Pi^m_\mu\Big((\theta\gamma_m\nabla_\nu\theta) - (\hat{\theta}\gamma_m\nabla_\nu\hat{\theta})\Big) - \varepsilon^{\mu\nu}(\theta\gamma^m\nabla_\mu\theta)(\hat{\theta}\gamma_m\nabla_\nu\hat{\theta}). \tag{3.342}$$

Knowing the fermionic symmetry transformations for the heterotic case and the Π and $P-$transformations for the flat case it is easy to guess the transformations for type II:

$$\begin{aligned}
\delta\Pi^m_\mu &= -\frac{i}{2}P^\rho{}_\nu\gamma^\nu\left(\kappa_\rho\gamma^m\nabla_\mu\theta\right) + \frac{1}{4}P^\rho{}_\nu\nabla_\mu\left(\gamma^\nu\beta^m_\rho\right) + \frac{i}{16}P^\rho{}_\nu P^{\sigma\tau}\nabla_\mu\left(\gamma^\nu c'_\tau\left(\kappa_\sigma\gamma^m\kappa_\rho\right)\right) \\
&\quad -\frac{i}{2}\bar{P}^\rho{}_\nu\hat{\gamma}^\nu\left(\hat{\kappa}_\rho\gamma^m\nabla_\mu\hat{\theta}\right) + \frac{1}{4}\bar{P}^\rho{}_\nu\nabla_\mu\left(\hat{\gamma}^\nu\hat{\beta}^m_\rho\right) + \frac{i}{16}\bar{P}^\rho{}_\nu\bar{P}^{\sigma\tau}\nabla_\mu\left(\hat{\gamma}^\nu\hat{c}'_\tau\left(\hat{\kappa}_\sigma\gamma^m\hat{\kappa}_\rho\right)\right) \\
\delta P^m_\mu &= -\frac{i}{4}P^\rho{}_\nu P^\sigma{}_\mu\gamma^\nu\left(\kappa_\rho\gamma^m\nabla_\sigma\theta\right) + \frac{1}{8}P^\rho{}_\nu P^\sigma{}_\mu\nabla_\sigma\left(\gamma^\nu\beta^m_\rho\right) + \frac{i}{32}P^\rho{}_\nu P^{\sigma\tau}P^\lambda{}_\mu\nabla_\lambda\left(\gamma^\nu c'_\tau\left(\kappa_\sigma\gamma^m\kappa_\rho\right)\right) \\
&\quad -\frac{i}{4}\bar{P}^\rho{}_\nu\bar{P}^\sigma{}_\mu\hat{\gamma}^\nu\left(\hat{\kappa}_\rho\gamma^m\nabla_\sigma\hat{\theta}\right) + \frac{1}{8}\bar{P}^\rho{}_\nu\bar{P}^\sigma{}_\mu\nabla_\sigma\left(\hat{\gamma}^\nu\hat{\beta}^m_\rho\right) + \frac{i}{32}\bar{P}^\rho{}_\nu\bar{P}^{\sigma\tau}\bar{P}^{\hat{\lambda}}{}_\mu\nabla_\lambda\left(\hat{\gamma}^\nu\hat{c}'_\tau\left(\hat{\kappa}_\sigma\gamma^m\hat{\kappa}_\rho\right)\right) \\
\delta\hat{\theta}^{\hat{\alpha}} &= \frac{1}{4}\bar{P}^\rho{}_\nu\hat{\gamma}^\nu\hat{\kappa}^{\hat{\alpha}}_\rho \\
\delta\hat{d}_{\mu\hat{\alpha}} &= -\frac{i}{2}\bar{P}^\rho{}_\nu\hat{\gamma}^\nu P^m_\mu(\gamma_m\hat{\kappa}_\rho)_{\hat{\alpha}} - \frac{i}{2}\bar{P}^\rho{}_\nu\hat{\gamma}^\nu(\gamma_m\nabla_\mu\hat{\theta})_{\hat{\alpha}}\hat{\beta}^m_\rho - \frac{1}{8}\bar{P}^\rho{}_\nu\bar{P}^{\sigma\tau}\hat{\gamma}^\nu\hat{c}'_\tau(\hat{\kappa}_\sigma\gamma^m\hat{\kappa}_\rho)(\gamma_m\nabla_\mu\hat{\theta})_{\hat{\alpha}} \\
&\quad -\frac{i}{4}\bar{P}^\rho{}_\nu\nabla_\mu(\hat{\gamma}^\nu\hat{\omega}_{\rho\hat{\alpha}}) + \frac{i}{8}\bar{P}^\rho{}_\nu\bar{P}^{\sigma\tau}\nabla_\mu\big(\hat{\gamma}^\nu\hat{c}'_\tau\hat{\beta}_{\sigma m}(\gamma^m\hat{\kappa}_\rho)_{\hat{\alpha}}\big)
\end{aligned} \tag{3.343}$$

$$\begin{aligned}
\delta\Pi^{mh}_\mu &= \frac{i}{2}P^\rho{}_\nu\gamma^\nu\left(\kappa_\rho\gamma^m\nabla_\mu\theta^h\right) - \frac{1}{4}P^\rho{}_\nu\nabla_\mu\left(\gamma^\nu\beta^m_\rho\right) + \frac{i}{16}P^\rho{}_\nu P^{\sigma\tau}\nabla_\mu\left(\gamma^\nu c'_\tau\left(\kappa_\sigma\gamma^m\kappa_\rho\right)\right) \\
&\quad +\frac{i}{2}\bar{P}^\rho{}_\nu\hat{\gamma}^\nu\left(\hat{\kappa}_\rho\gamma^m\nabla_\mu\hat{\theta}^h\right) - \frac{1}{4}\bar{P}^\rho{}_\nu\nabla_\mu\left(\hat{\gamma}^\nu\hat{\beta}^m_\rho\right) + \frac{i}{16}\bar{P}^\rho{}_\nu\bar{P}^{\sigma\tau}\nabla_\mu\left(\hat{\gamma}^\nu\hat{c}'_\tau\left(\hat{\kappa}_\sigma\gamma^m\hat{\kappa}_\rho\right)\right) \\
\delta P^{mh}_\mu &= \frac{i}{4}P^\rho{}_\nu P^\sigma{}_\mu\gamma^\nu\left(\kappa_\rho\gamma^m\nabla_\sigma\theta^h\right) - \frac{1}{8}P^\rho{}_\nu P^\sigma{}_\mu\nabla_\sigma\left(\gamma^\nu\beta^m_\rho\right) + \frac{i}{32}P^\rho{}_\nu P^{\sigma\tau}P^\lambda{}_\mu\nabla_\lambda\left(\gamma^\nu c'_\tau\left(\kappa_\sigma\gamma^m\kappa_\rho\right)\right) \\
&\quad +\frac{i}{4}\bar{P}^\rho{}_\nu\bar{P}^\sigma{}_\mu\hat{\gamma}^\nu\left(\hat{\kappa}_\rho\gamma^m\nabla_\sigma\hat{\theta}^h\right) - \frac{1}{8}\bar{P}^\rho{}_\nu\bar{P}^\sigma{}_\mu\nabla_\sigma\left(\hat{\gamma}^\nu\hat{\beta}^m_\rho\right) + \frac{i}{32}\bar{P}^\rho{}_\nu\bar{P}^{\sigma\tau}\bar{P}^{\hat{\lambda}}{}_\mu\nabla_\lambda\left(\hat{\gamma}^\nu\hat{c}'_\tau\left(\hat{\kappa}_\sigma\gamma^m\hat{\kappa}_\rho\right)\right) \\
\delta\hat{\theta}^{\hat{\alpha}h} &= -\frac{1}{4}\bar{P}^\rho{}_\nu\hat{\gamma}^\nu\hat{\kappa}^{\hat{\alpha}}_\rho \\
\delta\hat{d}^h_{\mu\hat{\alpha}} &= \frac{i}{2}\bar{P}^\rho{}_\nu\hat{\gamma}^\nu P^{mh}_\mu(\gamma_m\hat{\kappa}_\rho)_{\hat{\alpha}} + \frac{i}{2}\bar{P}^\rho{}_\nu\hat{\gamma}^\nu(\gamma_m\nabla_\mu\hat{\theta}^h)_{\hat{\alpha}}\hat{\beta}^m_\rho - \frac{1}{8}\bar{P}^\rho{}_\nu\bar{P}^{\sigma\tau}\hat{\gamma}^\nu\hat{c}'_\tau(\hat{\kappa}_\sigma\gamma^m\hat{\kappa}_\rho)(\gamma_m\nabla_\mu\hat{\theta}^h)_{\hat{\alpha}}
\end{aligned}$$

$$+\frac{i}{4}\bar{P}^{\rho}{}_{\nu}\nabla_\mu(\hat{\gamma}^\nu\hat{\omega}_{\rho\hat{\alpha}})+\frac{i}{8}\bar{P}^{\rho}{}_{\nu}\bar{P}^{\sigma\tau}\nabla_\mu\big(\hat{\gamma}^\nu\hat{c}'_\tau\hat{\beta}_{\sigma m}(\gamma^m\hat{\kappa}_\rho)_{\hat{\alpha}}\big) \tag{3.344}$$

$$\begin{aligned}
\delta\hat{\xi}^m &= \frac{1}{4}\bar{P}^{\lambda}{}_{\nu}\hat{\gamma}^\nu(P^m_\lambda+P^{mh}_\lambda)+\frac{i}{8}\bar{P}^{\rho}{}_{\nu}\bar{P}^{\lambda\tau}\hat{\gamma}^\nu\hat{c}'_\tau\big(\hat{\kappa}_\lambda\gamma^m(\nabla_\rho\hat{\theta}-\nabla_\rho\hat{\theta}^h)\big)+\frac{i}{16}\bar{P}^{\rho}{}_{\nu}\bar{P}^{\lambda\tau}\hat{\gamma}^\nu\hat{\gamma}'_\tau(\hat{\kappa}_\lambda\gamma^m\hat{\kappa}_\rho)\\
\delta\hat{\lambda}^{\hat{\alpha}} &= -\frac{i}{4}\bar{P}^{\rho}{}_{\nu}\hat{\gamma}^\nu(\nabla_\rho\hat{\theta}^{\hat{\alpha}}-\nabla_\rho\hat{\theta}^{\hat{\alpha}h})\\
\delta\hat{\chi}_{\hat{\alpha}} &= -\frac{1}{4}\bar{P}^{\rho}{}_{\nu}\hat{\gamma}^\nu(\hat{d}_{\rho\hat{\alpha}}+\hat{d}^h_{\rho\hat{\alpha}})+\frac{i}{8}\bar{P}^{\rho}{}_{\nu}\bar{P}^{\sigma\tau}\hat{\gamma}^\nu\hat{c}'_\tau\hat{\beta}_{\sigma m}\big(\gamma^m(\nabla_\rho\hat{\theta}-\nabla_\rho\hat{\theta}^h)\big)_{\hat{\alpha}}\\
&\quad-\frac{i}{8}\bar{P}^{\rho}{}_{\nu}\bar{P}^{\sigma\tau}\hat{\gamma}^\nu\hat{c}'_\tau(\gamma^m\hat{\kappa}_\sigma)_{\hat{\alpha}}(P_{\rho m}-P^h_{\rho m})-\frac{i}{8}\bar{P}^{\rho}{}_{\nu}\bar{P}^{\sigma\tau}\hat{\gamma}^\nu\hat{\gamma}'_\tau\hat{\beta}_{\sigma m}(\gamma^m\hat{\kappa}_\rho)_{\hat{\alpha}}
\end{aligned} \tag{3.345}$$

$$\begin{aligned}
\delta\hat{c}'^\nu &= \hat{\gamma}^\nu\\
\delta\hat{b}'_{\sigma\lambda} &= -\bar{P}^{\rho}{}_{\nu}\nabla_\sigma(\hat{\gamma}^\nu\hat{\beta}'_{\lambda\rho})+\frac{1}{2}\bar{P}^{\rho}{}_{\nu}\hat{\gamma}^\nu\nabla_\sigma\hat{\beta}'_{\lambda\rho}-\frac{i}{4}\bar{P}^{\rho}{}_{\nu}\hat{\gamma}^\nu\hat{\beta}_{\sigma m}\big(\hat{\kappa}_\lambda\gamma^m(\nabla_\rho\hat{\theta}-\nabla_\rho\hat{\theta}^h)\big)\\
&\quad+\frac{i}{8}\bar{P}^{\rho}{}_{\nu}\hat{\gamma}^\nu(\hat{\kappa}_\sigma\gamma^m\hat{\kappa}_\lambda)(P_{\rho m}-P^h_{\rho m})\\
\delta\hat{\gamma}'^\nu &= -\bar{P}^{\sigma\tau}\hat{\gamma}^\nu\nabla_\sigma\hat{c}'_\tau+\frac{1}{2}\bar{P}^{\sigma\tau}\nabla_\sigma(\hat{\gamma}^\nu\hat{c}'_\tau)\\
\delta\hat{\beta}'_{\sigma\lambda} &= \frac{i}{8}\bar{P}^{\rho}{}_{\nu}\hat{\gamma}^\nu\hat{\beta}_{\sigma m}(\hat{\kappa}_\lambda\gamma^m\hat{\kappa}_\rho)
\end{aligned} \tag{3.346}$$

$\hat{\gamma}^\nu$ is the transformation parameter for the right–moving sector. The fields and ghosts of the left–moving sector transform as in the heterotic case apart from the fact that one has to replace $\Pi_{\mu m}$ with $P_{\mu m}$.
For the type II case we now have two B–currents corresponding to the left– and right–moving sector:

$$\begin{aligned}
B_{\mu\nu} &= -\frac{1}{8}P^{\lambda}{}_{\mu}P^{\rho}{}_{\nu}(P_{\lambda m}+P^h_{\lambda m})\beta^m_\rho+\frac{i}{8}P^{\lambda}{}_{\mu}P^{\rho}{}_{\nu}(id_{\lambda\alpha}+id^h_{\lambda\alpha})\kappa^\alpha_\rho-\frac{i}{8}P^{\lambda}{}_{\mu}P^{\rho}{}_{\nu}(\nabla_\lambda\theta^\alpha+\nabla_\lambda\theta^{\alpha h})\omega_{\rho\alpha}\\
&\quad-\frac{1}{4}P^{\lambda}{}_{\mu}P^{\rho}{}_{\nu}P^{\sigma\tau}\beta'_{\lambda\rho}\nabla_\sigma c'_\tau-\frac{1}{8}P^{\lambda}{}_{\mu}P^{\rho}{}_{\nu}P^{\sigma\tau}c'_\tau\nabla_\sigma\beta'_{\lambda\rho}-\frac{1}{4}P^{\lambda}{}_{\mu}P^{\rho}{}_{\nu}b'_{\lambda\rho}\\
&\quad-\frac{1}{16}P^{\lambda}{}_{\mu}P^{\rho}{}_{\nu}P^{\sigma\tau}c'_\tau F_{\sigma\lambda\rho}-\frac{1}{16}P^{\lambda}{}_{\mu}P^{\rho}{}_{\nu}P^{\sigma\tau}\gamma'_\tau\Phi_{\sigma\lambda\rho}\\
\hat{B}_{\mu\nu} &= -\frac{1}{8}\bar{P}^{\lambda}{}_{\mu}\bar{P}^{\rho}{}_{\nu}(P_{\lambda m}+P^h_{\lambda m})\hat{\beta}^m_\rho+\frac{i}{8}\bar{P}^{\lambda}{}_{\mu}\bar{P}^{\rho}{}_{\nu}(i\hat{d}_{\lambda\hat{\alpha}}+i\hat{d}^h_{\lambda\hat{\alpha}})\hat{\kappa}^{\hat{\alpha}}_\rho-\frac{i}{8}\bar{P}^{\lambda}{}_{\mu}\bar{P}^{\rho}{}_{\nu}(\nabla_\lambda\hat{\theta}^{\hat{\alpha}}+\nabla_\lambda\hat{\theta}^{\hat{\alpha}h})\hat{\omega}_{\rho\hat{\alpha}}\\
&\quad-\frac{1}{4}\bar{P}^{\lambda}{}_{\mu}\bar{P}^{\rho}{}_{\nu}\bar{P}^{\sigma\tau}\hat{\beta}'_{\lambda\rho}\nabla_\sigma\hat{c}'_\tau-\frac{1}{8}\bar{P}^{\lambda}{}_{\mu}\bar{P}^{\rho}{}_{\nu}\bar{P}^{\sigma\tau}\hat{c}'_\tau\nabla_\sigma\hat{\beta}'_{\lambda\rho}-\frac{1}{4}\bar{P}^{\lambda}{}_{\mu}\bar{P}^{\rho}{}_{\nu}\hat{b}'_{\lambda\rho}\\
&\quad-\frac{1}{16}\bar{P}^{\lambda}{}_{\mu}\bar{P}^{\rho}{}_{\nu}\bar{P}^{\sigma\tau}\hat{c}'_\tau\hat{F}_{\sigma\lambda\rho}-\frac{1}{16}\bar{P}^{\lambda}{}_{\mu}\bar{P}^{\rho}{}_{\nu}\bar{P}^{\sigma\tau}\hat{\gamma}'_\tau\hat{\Phi}_{\sigma\lambda\rho}
\end{aligned} \tag{3.347}$$

In a lengthy calculation it can be shown that these currents are nilpotent under the fermionic transformations. As a second consistency check we verified the invariance of the action under the transformation, picking out the terms that transform into $\Pi_{\mu m}$ and $P_{\mu m}$. A tedious calculation, involving partial integration, the Fierz identity and various identities for the $P^{\mu\nu}$, showed that left– and right–moving sector completely decouple (as in the flat case) and that the correct currents are produced. Thus, gauging and gauge fixing can be performed as in the heterotic case. Calling (3.341) S_0 we can perform the Noether procedure as usual:

$$\delta S_0 = \int 2\nabla_\mu\gamma^\nu B^\mu{}_\nu+2\nabla_\mu\hat{\gamma}^\nu\hat{B}^\mu{}_\nu \tag{3.348}$$

We introduce gauge connections ${A_\mu}^\nu$ and ${\hat{A}_\mu}^\nu$ and extend the action by

$$S_1 = \int -2{A_\mu}^\nu {B^\mu}_\nu - 2{\hat{A}_\mu}^\nu {\hat{B}^\mu}_\nu \qquad (3.349)$$

where

$$\delta_0 {A_\mu}^\nu = \nabla_\mu \gamma^\nu \qquad \delta_0 {\hat{A}_\mu}^\nu = \nabla_\mu \hat{\gamma}^\nu \qquad (3.350)$$

The next step would be gauge fixing and reimplementation of the WZNW BRST invariance. We expect that the procedure is analogous to the the toy–model case but some difficulties may arise from the fact that the BRST variation of the composite B–field yields the energy–momentum tensor only on–shell.

Chapter 4

Summary and Outlook

In this diploma thesis on the covariant quantization of the superstring the following results were found:
We gave a WZNW formulation of the type II superstring using the Noether procedure to gauge the free field action given by Siegel. With that we generalized the WZNW formulation for the heterotic superstring, which was found by van Nieuwenhuizen and collaborators, to type II. This was possible by introducing auxiliary fields which separate the left– and right–moving sector off–shell. Gauging the symmetries of the free action forced us to introduce a new set of auxiliary fields which led to a formulation of the superstring as a gauged WZNW model.
In order to turn the Kazama algebra into a twisted $N = 2$ superconformal algebra a topological quartet has to be introduced. We pointed out that this quartet cannot be used to implement manifest worldsheet covariance. For this purpose two more pairs of ghosts have to be introduced. This was done explicitly, using a covariantized form of the Noether procedure, by gauging diffeomorphisms and the fermionic symmetry of our model. Due to the complexity of these calculations some of the computations, in particular gauge fixing, have been performed for a toy model only.
Some progress was made concerning the definition of the cohomology. One of the most difficult problems of this model for the covariant quantization of the superstring without pure spinor constraints is that the BRST operator has trivial cohomology. This is due to the fact that, apart from the constraint that relates the free field theory to the GS string, all the other conserved currents are set to 0 as well. Grassi and van Nieuwenhuizen suggested a method to solve this problem by introducing a second BRST operator which undoes the gauging of the other currents. Physical states are defined to lie in the relative cohomology of the two BRST operators. We generalized this method, which was only worked out for simple Lie algebras before, for an arbitrary set of constraints that generate a first class system and specialized it for the superstring. We expected to find a BRST operator which consists of the BRST operator of Berkovits' pure spinor formalism plus an expression which is exact with respect to the second BRST operator. Unfortunately we obtained a different result which we could not improve by deforming the BRST operator but later we found out that we can nevertheless get the desired physical spectrum.

There are many open problems, the most difficult of which is probably the definition of physical states. It is not yet clear if the definition of the physical states via a relative coho-

mology works for the superstring if we implement worldsheet diffeomorphism invariance. It would be interesting to work out an idea by van Nieuwenhuizen et al. that involves bosonization and a similarity transformation. It would also be interesting to further investigate our ansätze concerning the deformation of the BRST charges. A further problem are the h–fields of the gauged WZNW model which should decouple from the theory for physical states. Van Nieuwenhuizen and collaborators suggested an analogue of the Siegel gauge, demanding that the zero mode of the composite B–field, corresponding to the fermionic symmetry of the WZNW model, acting on physical states yields 0. No examples were computed so far to see if this really works.
Another problem related to the cohomology is the computation of correlation functions. Due to the huge amount of fields and ghosts in this model we have not been able to give a proper definition of the correlation function, neither at tree level nor at higher genus.
Concerning the implementation of world sheet diffeomorphism invariance for the type II superstring there are still some calculations missing. For type II the BRST variation of the B–field yields the energy–momentum tensor only on–shell. One may need to check how this affects the WZNW invariance of the gauged action and the gauge fixing procedure.
Since one of the most important motivations for the covariant quantization of the superstring are string theories in Ramond–Ramond backgrounds a goal is to couple this theory to a curved background.

Appendix A

Conventions

Our conventions are quite complementary to those of the standard string theory references. In particular, we use the NE–convention for superspace rather than the more common NW–convention. Since this thesis is based mostly on the work of van Nieuwenhuizen and collaborators it seemed practical and less confusing to adopt their conventions.

A.1 General Definitions

The worldsheet metric has signature $(-,+)$. We define light cone coordinates by:

$$\begin{aligned} \sigma^- &= \frac{1}{2}\left(\sigma^1 - \sigma^0\right) & \sigma^+ &= \frac{1}{2}\left(\sigma^1 + \sigma^0\right) \\ \partial = \partial_- &= \partial_1 - \partial_0 & \bar{\partial} = \partial_+ &= \partial_1 + \partial_0 \end{aligned} \tag{A.1}$$

After Wick–rotation to Euclidean space and introduction of complex coordinates we have with $\sigma^0 \to -i\sigma^2$:

$$\begin{aligned} z &= \frac{1}{2}\left(\sigma^1 + i\sigma^2\right) & \bar{z} &= \frac{1}{2}\left(\sigma^1 - i\sigma^2\right) \\ \partial = \partial_z &= \partial_1 - i\partial_2 & \bar{\partial} = \bar{\partial}_{\bar{z}} &= \partial_1 + i\partial_2 \end{aligned} \tag{A.2}$$

We define the conformal map from the closed string worldsheet to the complex plane as

$$z' = e^{-2iz} = e^{-i\sigma^1 + \sigma^2}. \tag{A.3}$$

We define chiral projection operators $P^{\mu\nu}$ and $\bar{P}^{\mu\nu}$ [2]:

$$\begin{aligned} P^{\mu\nu} &= g^{\mu\nu} - \varepsilon^{\mu\nu} \\ \bar{P}^{\mu\nu} &= g^{\mu\nu} + \varepsilon^{\mu\nu}, \end{aligned} \tag{A.4}$$

with the properties $P^{\mu\nu} A_\mu B_\nu = A_z B_{\bar{z}}$ and $\bar{P}^{\mu\nu} A_\mu B_\nu = A_{\bar{z}} B_z$. For further properties of these operators we refer to Appendix C. In Minkowski space we have:

$$\varepsilon^{\mu\nu} = \frac{\epsilon^{\mu\nu}}{\sqrt{-g}} \qquad \epsilon^{01} = 1 = -\epsilon^{10} \tag{A.5}$$

In Euclidean space we define:

$$\varepsilon^{\mu\nu} \;=\; -\frac{i\epsilon^{\mu\nu}}{\sqrt{g}} \qquad\qquad \epsilon^{12} \;=\; 1 \;=\; -\epsilon^{21} \tag{A.6}$$

The action in Minkowski space is:

$$S \;=\; \frac{1}{2\pi\alpha'}\int \mathrm{d}^2\sigma\sqrt{-g}\,\mathcal{L}, \tag{A.7}$$

with $\mathrm{d}^2\sigma \;=\; \mathrm{d}\sigma^0\mathrm{d}\sigma^1$. For the Euclidean action we get an extra minus from replacing the Minkowski metric by the Euclidean metric:

$$S^E \;=\; -\frac{1}{2\pi\alpha'}\int \mathrm{d}^2\sigma\sqrt{g}\,\mathcal{L}^E, \tag{A.8}$$

where $\mathrm{d}^2\sigma = \mathrm{d}\sigma^1\mathrm{d}\sigma^2$. $\mathcal{L}$ and $\mathcal{L}^E$ formally look the same, one only has to use the corresponding expressions for the ε–tensor and the metric.
Switching to complex coordinates the measure transforms as follows:

$$\mathrm{d}^2z \;=\; \mathrm{d}z\mathrm{d}\bar{z} \;=\; \frac{1}{4}\left(\mathrm{d}\sigma^1+i\mathrm{d}\sigma^2\right)\left(\mathrm{d}\sigma^1-i\mathrm{d}\sigma^2\right) \;=\; \frac{i}{2}\mathrm{d}\sigma^2\mathrm{d}\sigma^1 \;=\; \frac{i}{2}\mathrm{d}^2\sigma \tag{A.9}$$

The numerical values of the ε–tensor in complex/light cone coordinates are:

$$\varepsilon^{z\bar{z}} \;=\; -\frac{1}{2} \;=\; \varepsilon^{-+} \qquad\qquad \varepsilon_{z\bar{z}} \;=\; 2 \;=\; \varepsilon_{-+} \tag{A.10}$$

Whenever the constant α' is not written down explicitly it is set to $\alpha' = 2$.
In explicit calculations it is not necessary to specify whether we are in Euclidean or in Minkowski space. $S = \int\mathcal{L}$ is valid for both cases and the prefactors and measure part can be viewed as part of the $\int$–sign.

A.2 Superspace Conventions

As mentioned above we use Southwest–Northeast (NE) conventions for capital indices, where $M = (m,\alpha)$:

$$A_M B^M \;=\; (-)^M B^M A_M \tag{A.11}$$

It is useful to define two Kronecker deltas:

$$\delta_M^N \;=\; \delta^M{}_N \;=\; (-)^{MN}\delta_N{}^M, \tag{A.12}$$

where δ_M^N is numerically equal to the usual Kronecker delta. To see the necessity of this definition, take the metric and its graded inverse:

$$\begin{aligned}\mathcal{H}_{MP}\mathcal{H}^{PN} \;&=\; \delta_M{}^N \;=\; (-)^M\delta_M^N\\ (-)^P\mathcal{H}^{MP}\mathcal{H}_{PN} \;&=\; \delta^M{}_N \;=\; \delta_M^N\end{aligned} \tag{A.13}$$

For small Latin and Greek indices we are not as strict with the NE conventions. In particular we have:

$$\delta_\alpha^\beta \;=\; \delta_\alpha{}^\beta \;=\; \delta^\beta{}_\alpha \tag{A.14}$$

The graded commutator is always denoted with squared brackets:

$$[A,B] \;=\; AB-(-)^{AB}BA \tag{A.15}$$

A.3 Gamma Matrices and Spinors in 10 dimensions

This section mostly follows the arguments in [4].
At first we construct the Γ–matrices in $D=(9,1)$ by starting with the Pauli matrices,

$$\tau^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \tau^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \tau^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{A.16}$$

and working our way up to ten dimensions.
Now we define the well–known γ–matrices in $D=(3,1)$:

$$\gamma^k = \tau^k \otimes \tau^2 \qquad \gamma^4 = \mathbb{1} \otimes \tau^1 \qquad \gamma^5 = \mathbb{1} \otimes \tau^3, \tag{A.17}$$

where $k=\{1,2,3\}$. Here $\gamma^2, \gamma^4, \gamma^5$ are real and symmetric whereas γ^1, γ^3 are imaginary and antisymmetric 4×4–matrices.
The next step in the ladder is $D=(7,0)$ where we construct 7 symmetric purely imaginary 8×8–matrices λ^i:

$$\lambda^i = \left\{\gamma^2 \otimes \tau^2, \gamma^4 \otimes \tau^2, \gamma^5 \otimes \tau^2, \gamma^1 \otimes \mathbb{1}, \gamma^3 \otimes \mathbb{1}, \tau^2 \otimes \mathbb{1} \otimes \tau^1, \tau^2 \otimes \mathbb{1} \otimes \tau^3\right\} \tag{A.18}$$

Now we go to $D=(8,0)$ and construct 8 real block off–diagonal 16×16–matrices σ^μ:

$$\sigma^\mu = \left\{\lambda^i \otimes \tau^2, \mathbb{1} \otimes \tau^1\right\} \tag{A.19}$$

In $D=(8,0)$ we have a real block diagonal 16×16 chirality matrix χ which is the product of all σ^μ and which obeys $\chi^T=\chi$ and $\chi^2=1$. With that we can finally construct the ten real $D=(9,1)$ 32×32 Dirac matrices Γ^m:

$$\Gamma^m = \left\{\mathbb{1} \otimes (i\tau^2), \sigma^\mu \otimes \tau^1, \chi \otimes \tau^1\right\}, \tag{A.20}$$

where m runs from 0 to 9. The chirality matrix in $D=(9,1)$ is $\Gamma_\# = \Gamma^0 \ldots \Gamma^9 = \mathbb{1} \otimes \tau^3$. The charge conjugation matrix C, which satisfies $C\Gamma^m = -\Gamma^{m,T}C$, is given by $C=\Gamma^0$. We consider spinors $\Psi^T = (\lambda_L, \zeta_R)$ with spinor indices λ_L^α and $\zeta_{R,\dot\beta}$. Then the Γ–matrices have the following index structure:

$$\gamma^m = \begin{pmatrix} 0 & (\sigma^m)^{\alpha\dot\beta} \\ (\tilde\sigma^m)_{\dot\beta\gamma} & 0 \end{pmatrix} \qquad C = \begin{pmatrix} 0 & c_\alpha{}^{\dot\beta} \\ c^{\dot\beta}{}_\gamma & 0 \end{pmatrix}$$

$$\Gamma_\# = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \qquad \Gamma^9 = \begin{pmatrix} 0 & \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \\ \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} & 0 \end{pmatrix}, \tag{A.21}$$

where $\sigma^m = \{\mathbb{1}, \sigma^\mu, \chi\}$ and $\tilde\sigma^m = \{-\mathbb{1}, \sigma^\mu, \chi\}$, the matrices $c_\alpha{}^{\dot\beta}$ and $c^{\dot\beta}{}_\gamma$ are numerically equal to $\mathbb{1}$ and $-\mathbb{1}$, respectively. The form of $\Gamma_\#$ implies that the λ^α are chiral and the $\zeta_{\dot\beta}$ are antichiral.
In practical calculations we need the matrices $C\Gamma^m$[1]. Matrix multiplication yields:

$$C\Gamma^m = \begin{pmatrix} (\tilde\sigma^m)_{\alpha\beta} & 0 \\ 0 & -(\sigma^m)^{\dot\beta\dot\alpha} \end{pmatrix} \equiv \begin{pmatrix} \gamma^m_{\alpha\beta} & 0 \\ 0 & (\gamma^m)^{\beta\alpha} \end{pmatrix} \tag{A.22}$$

[1]One considers the expressions $\bar\Psi\Gamma^{m_1\cdots m_r}\Psi$, where $\Gamma^{m_1\cdots m_r}$ are antisymmetrized products of the Γ–matrices, which transform as tensors under Lorentz transformations. We have $\bar\Psi = \Psi^T\Gamma^0 = \Psi^T C$.

In our calculations we will only use the real symmetric 16×16 matrices $\gamma^m_{\alpha\beta} = \tilde{\sigma}^m_{\alpha\beta}$ and $\gamma_m^{\dot{\alpha}\dot{\beta}} = -\sigma^{m\,\dot{\alpha}\dot{\beta}}$. Now we explain why the dots can be omitted.
The Lorentz generators expressed in terms of the Γ–matrices are given by:

$$L^{mn} = \frac{1}{2}\left(\Gamma^m\Gamma^n - \Gamma^n\Gamma^m\right) = \begin{pmatrix} \frac{1}{2}\sigma^{m\,\alpha\dot{\beta}}\tilde{\sigma}^n_{\dot{\beta}\gamma} - m \leftrightarrow n & 0 \\ 0 & \frac{1}{2}\tilde{\sigma}^m_{\dot{\alpha}\beta}\sigma^{n\,\beta\dot{\gamma}} - m \leftrightarrow n \end{pmatrix} \tag{A.23}$$

Since this expression is block diagonal we find that the chiral spinors λ^α and the antichiral $\zeta_{\dot{\beta}}$ form separate representations for $SO(9,1)$. These representations are inequivalent because σ^m and $\tilde{\sigma}^m$ are equal except for $m = 0$ where $\sigma^0 = \mathbb{1}$ and $\tilde{\sigma}^0 = -\mathbb{1}$ and there is no matrix S satisfying $S\sigma^\mu = -\sigma^\mu S$ and $S\chi = -\chi S$ at the same time. It is easy to see that this is true: Assume that we found an S such that $S\sigma^\mu = -\sigma^\mu S$. Then $\chi = \sigma^1 \ldots \sigma^8$ implies $S\chi = (-)^8\chi S = \chi S$. This proves that we really have two inequivalent representations of $SO(9,1)$, which we denote by $\underline{16}$ and $\underline{16}'$.
In $D = (9,1)$ one cannot raise or lower spinor indices with the charge conjugation matrix because it is off–diagonal. Thus, we are free to define spinors κ_α and $\eta^{\dot{\alpha}}$ which transform under Lorentz transformations such that $\kappa_\alpha\lambda^\alpha$ and $\eta^{\dot{\alpha}}\chi_{\dot{\alpha}}$ remain invariant. If we denote the generators of λ^α by (γ^{kl}, γ^k) with $k, l = 1, \ldots, 8$ those for the $\chi_{\dot{\alpha}}$ are given by $(-\gamma^{kl,T}, -\gamma^{k,T}) = (\gamma^{kl}, -\gamma^k)$. (This follows from the form of the charge conjugation matrix.) Using the conditions for Lorentz invariance we imposed above we find that κ_α has $(\gamma^{kl}, -\gamma^k)$ and $\eta^{\dot{\alpha}}$ has (γ^{kl}, γ^k). Thus, κ_α and $\chi_{\dot{\alpha}}$ and λ^α and $\eta^{\dot{\alpha}}$ have the same transformation properties under Lorentz transformations. This is why we can omit the dots without causing confusion. We conclude that chiral spinors are given by λ^α and antichiral ones are given by χ_α.
Finally we give two important identities for the twenty real symmetric 16×16 matrices $\gamma^m_{\alpha\beta}$ and $\gamma^{m\,\alpha\beta}$:

$$\gamma^m_{\alpha\beta}\gamma^{n\,\beta\gamma} + \gamma^n_{\alpha\beta}\gamma^{m\,\beta\gamma} = -2\eta^{mn}\delta^\gamma_\alpha \tag{A.24}$$

$$\gamma_{m\,(\alpha\beta}\gamma^m_{\gamma)\delta} = 0 \tag{A.25}$$

In principle, both identities can be verified using the general form of the Fierz identity [46]

$$(\bar{\varphi}\Gamma^A\psi)(\bar{\chi}\Gamma^B\eta) = (-)^{\psi\bar{\chi}+\psi\eta+\bar{\chi}\eta}\frac{1}{2^{D/2}}\sum_I(\bar{\varphi}\Gamma^I\Gamma^B\eta)(\bar{\chi}\Gamma_I\Gamma^A\psi), \tag{A.26}$$

but since these calculations can turn out to be very tedious these identities will not be verified here. In fact, it is straight forward to prove the first relation directly by expressing the γ^m in terms of the σ^m and $\tilde{\sigma}^m$ and checking the relation component–wise using the Clifford algebra.

Appendix B

Fields and Ghosts

Since we are dealing with a large number of fields and ghosts we collect all the objects and their properties in the following tables.

Field	Grassmann Parity	Reality	Conf. Weight	Conjugate Momentum	Grassmann Parity	Reality	Conf. Weight	Central Charge
$x^{m(h)}$	0	real	0	$\Pi^{(h)}_{zm}$	0	real	1	1×10
$\theta^{\alpha(h)}$	1	herm.	0	$p^{(h)}_{z\alpha}$	1	antiherm.	1	-2×16
				$d^{(h)}_{z\alpha}$	1	antiherm.	1	

For the type II string the following definitions hold:

$$\Pi^m_\mu = \partial_\mu x^m - i\theta\gamma^m\partial_\mu\theta - i\hat{\theta}\gamma^m\partial_\mu\hat{\theta} \tag{B.1}$$

$$d_{\mu\alpha} = p_{\mu\alpha} - (\gamma^m\theta)_\alpha \left(i\partial_\mu x^m + \frac{1}{2}\theta\gamma^m\partial_\mu\theta + \frac{1}{2}\hat{\theta}\gamma^m\partial_\mu\hat{\theta} \right) \tag{B.2}$$

$$\hat{d}_{\mu\hat{\alpha}} = \hat{p}_{\mu\hat{\alpha}} - \left(\gamma^m\hat{\theta}\right)_{\hat{\alpha}} \left(i\partial_\mu x^m + \frac{1}{2}\theta\gamma^m\partial_\mu\theta + \frac{1}{2}\hat{\theta}\gamma^m\partial_\mu\hat{\theta} \right) \tag{B.3}$$

The OPEs for the elementary fields x^m, θ^α and $p_{z\alpha}$ and the corresponding h-fields are given by:

$$\begin{aligned} \partial x_m(z)\partial x_n(w) &\sim -\frac{\eta_{mn}}{(z-w)^2} & \partial x^h_m(z)\partial x^h_n w &\sim \frac{\eta_{mn}}{(z-w)^2} \\ p_{z\alpha}(z)\partial\theta^\beta(w) &\sim -\frac{\delta^\beta_\alpha}{(z-w)^2} & p^h_{z\alpha}(z)\partial\theta^{\beta h}(w) &\sim \frac{\delta^\beta_\alpha}{(z-w)^2} \end{aligned} \tag{B.4}$$

For practical calculations we mostly use the supersymmetric objects Π_{zm}, $\partial\theta^\alpha$ and $d_{z\alpha}$. These expressions can be grouped into "currents"[1] $J_M = (\Pi_{zm}, id_{z\alpha}, \partial\theta^\alpha)$ and $J^h_M = -(\Pi^h_{zm}, id^h_{z\alpha}, \partial\theta^{\alpha h})$. In condensed notation the relevant OPEs are:

$$J_M(z)J_N(w) \sim -\frac{J_K f^K{}_{MN}}{z-w} - \frac{\mathcal{H}_{MN}}{(z-w)^2}$$

[1] These objects are the conserved currents of the gauge symmetry described in section 3.3.

$$J^h_M(z)J^h_N(w) \sim \frac{J^h_K f^K{}_{MN}}{z-w} + \frac{\mathcal{H}_{MN}}{(z-w)^2} \tag{B.5}$$

The only non-vanishing structure constants are[2]:

$$\begin{aligned} f^m{}_{\alpha\beta} &= 2i\gamma^m_{\alpha\beta} = f^m_{\beta\alpha} \\ f^{\underline{\alpha}}{}_{\beta m} &= 2\gamma_{m\,\beta\alpha} = -f^{\underline{\alpha}}{}_{m\beta} \end{aligned} \tag{B.6}$$

The metric $\mathcal{H}_{MN}$ is:

$$\mathcal{H}_{MN} = \begin{pmatrix} \eta_{mn} & 0 & 0 \\ 0 & 0 & i\delta_\alpha^\beta \\ 0 & -i\delta^\alpha_\beta & 0 \end{pmatrix} \tag{B.7}$$

We can use its graded inverse

$$\mathcal{H}^{MN} = \begin{pmatrix} \eta^{mn} & 0 & 0 \\ 0 & 0 & -i\delta^\alpha_\beta \\ 0 & i\delta_\alpha^\beta & 0 \end{pmatrix}, \tag{B.8}$$

to pull indices: $J^M = J_N\mathcal{H}^{NM} = (\Pi^m_z, i\partial\theta^\alpha, d_{z\alpha})$.
Unpacking equations (B.5) yields the following operator algebra:

$$\begin{aligned} id_{z\alpha}(z)id_{z\beta}(w) &\sim -2i\frac{\gamma^m_{\alpha\beta}\Pi_{zm}(w)}{z-w} & id^h_{z\alpha}(z)id^h_{z\beta}(w) &\sim -2i\frac{\gamma^m_{\alpha\beta}\Pi^h_{zm}(w)}{z-w} \\ id_{z\alpha}(z)\Pi_{zm}(w) &\sim -2\frac{\gamma_{m\,\alpha\beta}\partial\theta^\beta}{z-w} & id^h_{z\alpha}(z)\Pi^h_{zm}(w) &\sim -2\frac{\gamma_{m\,\alpha\beta}\partial\theta^{\beta h}}{z-w} \\ \Pi_{zm}(z)\Pi_{zn}(w) &\sim -\frac{\eta_{mn}}{(z-w)^2} & \Pi^h_{zm}(z)\Pi^h_{zn}(w) &\sim \frac{\eta_{mn}}{(z-w)^2} \\ id_{z\alpha}(z)\partial\theta^\beta(w) &\sim -\frac{i\delta_\alpha^\beta}{(z-w)^2} & id^h_{z\alpha}(z)\partial\theta^{\beta h}(w) &\sim \frac{i\delta_\alpha^\beta}{(z-w)^2} \end{aligned} \tag{B.9}$$

Analogous OPEs hold for the right moving sector. One simply has to replace the fields by their hatted counterparts and worldsheet indices and derivatives by the antiholomorphic expressions. Metric and structure constants are numerically equal to those of the left moving sector.

Now we turn to the ghost fields of our model. The signs in the following table are chosen such that,

$$b_M(z)c^N(w) \sim \frac{\delta_M^N}{z-w}, \tag{B.10}$$

[2] In the following equation there appear underlined indices. They indicate that the indices are in the wrong position as compared to the capital Latin indices, e.g. M is an upper index but $\alpha \in M$ is downstairs. We will try to avoid this notation in order not to overload our formulas.

which is consistent with our convention $q(z)p(w) \sim -1/(z-w)$ where q is a field and p is its conjugate momentum.

Ghost	Grass. Parity	Reality	Conf. Weight	n_{gh}	Antighost	Grass. Parity	Reality	Conf. Weight	n_{gh}	Central Charge
$-\xi^m$	1	herm.	0	1	β_{zm}	1	antih.	1	-1	-2×10
λ^α	0	real	0	1	$\omega_{z\alpha}$	0	real	1	-1	2×16
χ_α	0	real	0	1	κ_z^α	0	real	1	-1	2×16
$-\xi'^m$	1	herm.	0	1	β'_{zm}	1	antih.	1	-1	-2×10
χ'_α	0	real	0	1	$\kappa_z'^\alpha$	0	real	1	-1	2×16
φ^m	0	herm	0	0	π_{zm}	0	antih.	1	0	1×10
$-i\varphi_\alpha$	1	real	0	0	$-i\pi_z^\alpha$	1	real	1	0	-2×16
$-c'^z$	1	herm.	-1	1	b'_{zz}	1	herm.	2	-1	-26
γ'^z	0	herm.	-1	2	β'_{zz}	0	antih.	2	-2	26
$-c^z$	1	herm.	-1	1	b_{zz}	1	herm.	2	-1	-26
γ^z	0	herm.	-1	2	β_{zz}	0	antih.	2	-2	26

Note that we included the fields (φ^m, π_{zm}) and $(\varphi_\alpha, \pi_z^\alpha)$ into this table although they are ghost number 0 fields with the "correct" statistics. However, considering the way these fields are introduced into our model, it seems more natural to place them among the ghosts.

Appendix C

Identities in 2 Dimensions

In two dimensions we have the following identities that involve the ε–tensor:

$$\varepsilon^{\mu\nu}\varepsilon^{\rho\lambda} = -g^{\mu\rho}g^{\nu\lambda} + g^{\mu\lambda}g^{\nu\rho} \tag{C.1}$$

$$\varepsilon_{\alpha\beta}\varepsilon^{\beta\gamma} = \delta_\alpha^\gamma \tag{C.2}$$

$$T^{[\mu_1\mu_2]\mu_3\cdots\mu_n} = \frac{1}{2}\varepsilon^{\mu_1\mu_2}\varepsilon_{\lambda_2\lambda_2}T^{\lambda_1\lambda_2\mu_3\cdots\mu_n}, \tag{C.3}$$

where $T^{\mu_1\cdots\mu_n}$ is an arbitrary tensor of rank n.
We define the chiral projection operators [2]:

$$\begin{aligned} P^{\mu\nu} &= g^{\mu\nu} - \varepsilon^{\mu\nu} \\ \bar{P}^{\mu\nu} &= g^{\mu\nu} + \varepsilon^{\mu\nu} = P^{\nu\mu} \end{aligned} \tag{C.4}$$

Using (C.2) it is easy to show that they indeed satisfy the properties of projection operators:

$$P^{\mu\lambda}P_\lambda{}^\nu = 2P^{\mu\nu} \tag{C.5}$$

$$P^{\mu\lambda}P^\nu{}_\lambda = 0 \tag{C.6}$$

The projector also satisfies the following useful identities:

$$P^{\lambda(\mu}P^{\nu)\rho} = g^{\mu\nu}P^{\lambda\rho} \tag{C.7}$$

$$P^{\mu[\nu}P^{\lambda]\rho} = \varepsilon^{\nu\lambda}P^{\mu\rho} \tag{C.8}$$

$$P^{[\mu\nu}P^{\lambda]\rho} = 0 \tag{C.9}$$

In the last equation the squared brackets refer to antisymmetrization with respect to μ and λ. The first two identities can be combined to $P^{\mu\nu}P^{\lambda\rho} = P^{\lambda\nu}P^{\mu\rho}$.

- Equation (C.7) can be verified by inserting the definition of $P^{\mu\nu}$ and using (C.1):

$$\begin{aligned} g^{\mu\nu}P^{\lambda\rho} - P^{\lambda(\nu}P^{\mu)\rho} &= g^{\mu\nu}g^{\lambda\rho} - g^{\mu\nu}\varepsilon^{\lambda\rho} - g^{\lambda(\nu}g^{\mu)\rho} + g^{\lambda(\nu}\varepsilon^{\mu)\rho} + \varepsilon^{\lambda(\nu}g^{\mu)\rho} - \varepsilon^{\lambda(\nu}\varepsilon^{\mu)\rho} \\ &\overset{(C.1)}{=} -g^{\mu\nu}\varepsilon^{\lambda\rho} + g^{\lambda(\nu}\varepsilon^{\mu)\rho} + \varepsilon^{\lambda(\nu}g^{\mu)\rho} \end{aligned}$$

  Now we observe that $-g^{\mu\nu}\varepsilon^{\lambda\rho} + g^{\lambda\nu}\varepsilon^{\mu\rho}$ is antisymmetric under the exchange of λ and μ which suggests to use (C.3):

$$-g^{\mu\nu}\varepsilon^{\lambda\rho} + g^{\lambda(\nu}\varepsilon^{\mu)\rho} = \frac{1}{2}\varepsilon^{\lambda(\mu}\varepsilon_{\alpha\beta}\left(-g^{\alpha\nu)}\varepsilon^{\beta\rho} + g^{\beta\nu)}\varepsilon^{\alpha\rho}\right)$$

$$
\begin{aligned}
&\stackrel{(C.2)}{=} -\frac{1}{2}\varepsilon^{\lambda(\mu}g^{\alpha\nu)}\delta^{\rho}_{\alpha} - \frac{1}{2}\varepsilon^{\lambda(\mu}g^{\beta\nu)}\delta^{\rho}_{\beta} \\
&= -\varepsilon^{\lambda(\mu}g^{\nu)\rho}
\end{aligned}
$$

This completes the proof of the identity (C.7).

- To prove (C.8)we use (C.3):

$$
\begin{aligned}
P^{\mu[\nu}P^{\lambda]\rho} &= \frac{1}{2}\varepsilon^{\nu\lambda}\varepsilon_{\sigma\tau}P^{\mu\tau}P^{\sigma\rho} \\
&= \frac{1}{2}\varepsilon^{\nu\lambda}\left(\varepsilon_{\sigma\tau}g^{\mu\tau} - \varepsilon_{\sigma\tau}\varepsilon^{\mu\tau}\right)P^{\sigma\rho} \\
&= \frac{1}{2}\varepsilon^{\nu\lambda}\left(\varepsilon_{\sigma}{}^{\mu} + \delta^{\mu}_{\sigma}\right)P^{\sigma\rho} = \frac{1}{2}\varepsilon^{\nu\lambda}P^{\mu}{}_{\sigma}P^{\sigma\rho} \\
&= \varepsilon^{\nu\lambda}P^{\mu\rho}
\end{aligned}
$$

- Since I could not think of an elegant proof for the identity (C.9) so I chose the direct way and checked it component–wise. Using (A.5) we can write $P^{\mu\nu}$ as a matrix:

$$
P^{\mu\nu} = \begin{pmatrix} g^{00} & g^{01} - \frac{1}{\sqrt{-g}} \\ g^{01} + \frac{1}{\sqrt{-g}} & g^{11} \end{pmatrix} \tag{C.10}
$$

Most components of (C.9) vanish trivially, a non–trivial one is for example

$$
P^{[11}P^{0]0} = P^{11}P^{00} - P^{01}P^{10} = g^{11}g^{00} - \left(\left(g^{01}\right)^2 + \frac{1}{g}\right) = 0, \tag{C.11}
$$

where we used $g = \det(g_{\mu\nu}) = 1/\det(g^{\mu\nu}) = g^{00}g^{11} - (g^{01})^2$.

C.1 Beltrami Differentials

Consider the following parameterization of the vielbeine:

$$
\begin{aligned}
e^{z} &= d\sigma^{z}e_{z}{}^{z} + d\sigma^{\bar{z}}e_{\bar{z}}{}^{z} = \left(d\sigma^{z} + d\sigma^{\bar{z}}\mu_{\bar{z}}{}^{z}\right)e_{z}{}^{z} \\
e^{\bar{z}} &= d\sigma^{z}e_{z}{}^{\bar{z}} + d\sigma^{\bar{z}}e_{\bar{z}}{}^{\bar{z}} = \left(d\sigma^{z}\bar{\mu}_{z}{}^{\bar{z}} + d\sigma^{\bar{z}}\right)e_{\bar{z}}{}^{\bar{z}}
\end{aligned} \tag{C.12}
$$

$\mu_{\bar{z}}{}^{z}$ and $\bar{\mu}_{z}{}^{\bar{z}}$ are called the Beltrami differentials. Using the above equation they can be expressed in terms of the vielbeine:

$$
\mu_{\bar{z}}{}^{z} = \frac{e_{\bar{z}}{}^{z}}{e_{z}{}^{z}} \qquad \bar{\mu}_{z}{}^{\bar{z}} = \frac{e_{z}{}^{\bar{z}}}{e_{\bar{z}}{}^{\bar{z}}} \tag{C.13}
$$

Thus, we can write the vielbein as follows:

$$
e_{\mu}{}^{a} = \begin{pmatrix} e_{z}{}^{z} & \bar{\mu}_{z}{}^{\bar{z}}e_{\bar{z}}{}^{\bar{z}} \\ \mu_{\bar{z}}{}^{z}e_{z}{}^{z} & e_{\bar{z}}{}^{\bar{z}} \end{pmatrix} \tag{C.14}
$$

Its determinant is given by:

$$
\det e = e\bar{e}\left(1 - \mu\bar{\mu}\right), \tag{C.15}
$$

where we set $e \equiv e_z{}^z$, $\bar{e} \equiv e_{\bar{z}}{}^{\bar{z}}$, $\mu \equiv \mu_{\bar{z}}{}^z$, $\bar{\mu} \equiv \bar{\mu}_z{}^{\bar{z}}$. Now we can express the metric in terms of the vielbeine[1]:

$$\begin{aligned} g_{\mu\nu} &= e_\mu{}^a g_{ab} e_\nu{}^b = \begin{pmatrix} e & \bar{\mu}\bar{e} \\ \mu e & \bar{e} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} e & \bar{\mu}\bar{e} \\ \mu e & \bar{e} \end{pmatrix}^T \\ &= e\bar{e} \begin{pmatrix} 2\bar{\mu} & 1+\mu\bar{\mu} \\ 1+\mu\bar{\mu} & 2\bar{\mu} \end{pmatrix} \end{aligned} \tag{C.16}$$

The determinant of the metric is:

$$\det g = -\det e^2 = -(e\bar{e})^2 (1-\mu\bar{\mu})^2 \tag{C.17}$$

The inverse of the metric is given by:

$$g^{\mu\nu} = \frac{1}{e\bar{e}(1-\mu\bar{\mu})^2} \begin{pmatrix} -2\mu & 1+\mu\bar{\mu} \\ 1+\mu\bar{\mu} & 2\bar{\mu} \end{pmatrix} \tag{C.18}$$

For the chiral projectors expressed in terms of the Beltrami differentials we get:

$$\begin{aligned} P^{\mu\nu} &= -\frac{1}{e\bar{e}(1-\mu\bar{\mu})^2} \begin{pmatrix} 2\mu & -(1+\mu\bar{\mu}) \\ -(1+\mu\bar{\mu}) & 2\bar{\mu} \end{pmatrix} + \frac{1}{e\bar{e}(1-\mu\bar{\mu})} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ &= \frac{2}{e\bar{e}(1-\mu\bar{\mu})^2} \begin{pmatrix} -\mu & 1 \\ \mu\bar{\mu} & -\bar{\mu} \end{pmatrix} \end{aligned} \tag{C.19}$$

Pulling one index with $g_{\mu\nu}$ we find:

$$P^\mu{}_\nu = \frac{2}{e\bar{e}(1-\mu\bar{\mu})} \begin{pmatrix} 1 & \mu \\ -\bar{\mu} & -\mu\bar{\mu} \end{pmatrix} \tag{C.20}$$

[1]Note that there is a small inconsistency in our conventions. Here we use $g_{ab} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ instead of $g_{ab} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$ of our conventions. In our calculations this does not matter because the factor always cancels in $\sqrt{g} g^{\mu\nu}$.

Appendix D

Mathematica File

The OPEs for the $N = 2$ and $N = 4$ algebra were computed with Mathematica using the OPE package by Chris Thielemans [36]. The file with the definitions of the fields and currents looks as follows:

Declaration of the Fields

```
Bosonic[Π[_]];
Fermionic[d[_], dθ[_]];
Bosonic[Πh[_]];
Fermionic[dh[_], dθh[_]];
Fermionic[β[_], ξ[_]];
Bosonic[κ[_], χ[_]];
Bosonic[ω[_], λ[_]];
Fermionic[β1[_], ξ1[_]];
Bosonic[κ1[_], χ1[_]];
Bosonic[πbos[_], φbos[_]];
Fermionic[πfer[_], φfer[_]];
Fermionic[b1, c1];
Bosonic[bet1, gam1];
Fermionic[b2, c2];
Bosonic[bet2, gam2];
```

Dummies

```
DefineDummy[i];
DefineDummy[μ];
dimension[i] = 10;
dimension[μ] = 16;
```

Basic OPEs

```
OPE[Π[m_], Π[n_]] := MakeOPE[{-Delta[m, n] One, 0}];
OPE[d[α_], dθ[β_]] := MakeOPE[{- Delta[α, β] One, 0}];
OPE[d[α_], d[β_]] := NewDummies[MakeOPE[{2 ⅈ γ[i[1]][α][β] Π[i[1]]}]];
OPE[Π[m_], d[α_]] := NewDummies[MakeOPE[{-2 ⅈ γ[m][α][μ[1]] dθ[μ[1]]}]];

OPE[Πh[m_], Πh[n_]] := MakeOPE[{Delta[m, n] One, 0}];
OPE[dh[α_], dθh[β_]] := MakeOPE[{Delta[α, β] One, 0}];
OPE[dh[α_], dh[β_]] := NewDummies[MakeOPE[{2 ⅈ γ[i[1]][α][β] Πh[i[1]]}]];
OPE[Πh[m_], dh[α_]] := NewDummies[MakeOPE[{-2 ⅈ γ[m][α][μ[1]] dθh[μ[1]]}]];

OPE[β[m_], ξ[n_]] := MakeOPE[{-Delta[m, n] One}];
OPE[κ[α_], χ[β_]] := MakeOPE[{Delta[α, β] One}];
OPE[ω[α_], λ[β_]] := MakeOPE[{Delta[α, β] One}];

OPE[β1[m_], ξ1[n_]] := MakeOPE[{-Delta[m, n] One}];
OPE[κ1[α_], χ1[β_]] := MakeOPE[{Delta[α, β] One}];
OPE[πbos[m_], φbos[n_]] := MakeOPE[{Delta[m, n] One}];
OPE[πfer[α_], φfer[β_]] := MakeOPE[{-Delta[α, β] One}];

OPE[b1, c1] := MakeOPE[{- One}];
OPE[bet1, gam1] := MakeOPE[{ One}];
OPE[b2, c2] := MakeOPE[{- One}];
OPE[bet2, gam2] := MakeOPE[{ One}];
```

Energy Momentum Tensor

```
Twzw := NewDummies[
   -1 / 2 NO[Π[i[1]], Π[i[1]]] + 1 / 2 NO[Πh[i[1]], Πh[i[1]]] + NO[d[μ[1]], dθ[μ[1]]] -
    NO[dh[μ[1]], dθh[μ[1]]] + NO[β[i[1]], Derivative[1][ξ[i[1]]]] +
    NO[ω[μ[1]], Derivative[1][λ[μ[1]]]] + NO[κ[μ[1]], Derivative[1][χ[μ[1]]]]];

Tco := NewDummies[NO[β1[i[1]], Derivative[1][ξ1[i[1]]]] +
    NO[κ1[μ[1]], Derivative[1][χ1[μ[1]]]] + NO[πbos[i[1]], Derivative[1][φbos[i[1]]]] +
    NO[πfer[μ[1]], Derivative[1][φfer[μ[1]]]]];

Tk1 := 2 NO[bet1, Derivative[1][gam1]] + NO[Derivative[1][bet1], gam1] +
   2 NO[b1, Derivative[1][c1]] + NO[Derivative[1][b1], c1];

Tk2 := 2 NO[bet2, Derivative[1][gam2]] + NO[Derivative[1][bet2], gam2] +
   2 NO[b2, Derivative[1][c2]] + NO[Derivative[1][b2], c2];
```

F− and Phi−currents

```
F := NewDummies[-ⅈ γ[i[1]][μ[1]][μ[2]] NO[β[i[1]], κ[μ[1]], dθ[μ[2]]] +
    ⅈ γ[i[1]][μ[1]][μ[2]] NO[β[i[1]], κ[μ[1]], dθh[μ[2]]] +
    ⅈ / 2 γ[i[1]][μ[1]][μ[2]] NO[κ[μ[1]], κ[μ[2]], Π[i[1]]] -
    ⅈ / 2 γ[i[1]][μ[1]][μ[2]] NO[κ[μ[1]], κ[μ[2]], Πh[i[1]]]];

Φ := NewDummies[ⅈ / 2 γ[i[1]][μ[1]][μ[2]] NO[β[i[1]], κ[μ[1]], κ[μ[2]]]];
```

B−current

```
Bwzw := NewDummies[-1 / 2 NO[Π[i[1]], β[i[1]]] - 1 / 2 NO[Πh[i[1]], β[i[1]]] +
    i / 2 NO[i d[μ[1]], κ[μ[1]]] + i / 2 NO[i dh[μ[1]], κ[μ[1]]] -
    i / 2 NO[dθ[μ[1]], ω[μ[1]]] - i / 2 NO[dθh[μ[1]], ω[μ[1]]]];

Bco := NewDummies[
   NO[β1[i[1]], Derivative[1][φbos[i[1]]]] + NO[κ1[μ[1]], Derivative[1][φfer[μ[1]]]]];

Bk1 := -2 NO[bet1, Derivative[1][c1]] - NO[c1, Derivative[1][bet1]] - b1;

Bnil := -1 / 2 NO[c1, F] - 1 / 2 NO[gam1, Φ];

Bk2 := -2 NO[bet2, Derivative[1][c2]] - NO[c2, Derivative[1][bet2]] - p b2;
```

BRST – current

```
Jwzw := NewDummies[-NO[Π[i[1]], ξ[i[1]]] + NO[Πh[i[1]], ξ[i[1]]] -
    NO[i d[μ[1]], λ[μ[1]]] + NO[ i dh[μ[1]], λ[μ[1]]] - NO[dθ[μ[1]], χ[μ[1]]] +
    NO[dθh[μ[1]], χ[μ[1]]] + i γ[i[1]][μ[1]][μ[2]] NO[β[i[1]], λ[μ[1]], λ[μ[2]]] +
    2 γ[i[1]][μ[1]][μ[2]] NO[κ[μ[1]], λ[μ[2]], ξ[i[1]]]];

Jco := NewDummies[-NO[πbos[i[1]], ξ1[i[1]]] - NO[πfer[μ[1]], χ1[μ[1]]]];

Jk1 := NO[b1, gam1];

Jk2 := NO[b2, gam2];

Jv := NO[c2, Twzw + 1 / 2 Tk2] + NO[gam2, Bwzw + Bnil + 1 / 2 Bk2];

Jvco := NO[c2, Tco] + NO[gam2, Bco];
```

Ghost Current

```
jwzw := NewDummies[NO[β[i[1]], ξ[i[1]]] + NO[ω[μ[1]], λ[μ[1]]] + NO[κ[μ[1]], χ[μ[1]]]];

jco := NewDummies[NO[β1[i[1]], ξ1[i[1]]] + NO[κ1[μ[1]], χ1[μ[1]]]];

jk1 := NO[b1, c1] + 2 NO[bet1, gam1];

jk2 := NO[b2, c2] + 2 NO[bet2, gam2];
```

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