# Aspects of Superstring Compactification

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**Abstract**

In this review we will discuss the conditions for superstring compactification on Calabi-Yau manifolds following from the conditions for unbroken supersymmetry in four dimensions. The generalization for superstrings with torsion will be given. Finally relations between Yukawa couplings in four dimensions will be derived from a ten-dimensional action. We will point out that the relations between these couplings and, in particular, their vanishing only depend on topological invariants of the internal manifold.

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# Contents

# 1 Introduction

This review focuses on superstring compactification. In the first section we will derive the conditions for compactification on Calabi-Yau manifolds from the conditions of unbroken supersymmetry in four dimensions. In the course of this discussion some basic notions of complex geometry will be introduced.
The second section gives a generalization of the conditions for unbroken supersymmetry for superstrings with torsion.
In the third section we will show how relations between Yukawa couplings in four dimensions can be obtained from a ten dimensional action. It will be shown that these relations only depend on topological invariants of the internal manifold.

In the first two sections we use the following conventions concerning indices: Capital latin letters $M, N, P, \ldots$ denote ten dimensional space time coordinates $X^M, M = 1, \ldots, 10$. Greek indices $\mu, \nu, \lambda, \ldots = 1, \ldots, 4$ will be used for coordinates $x^\mu$ of four dimensional space time. Indices of $K$ will be denoted with small latin letters $i, j, k \ldots = 5, \ldots, 10$. The coordinates of $K$ are $y^i$. These conditions cannot be maintained for the last section. Modifications of this conventions will be indicated when necessary.

# 2 Conditions for Calabi-Yau Compactification

Since superstring theory is a ten dimensional theory we have to find relations to four dimensional space if we want to discuss phenomenology. For that purpose we assume that our string theory lives on a ten dimensional manifold of the form $M^4 \times K$ where $M^4$ stands for the four dimensional Minkowski or (anti-)deSitter space and $K$ denotes an a priori arbitrary six dimensional manifold. It can be shown [2] that the conditions for unbroken supersymmetry imply that the cosmological constant vanishes and we therefore have Minkowski space.
The main challenge of compactification is to find a suitable six dimensional manifold which allows a low energy state in $M^4$ that resembles our four dimensional physics. This amounts to a certain conditions about topology and geometry of $K$. In particular, we want to find the correct particle spectrum. This question is linked to topological properties of the six-manifold. We can, for instance, find a relation between the number of fermion generations and the Euler number of $K$: $N_{gen} = |\chi(K)/2|$ [2]. Conditions like this reduce the choice of suitable manifolds but still the possibilities to choose $K$ are vast. Thus, we have to impose further constraints on K such as $N = 1$ low energy supersymmetry which will eventually lead us to Calabi-Yau manifolds.
There are several reasons that justify the choice of unbroken supersymmetry at the low energy level.

- **Gauge Hierarchy Problem:** The mass scale of weak symmetry breaking is very small compared to fundamental scales such as Planck mass. Quarks and leptons are extremely light particles. This implies that they do not get mass through the mechanism of compactification, i.e. they obey a massless Dirac equation in 10 dimensions and get their masses from spontaneous symmetry breaking. It is assumed that the $SU(2) \times U(1)$ Higgs doublet must remain massless at the compactification scale. There are mechanisms that yield massless charged fermions (cf. chapter 14 in [2]) but it is difficult to explain the existence of massless charged spin 0 fields. The only possibility to obtain such fields is to postulate unbroken supersymmetry of the low-energy world because the massless charged scalar fields can be obtained as supersymmetric partners of massless charged fermions. The small mass scale of $SU(2) \times U(1)$ breaking that we observe must be related to a small scale of supersymmetry breaking.

- **Solutions of the Equations of Motion:** States of unbroken supersymmetry always obey the equations of motion of the theory. In global supersymmetry the Hamiltonian is positive semidefinite and vanishes only when supersymmetry is unbroken.

- **Restriction for Phenomenology:** This hypothesis states that $N = 1$ supersymmetry in four dimensions gives just the right amount of restrictions to the theory without restricting phenomenology too severely. An unbroken supersymmetry algebra with $N \geq 2$ could not lead to a realistic model because this would imply a real representation of the gauge group. Since we observe chiral fermions in nature this cannot be the case. The relevant gauge groups are $SO(32)$ and $E_8 \times E_8$ whose gauge fields can create chiral fermions in four dimensions. In perturbative heterotic string theory $SO(32)$ is nonchiral and therefore not of interest.

## 2.1 SUGRA and $SU(3)$ Holonomy

### 2.1.1 Unbroken Supersymmetry

The large volume and low energy limit of superstring theory in four dimensions with unbroken supersymmetry should be compatible with supergravity. Thus, our conditions for unbroken supersymmetry can be obtained from transformations in supergravity.
Let us consider an infinitesimal supersymmetry parameter $\eta_\alpha(X^M)$. For each choice of $\eta_\alpha$ there is a conserved supercharge $Q$. We are interested in the class of supercharges that generate an unbroken supersymmetry. This means we are looking for a conserved supercharge $Q$ that annihilates the vacuum state $|\Omega\rangle$. This is equivalent to the statement $\langle\Omega|\{Q, U\}|\Omega\rangle = 0$ where $U$ is a generic operator which can be bosonic or fermionic. If $U$ is a bosonic operator the anticommutator $\{Q, U\}$ is fermionic and the vacuum expectation value above vanishes trivially. For a fermionic operator $U$ the anticommutator is a variation $\delta U$ of $U$ under the supersymmetry transformation generated by $Q$. Considering the classical limit $\delta U$ and $\langle\Omega|\delta U|\Omega\rangle$ coincide. For unbroken supersymmetry at tree level we must find a transformation such that $\delta U = 0$ for every fermionic operator $U$.
Now we consider the supersymmetry transformations of the elementary fermion fields of the low energy effective theory in ten dimensions. The conventions for the following formulas are those of [6]. We will denote the fields with a superscript 0 since it will be convenient to redefine them in section 3.

$$
\begin{aligned}
\delta\psi_M^0 &= \nabla_M^0\eta^0 + \frac{1}{48}e^{2\phi}\left(\gamma_M^0 H^0 - 12H_M^0\right)\eta^0 + (\text{Fermi})^2 \\
\delta\lambda^0 &= \nabla\!\!\!/^0\phi\eta^0 + \frac{1}{24}e^{2\phi}H^0\eta^0 + (\text{Fermi})^2 \\
\delta\chi^0 &= e^\phi F_{MN}\Gamma^{0MN}\eta^0 + (\text{Fermi})^2 \qquad (1)
\end{aligned}
$$

$\psi_M^0$ stands for the gravitino, $\lambda^0$ is the spin 1/2 dilatino, the superpartner of the dilaton, and $\chi^0$ denotes the gluino. Furthermore, $\phi$ is the dilaton field, $F_{MN}$ is the Yang-Mills field strength (we suppressed the gauge group indices here), and $H$ is the gauge invariant field strength of the antisymmetric tensor field $B_{MN}$. The $\Gamma^{0M}$ are the ten dimensional Dirac matrices that obey $\Gamma^{0M} = e^M_A \Gamma^{0A}$ where $e^M_A$ is the inverse vielbein and $\Gamma^{0A}$ are the 32 dimensional[1] constant matrices with $\{\Gamma^{0A}, \Gamma^{0B}\} = 2\eta^{AB}$. $\Gamma$s with multiple indices represent antisymmetrized products of Dirac matrices with unit weight. The expression (Fermi)$^2$ stands for quadratic terms in fermionic fields whose explicit form is not needed here.

Now we impose unbroken supersymmetry: $\delta\psi^0_M = \delta\lambda^0 = \delta\chi^0 = 0$. While we can arbitrarily specify the space time metric and the dilaton field, $F$ and $H$ have to obey a Bianchi identity. In ten dimensional field theory this would be $dH = \mathrm{tr} F \wedge F$ which gets corrections from string theory that yield:

$$dH = \mathrm{tr} R \wedge R - \frac{1}{30} \mathrm{Tr} F \wedge F \tag{2}$$

where $R$ and $F$ are the curvature two-form and the Yang-Mills field strength two-form respectively. In our conventions Tr denotes a trace in the adjoint of $E_8 \otimes E_8$ or $SO(32)$, while tr denotes a trace in the vector of $O(1,9)$.

Now we make three simplifications:

1. We neglect possible string theoretic corrections to the transformations (1).
2. The gauge invariant three form $H$ vanishes and consequently the Bianchi identity reduces to $dH = 0$.
3. The dilaton field $\phi$ is constant.

These are of course strong restrictions. In principle we would like to search for vacuum states not just in the limiting low energy field theory but in the exact string theory with all its massive degrees of freedom. A string theoretical generalization of the equations (1) would demand studying the supersymmetry variations of the whole infinite tower of massive fields. If the massive fields are integrated out we would have to study an infinite series of corrections to (1) of higher and higher order in the string coupling constant $\alpha'$. Although these equations are not known it is possible to show that the results obtained from the limiting low energy field theory are valid to all finite orders in $\alpha'$ [2].

It is possible to generalize the second and the third condition by allowing $H \neq 0$ and using a nonconstant dilaton field. Then we obtain a more general class of supersymmetric vacuum states. In this case we speak of "superstrings with torsion" which will be discussed in section 3.

With our simplifications the conditions for unbroken supersymmetry reduce to the following equations:

$$\begin{aligned} \delta\psi^0_M &= \nabla^0_M \eta^0 = 0 \\ \delta\chi^0 &= \Gamma^{0ij} F_{ij} \eta^0 = 0 \end{aligned} \tag{3}$$

We will concentrate on the first equation in (3) since it is related to the condition for Calabi-Yau manifolds. This equation states that the generator of unbroken supersymmetry must be covariantly constant. Using the relation $[\nabla_M, \nabla_N] = R_{MNPQ}\Gamma^{PQ}$ where $R_{MNPQ}$ is the Riemann curvature tensor in ten dimensions we get the following condition:

$$[\nabla^0_M, \nabla^0_N]\eta^0 = R_{MNPQ}\Gamma^{0PQ}\eta^0 = 0 \tag{4}$$

[1] The dimension of the spinor representation is $2^{D/2}$ where $D$ is the dimension.

The first four components $M = 1, \ldots, 4$ tell us that $\eta^0$ must be independent of the uncompactified coordinates. Thus, (4) constrains the properties of the manifold $K$.
Note, that the second equation in (3) only carries indices of the compact manifold. The corresponding contribution from Minkowski space would imply non-trivial electromagnetic backgrounds for the vacuum solutions.

#### 2.1.2 $SU(3)$ Holonomy

On an orientable $n$–dimensional Riemannian manifold $K$ the spin connection $\omega$ is an $SO(N)$ gauge field. If a physical field $\psi$ is parallel transported around a curve $\gamma$ the field transforms into $U\psi$ with $U = P \exp \int_\gamma \omega \cdot dx$ where $P$ is the path-ordering symbol. These $SO(N)$ matrices $U$ form a group which is called the holonomy group $H$ of the manifold.
The condition $\nabla_i^0 \eta^0 = 0$ is closely related to holonomy. A covariantly constant field $\eta^0$ always returns to its original value when parallel transported around a closed curve[2]. Thus, for the six dimensional manifold $K$ the condition (4) implies that the $SO(6)$ matrices $U$ of the holonomy group obey the condition $U\eta^0 = \eta^0$. This means we have to find a subgroup of $SO(6)$ where this condition is fulfilled. Group theory tells us that the Lie algebra of $SO(6)$ is isomorphic to that of $SU(4)$. The positive and negative chirality spinors of $SU(4)$ are the fundamental $\mathbf{4}$ and $\bar{\mathbf{4}}$ respectively. We assume that $\eta^0$ has positive chirality, which means that it belongs to $\mathbf{4}$ of $SU(4)$. $SU(4)$ has the stabilizer $SU(3)$ that leaves invariant a vector in $\mathbf{4}$. This subgroup is $SU(3)$. To see this we make an $SU(4)$ transformation that puts $\eta^0$ into the form

$$\eta^0 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \eta_0 \end{pmatrix} .$$

$SU(3)$ is the subgroup of $SU(4)$ that acts on the first three components of the vector. The $\mathbf{4}$ of $SU(4)$ contains precisely one $SU(3)$ singlet. Thus, a six dimensional manifold whose holonomy group is $SU(3)$ contains precisely one covariantly constant spinor field $\eta^0$ of positive chirality. The complex conjugate $\eta^0_-$ is a unique covariantly constant spinor field of negative chirality.
If we demand $N = 1$ supersymmetry in four dimensions the holonomy group of $K$ should be precisely $SU(3)$. This can be shown by arguments of group theory. The ten dimensional theory transforms under $SO(1,9)$, which has a 16 component positive chirality spinor. Under an $SO(1,3) \times SO(6)$ or $SO(1,3) \times SU(4)$ subgroup of $SO(1,9)$ the spinor decomposes as $\mathbf{16} \approx (\mathbf{2}, \mathbf{4}) \oplus (\mathbf{2}', \bar{\mathbf{4}})$. The $\mathbf{2}$ and $\mathbf{2}'$ denote the positive and negative chirality spinor representations of $SO(1,3)$ respectively. As we had argued before the $\mathbf{4}$ and $\bar{\mathbf{4}}$ both contain a single covariantly constant spinor field. It follows that the covariantly constant components of $\mathbf{16}$ will transform as $\mathbf{2} \oplus \mathbf{2}'$ under $SO(1,3)$. Since $\mathbf{2}$ and $\mathbf{2}'$ are complex conjugates of one another they correspond to a single four component Majorana spinor of $SO(1,3)$. Thus, we have shown, that $SU(3)$ holonomy in $K$ will yield unbroken $N = 1$ supersymmetry in four dimensions.

[2] Consider, for instance, the definition of the Riemann tensor by parallel transport of a vector around a closed loop. In flat space the Riemann tensor which can be written as a commutator of covariant derivatives vanishes and a vector that is parallel transported around a closed loop returns to its original value. Note that also the field strength tensor in gauge theory can be written as a commutator of covariant derivatives.

### 2.2 Manifolds with $SU(3)$ Holonomy

Now we have to find out which manifolds $K$ allow a metric of $SU(3)$ holonomy. Our starting point will be the fact that we have a covariantly constant field $\eta^0$. It will turn out to be useful to construct tensorial objects out of $\eta^0$ and its hermitian conjugate $\eta^{\dagger 0}$, for example the Kähler form $k_{ij}$ and the complex structure $J_i^{\ j} = k_{ik}g^{kj} = i\eta^{\dagger 0}\Gamma_i^{\ j}\eta^0$. We will introduce many notions of complex geometry, but we will not review the whole subject because this would go beyond the scope of the paper. For a solid introduction to complex geometry see for example [4].

#### 2.2.1 Complex Manifolds

We have to embed $SU(3)$ or $U(3)$ into $SO(6)$. This can be done in the following way. A $U(3)$ matrix is a $3\times 3$ matrix with complex components $a_{ij}$. It is possible to replace the imaginary unit $i$ by a real $2\times 2$ matrix that squares to one which we call the complex structure $I$:

$$I = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{5}$$

Instead of writing the elements of the $U(3)$ matrix as $a_{ij} = \mathrm{Re}a_{ij} + i\mathrm{Im}a_{ij}$ we can replace the imaginary unit by the real matrix above. With that $U$ becomes a real $6\times 6$ matrix $\tilde{U}$ with components $\tilde{a}_{ij} = \mathrm{Re}a_{ij} + I\mathrm{Im}a_{ij}$. If $U$ is unitary then $\tilde{U}$ is orthogonal. With that we have embedded $(S)U(3)$ in $SO(6)$. In particular, the matrix $U$ with $a_{ij} = \delta_{ij}$ can be replaced by the following real $6\times 6$ matrix:

$$\bar{I} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix} \tag{6}$$

Apart from the identity matrix $\bar{I}$ is the only $(S)U(3)$ invariant matrix in the fundamental **6** of $SO(6)$ that commute with $U(3)$. That there are only two matrices that have this property follows from the fact that a matrix the commutes with $U(3)$ must be proportional to unity in each irreducible $U(3)$ representation. The **6** of $SO(6)$ decompose as $\mathbf{6} \approx \mathbf{3} \oplus \bar{\mathbf{3}}$ and therefore there are only two matrices in **6** of $SO(6)$ that commute with $U(3)$. $\bar{I}$ can be defined uniquely (up to a factor) by saying it is real, traceless and commutes with $U(3)$.

#### 2.2.2 Almost Complex Structure

We mentioned in the previous section that we can use the covariantly constant field $\eta^0$ on the manifold $K$ of $SU(3)$ to define a tensor field

$$J_i^{\ j} = i\eta^{\dagger 0}\Gamma_i^{\ j}\eta^0 \tag{7}$$

where $\Gamma_i^{\ j}$ stands for the antisymmetrized product of two $\Gamma$ matrices. We can consider $J_i^{\ j}$ as a matrix that acts on the tangent vectors $v^i$ on the manifold by $v^i \to J^i_{\ j}v^j$ (we use the metric to shift the indices of $J$ to the required position). From that point of view $J$ is real, traceless (because it is antisymmetric) and $SU(3)$ invariant which are exactly the properties of the unique matrix $\bar{I}$ which makes $J$ equivalent to $\bar{I}$. In particlar, $J$ obeys $J^2 = -1$, which we will show now. A tensor field $J_i{}^j$ on a manifold that obeys $J^2 = -1$ is called *almost complex structure*. Since we do not need $SU(3)$ holonomy or the fact that the spinor is covariantly constant as indicated in (3) the following calculations are also true for the generalization to superstrings with torsion. Therefore we will skip the superscript 0 and define the tensor field $J$ as

$$J_i{}^j = i\eta^\dagger \Gamma_i{}^j \eta \tag{8}$$

with a Weyl spinor $\eta$ that is normalized by $\eta^\dagger\eta = 1$. The tensor is real for Euclidean signature and unitary representations:

$$\left(\eta^\dagger \Gamma^{ik}\eta\right)^* = \left(\eta^\dagger \Gamma^{ik}\eta\right)^\dagger = \eta^\dagger \gamma^\dagger_{[k}\gamma^\dagger_{i]}\eta = -\eta^\dagger \Gamma^{ik}\eta \tag{9}$$

Before we start with the actual and rather lengthy proof if $J_i{}^l J_l{}^k = -\delta_i^k$ we prove some formulas that we will use in the calculation.
We will use a special case of the Fierz rearrangement formulas which are given by:

$$(\bar{\varphi}\Gamma^A\psi)(\bar{\chi}\Gamma^B\eta) = (-)^{\psi\bar{\chi}+\psi\eta+\bar{\chi}\eta}\frac{1}{2^{D/2}}\sum_I(\bar{\varphi}\Gamma^I\Gamma^B\eta)(\bar{\chi}\Gamma_I\Gamma^A\psi) \tag{10}$$

Sums over collective indices $I = \{a_1 \dots a_n\}$ in $\sum_I \Gamma^I \otimes \Gamma_I$ are restricted to $a_i < a_{i+1}$ or, equivalently, have an implicit factor $1/n!$. Indices $n$ of $I_n$ indicate that the sum is restricted to $\Gamma$'s that are a product of $n$ different $\gamma$-matrices. The factor $(-)^{\psi\bar{\chi}+\psi\eta+\bar{\chi}\eta}$ takes into account that the spinors are complex and anticommuting. We will use the Fierz rearrangement formulas in the following form:

$$\eta\eta^\dagger = \frac{1}{2^{(D/2)}}\sum_I \Gamma^I \eta^\dagger \Gamma_I \eta \tag{11}$$

Now we prove the following relation:

**Lemma:**

$$\gamma^l \Gamma^I \gamma_l = (-)^r (D - 2r)\Gamma^I \tag{12}$$

**Proof:** (complete induction) The equation is obviously fulfilled for the case $r = 0$. For the case $r + 1$ we can write:

$$\begin{aligned}
\gamma^l \Gamma^{a_1 \dots a_{r+1}} \gamma_l &= \gamma^l \Gamma^{a_1\dots a_r}\left(-\gamma_l \gamma^{r+1} + 2\delta_i^{r+1}\right) \\
&\overset{(12)}{=} -(-)^r (D-2r)\Gamma^{a_1\dots a_{r+1}} + 2\gamma^{r+1}\Gamma^{a_1\dots a_r} \\
&= (-)^{r+1}(D-2r)\Gamma^{a_1\dots a_{r+1}} - 2(-)^{r+1}\Gamma^{a_1\dots a_{r+1}} \\
&= (-)^{r+1}\left(D - 2(r+1)\right)\Gamma^{a_1\dots a_{r+1}}
\end{aligned} \tag{13}$$

(12) is a special case of a more general formula for even dimensions given by:

$$\Gamma^{I_n}\Gamma^l\Gamma_{I_n} = (-)^{nL} k^D_{nL}\Gamma^L \tag{14}$$

with

$$k_{nL}^D = \sum_{j=0}^{n} (-)^j \binom{L}{j}\binom{D-L}{n-j} = \binom{D}{n} - 2\sum_{j\in 2\mathbb{Z}+1} (-)^j \binom{L}{j}\binom{D-L}{n-j} \tag{15}$$

For $n = 1$ we recover (12), for $n = 2$ we find $k_{2L}^D = \binom{D}{2} - 2L(D-L)$. Since we are dealing with chiral spinors only terms with an even number of $\gamma$ matrices between $\eta^\dagger$ and $\eta$ give a contribution[3]. Thus, the possible values for $\Gamma^I$ in six dimensions are $\mathbb{1}$, $\Gamma^{ab}$, $\Gamma^{abcd}$ and $\Gamma^{abcdef} = \gamma_*$.

**Lemma:**

$$\begin{aligned} \eta^\dagger\gamma_i\gamma_*\gamma^k\eta\eta^\dagger\gamma^*\eta &= -\eta^\dagger\gamma_i\gamma^k\eta\eta^\dagger\eta \\ \eta^\dagger\gamma_i\Gamma^{abcd}\gamma^k\eta\eta^\dagger\Gamma_{dcba}\eta &= -\eta^\dagger\gamma_i\Gamma^{ab}\gamma^k\eta\eta^\dagger\Gamma_{ba}\eta \end{aligned} \tag{16}$$

**Proof:** In six dimensions $-\Gamma^{abcd} = \Gamma^{ab}\gamma_*$ and $-\Gamma_{dcba} = \Gamma_{ba}\gamma^*$. Furthermore there exists a matrix $\tilde{\gamma} = i\gamma_* = -i\gamma^*$ which acts like the unit matrix between the chiral spinors $\eta^\dagger$ and $\eta$. Using the commutation relation $\gamma_*\gamma^a = (-)^{D-1}\gamma^a\gamma_*$ the above equations become

$$\begin{aligned} \eta^\dagger\gamma_i\gamma_*\gamma^k\eta\eta^\dagger\gamma^*\eta &= -\eta^\dagger\gamma_i\gamma^k\gamma_*\eta\eta^\dagger\gamma^*\eta = -i(-i)\eta^\dagger\gamma_i\gamma^k\tilde{\gamma}\eta\eta^\dagger\tilde{\gamma}\eta \\ \eta^\dagger\gamma_i\Gamma^{abcd}\gamma^k\eta\eta^\dagger\Gamma_{dcba}\eta &= -\eta^\dagger\gamma_i\Gamma^{ab}\gamma^k\gamma_*\eta\eta^\dagger\Gamma_{ba}\gamma^*\eta = -i(-i)\eta^\dagger\gamma_i\gamma^k\tilde{\gamma}\eta\eta^\dagger\Gamma_{ba}\tilde{\gamma}\eta \end{aligned} \tag{17}$$

which proves (16).

Using (16) we can prove the following identity:

$$\eta^\dagger\Gamma^{I_n}\eta\eta^\dagger\Gamma_{I_n}\Gamma^L\eta = c_n^D\eta^\dagger\eta\eta^\dagger\Gamma^L\eta \tag{18}$$

For the case of six dimensions we can derive $c_2^6 = 3$:

**Lemma:**

$$\eta^\dagger\Gamma^{ab}\eta\eta^\dagger\Gamma_{ba}\Gamma^L\eta = 3\eta^\dagger\eta\eta^\dagger\Gamma^L\eta^\dagger \tag{19}$$

**Proof:** Using (11) and (16) yields:

$$\eta^\dagger\eta\eta^\dagger\Gamma^L\eta = \frac{1}{8}\left[\eta^\dagger\Gamma^I\eta\eta^\dagger\Gamma_I\Gamma^L\eta\right] = \frac{1}{4}\left[\eta^\dagger\eta\eta^\dagger\Gamma^L\eta + \eta^\dagger\Gamma^{ab}\eta\eta^\dagger\Gamma_{ba}\Gamma^L\right] \tag{20}$$

This yields an equation which is solved by (19).
For completeness we also give the factors for 8 and 10 dimensions. In 8 dimensions we get: $8c_0 = c_0 + c_2 + \frac{1}{2}c_4$, $8c_2 = 28c_0 + 4c_2 - 2c_4$, $8c_4 = 70c_0 - 10c_2 + 3c_4$. For $c_0 = 1$ we get $c_2 = 0$, $c_4 = 14$ where we used:

| $k_{nL}^8$ | $L=0$ | $L=2$ | $L=4$ |
|---|---|---|---|
| $n=2$ | 28 | 4 | $-4$ |
| $n=4$ | 70 | $-10$ | 6 |

[3] In the Weyl representation all the $\gamma$ matrices are block off-diagonal

For 10 dimensions we find: $16c_0 = c_0 + c_2 + c_4$, $16c_2 = 45c_0 + 13c_2 - 3c_4$ and $16c_4 = 210c_0 - 14c_2 - 2c_4$, where we used

| $k^{10}_{nL}$ | $L=0$ | $L=2$ | $L=4$ |
|---|---|---|---|
| $n=2$ | 45 | 13 | $-3$ |
| $n=4$ | 210 | $-14$ | $-2$ |

The solution to this system is $c_2^{10} = 15$ and $c_4^{10} = 0$.
With these results we can compute the trace of $J^2$. For six dimensions we get $\mathrm{tr}J^2 = -6+12-12 = -6$, in 8 dimensions we have $\mathrm{tr}J^2 = -8+16-16 = -8$, for 10 dimensions we get $\mathrm{tr}J^2 = -10+20-40 = -30$.

Now we finally evaluate $J^2$. Using $\gamma^a\gamma^b = \frac{1}{2}[\gamma^a,\gamma^b] + \frac{1}{2}\{\gamma^a,\gamma^b\} = \Gamma^{ab} + g^{ab}$ we get:

$$J_i{}^l J_l{}^k = -\eta^\dagger\gamma_i\gamma^l\eta\eta^\dagger\gamma_l\gamma^k\eta + 2\eta^\dagger\gamma_i\gamma^k\eta\eta^\dagger\eta - \delta_i^k \tag{21}$$

The first summand can be rewritten using the relations derived above:

$$\begin{aligned}
\eta^\dagger\gamma_i\gamma^l\eta\eta^\dagger\gamma_l\gamma^k\eta &\overset{(11)}{=} \frac{1}{8}\eta^\dagger\gamma_i\gamma^l\Gamma^I\gamma_l\gamma^k\eta\eta^\dagger\Gamma_I\eta \\
&\overset{(12)}{=} \frac{1}{8}\Big[6\eta^\dagger\gamma_i\gamma^k\eta\eta^\dagger\eta + 2\eta^\dagger\gamma_i\Gamma^{ab}\gamma^k\eta\eta^\dagger\Gamma_{ba}\eta \\
&\quad -2\eta^\dagger\gamma_i\Gamma^{abcd}\gamma^k\eta\eta^\dagger\Gamma_{dcba}\eta - 6\eta^\dagger\gamma_i\gamma_*\gamma^k\eta\eta^\dagger\gamma^*\eta\Big] \\
&\overset{(16)}{=} \frac{1}{4}\Big[6\eta^\dagger\gamma_i\gamma^k\eta\eta^\dagger\eta + 2\eta^\dagger\gamma_i\Gamma^{ab}\gamma^k\eta\eta^\dagger\Gamma_{ab}\eta\Big]
\end{aligned} \tag{22}$$

For the last summand we need another identity for $n \in 2\mathbb{Z}$:

$$\begin{aligned}
\frac{1}{n!}\eta^\dagger\Gamma^{a_n\ldots a_1}\eta\eta^\dagger\gamma_i\Gamma_{a_1\ldots a_n}\gamma^k\eta &= \frac{1}{n!}\eta^\dagger\Gamma^{a_n\ldots a_1}\eta\eta^\dagger\gamma_i\left(\gamma^k\Gamma_{a_1\ldots a_n} - 2n\delta^k_{a_1}\gamma_{a_2}\ldots\gamma_n\right)\eta \\
= \eta^\dagger\Gamma^{I_n}\eta\eta^\dagger\gamma_i\gamma^k\Gamma_{I_n}\eta &- \frac{2}{(n-1)!}\eta^\dagger\left(\Gamma^{a_n\ldots a_2}\gamma^k - (n-1)\Gamma^{a_n\ldots a_3}g^{a_2k}\right)\eta\eta^\dagger\gamma_i\Gamma_{a_2\ldots a_n}\eta \\
= \eta^\dagger\Gamma^{I_n}\eta\eta^\dagger\gamma_i\gamma^k\Gamma_{I_n}\eta &- 2\eta^\dagger\Gamma^{I_{n-1}}\gamma^k\eta\eta^\dagger\gamma_i\Gamma_{I_{n-1}}\eta + \frac{2}{(n-2)!}\eta^\dagger\Gamma^{a_n\ldots a_3}\eta\eta^\dagger\gamma_i\Gamma^k{}_{a_3\ldots a_n}\eta
\end{aligned} \tag{23}$$

For $n=2$ and $D=6$ the r.h.s. can be written as $3X - 2X^2 + 2X$ with $X_i{}^k = \eta^\dagger\gamma_i\gamma^k\eta$ and $(X^2)_i{}^k = \eta^\dagger\gamma_i\gamma^l\eta\eta^\dagger\gamma_l\gamma^k\eta$. From (22) and (23) it follows that $(X^2)_i{}^k = \frac{1}{4}(6X_i{}^k + 2\eta^\dagger\Gamma^{I_2}\eta\eta^\dagger\gamma_i\Gamma_{I_2}\gamma^k\eta) = \frac{3}{2}X_i{}^k + \frac{1}{2}(5X - 2X^2)_i{}^k$. Thus, we have

$$\begin{aligned}
J^2 &= -\mathbb{1} - X^2 + 2X \\
&= -\mathbb{1} + 2X - 4X + X^2,
\end{aligned} \tag{24}$$

which implies $X^2 = 2X$ and $J^2 = -\mathbb{1}$.

The matrix $J$ cannot be diagonalized over the real numbers but it is possible to diagonalize it over the complex numbers. This means that at any point $y \in K$ we can find a basis $z^a, \bar{z}^a \quad a = 1,2,3$ which is unique up to a $U(3)$ transformation where $J$ takes the form $J_a{}^b = i\delta_a{}^b$ and $J_{\bar{a}}{}^{\bar{b}} = -i\delta_{\bar{a}}{}^{\bar{b}}$ which we call the canonical form of $J$. Here we see explicitly that J assigns the values $i$ and $-i$ to the $\mathbf{3}$ and $\bar{\mathbf{3}}$ of $U(3)$, respectively.

Having found local coordinates $z^a$ and $\bar{z}^a$ where $J$ takes the canonical form at a point $p$ we can ask now whether $J$ can be put into the canonical form in the whole open set around $p$. Coordinates of that type are called *local holomorphic coordinates.*

There is a similar problem in general relativity. Take a symmetric tensor field $g_{ij}$ on a manifold instead of the tensorfield $J_i{}^j$. At any point $p$ of the manifold $g_{ij}$, which may be the metric but could also be a generic symmetric tensor field on the manifold, takes the form $g_{ij} = \delta_{ij}$ in a particular coordinate system. A coordinate system where $g_{ij}$ takes the standard form also in a neighborhood of $p$ is called flat coordinate system. To determine whether such a coordinate system exists we construct a tensor which consists of derivatives of $g_{ij}$. This unique tensor is the curvature tensor $R_{ijkl}$. The necessary condition for the existence of flat coordinates is the vanishing of the curvature tensor. Conversely, if $R_{ijkl} = 0$ it can be shown that there exists a flat coordinate system at least locally.

In analogy to this problem we can construct a tensor field consisting of derivatives of $J_i{}^j$ which has to vanish when local holomorphic coordinates exist. This essentially unique tensor field is the *Nijenhuis tensor* defined by:

$$N_{ij}{}^k = J_i{}^l \left(\partial_l J_j{}^k - \partial_j J_l{}^k\right) - J_j{}^l \left(\partial_l J_i{}^k - \partial_i J_l{}^k\right) \tag{25}$$

According to a theorem by Newlander and Nierenberg it is possible to find local holomorphic coordinates in a whole neighborhood of $p \in K$ if $N = 0$ is given at $p$. If there exists an almost complex structure that fulfills this condition it is called complex structure. A manifold $K$ endowed with a complex structure is called complex manifold.

The local holomorphic coordinates $z^a$ of a complex manifold exist only locally. We can, however, cover a complex manifold by open sets $O_{(\alpha)}$ with local holomorphic coordinates $z^a_{(\alpha)}$ on each open set such that $J$ takes the standard form $J_a{}^b = i\delta_a{}^b$ and $J_{\bar{a}}{}^{\bar{b}} = -i\delta_{\bar{a}}{}^{\bar{b}}$. On the overlap region $O_{(\alpha)} \cap O_{(\beta)}$ of two open neighborhoods the corresponding coordinates $z^a_{(\alpha)}$ and $z^a_{(\beta)}$ must be related such that the standard structure of $J$ is preserved. This is fulfilled by the holomorphic (analytic) coordinate transformations $z^a \to \tilde{z}^b = \tilde{z}^b(z^a)$. If we had $\tilde{z}^b(z^a, \bar{z}^a)$ we would also get components like $J^{\bar{a}}{}_b$ and $J$ would no longer be of standard form. Conversely, we can say that if there exist local holomorphic coordinates $z^a_{(\alpha)}$ and $z^a_{(\beta)}$ that are holomorphic functions of each other on the overlap region $O_{(\alpha)} \cap O_{(\beta)}$ then there exists a complex structure $J$. Consequently we may define a complex manifold as a $2N$ dimensional manifold $K$ that can be covered by open sets $O_{(\alpha)}$ with local complex coordinates $z_{(\alpha)}$ with the property that on intersection regions $O_{(\alpha)} \cap O_{(\beta)}$ the $z^b_{(\beta)}$ are analytic functions of the $z_{(\alpha)}$.

Now we can show that a $2N$ dimensional manifold $K$ of $U(N)$ holonomy or a subgroup thereof is a complex manifold. At any point $x \in K$ tangent vectors which transform under $SO(2N)$ decompose as $N \oplus \bar{N}$ under $U(N)$. As we have already stated for the special case of $U(3)$ there is a unique matrix $J_i{}^j$ acting on the tangent vectors at $x$ which takes the values $i$ and $-i$ for $N$ and $\bar{N}$ respectively. $J$ defines an almost complex structure. Since $J$ is invariant under the holonomy group it is covariantly constant. Thus, all the covariant derivatives of $J$ vanish, and in particular the Nijenhius tensor is zero. Remember that we can use the covariant field $\eta$ to construct a complex structure: $J_i{}^j(y) = i\eta^\dagger \Gamma_i{}^j \eta$. Thus, a manifold of $U(N)$ holonomy is a complex manifold.

### 2.2.3 Kähler Manifolds

On a manifold with $U(N)$ or $SU(N)$ holonomy the Nijenhius tensor vanishes and the complex structure $J$ is covariantly constant. Now we define a covariantly constant two form, the Kähler form:

$$k_{ij} = J_i{}^k g_{kj} \tag{26}$$

$k_{ij}$ is covariantly constant since both the complex structure and the metric $g_{kj}$ (this is the metricity condition from general relativity!) are covariantly constant. A metric of $U(N)$ holonomy is called a Kähler metric, a manifold that admits such a metric is a Kähler manifold. The properties of such a manifold will be the subject of this section.

An $SO(2N)$ vector $v^i$ splits into $N \oplus \bar{N}$ under $U(N)$ which corresponds to splitting the vector into its holomorphic and antiholomorphic components $v^a$ and $v^{\bar{a}}$. The distinction between holomorphic and antiholomorphic indices is invariant under holomorphic changes of coordinates and does not depend on the choice of a particular holomorphic coordinate system. The metric tensor $g_{ij}$ at a point $p \in K$ is invariant under the action of $U(N)$ on the tangent space at $p$. It is impossible to form a $U(N)$ singlet from $N \otimes N$ or $\bar{N} \otimes \bar{N}$ for $N > 2$. This implies that $g_{ab} = g_{\bar{a}\bar{b}} = 0$ and, due to the symmetry of the metric, $g_{a\bar{b}} = g_{\bar{b}a}$. Thus, the components of the Kähler form in local holomorphic coordinates are:

$$k_{ab} = k_{\bar{a}\bar{b}} = 0 \qquad k_{a\bar{b}} = -k_{\bar{b}a} = i g_{a\bar{b}}$$

In a next step we introduce complex forms. Generalizing the notion of a real $p$-form to complex space we define a $(p,q)$-form $\psi_{a_1 \ldots a_p \bar{a}_1 \ldots \bar{a}_q}$ as a completely antisymmetric object in $p$ holomorphic and $q$ antiholomorphic indices.

The holomorphic exterior derivative $\partial$ maps a $(p,q)$-form to a $(p+1,q)$-form.

$$(\partial\psi)_{a_1 \ldots a_{p+1} \bar{a}_1 \ldots \bar{a}_q} = \frac{1}{p+1} \partial_{a_1} \psi_{a_2 \ldots a_{p+1} \bar{a}_1 \ldots \bar{a}_q} \pm \text{cyclic permutations} \tag{27}$$

Analogously there is an antiholomorphic exterior derivative $\bar{\partial}$ that maps $(p,q)$-forms to $(p,q+1)$-forms:

$$(\bar{\partial}\psi)_{a_1 \ldots a_p \bar{a}_1 \ldots \bar{a}_{q+1}} = \frac{1}{q+1} (-1)^p \partial_{\bar{a}_1} \psi_{a_1} \ldots a_p \bar{a}_2 \ldots \bar{a}_{q+1} \pm \text{cyclic permutations} \tag{28}$$

The operators $\partial$ and $\bar{\partial}$ are called Dolbeault operators. Note, that $\partial^2 = \bar{\partial}^2 = 0$. The exterior derivative $d$ that maps a $(p,q)$-form to a $(p+1,q+1)$-form is $d = \partial + \bar{\partial}$. The identity $\mathrm{d}^2 = 0$ implies that

$$\mathrm{d}^2 = \partial^2 + \bar{\partial}^2 + \partial\bar{\partial} + \bar{\partial}\partial = \partial\bar{\partial} + \bar{\partial}\partial = 0.$$

From this relation follows an important statement: If $\psi$ is a $(p,q)$ with $p > 0$ ($q > 0$) and $\partial\psi = 0$ ($\bar{\partial}\psi = 0$) then locally $\psi$ can be written as $\psi = \partial\chi$ ($\psi = \bar{\partial}\chi$) where $\chi$ is a $(p-1,q)$-form ($(p,q-1)$-form). This corresponds to the Poincaré lemma for real forms. For complex forms we get an additional statement: If $\psi$ is a $(p,q)$-form with $p,q > 0$ and $\partial\psi = \bar{\partial}\psi = 0$ then locally $\psi$ can be written as $\psi = \partial\bar{\partial}\chi$ with a $(p-1,q-1)$-form $\chi$.

The last statement enables us to give a general description of a metric of $U(N)$ holonomy. The Kähler form $k_{ij}$ is a covariantly constant $(1,1)$-form (remember $k_{ab} = k_{\bar{a}\bar{b}} = 0$!) that obeys $\partial k = \bar{\partial} k = 0$. Therefore $k$ can be expressed locally as:

$$k = i\partial\bar{\partial}\phi \tag{29}$$

where $\phi$ is a $(0,0)$-form, i.e. a scalar function. $\phi$ is called Kähler potential and it is defined uniquely up to a Kähler transformation $\tilde{\phi} = \phi + F + \bar{F}$ where $F$ and $\bar{F}$ are arbitrary holomorphic and antiholomorphic functions.
Using the relation $k_{a\bar{b}} = ig_{a\bar{b}}$ we find a general local expression for a metric of $U(N)$ holonomy:

$$g_{a\bar{b}} = g_{\bar{b}a} = \frac{\partial^2 \phi}{\partial z^a \partial \bar{z}^{\bar{b}}} \tag{30}$$

By inserting into the corresponding definitions we find simple expressions for the affine connection and the Riemann tensor:

$$\Gamma^a_{bc} = g^{a\bar{d}} \partial_b g_{c\bar{d}} \qquad \Gamma^{\bar{a}}_{\bar{b}\bar{c}} = g^{\bar{a}b} \partial_{\bar{b}} g_{\bar{c}d} \tag{31}$$

All the other components are 0. For the Riemann tensor we find:

$$R^{\bar{a}}_{\bar{b}c\bar{d}} = \partial_c \Gamma^{\bar{a}}_{\bar{b}\bar{d}} \tag{32}$$

where $R^{\bar{a}}_{\bar{b}c\bar{d}} = g^{a\bar{a}} R_{a\bar{b}c\bar{d}}$. The other non-vanishing components of the Riemann tensor can be obtained from the equation above using the symmetry properties $R_{a\bar{b}c\bar{d}} = R_{c\bar{b}a\bar{d}} = R_{a\bar{d}c\bar{b}}$.
The Ricci tensor on a Kähler manifold is

$$R_{\bar{b}c} = R^{\bar{a}}_{\bar{b}\bar{a}c} = -\partial_c \Gamma^{\bar{a}}_{\bar{b}\bar{c}} = -\partial_{\bar{b}} \partial_c \ln \det g \tag{33}$$

where we used the identity $\Gamma^{\bar{a}}_{\bar{b}\bar{a}} = \partial_{\bar{b}} \ln \det g$ .
It follows from the form of the affine connection that the standard complex structure $J$ is covariantly constant:

$$\begin{aligned}
\nabla_c J_a{}^b &= \partial_c J_a{}^b + \Gamma^b_{cd} J_a{}^d - \Gamma^d_{ca} J_d{}^b \\
&= i\partial_c \delta_a{}^b + i\Gamma^b_{cd} \delta_a{}^d - i\Gamma^d_{ca} \delta_b{}^d \\
&= i(\Gamma^b_{ca} - \Gamma^b_{ca}) \; = \; 0
\end{aligned}$$

The same argument holds for $J_{\bar{a}}{}^{\bar{b}}$. Since the subgroup of $SO(2N)$ under which $J$ is invariant is $U(N)$, covariant constancy of $J$ means that the holonomy group of a Kähler manifold is at most $U(N)$. This proves that the metrics of $U(N)$ holonomy are precisely the Kähler metrics.

#### 2.2.4 Ricci flat Kähler Manifolds

So far we have only considered manifolds with $U(N)$ holonomy which led us to Kähler manifolds. Since our condition for $N = 1$ supersymmetry implies manifolds with $SU(N)$ or, in particular, $SU(3)$ holonomy we expect further restrictions for $K$ which will turn out to be a Calabi-Yau manifold.
The question is whether it is possible to find a metric that has $SU(N)$ holonomy. The spin connection on a Kähler manifold is a $U(N)$ or equivalently $SU(N) \times U(1)$ gauge field. We may call the $U(1)$ part, which is abelian, $A$. A metric on $K$ would be of $SU(N)$ holonomy if the gauge invariant field strength $F = dA$ vanished. This is only possible if the $U(1)$ part of the spin connection is topologically trivial. The closed two form $F$ defines an element of the second de Rham cohomology group with real coefficients $H^2(K;\mathbb{R})$. This element is a topological invariant, called the first Chern class $c_1(K)$ of $K$. More information on characteristic

classes can be found for example in chapter 12 of [2] or chapter 11 of [4]. Only if the first Chern class vanishes the manifold $K$ allows a metric of $SU(N)$ holonomy. Conditions like this are called topological obstructions.
Calabi conjectured and Yau showed that the metrics of $SU(N)$ holonomy correspond precisely to Kähler manifolds with vanishing first Chern class and that there is up to scaling precisely one metric of $SU(N)$ holonomy for any given complex structure on $K$ and any given Kähler class. Two metrics $g$ and $g'$ are said to be of the same Kähler class if they obey $g'_{a\bar{b}} = g_{a\bar{b}} + \partial_a \partial_{\bar{b}} \psi$ with an arbitrary scalar field $\psi$.
Now we will show that a metric of $SU(N)$ holonomy is Ricci flat. As we have shown before a manifold of $SU(N)$ holonomy admits a covariantly constant spinor field $\eta^0$ that obeys:

$$R_{ijkl}\Gamma^{kl}\eta^0 = 0 \tag{34}$$

Multiplying the equation with $\Gamma^j$ from the left we get:

$$\Gamma^j \Gamma^{kl} R_{ijkl} \eta^0 = 0$$

Now we use two identities: $\Gamma^j\Gamma^{kl} = \Gamma^{jkl} + g^{jk}\Gamma^l - g^{jl}\Gamma^k$ and $R_{ijkl} + R_{klij} + R_{iljk} = 0$. This implies:

$$\Gamma^k R_{ik} \eta^0 = 0 \tag{35}$$

From this equation it follows that $R_{ik} = 0$. Thus, a metric of $SU(N)$ holonomy is necessarily Ricci flat.
The converse of this statement is also true which we will show now. When we embed $U(1)$ in $SO(N)$ the $U(1)$ generator is the complex structure $J_i^{\ j}$ with non-vanishing components $J_a^{\ b} = i\delta_a^{\ b}$, $J_{\bar{a}}^{\ \bar{b}} = -i\delta_{\bar{a}}^{\ \bar{b}}$. Given an $SO(2N)$ generator, which is an antisymmetric matrix $M_{kl}$, its $U(1)$ part is $\mathrm{tr} JM = M^k_{\ l} J^l_{\ k}$. Under parallel transport around a small loop in the $ij$ plane vectors are rotated by a matrix $R_{(ij)}$ whose $kl$ element is $R_{ijkl}$.[4] The $U(1)$ part of this matrix is $F_{ij} = \mathrm{tr} JR_{(ij)} = R_{ijkl}J^{lk}$. This is the $U(1)$ part of the rotation undergone by tangent vectors that are transported around a small loop in the $ij$ plane, which corresponds to the field strength of the $U(1)$ part of the spin connection. Since we are on a Kähler manifold the non-vanishing components of the spin connection are:

$$F_{a\bar{b}} = -F_{\bar{b}a} = R_{a\bar{b}\ \ l}^{\ \ k} J^l_{\ k} = iR_{a\bar{b}\ \ c}^{\ \ c} - iR_{a\bar{b}\ \ \bar{c}}^{\ \ \bar{c}} \tag{36}$$

Now we use the symmetry properties of the Riemann tensor:

$$R_{a\bar{b}\ \ c}^{\ \ c} = R_{a\bar{b}\bar{d}c} g^{\bar{d}c} = -R_{a\bar{b}c\bar{d}} g^{c\bar{d}} = -R_{a\bar{b}\ \ \bar{d}}^{\ \ \bar{d}} \tag{37}$$

Now we get, using a Kähler identity $R_{a\ \bar{b}c}^{\ c} = -R_{a\bar{b}\ \ c}^{\ \ c}$:

$$F_{a\bar{b}} = 2iR_{a\bar{b}\ \ c}^{\ \ c} = -2iR_{a\bar{b}} \tag{38}$$

Thus, a metric of $SU(N)$ holonomy is the same thing as a Ricci-flat Kähler metric.
On a manifold with $SU(N)$ holonomy there alway exists a holomorphic $N$-form. The fundamental representation of $SO(2N)$ splits as $N \oplus \bar{N}$ under $SU(N)$. Out of the antisymmetric combination of $N$ copies of the $N$ of $SU(N)$ we can form an $SU(N)$ invariant $\varepsilon_{a_1 \ldots a_n}$ which

[4]Note that the rotation of a vector $V^k$ under parallel transport around a closed loop can be described by $\delta V^k = [\nabla_i, \nabla_j]V^k = R_{ij\ \ l}^{\ \ k} V^l$.

is not a $U(N)$ invariant. If we consider an $N$-form $\omega$ on a manifold of $SU(N)$ holonomy, it has a component that is neutral under the holonomy group and therefore it is invariant under parallel transport in case of a manifold with $SU(N)$ holonomy. The covariantly constant $N$ forms of manifolds with $SU(N)$ holonomy are forms of type $(N,0)$ and $(0,N)$, since it its from $N\otimes\ldots\otimes N$ and $\bar{N}\otimes\ldots\otimes\bar{N}$ that one can form an $SU(N)$ singlet.

# 3 Superstrings with Torsion

## 3.1 Conditions for Unbroken Supersymmetry

In this section we generalize the conditions of unbroken supersymmetry in four dimensions for configurations with non-vanishing torsion $H$ and a non-constant dilaton field $\phi$. For this case it is convenient to rewrite the equations (1) in terms of the following transformations:

$$\begin{aligned}
g_{MN} &= e^{-2\phi}g^0_{MN} \\
\eta &= e^{-\phi/2}\eta^0 \\
\psi_M &= e^{-\phi/2}\left(\psi^0_M - \frac{1}{2}\gamma^0_M\lambda^0\right) \\
\lambda &= e^{\phi/2}\lambda^0 \\
\gamma_M &= e^{-\phi}\lambda^0
\end{aligned} \tag{39}$$

The three form $H_{MNP}$ with lower indices is kept fixed. If $H$ does not have indices it is contracted with $\gamma$ matrices according to $H = H_{MNP}\Gamma^{MNP}$, which transforms as $H = e^{3\phi}H^0$ due to the transformation rules of the $\gamma$ matrices. We get the transformed equations for $N=1$ supersymmetry by inserting the inverted relations (39) into (1). Before we can start to transform the dilatino equation we have to take into account that the covariant derivative $\nabla^0_M$ transforms under the change of the metric:

$$\tilde{\nabla}_M\varepsilon = \nabla_M\varepsilon + \frac{1}{2}\Omega^{-1}\Gamma_M{}^N(\nabla_N\Omega)\varepsilon \qquad \text{for} \qquad \tilde{g}_{MN} = \Omega^2 g_{MN} \tag{40}$$

With $\Omega^2 = e^{2\phi}$ and $\Gamma_M{}^{N0} = \Gamma_M{}^N$ we get from this relation:

$$\nabla^0_M\varepsilon = \nabla_M\varepsilon + \frac{1}{2}\Gamma_M{}^N(\nabla_N\phi)\varepsilon \tag{41}$$

Furthermore we will use the following formulas:

$$\begin{aligned}
\gamma^M\Gamma_M{}^N(\nabla_N\phi) &= 0 \\
\Gamma_M{}^N\nabla_N\phi &= \gamma_M\gamma^N\nabla_N\phi - \not{\nabla}_M\phi
\end{aligned} \tag{42}$$

Now we insert these transformations into the dilatino equation (46):

$$\begin{aligned}
e^{-\phi/2}\delta\lambda &= \left[e^{-\phi}\gamma^M\left(\nabla_M\phi + \frac{1}{2}\overbrace{\Gamma_M{}^N(\nabla_N\phi)\,\phi}^{0\text{ due to (42)}}\right)\right]e^{\phi/2}\eta + \frac{1}{24}e^{2\phi}\left(e^{-3\phi}H\right)\left(e^{\phi/2}\eta\right) \\
\delta\lambda &= \not{\nabla}\phi\eta + \frac{1}{24}H\eta
\end{aligned} \tag{43}$$

Now we transform the gravitino equation (45). The left hand side becomes using the relation for $\delta\lambda$ we just derived:

$$e^{\phi/2}\delta\psi_M + \frac{1}{2}e^{\phi/2}\gamma_M\delta\lambda = e^{\phi/2}\delta\psi_M + \frac{1}{2}e^{\phi/2}\gamma_M\left((\nabla\!\!\!/\phi) + \frac{1}{24}H\eta\right)$$

For the right hand side of (45) we get:

$$\begin{aligned} &\nabla_M\left(e^{\phi/2}\eta\right) + \frac{1}{2}\Gamma_M{}^N(\nabla_N\phi)e^{\phi/2}\eta + \frac{1}{48}e^{2\phi}\left(e^{\phi}\gamma_M e^{-3\phi}H - e^{\phi}e^{-3\phi}12H\right)e^{\phi/2}\eta \\ = \;& \frac{1}{2}(\nabla_M\phi)e^{\phi/2}\eta + e^{\phi/2}\nabla_M\eta + \frac{1}{2}\Gamma_M{}^N(\nabla_N\phi)e^{\phi/2}\eta + \frac{1}{48}e^{\phi/2}(\gamma_M H - 12H)\eta \end{aligned}$$

Now we can combine the left and the right hand side of the equation and cancel a factor $e^{\phi/2}$ which yields:

$$\begin{aligned} \delta\psi_M + \frac{1}{2}\gamma_M\gamma^N(\nabla_N\phi)\eta &= \frac{1}{2}(\nabla_M\phi)\eta + \nabla_M\eta + \frac{1}{2}\Gamma_M{}^N(\nabla_n\phi)\eta - \frac{1}{4}H\eta \\ \delta\psi_M + \frac{1}{2}\gamma_M\gamma^N(\nabla_N\phi)\eta &= \frac{1}{2}(\gamma_M\gamma^N - \delta_M^N)\nabla_N\phi\eta - \frac{1}{4}H\eta \end{aligned}$$

Cancellation of various terms leads to the following result:

$$\delta\psi_M = \nabla_M\eta - \frac{1}{4}H_M\eta \tag{44}$$

The equation for the gluino does not change its form in the transformation.
Thus, we obtain the **conditions for unbroken supersymmetry for superstrings with torsion**:

$$\delta\psi_M = \nabla_M\eta - \frac{1}{4}H_M\eta = 0 \tag{45}$$

$$\delta\lambda = (\nabla\!\!\!/\phi)\eta + \frac{1}{24}H\eta = 0 \tag{46}$$

$$\delta\chi = F_{MN}\Gamma^{MN}\eta = 0 \tag{47}$$

Note, that $\eta$ is a chiral spinor. This is pure convention, we could also have chosen the antichiral spinor $\eta_-$.

### 3.2 Complex Structure and Kähler form

Now we generalize the notions of the previous section in order to investigate if the manifold implied by the equations (45)-(47) is complex. Equation (45) states that the chiral spinor $\eta$ is covariantly constant with respect to a connection with torsion $-H_{mnp}$, where the latin indices belong to the manifold $K$. We will call such a spinor $H$-covariantly constant. For compactification with a maximally symmetric space time $M^4$ one should assume that the dilaton only depends on coordinates of the compact manifold $\phi = \phi(y)$ and that the components of $H$ tangent to $M^4$ vanish. Thus, in four dimensions equation (45) reduces to:

$$\delta\psi_\mu = \nabla_\mu\eta = 0 \tag{48}$$

Having a chiral spinor $\eta$ we can define an almost complex structure in complete analogy to the previous section:

$$J_m{}^n = i\eta^\dagger\Gamma_m{}^n\eta \tag{49}$$

As we have already shown $J$ fulfills

$$J_m{}^n J_n{}^p = -\delta_m^p \tag{50}$$

It follows from the definition of $J$ and from (45) that $J_m{}^n$ is $H$-covariantly constant:

$$\nabla_m J_n{}^p - H_{sm}{}^p J_n{}^s - H^s{}_{mn} J_s{}^p = 0 \tag{51}$$

The Nijenhuis tensor is given by

$$\begin{aligned} N_{mn}{}^p &= J_m{}^q J_{[n}{}^p{}_{,q]} - J_n{}^q J_{[m}{}^p{}_{,q]} \\ &= \frac{1}{2}\left(J_m{}^q \left(\nabla_q J_n{}^p - \nabla_n J_q{}^p\right) - J_n{}^q \left(\nabla_q J_m{}^p - \nabla_m J_q{}^p\right)\right) \end{aligned} \tag{52}$$

Note that we could replace the partial derivative in (25) by a covariant derivative since $N$ is a tensor. Brackets around groups of indices mean that the expression is antisymmetrized with respect to these indices. In a rather lengthy calculation we can rewrite the Nijenhuis tensor using:

$$\begin{aligned} N_{mnp} &= N_{mn}{}^l g_{lp} \\ \nabla_m J_n{}^l &= H_{sm}{}^l J_n{}^s + H^s{}_{mn} J_s{}^l \\ J_m{}^n J_n{}^p &= -\delta_m^p \end{aligned} \tag{53}$$

Furthermore we use the fact that $J_m{}^n$ is antisymmetric with respect to the indices $m$ and $n$. With these manipulations the Nijenhuis tensor can be written as:

$$N_{mnp} = \left(H_{mnp} - 3J_{[m}{}^q J_n{}^r H_{p]qr}\right) \tag{54}$$

If we can show that this expression vanishes the manifold $K$ is complex. This will be done by using the equations of motion an $\gamma$ matrix identities.
First let us write down some commutation and anticommutation relations for $\gamma$ matrices. The complete collection of formulas can be found in [1]. Note that $\Gamma$s with multiple indices are antisymmetrized products of $\gamma$ matrices and $\Gamma^m \equiv \gamma^m$.

$$\begin{aligned} [\Gamma_m, \Gamma^r] &= 2\Gamma_m{}^r \\ \{\Gamma_m, \Gamma^r\} &= 2\delta_m^r \\ \{\Gamma_{mnp}, \Gamma^r\} &= 6\delta^r_{[m}\Gamma_{np]} \\ \{\Gamma_{mnp}, \Gamma^{rst}\} &= 18\delta^{[r}_{[m}\Gamma_{np]}{}^{st]} - 12\delta^{rst}_{[mnp]} \end{aligned} \tag{55}$$

Furthermore we can rewrite the equation (46) in terms of commutators and anticommutators with a $\gamma$ matrix:

**Lemma:** The equation (46) can be written as:

$$\eta^\dagger \left( [\gamma_n, \nabla\!\!\!\!/\,\phi] + \frac{1}{24}\{\gamma_n, H\} \right) \eta \tag{56}$$

**Proof:** Multiplication of the equation (46) with $\gamma_n$ yields:

$$\gamma_n \left(\nabla\!\!\!\!/\,\phi\right) \eta = -\frac{1}{24}\gamma_n H \eta \tag{57}$$

The hermitian conjugate of this relation is given by

$$\begin{aligned} \eta^\dagger \left(\nabla\!\!\!\!/\,\phi\right)^\dagger \gamma_n &= -\frac{1}{24}\eta^\dagger H^\dagger \gamma_n \\ \eta^\dagger \left(\nabla\!\!\!\!/\,\phi\right) \gamma_n &= \frac{1}{24}\eta^\dagger H \gamma_n \end{aligned} \tag{58}$$

where we used $H^\dagger = -H$, which follows from the fact that the hermitian conjugate of the product of three $\gamma$ matrices yields a minus. Multiplication of (57) with $\eta^\dagger$ from the left and (58) with $\eta$ from the right and subtraction of the two equations proves (56). This relation also holds true for (anti-)commutators with $\Gamma_{mnp}$, which we will need soon.
Using the fourth relation in (55) and Fierz rearrangement is is possible to show that the Nijenhuis tensor (52) can be written as:

$$\begin{aligned} N_{mnp} &= -\frac{1}{12}\eta^\dagger \left\{H, \Gamma_{mnp} + 3i\gamma_{[m}J_{np]}\right\}\eta \\ &\overset{(56)}{=} -\frac{1}{12}\eta^\dagger \left[-\frac{1}{24}\nabla\!\!\!\!/\,\phi, \Gamma_{mnp} + 3i\gamma_{[m}J_{np]}\right]\eta \\ &= 0 \end{aligned} \tag{59}$$

Together with $J^2 = -\mathbb{1}$ this result states that the manifold is complex.
For the proof of the relation between (54) and (59) it is more convenient to start with the result and go back to (54).
First we compute the anticommutator of $H$ with $\Gamma_{mnp}$ using the fourth relation in (55):

$$-\frac{1}{12}\eta^\dagger \left\{H, \Gamma_{mnp}\right\}\eta = H_{mnp} - \frac{3}{2}H_{rst}\eta^\dagger \delta^{[r}_{[m}\Gamma_{np]}{}^{st]}\eta \tag{60}$$

For the second anticommutator we use Fierz rearrangement. With the definition of the almost complex structure we get:

$$\begin{aligned} \frac{i}{4}\eta^\dagger \left\{H_{rst}\Gamma^{rst}, \Gamma_{[m}J_{np]}\right\}\eta &= \frac{1}{4}H_{rst}\eta^\dagger \left\{\Gamma^{rst}, \Gamma_{[m}\eta^\dagger \Gamma_{np]}\eta\right\}\eta \\ &= \frac{1}{4}H_{rst}\left(\eta^\dagger \Gamma^{rst}\Gamma_{[m}\eta\eta^\dagger \Gamma_{np]}\eta + \eta^\dagger \Gamma_{[m}\Gamma^{rst}\eta\eta^\dagger \Gamma_{np]}\eta\right) \\ &= \frac{1}{4}H_{rst}\,\eta^\dagger\{\Gamma^{rst}, \Gamma_{[m}\}\eta\eta^\dagger \Gamma_{np]}\eta \\ &\overset{(55)}{=} \frac{3}{2}\eta^\dagger \delta^{[r}_{[m}\Gamma^{st]}\eta\;\eta^\dagger \Gamma_{np]}\eta\;H_{rst} \end{aligned} \tag{61}$$

Now we use Fierz rearrangement:

$$\begin{aligned} \frac{3}{2}\eta^\dagger \delta^{[r}_{[m}\gamma^s\gamma^{t]}\eta\;\eta^\dagger \Gamma_{np]}\eta\;H_{rst} &= \frac{3}{16}\;\eta^\dagger \delta^{[r}_{[m}\gamma^s\gamma^{t]}\Gamma^I\Gamma_{np]}\eta\;\eta^\dagger \Gamma_I\eta H_{rst} \\ &= \frac{3}{8}\left(\eta^\dagger \delta^{[r}_{[m}\gamma^s\gamma^{t]}\Gamma_{np]}\eta\eta^\dagger\eta\;H_{rst} + \eta^\dagger \delta^{[r}_{[m}\gamma^s\gamma^{t]}\Gamma^{ab}\Gamma_{np]}\eta\eta^\dagger \Gamma_{ba}\eta\;H_{rst}\right) \\ &= \frac{3}{8}\Big(4\eta^\dagger \delta^{[r}_{[m}\gamma^s\gamma^{t]}\Gamma_{np]}\eta\eta^\dagger\eta\;H_{rst} + 2\eta^\dagger g^{sa}\gamma^b\gamma^t\Gamma_{[mn}\eta\eta^\dagger \Gamma_{ba}\eta\;H_{p]st} \\ &\quad +2\eta^\dagger g^{ta}\gamma^s\gamma^b\Gamma_{[mn}\eta\eta^\dagger \Gamma_{ba}\eta\;H_{p]st}\Big) \end{aligned} \tag{62}$$

[the first term looks like it could cancel the 3/2 term from the first anticommutator but it may yield commutation terms, for the second one I need some Fierz identity like in the $J^2$

case, but the problem here is that I have to separate the indices in $\Gamma_{mn}$ since in the result the indices $mnp$ belong to different $J$s, looks like I have to insert some unity from the FI between the 2 $\gamma$s...]
With $N_{mnp} = 0$ we are dealing with a complex manifold. Thus, we can introduce complex coordinates $z$ and $\bar{z}$ and (anti-)holomorphic indices denoted by $a, b, c \ldots$ and $\bar{a}, \bar{b}, \bar{c} \ldots$ with letters from the beginning of the alphabet. The metric $g$ is hermitian, which follows from (50) and Fierz rearrangement. The condition for a hermitian metric is given by $g(JX, JY) = g(X, Y)$. In components we get:

$$J_m{}^k J_n{}^l g_{kl} = J_m{}^k J_{nk} = -J_m{}^k J_{kn} = -(-g_{mn}) = g_{mn} \tag{63}$$

This means that in complex coordinates the metric is of type $(1,1)$, i.e. it has one holomorphic and one antiholomorphic index. By multiplication with $J_m{}^n$ we can define the fundamental form:

$$\begin{aligned} J &= \frac{1}{2} J_m{}^n g_{np} \mathrm{d}x^m \wedge \mathrm{d}x^p \\ &= i\, g_{a\bar{b}} \mathrm{d}z^a \wedge \mathrm{d}\bar{z}^{\bar{b}} \end{aligned} \tag{64}$$

If $\mathrm{d}J = 0$ the manifold is Kähler and $J$ is the Kähler form.
Now we will express the torsion in terms of the fundamental form. In fact, we can find a solution for $H$ in terms is the almost complex structure, using (54). We prove the following lemma:

**Lemma:** Multiplying (51) with $J_k{}^l$ yields:

$$J_k{}^l \nabla_m J_n{}^p = J_k{}^l H_{sm}{}^p J_n{}^s - H^s{}_{mn} J_s{}^p \tag{65}$$

We will need two special cases of this equation in this review:

- **Case $l = m$:** Using (54) we get:

$$J_k{}^m \nabla_m J_n{}^p = H^p{}_{nk} \tag{66}$$

- **Case $m = p, l = n$:**

$$J_k{}^n \nabla_p J_n{}^p = -\eta^\dagger \Gamma_k{}^n \eta \eta^\dagger H_{nsp} \Gamma^{sp} \eta \tag{67}$$

**Proof:** For (66) we insert $H_{mnp} = 3J_{[m}{}^q J_n{}^r H_{p]qr}$ which follows from (54). Now we get using (50):

$$\begin{aligned} J_k{}^m \nabla_m J_n{}^p &= g^{pr} J_k{}^m H_{smr} J_n{}^s - J_k{}^m H_{rmn} J_s{}^p \\ &= 3g^{pr} J_k{}^m \left( J_{[s}{}^u J_m{}^v H_{r]uv} \right) J_n{}^s - 3g^{sr} J_k{}^m \left( J_{[r}{}^u J_m{}^v H_{n]uv} \right) J_s{}^p \\ &= \frac{1}{6} \left( 3H^p{}_{nk} - 3H_n{}^p{}_k \right) \;=\; H^p{}_{nk} \end{aligned} \tag{68}$$

For (67) we get with (50):

$$\begin{aligned} J_k{}^n \nabla_p J_n{}^p &= J_k{}^n H_{sp}{}^p J_n{}^s + J_k{}^n H^s{}_{pn} J_s{}^p \\ &= -H_{kp}{}^p - \eta^\dagger \Gamma_k{}^n \eta \eta^\dagger \Gamma^{sp} H_{nsp} \eta \end{aligned} \tag{69}$$

${H_{kp}}^p = 0$ due to its antisymmetry and (67) is proved.

With (66) we have found a way to express $H$ in terms of the complex structure. In terms of the fundamental form this result looks as follows:

$$H = \frac{1}{2} i(\bar{\partial} - \partial)J \tag{70}$$

From (66), which we use in the form ${J_k}^m \nabla_m J_{np} = -H_{knp}$ it follows that the index structure of $H$ looks like this:

$$\begin{array}{ll} {J_a}^b \nabla_b J_{c\bar{d}} = -H_{ac\bar{d}} & {J_{\bar{a}}}^{\bar{b}} \nabla_{\bar{b}} J_{c\bar{d}} = -H_{\bar{a}c\bar{d}} \\ {J_a}^b \nabla_b J_{\bar{d}c} = -H_{a\bar{d}c} & {J_{\bar{a}}}^{\bar{b}} \nabla_{\bar{b}} J_{\bar{d}c} = -H_{\bar{a}\bar{d}c} \end{array}$$

Using ${J_a}^b = i{\delta_a}^b$ and $J_{c\bar{d}} = ig_{c\bar{d}}$ as well as their complex conjugates we get:

$$\begin{array}{ll} H_{ac\bar{d}} = \nabla_a g_{c\bar{d}} & H_{\bar{a}c\bar{d}} = -\nabla_{\bar{a}} g_{c\bar{d}} \\ H_{a\bar{d}c} = -\nabla_a g_{\bar{d}c} & H_{\bar{a}\bar{d}c} = \nabla_{\bar{a}} g_{\bar{d}c} \end{array}$$

Collecting the terms we find (70). The component expression of $H$ is given by:

$$H_{\bar{a}\bar{b}c} = -g_{c[\bar{a},\bar{b}]} = -\frac{1}{2}(\bar{\nabla}_{\bar{b}} g_{c\bar{a}} - \bar{\nabla}_{\bar{a}} g_{c\bar{b}}) \tag{71}$$

Thus, given any hermitian metric on a complex manifold one can construct the torsion (70) such that the torsion plus Christoffel connection has $U(3)$ holonomy and the corresponding covariant derivative annihilates the complex structure.
The curl of (70) and (2) lead to an equation for the fundamental form:

$$\mathrm{d}H = i\partial\bar{\partial}J = \mathrm{tr} R \wedge R - \frac{1}{30}\mathrm{Tr} F \wedge F \tag{72}$$

Here we used $\mathrm{d} = \partial + \bar{\partial}$ and $\bar{\partial}\partial = -\partial\bar{\partial}$.

### 3.3 Holomorphic $n$-form

Now we want to prove the existence of a holomorphic $n$-form, i.e. a $\bar{\partial}$ closed form of type $(n,0)$ where $n$ is the number of complex dimensions. We rewrite (46) using (56), (67), (55) and the definition of the almost complex structure:

$$\begin{aligned} 0 &\overset{(56)}{=} 4i{J_m}^n \eta^\dagger \left( [\gamma_n, \nabla\!\!\!\!/ \phi] + \frac{1}{24}\{\gamma_n, H\} \right) \eta \\ &\overset{(55)}{=} 4i{J_m}^n \eta^\dagger 2{\Gamma_n}^s \nabla_s \phi \eta + \frac{1}{24} 4i {J_m}^n \eta^\dagger 6 \delta_n^{[r} \Gamma^{st]} H_{rst} \eta \\ &\overset{(67)}{=} -8\nabla_m \phi + {J_m}^n \nabla_p {J_n}^p \\ &= -8\nabla_m \phi + H_{nsp} {J_m}^n J^{sp} \end{aligned} \tag{73}$$

In the following computations we will need (73) for holomorphic and antiholomorphic indices. Using ${J_{\bar{a}}}^{\bar{b}} = -i{\delta_{\bar{a}}}^{\bar{b}}$ we get:

$$8\nabla_{\bar{b}}\phi = -i{\delta_{\bar{b}}}^{\bar{a}} H_{\bar{a}sp} J^{sp} = -iH_{\bar{b}sp} J^{sp} \tag{74}$$

For the case $s = \bar{a},\ p = c$ we get:

$$8\nabla_{\bar{b}}\phi = -iH_{\bar{b}\bar{a}c}J^{\bar{a}c} = H_{\bar{b}\bar{a}c}g^{\bar{a}b} = H_{\bar{b}\bar{a}}{}^{\bar{a}} = H_{\bar{b}}{}^{c}{}_{c} \tag{75}$$

For $s = a,\ p = \bar{c}$ we get:

$$8\nabla_{\bar{b}}\phi = -iH_{\bar{b}a\bar{c}}J^{a\bar{c}} = -H_{\bar{b}a\bar{c}}g^{a\bar{c}} = -H_{\bar{b}a}{}^{a} = -H_{\bar{b}}{}^{\bar{c}}{}_{\bar{c}} \tag{76}$$

Combining the last two equations we get the following identities: $H_{\bar{b}\bar{a}}{}^{\bar{a}} = H_{\bar{b}}{}^{c}{}_{c} = -H_{\bar{b}a}{}^{a} = -H_{\bar{b}}{}^{\bar{c}}{}_{\bar{c}}$.

With the negative chirality spinor $\eta_-$ the expression

$$\omega = e^{8\phi}\eta_-^T\Gamma_{a_1\ldots a_n}\eta_-\mathrm{d}z^{a_1}\ldots\mathrm{d}z^{a_n} \tag{77}$$

is a holomorphic $n$ form.

Now we prove that (77) is really a holomorphic $n$-form. With (45), (66) and (73) we can compute the covariant derivative of $\omega$:

$$\begin{aligned}
\nabla_{\bar{b}}\omega_{a_1\ldots a_n} &= \frac{1}{2}H^{c}_{\bar{b}a_1}\omega_{ca_2\ldots a_n} + \ldots + \frac{1}{2}H^{c}_{\bar{b}a_n}\omega_{a_1\ldots a_{n-1}c} + 8\nabla_{\bar{b}}\phi\omega_{a_1\ldots a_n} \\
&= -(-)^n n\frac{1}{2}H^{c}_{\bar{b}[a_1}\omega_{a_2\ldots a_n]c} + 8\nabla_{\bar{b}}\phi\omega_{a_1\ldots a_n} \\
&= (\frac{1}{2}H_{\bar{b}a\bar{c}}g^{a\bar{c}} + 8\nabla_{\bar{b}}\phi)\omega_{a_1\ldots a_n} \\
&\overset{(76)}{=} 4\nabla_{\bar{b}}\phi\,\omega_{a_1\ldots a_n}
\end{aligned} \tag{78}$$

$\bar{\partial}\omega$ is related to the covariant derivative by:

$$\begin{aligned}
\partial_{\bar{b}}\omega_{a_1\ldots a_n} &= \nabla_{\bar{b}}\omega_{a_1\ldots a_n} - (-)^n n\Gamma^{c}_{\bar{b}[a_1}\omega_{a_2\ldots a_n]c} \\
&= \nabla_{\bar{b}}\omega_{a_1\ldots a_n} + \Gamma^{c}_{\bar{b}c}\omega_{a_1\ldots a_n}
\end{aligned} \tag{79}$$

Here $\Gamma^{c}_{\bar{b}c}$ is the connection. In the Kähler case the second term is absent. Comparing (78) and (79) (note that this only works properly if the manifold is not Kähler) we find:

$$\Gamma^{c}_{\bar{b}c} = -4\nabla_{\bar{b}}\phi \tag{80}$$

$$\bar{\partial}\omega = 0 \tag{81}$$

Thus, we proved that $\omega$ is indeed a holomorphic $n$-form.

The holomorphic $n$-form is is completely determined (up to an overall constant), both locally and globally, by the complex structure alone. It does *not* depend on either the scalar field $\phi$ or the metric.
The norm of $\omega$ given by

$$||\omega|| = \left(\omega_{a_1\ldots a_n}\bar{\omega}_{\bar{b}_1\ldots\bar{b}_n}g^{a_1\bar{b}_1}\ldots g^{a_n\bar{b}_n}\right)^{1/2} \tag{82}$$

is an everywhere non-zero, finite positive real number. To see this we use the relation

$$\begin{aligned}
\nabla_{\bar d}\bar\omega_{\bar a_1\ldots\bar a_n} &= \left(\frac{1}{2}H_{\bar b\bar a c}g^{\bar a c} + 8\nabla_{\bar b}\phi\omega_{a_1\ldots a_n}\right) \\
&\overset{(74)}{=} 12\nabla_{\bar d}\phi\bar\omega_{\bar a_1\ldots\bar a_n}
\end{aligned} \tag{83}$$

With that we can compute the covariant derivative of $||\omega||$:

$$\begin{aligned}
\nabla_{\bar a}||\omega|| &= \frac{1}{2}\left(\nabla_{\bar a}\omega\,\bar\omega + \omega\,\nabla_{\bar a}\bar\omega\right) \\
&= \frac{1}{2}\frac{1}{||\omega||}\left(4(\nabla_{\bar a}\phi)\omega\bar\omega + 12\omega(\nabla_{\bar a}\phi)\bar\omega\right) \\
&= 8||\omega||\nabla_{\bar a}\phi
\end{aligned} \tag{84}$$

Integrating this result we get:

$$||\omega|| = e^{8(\phi+\phi_0)}, \tag{85}$$

where $\phi_0$ is an integration constant. For a finite $\phi$ (85) is everywhere positive and finite. Taking the logarithm on both sides we can express $\phi$ in terms of $\omega$, i.e. in terms of the complex structure and the metric (up to a constant).

$$\phi = \frac{1}{8}\ln||\omega|| + \phi_0 \tag{86}$$

We can write the dilatino equation (73) as an equation for $J$:

$$\mathrm{d}^\dagger J + i(\partial - \bar\partial)\ln||\omega|| = 0 \tag{87}$$

$\mathrm{d}^\dagger$ is the adjoint of d with $\mathrm{d}^\dagger J = \nabla^m J_{nm}\mathrm{d}x^n$.
Plugging these definitions into (73) we get:

$$\begin{aligned}
0 &= -8\nabla_m\phi + J_m{}^n\nabla_p J_n{}^p \\
&= -8J_n{}^m\nabla_m\phi - \nabla^p J_{np} \\
&= -8J_n{}^m\nabla_m\phi\mathrm{d}x^n - \mathrm{d}^\dagger J
\end{aligned} \tag{88}$$

In the fourth line we multiplied the expression with $J_s{}^m$. With $||\omega|| = 8\phi$ and $J_a{}^b = i\delta_a{}^b$ and its complex conjugate we get (87).
Next we show that the existence of an everywhere non-vanishing holomorphic $n$-form implies that the Hodge number $h^{n,0}$ is precisely one. Since $||\omega||^2 > 0$ the integral over the manifold

$$\int \omega\wedge\bar\omega = n!\int \mathrm{d}^n z\mathrm{d}^n\bar z\sqrt{g}||\omega||^2 > 0 \tag{89}$$

is non-zero. $\bar\omega$ is of maximal degree of $\mathrm{d}z$ and therefore $\bar\partial$ closed. Using Stokes' theorem this implies that neither $\omega$ nor $\bar\omega$ are exact and therefore $h^{n,0}$ must be at least one. To show this explicitly we assume that we can write $\omega = \partial\alpha$ where $\alpha$ is a $(n-1,0)$-form. Thus, we can write:

$$\int \omega\wedge\bar\omega = \int \partial\alpha\wedge\bar\omega = -\int \alpha\wedge *\partial\underbrace{*\bar\omega}_{(n,0)-\text{form}} = 0, \tag{90}$$

but this contradicts $\int \omega \wedge \bar{\omega} > 0$ and therefore $\omega$ cannot be exact. In an analogous procedure one can show that also $\bar{\omega}$ is not exact. Since $\omega$ is everywhere non-vanishing any holomorphic $n$-form must be $\omega$ times a holomorphic function. Every global holomorphic function on a compact manifold is a constant and therefore all holomorphic $n$-forms must be multiples of $\omega$ which implies that $h^{n,0}$ is precisely one. The condition $h^{n,0} = 1$ is a topological restriction on superstring compactification. Manifolds with $h^{n,0} \neq 1$ like $S^3 \times S^3$ or $S^5 \times S^1$ are therefore ruled out. The first Chern class of the manifold vanishes. The real cohomology class $c_1$ is represented by the $U(1)$ part (defined by ${J_m}^n$) of any connection that commutes with the complex structure. Equation (45) implies that the $H$ connection has $SU(3)$ holonomy. Thus, there is no $U(1)$ part and $c_1$ vanishes.
Finally we can write the gluino equation (47) is a more convenient form:

$$\begin{aligned} J^{a\bar{b}} F_{a\bar{b}} &= i\eta^\dagger F_{mn}\Gamma^{mn}\eta \;=\; 0 \\ F_{\bar{a}\bar{b}} &= \frac{1}{8}\eta^\dagger\{\Gamma_{\bar{a}\bar{b}}, F_{mn}\Gamma^{mn}\}\eta \;=\; 0 \;=\; F_{ab} \end{aligned} \tag{91}$$

The second line can be proved using the identity $\{\Gamma_{\bar{a}\bar{b}}, \Gamma^{mn}\} = 16\left(\delta^m_{\bar{a}}\delta^n_{\bar{b}} - \delta^n_{\bar{a}}\delta^m_{\bar{b}}\right)$ which can be verified directly be repeated use of the identity $\{AB, C\} = \{A, C\}B - A\{B, C\}$.

### 3.4 Summary

Let us summarize the necessary and sufficient conditions for spacetime supersymmetry for superstrings with torsion:

1. The internal manifold must be complex.
2. The fundamental form $J_{a\bar{b}} = ig_{a\bar{b}}$ must obey

$$\partial\bar{\partial}J = i\frac{1}{30}\mathrm{Tr}F \wedge F - i\mathrm{tr}R \wedge R \tag{92}$$

$$\mathrm{d}^\dagger J = -i(\partial - \bar{\partial})\ln||\omega||. \tag{93}$$

3. The Yang-Mills field strength must satisfy

$$J^{a\bar{b}}F_{a\bar{b}} = 0$$

$$F_{ab} = F_{\bar{a}\bar{b}} = 0 \tag{94}$$

$H$, $\phi$ an the physical metric $g^0$ are given by:

$$H = \frac{1}{2}i(\bar{\partial} - \partial)J \tag{95}$$

$$\phi = \frac{1}{8}\ln||\omega|| + \phi_0 \tag{96}$$

$$g^0_{MN} = e^{2\phi_0}||\omega||^{1/4}g_{MN} \tag{97}$$

# 4 Yukawa Couplings

## 4.1 Introduction

In this section we will discuss the derivation of Yukawa couplings in four dimensions from cubic (or higher) terms in the ten dimensional action. It will turn out that the derived coupling constants are given by topological quantities that characterize the internal manifold $K$. The topological arguments presented here can only give relations among Yukawa couplings, since the couplings are only given up to normalization constants that involve the metric. We will, however, be able to predict the vanishing of Yukawa couplings resulting from topological reasons (and not due to symmetries). The reason why this is of interest is that phenomena in the low energy world such as the lightness of the electron and the first generation quarks or the small Cabibbo angle are possibly related to the vanishing of some Yukawa couplings at tree level.
We will discuss this topic for superstring compactification on Calabi-Yau manifolds. Many arguments also work for Kähler manifolds that are not Ricci-flat. Generalizations to superstrings with torsion as given in section 3 will not be taken into account. The aim of this section is not to give a complete review of this subject but to show by examples how four-dimensional Yukawa couplings are obtained from the ten-dimensional action and to find a relation to topological invariants of the internal manifold.
Unfortunately it will not be possible to keep up the index conventions used in the sections 2 and 3. Changes concerning conventions will be indicated whenever it becomes necessary.

## 4.2 Yukawa couplings from a ten dimensional action

We can derive four dimensional Yukawa couplings from cubic or higher terms in the ten dimensional action. As an example we consider the following term:

$$L_I = \int d^{10}X\sqrt{-g}\bar{\psi}_a(X)\gamma^m A_{mb}(X)\psi_c(X)f^{abc} \tag{98}$$

where $a, b, c$ are $E_8 \times E_8$ indices. Now we can expand the fields in harmonics of the internal manifold $K$:

$$\psi(X) = \sum_i \psi^i(x) \otimes \psi^i(y) \tag{99}$$

$$A_m(X) = \sum_i A^i_m(x)\phi^i(y) + \sum_j \phi^j(x)A^j_m(y) \tag{100}$$

Here we suppressed the gauge indices. For obvious reasons it is no longer possible to maintain the index conventions for the labeling of the the manifolds $K$ and Minkowski space. On $K$ the fields obey the following equations of motion:

$$\gamma^m\nabla_m\psi^i(y) = \lambda^i\psi^i(y) \tag{101}$$

$$\nabla^2A^i_m(y) - R^n_m A^i_n(y) + 2F^n_m A^i_n(y) = \lambda^i A_m(y) \tag{102}$$

$$\nabla^2\phi^j(y) = \lambda^j\phi^j(y) \tag{103}$$

Each mode on $K$ corresponds to a four dimensional field with mass proportional to $\lambda^i$. In order to get properly normalized kinetic terms in four dimensions we choose the normalization of the internal fields to be:

$$\int_K d^6y\sqrt{g}\phi^{*i}(y)\phi^j(y) = \delta^{ij} \tag{104}$$

Since $K$ has Euclidean signature we have $\sqrt{g}$ instead of $\sqrt{-g}$. Inserting the decompositions of the fields into (98) we get terms of the following form (This expression comes from the contribution of the second summand in (100)):

$$L_I = \int_K d^6y\sqrt{g}\psi^{0\dagger}(y)A^0(y)\psi^0(y)\int d^4x\sqrt{-g}\bar{\psi}^0(x)\phi^0(x)\psi^0(x) \tag{105}$$

The superscript 0 indicates that we are dealing with zero mode wave functions ($\lambda^i = 0$). The four dimensional integral describes a cubic interaction term. Thus, we conclude that the integral over the internal manifold is the effective Yukawa coupling:

$$g = \int d^6y\sqrt{g}\psi^{0\dagger}(y)A^0\psi^0(y) \tag{106}$$

### 4.3 Relation to Topological Invariants

Before we go on with our discussion of Yukawa couplings we will briefly discuss the relation between the gauge group and the holonomy group in the ten dimensional theory. Note that the gauge group is $E_8 \times E_8$. Since we are on a Calabi-Yau manifold the metric has $SU(3)$ holonomy which determines the topological class of the Kähler form and the complex structure on $K$. We have not yet considered the topological class of the bundle $X$ that describes the gauge group. For vanishing torsion $H = 0$ the relation $\mathrm{tr}R \wedge R - \frac{1}{30}\mathrm{Tr}F \wedge F = 0$ must be fulfilled in each point at $K$. Having fixed the complex structure and the gauge group we are left with a finite number of adjustable parameters to fulfill this identity which in turn represents an infinite set of conditions. Thus, we require some relation between $F$ and $R$ which is achieved by regarding the spin connection of $K$ as a gauge field. Then the holonomy group is equal to the gauge group and the Yang-Mills field strength is equal to the Riemann curvature tensor. This construction is known as the embedding of the spin connection into the gauge group. As a consequence we need to find and $SU(3)$ subgroup of $E_8 \times E_8$. We pick a minimal embedding and select a maximal subgroup $SU(3) \times E_6$ of $E_8$. The adjoint representation of $E_8$ under $SU(3) \times E_6$ is given by:

$$\mathbf{248} = (\mathbf{3}, \mathbf{27}) \oplus (\bar{\mathbf{3}}, \bar{\mathbf{27}}) \oplus (\mathbf{8}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{78}) \tag{107}$$

$(\mathbf{8}, \mathbf{1})$ and $(\mathbf{1}, \mathbf{78})$ are the adjoint representations of of $SU(3)$ and $E_6$, respectively. $(\mathbf{3}, \mathbf{27})$ is the tensor product of the fundamental $\mathbf{3}$ of $SU(3)$ with the fundamental $\mathbf{27}$ of $E_6$.
Now we go back to Yukawa couplings. The gauge field $A^a_m$ in (106) is a one-form with a gauge index $a$ which labels the $\mathbf{3}$ and $\bar{\mathbf{3}}$ of $SU(3)$. The $\mathbf{3}$ and $\bar{\mathbf{3}}$ label the holomorphic tangent and cotangent bundles $T$ and $\bar{T}$. Thus, $A^a_m$ is a one-form with values in $T$ and $\bar{T}$.
For one-forms with values in $T$ there is a natural exterior derivative given by:

$$\bar{\partial}A = \bar{\partial}_{\bar{n}}A^a_{\bar{m}}\mathrm{d}z^{\bar{n}} \wedge \mathrm{d}z^{\bar{m}}\frac{\partial}{\partial z^a}, \tag{108}$$

where $a, b, c$ are used for tangent space indices. We can also define the adjoint of $\bar{\partial}$ for a given Kähler metric:

$$(\bar{\partial}^* A)^b = \nabla^{\bar{m}}A^b_{\bar{m}} \tag{109}$$

Here $\nabla^m$ is the usual covariant derivative. With the embedding of the spin connection into the gauge group given by $(A_{Bm})^b_a = \Gamma^b_{ma}$, $(A_{B\bar{m}})^{\bar{b}}_{\bar{a}} = \Gamma^{\bar{b}}_{\bar{m}\bar{a}}$ (108) and (109) become equations

for gauge covariant curl and divergence of $A$.
With $R = F$ and $\lambda^i = 0$ it follows from (102) that $A$ is harmonic:

$$\Delta A^b = (\bar{\partial}^* \bar{\partial} + \bar{\partial}\bar{\partial}^*) A^b = 0 \tag{110}$$

The zero modes of the $\Delta$ operator on a one-form determine the elements of the cohomology class $H^1(T)$ which is a topological invariant.
We can construct zero modes of the Dirac equation (101) using supersymmetry:

$$\psi^a = A^a_{\bar{m}} \gamma^{\bar{m}} \eta^0 \tag{111}$$

where $\eta^0$ is the spinor field given in section 2. Using the relation $\gamma^m \eta^0 = 0$ we find that $\psi^a$ is a zero mode of the Dirac equation:

$$\gamma^m \nabla_m \psi^a = \gamma^m \nabla_m (A^a_{\bar{m}} \gamma^{\bar{m}} \eta^0) = A^a_{\bar{m}} (-\gamma^{\bar{m}} \gamma^m + g^{\bar{m}m}) \nabla_m \eta^0 = 0 \tag{112}$$

We cannot use $\eta^0_-$ since $\gamma^{\bar{m}} \eta^0_- = 0$. Thus, the harmonic elements of $H^1(T)$ lead only to left-chiral fermions in the $\mathbf{3}$ of $SU(3)$.
For the case of a Ricci flat metric in $K$, there is a map from harmonic elements of $H^1(T)$ to harmonic $(2,1)$-forms given by the holomorphic and covariantly constant three-form $\omega$:

$$G_{\bar{m}np} = A^a_{\bar{m}} \omega_{anp} \tag{113}$$

Since $A$ is harmonic we find that $(\bar{\partial} + \bar{\partial}^*)^2 G = 0$. On a Kähler manifold we have $(\bar{\partial} + \bar{\partial}^*)^2 = (\partial + \partial^*)^2 = \frac{1}{2}(\mathrm{d} + \mathrm{d}^*)^2$, which implies that $G$ is harmonic. It can be shown (see for example [2]) that the number of fermion generations is related to the zero modes of the Dirac operator. With the the relation (113) we conclude that the number of fermion families is given by $\mathrm{dim} H^{2,1}$.
Antifamilies can be related with elements of $H^1(\bar{T})$. The $\bar{\partial}$ harmonic elements obey:

$$(\bar{\partial} A)_a = (\bar{\partial}^* A)_a = 0 \tag{114}$$

From this we can immediately conclude that the zero modes of the Yang-Mills field in the $\bar{\mathbf{3}}$ are given by:

$$(\bar{\partial}^* \bar{\partial} + \bar{\partial}\bar{\partial}^*) A_a = 0 \tag{115}$$

A solution which is given on all Kähler manifolds is $A_{\bar{m}a} = e_{\bar{m}a}$, where $e$ is the vielbein. Superpartners of $A_{\bar{m}a}$ are left handed fermions in $\bar{\mathbf{3}}$ representation. We can relate the elements of $H^1(\bar{T})$ to harmonic $(1,1)$-forms by:

$$V_{m\bar{n}} = -i A_{\bar{n}a} e^a_m \tag{116}$$

We conclude that the number of antifamilies is given by $\mathrm{dim} H^{1,1}$.
Thus, we find that the net number of fermion families is given by $h^{2,1} - h^{1,1}$, where the $h$ are the Hodge numbers. For a Calabi-Yau manifold it can be shown [2] that $|\chi/2| = h^{2,1} - h^{1,1}$, which states that the net number of fermion families is linked to the Euler characteristic of the internal manifold $K$.
These results have influence on the existence of certain Yukawa couplings. There will be no cubic couplings between families and antifamilies because there is no $SU(3)$ invariant way to

combine two elements of $H^1(T)$ with one element of $H^1(\bar{T})$.
There can be cubic interactions between the $\mathbf{\bar{27}}$ given by:

$$L_{ijk} = \bar{g}_{ijk} \int d^4x \sqrt{-g} \bar{\psi}_i^{\bar{A}} \gamma^{\bar{m}} A_{\bar{m}j}^{\bar{B}} \psi_k^{\bar{C}} d_{\bar{A}\bar{B}\bar{C}} \tag{117}$$

with

$$\bar{g}_{ijk} = \int_K d^6y \sqrt{g} \psi_{ai}^{\dagger} \gamma^{\bar{m}} A_{\bar{m}bj} \psi_{ck} \varepsilon^{abc}. \tag{118}$$

Index conventions in this expression are as follows: $\bar{A}, \bar{B}, \bar{C}$ are $E_6$ indices in the $\mathbf{\bar{27}}$ representation, $d_{\bar{A}\bar{B}\bar{C}}$ is the symmetric cubic invariant in the $\mathbf{\bar{27}}$, $a, b, c$ are $SU(3)$ indices and $\psi_{ai}$ denotes the $i^{\text{th}}$ zero mode wave function in the $\mathbf{3}$ representation.
Now we insert (111) into (118) which yields:

$$\bar{g}_{ijk} = \int d^6y \sqrt{g} \eta^{T0} \gamma^{\bar{m}} A_{\bar{m}ai} \gamma^{\bar{n}} A_{\bar{n}bj} A_{\bar{p}ck} \gamma^{\bar{p}} \eta^0 \varepsilon^{abc} \tag{119}$$

The holomorphic three-form $\omega$ can be expressed in terms of the spinor field $\eta^0$ by $\omega_{mnp} = \eta^{T0} \gamma_m \gamma_n \gamma_p \eta^0$, which leads to the following expression for the Yukawa coupling:

$$\begin{aligned} \bar{g}_{ijk} &= \int d^6y \sqrt{g} \omega^{\bar{m}\bar{n}\bar{p}} A_{\bar{m}ai} A_{\bar{n}bj} A_{\bar{p}ck} \varepsilon^{abc} \\ &= \int_K \omega \wedge A_{ai} \wedge A_{bj} \wedge A_{ck} \varepsilon^{abc} \end{aligned} \tag{120}$$

with $A_{ai} = A_{\bar{m}ai} \mathrm{d}z^{\bar{m}}$. This expression is symmetric in the indices $ijk$, as it is required by supersymmetry[5]. For a Calabi-Yau manifold covariant constancy of $\omega$ implies that $\omega_{mnp} = \varepsilon_{mnp}$. Now we use (116) which yields: $V_i = A_{\bar{m}ai} e_n^a \mathrm{d}z^{\bar{m}} \wedge \mathrm{d}z^n$. Inserting this into our expression for the Yukawa coupling we get:

$$\bar{g}_{ijk} = \int_k V_i \wedge V_j \wedge V_k. \tag{121}$$

For a general hermitian metric the the integrands in (120) and (121) will not agree, since we need the Calabi-Yau condition to set $\omega = \varepsilon$. The integrals, however, are topological invariants and therefore the right hand sides of (120) and (121) will agree. If we normalize the $V_i$ such that they represent integral cohomology classes the integral (121) is determined only by topology, in terms of the cohomology ring of the manifold $K$. This means that it is invariant under $\delta V_i = \mathrm{d}\lambda_i$. Note that we do not need the metric of $K$ to compute the integral (121). In terms of homology (121) counts the number of points of intersection of the two complex dimensional hypersurfaces associated with $V_i$, $V_j$ and $V_k$.
Normalization of the $V$ in terms of topology will violate the normalization condition we required in (104). To fulfill this condition we need to know the metric and the wave functions. Nevertheless topological considerations suffice to give certain relations among Yukawa couplings and to see if $\bar{g}_{ijk}$ vanishes for certain $i, j, k$.

[5] We need a symmetric expression to relate our expressions for Yukawa couplings to the four dimensional superpotential with $W = \lambda_{ijk} \Phi_i \Phi_j \Phi_k$ with the chiral superfield $\Phi$. $W$ is obviously symmetric in the indices $i, j, k$.

The Yukawa couplings among families can be computed in a similar procedure:

$$g_{ijk} = \int_K \omega \wedge A_i^a \wedge A_j^b \wedge A_k^c \wedge \varepsilon_{abc} \tag{122}$$

where $A_i^a = A_{\bar{m}i}^a \mathrm{d}z^{\bar{m}}$ is the $i^{\text{th}}$ zero mode associated with the **3** representation. (122) depends on the complex structure of $K$ via $\omega$ but not in the metric. The expression is symmetric in the indices $i, j, k$. The integral can be evaluated without knowing the precise form of the $A_i^a$ – we only need to know their cohomology class in $H^1(T)$. Again we can only get relations among Yukawa couplings from (122) since the proper normalization of the $A_i^a$ does not follow from topology.

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